ANNALES DE L'INSTITUT FOURIER

SEAN DINEEN

Holomorphic functions on locally convex topological vector spaces. II. Pseudo convex domains

Annales de l'institut Fourier, tome 23, nº 3 (1973), p. 155-185

<http://www.numdam.org/item?id=AIF_1973__23_3_155_0>

Ann. Inst. Fourier, Grenoble
23,3 (1973), 155-185

HOLOMORPHIC FUNCTIONS ON LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

II. Pseudo convex domains (*) (**)

by Sean DINEEN

In this article we investigate the problem of when pseudo-convex domains are domains of holomorphy. This problem was originally posed for infinite dimensional locally convex spaces by Bremmerman ([4], [5], [6], [7], [8]). The problem has been solved in the affirmative for tube domains [5], for open subsets of C^n (n = countable infinity) [18], for Riemann domains over C^n [27] and for open subsets of a Banach space E with a basis whose intersection with each finite dimensional subspace of E is Runge [16]. We use the method of Hirschowitz [18] to circumvent the problem of having no continuous norm and show that if $U \subset E$ is pseudo convex and p is a continuous semi norm on E such that $\{y, p(y) < \delta\} \subset U$ then $U + \{y, p(y) = 0\} = U$. In this way the characterisation problem on U can be transferred to a space on which there exists a continuous norm. By this method we are able to prove generalisations of the Cartan-Thullen-Oka-Norguet-Bremmerman theorem in a variety of cases which include the following :

a) U open in $\prod_{n=1}^{\infty} E_n$, E_n a Frechet space with a basis for each n and $U \cap F$ Runge for each finite dimensional subspace F of E.

b) U open in $\sum_{i=1}^{\infty} E_i$ where each E_i is a Frechet space with a basis and $U \cap F$ is Runge for each finite dimensional subspace F of E.

(¹) A number of the results contained in this paper were announced in C.R. Acad. Sc., Paris, t. 274, 544-547 (1972).

(²) Part I has appeared in Ann. Inst. Fourier, t. 23, 1 (1973).

c) U an open subset of a nuclear space and $U \cap F$ Runge for each finite dimensional subspace F of E.

It has been shown in [19] that the theorem does not hold in its full generality for all locally convex topological vector spaces. Although complete results on this problem are not yet available an examination of the results obtained here suggest that the following properties of the locally convex space E will have some bearing on the final solution :

1) Countability conditions on E (e.g. is E separable, Lindelof).

2) Geometric position of $E/p^{-1}(0)$ in E (p is a continuous semi-norm on E) (e.g. is $E/p^{-1}(0)$ complemented topologically in E ?)

3) Geometric properties of $(E/p^{-1}(0), p)$ (p is a continuous semi-norm on E) (e.g. does $(E/p^{-1}(0), p)$ have a basis ?).

Unless otherwise stated our notation is the same as [15]. For background information on pseudo-convexity and plurisubharmonic functions we refer to [18], [9], [24], [26], [31], [33].

I thank Phillip Boland and Phillip Noverraz for many helpful conversations and correspondance.

1. *The Cartan-Thullen-Oka-Bremmerman-Norguet theorem.*

We consider the following properties on an open subset of a locally convex space E.

(H1) U *is a domain of existence of a holomorphic function (i.e. there exists* $f \in \mathcal{H}(U)$ *such that there is no* U_1, U_2 *connected open in* E, $\phi \neq U_2 \subset U_1 \cap U, U_1 \not\subset U$, *and* $f_1 \in \mathcal{H}(U_1)$ *such that*

$$f|_{U_2} = f_1|_{U_2}).$$

(H2) *For each sequence,* $(\xi_n)_{n=1}^{\infty}$, *of elements of* U, $\xi_n \to \xi$, $\xi \in \delta U$ (δU *is the boundary of* U *in* E) *there exists* $f \in \mathcal{H}(U)$ *such that* $\sup_n |f(\xi_n)| = \infty$.

(H3) U *is a domain of holomorphy (i.e. there exists no* U_1, U_2 *open connected in* E *such that* $\phi \neq U_2 \subset U_1 \cap U, U_1 \not\subset U$ *and for each* $f \in \mathcal{H}(U)$ *there exists* $f_1 \in \mathcal{H}(U_1)$ *such that* $f_1|_{U_2} = f|_{U_2}$.

(H4) U *is holomorphically convex (i.e. the holomorphic hull*(¹).

(¹) $\hat{A}_{\mathcal{H}(U)}$ will denote the holomorphic hull of A with respect to $\mathcal{H}(U)$.

of each compact subset of U *is a precompact subset of* U(1).

(H5) U *is holomorphically convex and if* K *is a compact subset of* U *and* V *is a balanced open neighbourhood of* 0 *in* E *such that* $K + V \subset U$, *then* $\hat{K}_{\mathscr{H}(U)} + V \subset U$.

(H6) U *is a plurisubharmonically convex (i.e. the plurisubharmonic hull of each compact subset of* U *is a precompact subset of* U)

(H7) U *is pseudo convex (i.e.* $U \cap F$ *is pseudo convex for each finite dimensional subspace* F *of* E).

(H8) U *is polynomially convex (i.e. the polynomial hull of each compact subset* K *of* U, $\hat{K}_{\mathscr{P}(E)}$ *is a precompact subset of* U).

(H9) U *is finitely polynomially convex if* $U \cap F$ *is polynomially convex for each finite dimensional subspace* F *of* E).

It shall be necessary to consider locally convex spaces which may not be Hausdorf, however, if F is the closure of 0 in E and π denotes the quotient mapping of E onto $\frac{E}{F}$ then (H_i) $(i = 1, \ldots, 8)$ is true for an open subset U of E if and only if the same is true of $\pi(U)$(2). Thus we restrict ourselves to Hausdorff locally convex spaces in the proofs but use the fact that the results proved are valid for non-Hausdorff spaces.

If E is a locally convex space such that each open subset of E which satisfies H(7) (resp H(9)) also satisfies H(i) ($i = 1,2,3,4,5$) (resp satisfies H(i), $i = 1,2,3,4,5,6,8$) then we say E is a CTONB(i) (resp CTONBR(i)) space. The following proposition can be combined with CTONB(i) (resp CTONBR(i)) spaces to prove the usual Cartan-Thullen-Oka-Norguet-Bremmerman (. . . . . Runge) theorem.

PROPOSITION 1.1. – *For any* l.c.s. E *and any open subset* U *of* E *we have the following :*

---

(1) K is a precompact subset of U if it is a precompact subset of E and there exists p a continuous semi-norm on E such that $\inf\limits_{\substack{x \in K \\ y \in \complement U}} p(x - y) > 0$.

(2) This can easily be proved by using the techniques we develop in the remainder of this section.

$$(H1) \Rightarrow (H2) \Rightarrow (H3) \Rightarrow (H4) \Leftrightarrow (H5) \Rightarrow (H6) \Rightarrow (H7)$$

$$(H8) \Rightarrow (H4) \text{ and } (H9) \Rightarrow (H7).$$

Proof. – (H1) ⇒ (H2) can easily be proved by using Cauchy's inequalities (see [12] for Banach spaces).

$$(H5) \Rightarrow (H4) \Rightarrow (H6) \Rightarrow (H7)\,, (H8) \Rightarrow (H4) \text{ and } (H9) \Rightarrow (H7)$$

are either trivial or well known ([9], [31]).

Suppose (H3) is not true then there exists U_1, U_2 open connected in E such that (H3) fails to hold for the pair U_1, U_2.

Let $V_1 = \{ x \in U \cap U_1 \,, f(x) = f_1(x)$ for all $f \in \mathscr{H}(U)$,

$$f_1 \in \mathscr{H}(U_1) \text{ such that } f|_{U_2} = f_1|_{U_2}\}$$

By definition $U_2 \subset V_1$. Since all holomorphic functions are continuous V_1 is a closed subset of $U \cap U_1$. By using the Taylor series expansion and the fact that analytic continuation is unique we find that V_1 is an open subset of $U \cap U_1$. Let V be the connected component of V_1 which contains U_2. We now consider V as an open subset of U_1. V is not a component of U_1 since U_1 is connected and $U_1 \not\subset U$. Since E is a l.c.s. this implies that V is not sequentially closed in U_1. Let $(x_n)_{n=1}^{\infty}$ be a sequence in V,

$$x_n \to x \in U_1 \cap \complement U$$

Since $x \notin V \subset U \cap U_1$, $x \notin U$. Hence $x \in \delta U$. Now f_1 is continuous at x (since $x \in U_1$). Hence

$$\text{i.e.} \quad \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f_1(x_n) = f_1(x)$$

exists and is finite

$$\sup_n |f(x_n)| < \infty \text{ for all } f \in \mathscr{H}(U).$$

Thus (H2) ⇒ (H3).

(H3) ⇒ (H4). Suppose (H4) is not true. Let K be compact in U and $x_\alpha \in \hat{K}_{\mathscr{H}(U)}$, $x_\alpha \to \delta U$(1) as $\alpha \to \infty$ (this is possible since $\hat{K}$ is

---

(1) $x_\alpha \to \delta U$ if $(x_\alpha)_{\alpha \in A}$ is a precompact subset of E and $\inf\limits_{\substack{\alpha \in A \\ y \in U)}} p(x_\alpha - y) = 0$ for each continuous semi-norm on E.

contained in the convex hull of K and hence is always a precompact subset of E). Let p be a continuous semi-norm on E such that

$$d_p\,(\mathrm{K}, \mathcal{C}\mathrm{U}) = \inf_{\substack{x \in \mathrm{K} \\ y \in \mathcal{C}\mathrm{U}}} p(x-y) = \delta > 0$$

Let $y \in$ E, $p(y) < \delta$. Choose $\delta_1 > 1$ such that

$$\mathrm{K}_1 = \mathrm{K} + \{\lambda y | \; |\lambda| \leqslant \delta_1\} \subset \mathrm{U}$$

(hence K_1 is also a compact subset of U). Let W be a convex balanced neighbourhood of 0 such that

$$\|f\|_{\mathrm{K}_1 + \delta_1 \mathrm{W}} = \mathrm{M} < \infty$$

for a given preassigned $f \in \mathcal{H}(\mathrm{U})$. For any $\alpha, \lambda \in \mathbf{C}$, $|\lambda| \leqslant 1$ and $\omega \in \mathrm{W}$ we have

$$\left| \frac{\hat{d}^n f(x_\alpha)}{n!} (\lambda y + \omega) \right| \leqslant \left\| \frac{\hat{d}^n f(x)}{n!} (\lambda y + \omega) \right\|_{x \in \mathrm{K}}$$

$$= \sup_{x \in \mathrm{K}} \left| \frac{1}{2\pi} \int_{|\lambda_1| = \delta_1} \frac{f(x + \lambda_1(\lambda y + \omega))}{\lambda_1^{n+1}} \, d\lambda_1 \right|$$

Now $x + \lambda_1 \lambda y + \lambda_1 \omega \in \mathrm{B}\,\mathrm{K} + \{\alpha y, |\alpha| \leqslant \delta_1\} + \delta_1 \mathrm{W}$ and this implies

$$\left| \frac{\hat{d}^n f(x_\alpha)}{n!} (\lambda y + \omega) \right| \leqslant \sup_{x \in \mathrm{K} + \{\lambda y, |\alpha| \leqslant \delta_1\} + \mathrm{W}} |f(x)| \,.\, \delta_1^{-n} \leqslant \frac{\mathrm{M}}{\delta_1^n}$$

Hence $$\sum_{n=0}^{\infty} \left| \frac{\hat{d}^n f(x_\alpha)}{n!} (\lambda y + \omega) \right| \leqslant \mathrm{M} \cdot \sum_{n=0}^{\infty} \frac{1}{\delta_1^n} < \infty$$

Since α, λ and ω were arbitrary we have

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(x_\alpha)}{n!} \right\|_{\{\lambda y, |\lambda| \leqslant 1\} + \mathrm{W}} < \infty$$

which means that the Taylor series of f at x_α converges in a W-neighbourhood of y. Choose x_α such that $d_p(x_\alpha, \mathcal{C}\mathrm{U}) < \delta/2$ then if

$$\tilde{f}(x + x_{\alpha_1}) = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(x_{\alpha_1})}{n!} (x)$$

$\widetilde{f}$ is holomorphic in a p_δ-neighbourhood of x_{α_1}.

Let V be a convex neighbourhood of x_α contained in

$$\mathrm{U} \cap \{x\,,\, p(x - x_\alpha) < \delta\}.$$

Taking $\mathrm{V} = \mathrm{U}_2$ and $\mathrm{U}_1 = \{x\,,\, p(x - x_\alpha) < \delta\}$ we find that (H3) is not true. Hence (H3) ⇒ (H4), (H4) ⇒ (H5).

If (H5) were not true then there would exist K compact in U and $y \in \mathrm{E}$ such that $\mathrm{K}_1 = \mathrm{K} + \{\lambda y_1 \ |\lambda| \leqslant 1\} \subset \mathrm{U}$ and is compact there but $\hat{\mathrm{K}} + \{\lambda y_1 \ |\lambda| \leqslant 1\} \not\subset \mathrm{U}$. The method used to show (H3) ⇒ (H4) can now be used to show $\hat{\mathrm{K}}_1$ is not a precompact subset of U. This contradicts (H4) and hence (H4) ⇒ (H5).

Remark. – If E is quasi-complete then we can replace the condition "precompact set" by "compact set" in proposition 1.1.

The remainder of this section will be devoted to showing certain spaces are CTONB(i) or CTONBR(i) spaces. We first consider locally convex spaces for which we can prove the required result directly and then proceed to investigate certain kinds of projective limits of CTONB(i) and CTONBR(i) spaces.

LEMMA 1.1. – *Let* U *be a connected pseudo convex open subset of* E, E *an arbitrary* l.c.s. *Let p be a continuous semi-norm on* E *such that* $\{y\,,\, p(y) \leqslant \delta\} \subset \mathrm{U}$ *for some* $\delta > 0$ then

$$\mathrm{U} = \mathrm{U} + \{y\,,\, p(y) = 0\}$$

Proof. – Let $x \in \mathrm{U}$ and $y \in \mathrm{E}\,, p(y) = 0$, be arbitrary. Since U is connected there exists a finite dimensional subspace F of E such that

i) $x\,, y \in \mathrm{F}$

ii) 0 and x belong to the same connected component of $\mathrm{U} \cap \mathrm{F}$.

F is finite dimensional and hence every semi-norm on F is continuous. By definition $\mathrm{U} \cap \mathrm{F}$ is pseudo-convex since U is pseudo convex and hence the connected component of $\mathrm{U} \cap \mathrm{F}$ containing 0 say U_1, is again a pseudo convex subset of F.

Let $\widetilde{p}$ denote the restriction of p to F.

Hence $$U_1 \supset \{\omega \in F , p(\omega) \leqslant \delta\} \cdot$$

The method used in [18] can now be applied to show

$$U_1 + \{\omega \in F , \widetilde{p}(\omega) = 0\} = U_1$$

Hence $$x + \lambda y \in U_1 \subset U \quad \text{for all} \quad \lambda \in C.$$

Since x and y were arbitrarily chosen this implies

$$U + \{y , p(y) = 0\} = U.$$

Remark. – If π denotes the quotient mapping from E onto $E/p^{-1}(0)$ then the proceeding lemma is equivalent to showing

$$U = \pi^{-1}(\pi(U)).$$

LEMMA 1.2. – *Let π be a linear mapping from the vector space* E *onto* F. *Let* U *be an open*(¹)*subset of E such that* $U = \pi^{-1}(\pi(U))$ *then* $\pi(U)$ *is pseudo convex* (*resp.* $\pi(U) \cap F_1$ *is Runge*(²). *for each finite dimensional subspace* F_1 *of* F), *if* U *is pseudo convex* (*resp.* $U \cap G$ *is Runge for each finite dimensional subspace* G *of* E).

Proof. – It suffices to show that if F_1 is a finite dimensional subspace of F there exists a subspace G of E such that π ;

$$U \cap G \to \pi(U) \cap F_1$$

is a linear isomorphism.

Suppose F_1 is spanned by $\xi_1 , \dots , \xi_n$. Choose $\eta_1 , \dots , \eta_n \in E$ such that $\pi(\eta_i) = \xi_i$ for $i = 1 , \dots , n$. Hence π ; $G \to F_1$ is an isomorphism where G is the n-dimensional subspace of E spanned by $\eta_1 , \dots , \eta_n$ (linear mapping of one n dimensional subspace onto another is always an isomorphism).

(¹) For vector spaces we say U is open if and only if $U \cap F$ is open in F for each finite dimensional subspace F of E.

(²) U is Runge if the polynomials on E are dense in $\mathcal{H}(U)$ when $\mathcal{H}(U)$ is endowed with the compact open topology.

Now $U \cap G \subset U$ and $U \cap G \subset G$, hence

$$\pi(U \cap G) \subset \pi(U) \cap \pi(G) = \pi(U) \cap F_1 .$$

Conversly if $y \in \pi(U) \cap F_1$, $y = \sum_{i=1}^{n} \alpha_i \xi_i$. Let $x = \sum_{i=1}^{n} \alpha_i \eta_i$ then $x \in G$ and $\pi(x) = y$. Hence $x \in \pi^{-1}(\pi(U)) = U$ and thus $x \in U \cap G$. This means $\pi(U \cap G) = \pi(U) \cap F_1$ and we have completed the proof.

We now restrict ourselves to specific locally convex spaces.

DEFINITION. – *A basis*(¹), $(U_n)_{n=1}^{\infty}$, *in* L.C.S. E *is a strong basis if there exists a set of continuous semi-norms on* E, $(p_\alpha)_{\alpha \in A}$ *which defines the topology of* E *such that*

$$p_\alpha \left(\sum_{n=1}^{\infty} \alpha_n U_n \right) = \operatorname*{Sup}_{n} p_\alpha \left(\sum_{j=1}^{n} \alpha_j U_j \right)$$

for each $\alpha \in A$.

Any basis in a barrelled locally convex space is a strong basis. In particular any basis in a Frechet space is a strong basis. We let E^n (resp. E_n) denote the closed vector space spanned by $U_1, \ldots, U_n$ (resp $(U_j)_{j=1}^{\infty}$).

PROPOSITION 1.2. – *A metrizable locally convex space* E *which has a strong basis is a CTONBR(1) space.*

The proof is rather long and is divided into a number of lemmas.

Let U denote a finitely polynomially convex subset of E. Let $(U_n)_{n=1}^{\infty}$ be a strong basis in E and let $(p_n)_{n=1}^{\infty}$ denote the corresponding family of semi-norms. Without loss of generality we can suppose $(p_n)_{n=1}^{\infty}$ is an increasing family of semi-norms and by lemma 1.2 we can also assume that

$$U = \pi^{-1}(\pi(U))$$

where

$$\pi ; E \to E/p_1^{-1}(0) \Rightarrow \pi(E).$$

(¹) A basis is always taken to be a Schauder basis (i.e. the projection onto each coordinate is continuous).

if p is a continuous semi-norm on E we denote by $\widetilde{p}$ the corresponding semi-norm on $\pi(\mathrm{E})$. (Note that $\widetilde{p}_j$ is a continuous norm on $\pi(\mathrm{E})$ for each j and $(\widetilde{p}_j)_{j=1}^{\infty}$ defines the quotient topology on $\pi(\mathrm{E})$.

LEMMA 1.3. – *Let* $x = \alpha_1 + \alpha_2 \in \pi(\mathrm{E})$ *where* $\alpha_1 \in \pi(\mathrm{E}^n)$ *and*

$$\alpha_2 \in \pi(\mathrm{E}_n)$$

then $\widetilde{p}_j(\alpha_1) \leqslant \widetilde{p}_j(x)$ *for each positive integer* j.

Proof. – Choose $\beta_1 \in \mathrm{E}^n$, $\beta_2 \in \mathrm{E}_n$ such that $\pi(\beta_j) = \alpha_j$ for

$$j = 1,2.$$

Let $\gamma \in \mathrm{E}$ be chosen such that

$$p_1(\beta_1 - \gamma) = 0$$

Hence $\gamma - \beta_1 = \eta_1 + \eta_2$ where $\eta_1 \in \mathrm{E}^n$ and $\eta_2 \in \mathrm{E}_n$,

$$p_1(\eta_1) = 0 \quad \text{and} \quad p_1(\eta_2) = 0.$$

This means that $p_1(\gamma - \eta_2 - \beta_1) = 0$ and

$$p_j(\gamma - \eta_2) = p_j(\beta_1 + \eta_1) \leqslant p_j(\beta_1 + \eta_1 + \eta_2) = p_j(\gamma)$$

Hence $\widetilde{p}_j(\alpha_1) = \inf\limits_{\gamma \in \pi^{-1}(\alpha_1) \cap \mathrm{E}^n .} p_j(\gamma)$

Now suppose $\gamma \in \mathrm{E}$, $\gamma = \gamma_1 + \gamma_2$, $\gamma_1 \in \mathrm{E}^n$, $\gamma_2 \in \mathrm{E}_n$ and

$$p_1(\beta_1 + \beta_2 - \gamma_1 - \gamma_2) = 0.$$

We then have $p_1(\beta_1 - \gamma_1) = 0$ and $p_1(\beta_2 - \gamma_2) = 0$

Hence $\widetilde{p}_j(\alpha_1) \leqslant p_j(\gamma_1) \leqslant p_j(\gamma_1 + \gamma_2)$ which implies

$$\widetilde{p}_j(\alpha_1) \leqslant \inf_{\substack{\gamma = \gamma_1 + \gamma_2 \in \mathrm{E} \\ \gamma \in \pi^{-1}(\beta_1 + \beta_2)}} p_j(\gamma_1 + \gamma_2) = \widetilde{p}_j(\pi(\beta_1 + \beta_2)) = \widetilde{p}_j(x).$$

LEMMA 1.4. – *Let* P *be a continuous polynomial on* $\pi(\mathrm{E}^n)$ *then there exists a* p_1 continuous polynomial on E, Q, *such that*

$$\mathrm{Q}(x + y) = \mathrm{Q}(x) \text{ for all } x \in \mathrm{E}^n,\ y \in \mathrm{E}_n \tag{1}$$

$$\mathrm{P} \circ \pi |_{\mathrm{E}^n} = \mathrm{Q} |_{\mathrm{E}^n} \tag{2}$$

Proof. – Define $\widetilde{Q}$ on E^n by $\widetilde{Q} = P \circ \pi|_{E^n}$. Now extend $\widetilde{Q}$ to E to get Q by

$$Q\left(x = \sum_{i=1}^{\infty} \alpha_i U_i\right) = \widetilde{Q}\left(\sum_{i=1}^{\infty} \alpha_i U_i\right)$$

(1) and (2) are immediately satisfied. It remains to show Q is a p_1-continuous polynomial on E.

Let $(x_m)_{m=1}^{\infty} \in E$ be arbitrary and suppose $p_1(x_m) \to 0$ as $m \to \infty$ For each m, $x_m = y_m + z_m$, $y_m \in E^n$, $z_m \in E_n$. And $p_1(y_m) \to 0$ as $m \to \infty$.

By construction

$$Q(x_m) = Q(y_m) = (P \circ \pi)\ (y_m) = P(\pi(y_m))$$

Now $\widetilde{p}_1$ is a continuous norm on $\pi(E)$ and hence its restriction to $\pi(E^n)$ is also continuous. Hence P is $\widetilde{p}_1$ continuous on $\pi(E^n)$ and since $\widetilde{p}_1(\pi(x)) \leqslant p_1(x)$ for all $x \in E$ this implies $P(\pi(y_m)) \to 0$ as $m \to \infty$ whenever $p_1(y_m) \to 0$ as $m \to \infty$. Hence Q is p_1 continuous on E. This completes the proof.

Now suppose P is a continuous polynomial on $\pi(E^n)$. Let Q be the p_1-continuous polynomial on E associated with P in the previous lemma.

We define $\widetilde{P}$ on $\pi(E)$ in the following manner,

$$\widetilde{P}(x) = Q(\pi^{-1}(x)) \quad \text{for all} \quad x \in \pi(E).$$

$\widetilde{P}$ is well defined for if $y, \omega \in \pi^{-1}(x)$ then $\pi(y) = \pi(\omega) = x$ i.e. $\pi(y - \omega) = 0$.

Since Q is p_1-continuous on E we have $Q(z + (y - \omega)) = Q(z)$ for all $z \in E$.

Hence $Q(y) = Q(\omega)$.

Also if $y_n \in \pi(E)$, $\widetilde{p}_1(y_n) \to 0$ as $n \to \infty$ then there exists $z_n \in E$, $\pi(z_n) = y_n$ and $p_1(z_n) \to 0$ as $n \to \infty$. Hence $Q(z_n) \to Q(0)$ as $n \to \infty$, this implies that $\widetilde{P}(y_n) = Q(z_n) \to Q(0) = (0)$ as $n \to \infty$ and we have shown that $\widetilde{P}$ is $\widetilde{p}_1$ continuous on $\pi(E)$.

LEMMA 1.5. – *An open subset* U *of a separable metrizable* l.c.s. E *is the domain of existence of a holomorphic function if and* *only if there exists* $(V_n)_{n=1}^{\infty}$ *an increasing sequence of open subsets of* U *which covers* U *and is such that* $\hat{V}_n$ *is bounded away from the boundary of* U *for each n.*

Proof. – (see also [12]). Let $(p_n)_{n=1}^{\infty}$ be an increasing sequence of continuous semi-norms on E which defines the topology of E. Suppose U is the domain of existence of f. For each K compact in U choose j_K a positive integer and $\alpha_K > 0$ such that

$$\|f\|_{K+2\alpha_K \{x\,;\,p_{j_K}(x) \leqslant 1\}} < \infty.$$

Since E is separable we can choose $(K_n)_{n=1}^{\infty}$ a sequence of compact subsets of U such that $\bigcup_{n=1}^{\infty} (K_n + \alpha_{K_n} \{x\,,\,p_{j_{K_n}}(x) \leqslant 1\}) \supset U$. By using Cauchy's inequality and the fact that U is the domain of existence of f we find that

$$\inf_{\substack{x \in \hat{W}_N \\ y \in \complement U}} p_N(x-y) \geqslant \inf_{i=1,\ldots,N} \alpha_{K_i}$$

where $$W_N = \bigcup_{i=1}^{N} K_i + \{x\,,\,p_{j_{K_i}}(x) \leqslant \alpha_{K_i}\}$$

and $$N = \sup_{i=1,\ldots,N} j_{K_i}$$

The sequence $(W_n^0)_{n=1}^{\infty}$ has all the required properties. Conversely suppose $(V_n)_{n=1}^{\infty}$ was an increasing sequence of open subsets of U which covered U and was such that $\hat{V}_n$ was bounded away from the boundary of U for each n.

Let M be a countable dense subset of U and let $(\xi_n)_{n=1}^{\infty}$ be a sequence of elements in M containing each point of M infinitely often. For each ξ_n let $A_n = \{x \in E\,,\,d(x\,,\,\xi_n) < d(\xi_n\,,\,\complement U)$

where $$d(x\,,y) = \sum_{n=1}^{\infty} \frac{1}{n^2}\,\frac{p_n(x-y)}{1+p_n(x-y)}\,.$$

Let $C_2 = V_2$.

Choose $z_2 \in A_2 \cap \complement V_2$. Choose k_2 such that $\hat{V}_{k_2} \supset C_2 \cup \{z_2\}$ and let $C_3 = \hat{V}_{k_2}$ and so on by means of an obvious inductive process. For each n there exists $f_n \in \mathcal{H}(U)$ such that

$$\|f_n\|_{\hat{V}_{k_n}} < \frac{1}{2^n} \text{ and } |f_n(z_{n+1})| > 2^n + |\sum_{i=2}^{n-1} f_i(z_{n+1})| \cdot$$

The function $f = \sum_{n=2}^{\infty} f_n \in \mathcal{H}(U)$ and has U as its natural domain of existence (¹).

Lemma 1.6. – *Let* U *be an open subset of a metrizable locally convex space* E *and let* F *be a closed subspace of* E. *If* π_1 *denotes the quotient mapping of* E *onto* E/F *and* $U = \pi^{-1}(\pi_1(U))$ *then* U *is the domain of existence of a holomorphic function if the same is true of the open set* $\pi_1(U)$.

Proof. – Now E/F is metrizable and lemma 1.5. implies that if $\pi_1(U)$ is the domain of existence of a holomorphic function then there exists $f \in \mathcal{H}(\pi_1(U))$ such that $\|f\|_{V \cap U} = \infty$ for any open subset V which intersects $\delta(\pi(U))$.

Let $g = f \circ \pi$, then $g \in \mathcal{H}(U))$.

Now if $\xi \in \partial U$ and W is a neighbourhood of ξ in E then $\pi_1(W)$ is an open subset of $\frac{E}{F}$ which contains $\pi_1(\xi) \in \delta(\pi_1(U))$.

Hence $$\|g\|_{W \cap U} = \infty.$$

This implies that U is the domain of existence of g.

Proof of proposition 1.2. – Let U be a connected open pseudoconvex subset of E. By lemma 1.1. we can suppose $U = \pi^{-1}(\pi(U))$ where π is the quotient mapping from E onto $E/p_1^{-1}(0)$. By lemma 1.6. it suffices to show $\pi(U)$ is the domain of existence of a holomorphic function.

For each compact subset K of $\pi(U)$ contained in $\pi(E^n)$ for some n and each integer j such that

---

(¹) An examination of the construction we have used shows that if U_1 is any open subset of E such that

$$U_1 \cap \delta U \neq \phi \text{ then } \|f\|_{U \cap U_1} = \infty.$$

$$d_j(\mathrm{K}) = \inf_{\substack{x \in \mathrm{K} \\ y \in \mathcal{C}(\pi(\mathrm{U}))}} \widetilde{p}_j(x - y) > 0$$

we let $$\mathrm{K}_{(j)} = \mathrm{K} + \{y \, , \widetilde{p}_j(y) < \frac{1}{4} d_j(\mathrm{K})\}$$

The set of all such $\mathrm{K}_{(j)}$'s forms an opening covering of $\pi(\mathrm{U})$. Since E has a basis it is separable and hence we can subtract from the set of all $\mathrm{K}_{(j)}$'s a countable open covering of $\pi(\mathrm{U})$, say $(\mathrm{K}_{(j_n)})_{n=1}^{\infty}$.

By lemma 1.3 it suffices to show $\widehat{\bigcup_{n=1}^{m} \mathrm{K}_{j_n}}$ is bounded away from the boundary of $\pi(\mathrm{U})$ for each m. If this were not so there would exist $(\xi_l)_{l=1}^{\infty} \in \pi(\mathrm{U})$, $\xi_l \to \xi \in \delta(\pi(\mathrm{U}))$

and $$\sup |f(\xi_l)| \leqslant \|f\|_{\bigcup_{n=1}^{m} \mathrm{K}_{j_n}}$$

for all $$f \in \mathcal{H}(\pi(\mathrm{U})).$$

By the same procedure as used in [16] we can suppose $\xi = 0$ and $\xi_j \in \bigcup_{n=1}^{\infty} \mathrm{E}^n$ for each integer j.

Let

$$\theta = \sup_{i=1,\dots,m} . \, j_i$$

Choose M a positive integer such that

(a) $\mathrm{K}_i \subset \mathrm{E}^{\mathrm{M}}$ for $i = 1, \dots, m$.

(b) there exists $\xi_k \in \widehat{\left(\bigcup_{n=1}^{m} \mathrm{K}_{j_n}\right)} \cap \mathrm{E}^{\mathrm{M}}$

for which $\widetilde{p}_\theta(\xi_k) < \frac{1}{4} \inf_{i=1,\dots,m} . \, d_{j_i}(\mathrm{K}_i) = \alpha/4$.

Let P be a continuous polynomial on $\pi(\mathrm{E}^{\mathrm{M}})$ and let P denote the extension to $\pi(\mathrm{E})$ of P which we have previously discussed.

Hence $$|\mathrm{P}(\xi_k)| = |\widetilde{\mathrm{P}}(\xi_k)| \leqslant \sup_{n=1,\dots,m} . \, \|\widetilde{\mathrm{P}}\|_{\mathrm{K}_{j_n}}$$

Now if $\omega_0 \in \mathrm{K}_i$, $\omega_1 \in \pi(\mathrm{E}^{\mathrm{M}})$, $\omega_2 \in \pi(\mathrm{E}_{\mathrm{M}})$ and

$$\widetilde{P}_{j_i}(\omega_1 + \omega_2) < \frac{1}{4}\, d_{j_i}(K_i)$$

then $$\widetilde{P}(\omega_0 + \omega_1 + \omega_2) = P(\omega_0 + \omega_1)$$

and by lemma 1.3

$$\widetilde{P}_{j_i}(\omega_1) \leqslant \widetilde{P}_{j_i}(\omega_1 + \omega_2) \leqslant \frac{1}{4}\, d_{j_i}(K_i)$$

Hence

$$|P(\xi_k)| \leqslant \sup_{i=1,\dots,m} \|P\|_{K_{j_i} \cap \pi(E^M)} .$$

Now

$$d_\theta \left(\bigcup_{i=1}^{\infty} K_{j_i} \cap \pi(E^M)\right)(^1)$$

$$\geqslant \inf_{i=1,\dots,m}. \ d_\theta(K_{j_i} \cap \pi(E^M))$$

$$\geqslant \inf_{i=1,\dots,m}. \ d_{ji}(K_{j_i} \cap \pi(E^M)) = \frac{3}{4}\alpha .$$

Since $\widetilde{p}_\theta(\xi_k) < \alpha/4$ this contradicts the fact that $\pi(U) \cap \pi(E^M)$ is pseudo convex and hence holomorphically convex (we need the fact that $\widetilde{p}_\theta$ was a norm on $\pi(E)$ in order to insure that $K_{j_i} \cap \pi(E^M)$ was a compact subset of a finite dimensional space)

Remark. – If there existed a continuous norm on E or if we knew that $\pi(E)$ had a strong basis then the proof of proposition 1.2 could be considerably shortened and would in fact be more or less the same as that given for Banach spaces with a basis in [16].

An examination of the final part of the proof of proposition 1.2 shows that we have in fact proved the following result.

PROPOSITION 1.3. – *A metrizable space with a strong basis is a* CTONBR (8) *space.*

LEMMA 1.1. – *If* U *is an open subset of a Lindelof*(2) L.C.S. *with a strong basis* E *then* E *can be endowed with the structure of*

---

(1) $d_\theta(x) = d_\theta(x, 0) = \widetilde{p}_\theta(x)$

(2) A topological space X is Lindelof if every open cover of X contains a countable subcover.

a locally convex semi-metrizable space with a strong basis weaker than the original topology on E *such that* U *is open with respect to the new structure.*

Proof. – Let $(p_\alpha)_{\alpha \in A}$ be the set of continuous semi-norms on E associated with a strong basis. For each $x \in U$ choose $\alpha_x \in A$ such that $V_x = x + \{y , p_{\alpha_x}(y) < \delta_x\} \subset U$ for some $\delta_x > 0$. $\bigcup_{x \in U} V_x$ is an open covering of U.

Since E is Lindelof U is also Lindelof and hence we can choose $(x_n)_{n=1}^\infty$ a sequence of elements of U such that $\bigcup_{n=1}^\infty V_{x_n} = U$. Now $(E , (p_{x_n})_{n=1}^\infty)$ is a semi-metrizable l.c.s. with a strong basis and and U is open in $(E , (p_{x_n})_{n=1}^\infty)$. This completes the proof.

PROPOSITION 1.4. – *A Lindelof space with a strong basis is a* CTONBR(2) *and a* CTONBR(8) *space.*

Proof. – Let U denote a pseudo convex open subset of E such that $U \cap F$ is Runge for each finite dimensional subspace F of E. Let m denote the topology of E and let $i(m)$ denote the semi-metrizable topology on E as constructed in the previous lemma. It is immediate by proposition 1.3. that U is polynomially convex and hence E is a CTONBR(8) space.

By proposition 1.2 U is the domain of existence of a holomorphic function when U is endowed with the $i(m)$ topology. Now if $x_n \in U , x_n \overset{m}{\to} x \in \delta U$ then $x_n \overset{i(m)}{\to} x$ and hence there exists $f \in \mathcal{H}(U)$ such that $\sup |f(x_n)| = \infty$. Hence E is a CTONBR(2) space.

Examples of spaces which satisfy the condition of proposition 1.4.

1) E Lindelof, barrelled and possessing a basis.

2) $E = \sum_{i=1}^\infty E_i$, where E_i is a Frechet space with a basis for each i.

We now consider CTONBR(4) spaces.

PROPOSITION 1.5. – *If* E *is a* l.c.s. *such that for each compact subset* K *of* E *there exists a closed complemented subspace of* E *which contains* K *and is a* CTONBR(4) *space then* E *is a* CTONBR(4) *space.*

Proof. – Let U be an open finitely polynomially convex subset of E. For K compact in U choose E(K) a closed complemented subspace of E which is a CTONBR(4) space and which contains K. Hence K is a compact subset of U ∩ E(K) and thus

$$\hat{K}_{\mathfrak{P}(E(K))} = \hat{K}_{\mathfrak{P}(E)}$$

is a precompact subset of U. This completes the proof.

Examples of spaces which satisfy the criterion of proposition 1.5 :

1) E an arbitrary L^p space, $(1 \leqslant p < \infty)$ (note, E need not be complete nor separable)

2) $E = \sum_{i \in A}^{\infty} E_i$, where A is an arbitrary indexing set and E_i is a Frechet space with a basis for each i.

We now consider projective limits of various kinds.

Let $(E_i)_{i \in A}$ be a set of l.c.s. spaces.

Let E be a vector space and let π_i be a linear mapping from E onto E_i for each $i \in A$. *We say* E *is the projective limit of* $(E_i)_{i \in A}$ *by means of the mappings* $(\pi_i)_{i \in A}$ *if* E *has the weakest locally convex topology for which all the functions* π_i *are continuous. We write* $E = \underset{\substack{\longleftarrow \\ i \in A}}{\lim} (E_i, \pi_i)$. *The projective limit is said to be directed if* A *is directed and for each* $i, j \in A$ *there exists* $k \in A, k \geqslant i, k \geqslant j$ *and continuous linear mappings* π_i^k , π_j^k *such that the following diagram is well defined and commutative*

$$\begin{tikzcd}
& & E_i \\
E \arrow[r, "\pi_k"] \arrow[urr, bend left] \arrow[drr, bend right, "\pi_j"'] & E_k \arrow[ur, "\pi_i^k"] \arrow[dr, "\pi_j^k"] & \\
& & E_j
\end{tikzcd}$$