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FOLIATIONS AND SPINNABLE STRUCTURES ON MANIFOLDS

by Itiro TAMURA

Introduction.

In a previous paper, it was shown by using the methods of differential topology that every odd dimensional sphere has a codimension-one foliation (Tamura [12]). This result and the techniques used there, are powerful tools in carrying out the construction of codimension-one foliations for compact odd dimensional differentiable manifolds.

In this paper the concept of a spinnable structure on a differentiable manifold is introduced in order to clarify the implication of the differential topological arguments mentioned above (see Definition 1). Roughly speaking, a differentiable manifold is spinnable if it can spin around an axis as if the top spins. The choice of axes of spinnable structures is essential for constructing a codimension-one foliation. For example, the existence of a codimension-one foliation follows from the existence of a spinnable structure having an odd dimensional sphere as axis (see Theorem 5). Thus the surgery of axes plays an important role in our study. The middle dimensional surgery is accomplished by means of spinnable structures of special kinds on odd dimensional spheres (Theorems 1, 2). Making use of the surgery, it is shown that every $(m-1)$-connected closed $(2m+1)$-dimensional differentiable manifold admits a spinnable structure having S^{2m-1} as axis for $m \geqslant 3$ and, thus, has a codimension-one foliation (see Theorems 8, 9).

The main results of this paper were announced with sketches of proofs in a short note (Tamura [13]).

1. Definition of spinnable structure.

We shall restrict our attention to the differentiable case, although definitions and theorems of this paper are valid in the piecewise linear category.

DEFINITION 1. — *An m-dimensional differentiable manifold M^m is called spinnable if there exists an $(m-2)$-dimensional submanifold X satisfying the following conditions:*

(i) *The normal bundle of X is trivial.*

(ii) *Let $X \times D^2$ be a tubular neighborhood of X, then $C = M^m - X \times \operatorname{Int} D^2$ is the total space of a differentiable fibre bundle over a circle, say ξ.*

(iii) *Let $p : C \to S^1$ be the projection of ξ, then the following diagram commutes:*

$$\begin{tikzcd} X \times S^1 \arrow[r, "\iota"] \arrow[dr, "p'"'] & C \arrow[d, "p"] \\ & S^1, \end{tikzcd}$$

where ι denotes the inclusion map and p' denotes the natural projection onto the second factor.

The submanifold X is called an *axis* and a fibre F of ξ is called a *generator*. F is an $(m-1)$-dimensional submanifold of M^m. Obviously $\partial F = X$ holds if $\partial M^m = \emptyset$. The fibre bundle $\xi = \{C, p, S^1, F\}$ is called a *spinning bundle*, and the pair (X, ξ) is called a *spinnable structure* on M^m. Notice that M^m is obtained from C by identifying (x, θ) with (x, θ') for each $(x, \theta), (x, \theta') \in X \times S^1$.

Example 1. — The m-sphere S^m is spinnable with (naturally imbedded) S^{m-2} as axis and D_+^{m-1} as generator, where D_+^{m-1} denotes the upper hemi-sphere of S^{m-1}.

The concept of spinnable structure originated with J. W. Alexander [2] in 1923, and the terms of « axis », « generator » were used by him. Recently H. E. Winkelnkemper [14] established the existence theorem of spinnable structures in which termes of « binding », « page » are used. See also A. H. Durfee and H. B. Lawson [4].

Example 2. — Let $S^{2n+1} = (S^n \times D^{n+1}) \cup (D^{n+1} \times S^n)$ be the natural decomposition. Let Δ denote the diagonal of $S^n \times S^n$ and let N be a tubular neighborhood of Δ in $S^n \times S^n$. As is well known, N is the total space of D^n-bundle over S^n associated with the tangent bundle of S^n and ∂N is the Stiefel manifold $V_{n+1,2}$. Since Δ is isotopic to $S^n \times 0$ (resp. $0 \times S^n$), it follows that $S^n \times D^{n+1} = N \times I$, $D^{n+1} \times S^n = N \times I$. Thus the following decomposition holds :

$$S^{2n+1} = (V_{n+1,2} \times D^2) \cup (N \times I) \cup (N \times I).$$

This implies that S^{2n+1} admits a spinnable structure with $V_{n+1,2}$ as axis and N as generator.

2. A spinnable structure on S^{4n+1}.

In sections 2 and 3, we shall construct spinnable structures on S^{4n+1} $(n \geqslant 2)$ and on S^{4n-1} $(n \geqslant 2)$ respectively, which will be used to perform a surgery on axes in section 6. In the following homology group $H_q(\ )$ means always the integral homology group.

Let $n \geqslant 2$ and let $*$ be a point of S^{2n}. Let a and b denote $2n$-dimensional submanifolds $S^{2n} \times \{*\}$ and $\{*\} \times S^{2n}$ of $S^{2n} \times S^{2n}$ respectively, and let d (resp. d') denote the diagonal $\{(x, x) ; x \in S^{2n}\}$ (resp. $\{(x, -x) ; x \in S^{2n}\}$) of $S^{2n} \times S^{2n}$. Let us choose orientations of $S^{2n} \times S^{2n}$ and of submanifolds a, b, d and d' so that

$$[a],\quad [b],\quad [d],\quad [d'] \in H_{2n}(S^{2n} \times S^{2n}),$$
$$[d] = [a] + [b],\quad [d'] = [a] - [b],\quad [a] \circ [b] = 1,$$

where $[a]$, $[b]$ etc. denote homology classes represented by a, b etc., and $[a] \circ [b]$ denotes the intersection number of $[a]$ and $[b]$. Then it follows that

$$[d] \circ [d] = 2, \qquad [d'] \circ [d'] = -2.$$

Let $S_i^{2n} \times D_i^{2n+1}$ $(i = 1, 2, \ldots, 17)$ be 17 copies of $S^{2n} \times D^{2n+1}$ and let

$$V = (S_1^{2n} \times D_1^{2n+1}) \natural (S_2^{2n} \times D_2^{2n+1}) \natural \cdots \natural (S_{17}^{2n} \times D_{17}^{2n+1})$$

be the boundary connected sum of $S_i^{2n} \times D_i^{2n+1}$ $(i = 1, 2, \ldots,$

17). Then S^{4n+1} has the natural decomposition:

$$S^{4n+1} = V_{+} \cup V_{-},$$

where V_{+} and V_{-} are copies of V.

Let a_i, b_i, d_i and d'_i denote oriented submanifolds of $S^{2n}_i \times S^{2n}_i$ corresponding to a, b, d and d' of $S^{2n} \times S^{2n}$ ($i = 1, 2, \ldots, 17$). Further let b'_4 (resp. b'_{12}) denote $\{*'\} \times S^{2n}_4$ (resp. $\{*''\} \times S^{2n}_{12}$), where $*'$ (resp. $*''$) is a point of S^{2n}_4 (resp. S^{2n}_{12}) different from $*$. We may suppose that

$$a_i, \quad b_i, \quad d_i, \quad d'_i, \quad b'_4, \quad b'_{12} \subset \partial V_{+}.$$

Now let us define subsets K_1, K_2 of ∂V_{+} by

$$\begin{aligned} K_1 = {} & (d'_1 \# b'_4) \cup d'_2 \cup (d'_3 \# b_2) \cup (d'_4 \# b_3) \cup (d'_5 \# b_4) \\ & \cup (d'_6 \# b_5) \cup (d'_7 \# b_6) \cup (d'_8 \# b_7), \\ K_2 = {} & (d_9 \# b'_{12}) \cup d_{10} \cup (d_{11} \# b_{10}) \cup (d_{12} \# b_{11}) \cup (d_{13} \# b_{12}) \\ & \cup (d_{14} \# b_{13}) \cup (d_{15} \# b_{14}) \cup (d_{16} \# b_{15}) \cup (d_{17} \# b_{16}), \end{aligned}$$

where $\#$ denotes the connected sum in ∂V_{+} formed by using a tube which is disjoint from other submanifolds (Fig. 1).

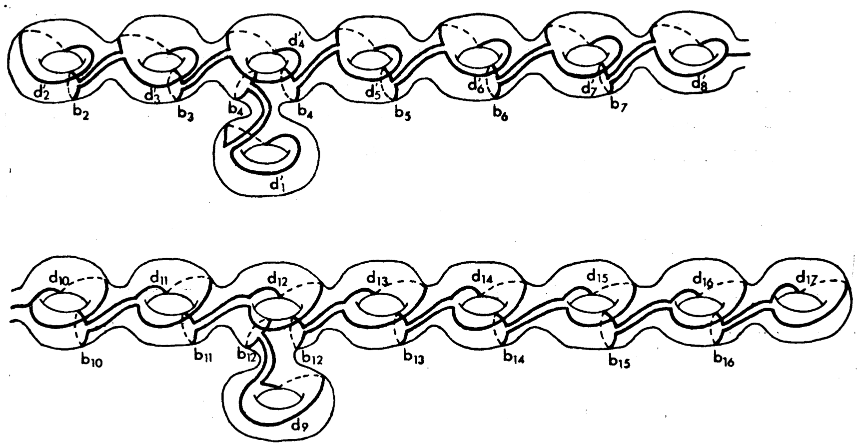

Fig. 1.

Let G_1 and G_2 be smooth regular neighborhoods of K_1 and K_2 in ∂V_{+} respectively and let $G = G_1 \natural G_2$ be the boundary connected sum of G_1 and G_2 in ∂V_{+}. Then G_1

and G_2 have the homotopy type of bouquets of $2n$-spheres. It is easy to see that the homomorphisms

$$H_{2n}(G) \to H_{2n}(V_+), \qquad H_{2n}(G) \to H_{2n}(V_-),$$

which are induced by the inclusion maps $G \to V_+$, $G \to V_-$, are bijective. Thus the inclusion maps

$$G \to V_+, \qquad G \to V_-$$

are homotopy equivalences. Since $G' = \partial V_+ - \operatorname{Int} G$ is simply connected, it follows by the Poincaré-Lefschetz duality theorem that the inclusion maps

$$G' \to V_+, \qquad G' \to V_-$$

are also homotopy equivalences. Therefore, according to the relative h-cobordism theorem (Smale [11]), we have

$$G = G', \qquad V_+ = G \times I, \qquad V_- = G \times I.$$

This observation implies that S^{4n+1} admits a spinnable structure with G as generator and ∂G as axis.

Now let us study the generator G and the axis ∂G. It is obvious that $H_{2n}(G)$ is a free abelian group of rank 17 whose generators are

$$[d_1' \# b_4'], \qquad [d_2'], \qquad [d_3' \# b_2], \qquad \ldots, \qquad [d_{17} \# b_{16}],$$

and that the matrix of intersection numbers of these generators is given by

$$\begin{pmatrix} -E_8 & \\ & E_9 \end{pmatrix},$$

where

$$E_8 = \begin{bmatrix} 2 & & & 1 & & & & \\ & 2 & 1 & & & 0 & & \\ & 1 & 2 & 1 & & & & \\ 1 & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 & \\ & 0 & & & & 1 & 2 & 1 \\ & & & & & & 1 & 2 \end{bmatrix}, \qquad E_9 = \begin{bmatrix} 2 & & & 1 & & & & & \\ & 2 & 1 & & & & & 0 & \\ & 1 & 2 & 1 & & & & & \\ 1 & & 1 & 2 & 1 & & & & \\ & & & 1 & 2 & 1 & & & \\ & & & & 1 & 2 & 1 & & \\ & & & & & 1 & 2 & 1 & \\ & 0 & & & & & 1 & 2 & 1 \\ & & & & & & & 1 & 2 \end{bmatrix}.$$

The matrix E_8 is the well known positive definite unimodular matrix, and the rank of E_9 is 8 and its elementary divisor is (1, 1, 1, 1, 1, 1, 1, 1).

Consider the homology exact sequence of $(G, \partial G)$:

$$\cdots \to H_q(\partial G) \to H_q(G) \to H_q(G, \partial G) \to H_{q-1}(\partial G) \to \cdots.$$

It follows by the Poincaré-Lefschetz duality theorem that

$$H_q(G, \partial G) \cong H^{4n-q}(G),$$

and that the homomorphism

$$H_{2n}(G) \to H_{2n}(G, \partial G) \cong \mathrm{Hom}\,(H_{2n}(G), \mathbf{Z})$$

in the above homology exact sequence is determined by the matrix

$$\begin{pmatrix} -E_8 & \\ & E_9 \end{pmatrix}.$$

Thus the following holds:

$$H_q(\partial G) = \begin{cases} \mathbf{Z} & q = 0, \quad 2n-1, \quad 2n, \quad 4n-1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, G_2 is diffeomorphic to a handlebody formed from $-G_1$ by attaching a $2n$-handle representing $d_{17} \# b_{16}$:

$$G_2 = (-G_1) \bigcup_g (D^{2n} \times D^{2n}),$$

where $g\colon \partial D^{2n} \times D^{2n} \to \partial(-G_1)$ is an attaching map. Since ∂G_1 is the Milnor sphere (Milnor-Kervaire [8]), it is easy to see that

$$G_2 = (-G_1) \natural \left(D^{4n} \bigcup_g (D^{2n} \times D^{2n})\right).$$

According to the h-cobordism theorem, $B = D^{4n} \bigcup_g (D^{2n} \times D^{2n})$ is the total space of a $2n$-disk bundle over S^{2n}.

We have

$$\begin{aligned}\partial G &= \partial(G_1 \natural G_2) = \partial(G_1 \natural (-G_1) \natural B) = \partial G_1 \# (-\partial G_1) \# \partial B \\ &= \partial B.\end{aligned}$$

Since B is parallelizable and $H_{2n-1}(\partial B) \cong Z$, it follows by the standard arguments in differential topology (see Tamura [12]) that

$$B = S^{2n} \times D^{2n}, \qquad \partial B = S^{2n} \times S^{2n-1}.$$

Hence, letting $A = G_1 \natural (-G_1)$, we have

$$G = (S^{2n} \times D^{2n}) \natural A, \qquad \partial G = S^{2n} \times S^{2n-1}.$$

Thus the following theorem holds.

THEOREM 1. — S^{4n+1} $(n \geqslant 2)$ *admits a spinnable structure with* $(S^{2n} \times D^{2n}) \natural A$ *as generator and* $S^{2n} \times S^{2n-1}$ *as axis.*

This theorem is a key to the construction of codimension-one foliations of odd dimensional spheres. The above proof is a slightly different version of the proof given in a previous paper (Tamura [12]). An approach to this theorem by means of the Milnor fibering is considered by A. H. Durfee [3].

3. A spinnable structure on S^{4n-1}.

Let $n \geqslant 2$ and let $*$ be a point of S^{2n-1}. Denote $S^{2n-1} \times \{*\}$, $\{*\} \times S^{2n-1}$ and the diagonal $\{(x, x)\,;\, x \in S^{2n-1}\}$ of $S^{2n-1} \times S^{2n-1}$ by $\bar{a}$, $\bar{b}$ and $\bar{d}$ respectively. Let us choose orientations of $S^{2n-1} \times S^{2n-1}$ and of submanifolds $\bar{a}$, $\bar{b}$ and $\bar{d}$ so that

$$[\bar{a}], \quad [\bar{b}], \quad [\bar{d}] \in H_{2n-1}(S^{2n-1} \times S^{2n-1}),$$
$$[\bar{d}] = [\bar{a}] + [\bar{b}], \qquad [\bar{a}] \circ [\bar{b}] = 1.$$

Let $S_i^{2n-1} \times D_i^{2n}$ $(i = 1, 2, 3, 4, 5)$ be 5 copies of $S^{2n-1} \times D^{2n}$ and let

$$\bar{V} = (S_1^{2n-1} \times D_1^{2n}) \natural (S_2^{2n-1} \times D_2^{2n}) \natural \cdots \natural (S_5^{2n-1} \times D_5^{2n})$$

be their boundary connected sum. Then S^{4n-1} has the natural decomposition :

$$S^{4n-1} = \bar{V}_+ \cup \bar{V}_-,$$

where $\bar{V}_+$ and $\bar{V}_-$ are copies of $\bar{V}$.

Let $\bar{a}_i$, $\bar{b}_i$ and $\bar{d}_i$ denote oriented submanifolds of $S_i^{2n-1} \times S_i^{2n-1}$ corresponding to $\bar{a}$, $\bar{b}$ and $\bar{d}$ of $S^{2n-1} \times S^{2n-1}$ respectively $(i = 1, 2, 3, 4, 5)$. We may suppose that

$$\bar{a}_i, \qquad \bar{b}_i, \qquad \bar{d}_i \subset \partial \bar{V}_+.$$

Let us define subsets $\overline{K}_1$, $\overline{K}_2$ of $\partial\overline{V}_+$ by

$$\overline{K}_1 = \bar{d}_1 \cup (\bar{d}_2 \# \bar{b}_1),$$
$$\overline{K}_2 = \bar{d}_3 \cup (\bar{d}_4 \# \bar{b}_3) \cup (\bar{d}_5 \# \bar{b}_4).$$

Let $\overline{G}_1$ and $\overline{G}_2$ be smooth regular neighborhoods of $\overline{K}_1$ and $\overline{K}_2$ in $\partial\overline{V}_+$ respectively and let $\overline{G} = \overline{G}_1 \natural \overline{G}_2$. Then it is easy to see that the inclusion maps

$$\overline{G} \to \overline{V}_+, \qquad \overline{G} \to \overline{V}_-$$

are homotopy equivalences. Thus, according to the relative h-cobordism theorem, we have

$$\overline{V}_+ = \overline{G} \times I, \qquad \overline{V}_- = \overline{G} \times I.$$

This observation implies that S^{4n-1} admits a spinnable structure with $\overline{G}$ as generator and $\partial\overline{G}$ as axis.

Now let us study the generator $\overline{G}$ and the axis $\partial\overline{G}$. It is obvious that $H_{2n-1}(\overline{G}_2)$ is a free abelian group of rank 3 generated by $[\bar{d}_3]$, $[\bar{d}_4 \# \bar{b}_3]$, $[\bar{d}_5 \# \bar{b}_4]$ and that the matrix of intersection numbers of these generators is given by

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let us choose new generators $[\bar{d}_3]$, $[\bar{d}_4 \# \bar{b}_3]$, $[{}_3\bar{d} \# \bar{d}_5 \# \bar{b}_4]$ for $H_{2n-1}(\overline{G}_2)$. Then, since

$$[\bar{d}_3 \# \bar{d}_5 \# \bar{b}_4] \circ [\bar{d}_3] = 0, \quad [\bar{d}_3 \# \bar{d}_5 \# \bar{b}_4] \circ [\bar{d}_4 \# \bar{b}_3] = 0,$$

we can cancel geometrical intersections $(\bar{d}_3 \# \bar{d}_5 \# \bar{b}_4) \cap \bar{d}_3$, $(\bar{d}_3 \# \bar{d}_5 \# \bar{b}_4) \cap (\bar{d}_4 \# \bar{b}_3)$ by an isotopic deplacement of $\bar{d}_3 \# \bar{d}_5 \# \bar{b}_4$. Thus it follows by the h-cobordism theorem that

$$\overline{G}_2 = \overline{G}_1 \natural \overline{B},$$

where $\overline{B}$ denotes a tubular neighborhood of $\bar{d}_3 \# \bar{d}_5 \# \bar{b}_4$.

The normal bundle of $\bar{d}_3 \# \bar{d}_5 \# \bar{b}_4$ is stably trivial and its characteristic map, an element of $\pi_{2n-2}(SO(2n-1))$, is the sum of characteristic maps of normal bundles of $\bar{d}_3$ and $\bar{d}_5$.

As is well known, the kernel of

$$\pi_{2n-2}(\mathrm{SO}(2n-1)) \to \pi_{2n-2}(\mathrm{SO})$$

is $\mathbf{Z}_2$ or 0. This implies that

$$\overline{\mathrm{B}} = \mathrm{S}^{2n-1} \times \mathrm{D}^{2n-1}.$$

Let $\overline{\mathrm{A}} = \overline{\mathrm{G}}_1 \natural \overline{\mathrm{G}}_1$. Then, since $\partial\overline{\mathrm{G}}_1$ is the Kervaire sphere or the natural sphere (Milnor-Kervaire [8]), we have $\partial\overline{\mathrm{A}} = \mathrm{S}^{4n-3}$.

Thus the following theorem holds.

THEOREM 2. — *S^{4n-1} $(n \geqslant 2)$ admits a spinnable structure with $(\mathrm{S}^{2n-1} \times \mathrm{D}^{2n-1}) \natural \overline{\mathrm{A}}$ as generator and $\mathrm{S}^{2n-1} \times \mathrm{S}^{2n-2}$ as axis.*

4. Foliations of odd dimensional spheres.

The following lemma is well known (Lawson [7], Lemma 1).

LEMMA 1. — *Let E be a differentiable manifold with or without boundary which is the total space of a differentiable fibering over S^1. Then there exists a codimension-one foliation of E such that $\partial\mathrm{E}$ is a sum of leaves in case where $\partial\mathrm{E} \neq \emptyset$.*

The existence of a spinnable structure is closely related to the existence of a codimension-one foliation. The following lemma is a direct consequence of lemma 1.

LEMMA 2. — *Let M be a spinnable differentiable manifold with axis X. Suppose that $\mathrm{X} \times \mathrm{D}^2$ has a codimension-one foliation such that $\mathrm{X} \times \mathrm{S}^1$ is a sum of leaves. Then M has a codimension-one foliation.*

In the following we shall prove the existence of codimension-one foliations for odd dimensional spheres, making use of Theorem 1 and Example 2 (or Theorem 2). Starting point is the existence of codimension-one foliation of S^5 which was proved by H. B. Lawson [7] using the Milnor fibering of $z_0^3 + z_1^3 + z_2^3 = 0$. The following simple proof is due to Tadayoshi Mizutani [9].

THEOREM 3 (Lawson). — *S^5 has a codimension-one foliation.*

Proof. — Consider the standard fibering $p : S^5 \to CP^2$. Let T^2 be a torus imbedded in $CP^2 - CP^1$ and let $T^2 \# CP^1 = T^2 \# S^2$ be the connected sum in CP^2. Let N be a tubular neighborhood of $T^2 \# CP^1$ in CP^2. Then it is obvious that the restriction of the fibering $p : S^5 \to CP^2$ on $CP^2 - \mathrm{Int}$ N is trivial. Thus we have

$$S^5 = p^{-1}(N) \cup p^{-1}(CP^2 - \mathrm{Int}\, N)$$
$$= p^{-1}(N) \cup (CP^2 - \mathrm{Int}\, N) \times S^1.$$

Since $T^2 = S^1 \times S^1$, $p^{-1}(N)$ is the total space of a fibering over S^1. Thus, by Lemma 1, both $p^{-1}(N)$ and $(CP^2 - \mathrm{Int}\, N) \times S^1$ have the codimension-one foliation with $p^{-1}(\partial N)$ as a leaf. This completes the proof.

THEOREM 4. — *Every odd dimensional sphere has a codimension-one foliation.*

Proof. — S^3 has the well known Reeb foliation (Reeb [10]). Now, beginning with the foliation given in Theorem 3, we proceed inductively.

Suppose that, for $2 \leqslant r < m$, S^{2r+1} has a codimension-one foliation. As is well known, there exists a smooth closed curve in S^{2r+1} which is transverse to the leaves (Haefliger [6]). Thus, by modifying the foliation of the complement of an open tubular neighborhood of the curve, we have a codimension-one foliation of $S^{2r-1} \times D^2$ having the boundary $S^{2r-1} \times S^1$ as a leaf (cf. Lemma 1).

In case where m is even, say $m = 2n$, S^{4n+1} admits a spinnable structure with $S^{2n} \times S^{2n-1}$ as axis by Theorem 1. Further $S^{2n} \times S^{2n-1} \times D^2$ admits a codimension-one foliation having its boundary as a leaf which is induced by the projection $S^{2n} \times S^{2n-1} \times D^2 \to S^{2n-1} \times D^2$ from the foliation of $S^{2n-1} \times D^2$. Thus S^{4n+1} admits a codimension-one foliation by Lemma 2.

In case where m is odd, say $m = 2n - 1$, S^{4n-1} admits a spinnable structure with $V_{2n,2}$ (or $S^{2n-1} \times S^{2n-2}$) as axis by Example 2 (or Theorem 2). Further $V_{2n,2} \times D^2$ (or $S^{2n-1} \times S^{2n-2} \times D^2$) admits a codimension-one foliation having its boundary as a leaf which is induced by the projection $V_{2n,2} \times D^2 \to S^{2n-1} \times D^2$ (or $S^{2n-1} \times S^{2n-2} \times D^2 \to S^{2n-1} \times D^2$)

from the foliation of $S^{2n-1} \times D^2$. Thus S^{4n-1} admits a codimension-one foliation by Lemma 2. This completes the proof.

The following corollary is an immediate consequence of Theorem 4.

COROLLARY. — $S^{2m+1} \times D^2$ *admits a codimension-one foliation having its boundary as a leaf.*

Now let us consider a spinnable structure of a special kind.

DEFINITION 2. — *An m-dimensional differentiable manifold is called specially spinnable if it has* S^{m-2} *as axis.*

By Lemma 2 and the above corollary, we obtain the following theorem.

THEOREM 5. — *Every specially spinnable odd dimensional differentiable manifold has a codimension-one foliation.*

Let M^m be a specially spinnable m-dimensional differentiable manifold. Then $M^m = (S^{m-2} \times D^2) \cup C$. Thus m-dimensional differentiable manifold $(S^1 \times D^{m-1}) \cup C$ which is obtained by performing a surgery on S^{m-2} is the total space of a fibering over S^1 having $F \cup D^{m-1}$ as fibre. This yields the following.

THEOREM 6. — *Every specially spinnable differentiable manifold is obtained from a fibering over* S^1 *by performing a surgery on a cross-section.*

5. Existence of spinnable structures.

The following existence theorem of a spinnable structure on a closed odd dimensional differentiable manifold is due to H. E. Winkelnkemper [14]. In a previous note (Tamura [13]), this theorem was proved under an additional hypothesis that $H_m(M)$ is torsion free.

THEOREM 7. — *Let* M *be a simply connected closed* $(2m+1)$-*dimensional differentiable manifold* $(m \geqslant 3)$. *Then* M *admits a spinnable structure such that the generator has the homotopy* *type of an m-complex and the inclusion map of the generator into* M *is a homotopy m-equivalence.*

Proof. — Let $f: \mathrm{M} \to \mathbf{R}$ be a nice function on M and let

$$\mathrm{W} = f^{-1}([0, m + (1/2)]), \qquad \mathrm{W}' = f^{-1}([m + (1/2), 2m + 1]).$$

Then we have

$$\mathrm{M} = \mathrm{W} \cup \mathrm{W}', \qquad \partial\mathrm{W} = \partial\mathrm{W}' = \mathrm{W} \cap \mathrm{W}' = f^{-1}(m + (1/2)).$$

It is obvious that W and W′ have the homotopy type of simply connected m-dimensional finite complexes and that the homomorphisms

$$\iota_* : \mathrm{H}_q(\mathrm{W}) \to \mathrm{H}_q(\mathrm{M}), \qquad \iota'_* : \mathrm{H}_q(\mathrm{W}') \to \mathrm{H}_q(\mathrm{M})$$

induced by the inclusion maps $\iota : \mathrm{W} \to \mathrm{M}$, $\iota' : \mathrm{W}' \to \mathrm{M}$, are bijective for $q = 0, 1, \ldots, m - 1$ and surjective for $q = m$. Further, by the homology exact sequence of $(\mathrm{W}, \partial\mathrm{W})$, $(\mathrm{W}', \partial\mathrm{W})$ and the Poincaré-Lefschetz duality theorem, it follows that the homomorphisms

$$\bar{\iota}_* : \mathrm{H}_q(\partial\mathrm{W}) \to \mathrm{H}_q(\mathrm{W}), \qquad \bar{\iota}'_* : \mathrm{H}_q(\partial\mathrm{W}) \to \mathrm{H}_q(\mathrm{W}')$$

induced by the inclusion maps $\bar{\iota} : \partial\mathrm{W} \to \mathrm{W}$, $\bar{\iota}' : \partial\mathrm{W} \to \mathrm{W}'$, are bijective for $q = 0, 1, \ldots, m - 1$, and that the following exact sequences hold:

$$0 \to \mathrm{H}^m(\mathrm{W}) \to \mathrm{H}_m(\partial\mathrm{W}) \xrightarrow{\bar{\iota}_*} \mathrm{H}_m(\mathrm{W}) \to 0,$$
$$0 \to \mathrm{H}^m(\mathrm{W}') \to \mathrm{H}_m(\partial\mathrm{W}) \xrightarrow{\bar{\iota}'_*} \mathrm{H}_m(\mathrm{W}') \to 0.$$

Since $\mathrm{H}_m(\mathrm{W})$, $\mathrm{H}_m(\mathrm{W}')$ are torsion free, the above exact sequences split:

$$\mathrm{H}_m(\partial\mathrm{W}) \cong \mathrm{H}_m(\mathrm{W}) \oplus \mathrm{H}^m(\mathrm{W}),$$
$$\mathrm{H}_m(\partial\mathrm{W}) \cong \mathrm{H}_m(\mathrm{W}') \oplus \mathrm{H}^m(\mathrm{W}').$$

Thus, in particular, $\mathrm{H}_m(\mathrm{W})$ and $\mathrm{H}_m(\mathrm{W}')$ have the same rank, say r.

Let

$$\begin{aligned}\mathrm{S}^{2m+1} = {} & ((\mathrm{S}_1^m \times \mathrm{D}_1^{m+1}) \natural (\mathrm{S}_2^m \times \mathrm{D}_2^{m+1}) \natural \cdots \natural (\mathrm{S}_r^m \times \mathrm{D}_r^{m+1})) \\ & \cup ((\mathrm{D}_1^{m+1} \times \mathrm{S}_1^m) \natural (\mathrm{D}_2^{m+1} \times \mathrm{S}_2^m) \natural \cdots \natural (\mathrm{D}_r^{m+1} \times \mathrm{S}_r^m))\end{aligned}$$

be the natural decomposition and let

$$\hat{W} = W \natural (S_1^m \times D_1^{m+1}) \natural (S_2^m \times D_2^{m+1}) \natural \cdots \natural (S_r^m \times D_r^{m+1}),$$
$$\hat{W}' = W' \natural (D_1^{m+1} \times S_1^m) \natural (D_2^{m+1} \times S_2^m) \natural \cdots \natural (D_r^{m+1} \times S_r^m).$$

Then $M = M \# S^{2m+1}$ has a decomposition

$$M = \hat{W} \cup \hat{W}'.$$

Let $\eta : \hat{W} \to M$, $\eta' : \hat{W}' \to M$, $\bar{\eta} : \partial\hat{W} \to \hat{W}$ and $\bar{\eta}' : \partial\hat{W} \to \hat{W}'$ be the inclusion maps. Then it is obvious that the homomorphisms

$$\eta_* : H_q(\hat{W}) \to H_q(M), \qquad \eta'_* : H_q(\hat{W}') \to H_q(M),$$
$$\bar{\eta}_* : H_q(\partial\hat{W}) \to H_q(\hat{W}), \qquad \bar{\eta}'_* : H_q(\partial\hat{W}) \to H_q(\hat{W}')$$

are bijective for $q = 0, 1, \ldots, m-1$, and that

$$H_m(\hat{W}) \cong H_m(W) \oplus H_m((S_1^m \times D_1^{m+1}) \natural (S_2^m \times D_2^{m+1}) \natural \cdots \natural (S_r^m \times D_r^{m+1})),$$
$$H_m(\hat{W}') \cong H_m(W') \oplus H_m((D_1^{m+1} \times S_1^m) \natural (D_2^{m+1} \times S_2^m) \natural \cdots \natural (D_r^{m+1} \times S_r^m)),$$
$$H_m(\partial\hat{W}) \cong H_m(\hat{W}) \oplus H^m(\hat{W}) \cong H_m(\hat{W}') \oplus H^m(\hat{W}').$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ (resp. $\alpha'_1, \alpha'_2, \ldots, \alpha'_r$) be a system of generators of $H_m(W)$ (resp. $H_m(W')$) and let β_i (resp. β'_i) be a homology class of $H_m(\hat{W})$ (resp. $H_m(\hat{W}')$) represented by $S_i^m \times 0$ (resp. $0 \times S_i^m$). Then $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_r$ (resp. $\alpha'_1, \alpha'_2, \ldots, \alpha'_r, \beta'_1, \beta'_2, \ldots, \beta'_r$) form a system of generators of $H_m(\hat{W})$ (resp. $H_m(\hat{W}')$). By the splitting as above, we may suppose that $\alpha_i, \alpha'_i, \beta_i$ and β'_i are elements of $H_m(\partial\hat{W})$.

Now let $g : \partial\hat{W} \to \mathbf{R}$ be a nice function and let F be a compact $2m$-dimensional submanifold of $\partial\hat{W} = \partial\hat{W}'$ formed from $g^{-1}([0, m-(1/2)])$ by adding m-handles such that (i) the minimal number of m-handles which make

$$H_{m-1}(F) \cong H_{m-1}(\partial\hat{W}),$$

(ii) m-handles which represent homology classes

$$\alpha_1 + \beta'_1, \quad \alpha_2 + \beta'_2, \quad \ldots, \quad \alpha_r + \beta'_r,$$
$$\alpha'_1 + \beta_1, \quad \alpha'_2 + \beta_2, \quad \ldots, \quad \alpha'_r + \beta_r$$

of $H_m(\partial\hat{W})$. Then it is easy to see that the homomorphisms

$$\lambda_* : H_m(F) \to H_m(\hat{W}), \qquad \lambda'_* : H_m(F) \to H_m(\hat{W}')$$

induced by the inclusion maps $\lambda : F \to \hat{W}$, $\lambda' : F \to \hat{W}'$, are bijective. Further, since the homomorphism $H_q(F) \to H_q(\partial \hat{W}$ induced by the inclusion map $F \to \partial\hat{W}$ is bijective for $q = 0, 1, \ldots, m-1$, the homomorphisms

$$\lambda_* : H_q(F) \to H_q(\hat{W}), \qquad \lambda'_* : H_q(F) \to H_q(\hat{W}')$$

are bijective for $q = 0, 1, \ldots, m-1$. Thus the inclusion maps

$$\lambda : F \to \hat{W}, \qquad \lambda' : F \to \hat{W}'$$

are homotopy equivalences. Since F, $\hat{W}$ and $\hat{W}'$ are simply connected, it follows by the relative h-cobordism theorem that

$$\hat{W} = F \times I, \qquad \hat{W}' = F \times I.$$

This observation implies that M admits a spinnable structure with F as generator. It is clear that the inclusion map $F \to M$ is a homotopy m-equivalence.

Remark. — The hypothesis on simply connectedness in Theorem 7 may be omitted by replacing the h-cobordism with the s-cobordism.

6. Foliations of $(m-1)$-connected $(2m+1)$-dimensional differentiable manifolds.

In this section we shall prove without using any classification theorem of differentiable manifolds that every $(m-1)$-connected closed $(2m+1)$-dimensional differentiable manifold $(m \geqslant 3)$ has a codimension-one foliation. For simply connected closed 5-dimensional differentiable manifolds, the existence of codimension-one foliations is already shown by making use of the classification theorem (A'Campo [1], Fukui [5]).

THEOREM 8. — *Let* M^{2m+1} *be an* $(m-1)$*-connected closed* $(2m+1)$*-dimensional differentiable manifold* $(m \geqslant 3)$. *Then* M^{2m+1} *is specially spinnable.*

Proof. — According to Theorem 7, M^{2m+1} has a spinnable structure with F as generator such that the inclusion map $F \to M^{2m+1}$ is a homotopy m-equivalence. Since M^{2m+1} is $(m-1)$-connected, it follows that F is a handlebody consisting of m-handles :

$$F = D^{2m} \cup (D_1^m \times D_1^m) \cup (D_2^m \times D_2^m) \cup \cdots \cup (D_s^m \times D_s^m).$$

First let us consider the case where m is even, say $m = 2n$. Let $M^{4n+1} = \hat{W} \cup \hat{W}'$ be the decomposition used in the proof of Theorem 7 and let $S^{4n+1} = V_+ \cup V_-$ be the decomposition of section 2. Then M^{4n+1} has a decomposition as follows :

$$M^{4n+1} = (\hat{W} \natural V_+) \cup (\hat{W}' \natural V_-).$$

For G, b_{17} (see section 2) and F, we may suppose that

$$G, \quad b_{17}, \quad F \subset \partial(\hat{W} \natural V_+) = \partial(\hat{W}' \natural V_-).$$

Let

$$L = D^{4n} \cup (D_1^{2n} \times 0) \cup (D_2^{2n} \times 0) \cup \cdots \cup ((D_s^{2n} \times 0) \natural b_{17})$$

and let N(L) be a smooth regular neighborhood of L in $\partial(\hat{W} \natural V_+)$ (see Fig. 2). Then $F' = N(L) \cup G$ is a compact $4n$-dimensional differentiable submanifold of $\partial(\hat{W} \natural V_+)$ having the homotopy type of a bouquet of $2n$-spheres. It is easy to see that the inclusion maps

$$F' \to \hat{W} \natural V_+, \qquad F' \to \hat{W}' \natural V_-$$

are homotopy equivalences. This implies that M^{4n+1} has a spinnable structure with F′ as axis.

Since $G = (S^{2n} \times D^{2n}) \natural A$ and $\partial A = S^{4n-1}$ (Theorem 1), it is easily obssrved that

$$\begin{aligned}\partial F' &= \partial(N(L) \cup (S^{2n} \times D^{2n}) \natural A) = \partial(N(L) \cup (S^{2n} \times D^{2n})) \\ &= \partial(D^{4n} \cup (D_1^{2n} \times D_1^{2n}) \cup (D_2^{2n} \times D_2^{2n}) \\ &\qquad \cup \cdots \cup (D_{s-1}^{2n} \times D_{s-1}^{2n})).\end{aligned}$$

Thus, by repeating this process for each $2n$-handles of F, we can construct a spinnable structure on M^{4n+1} having a

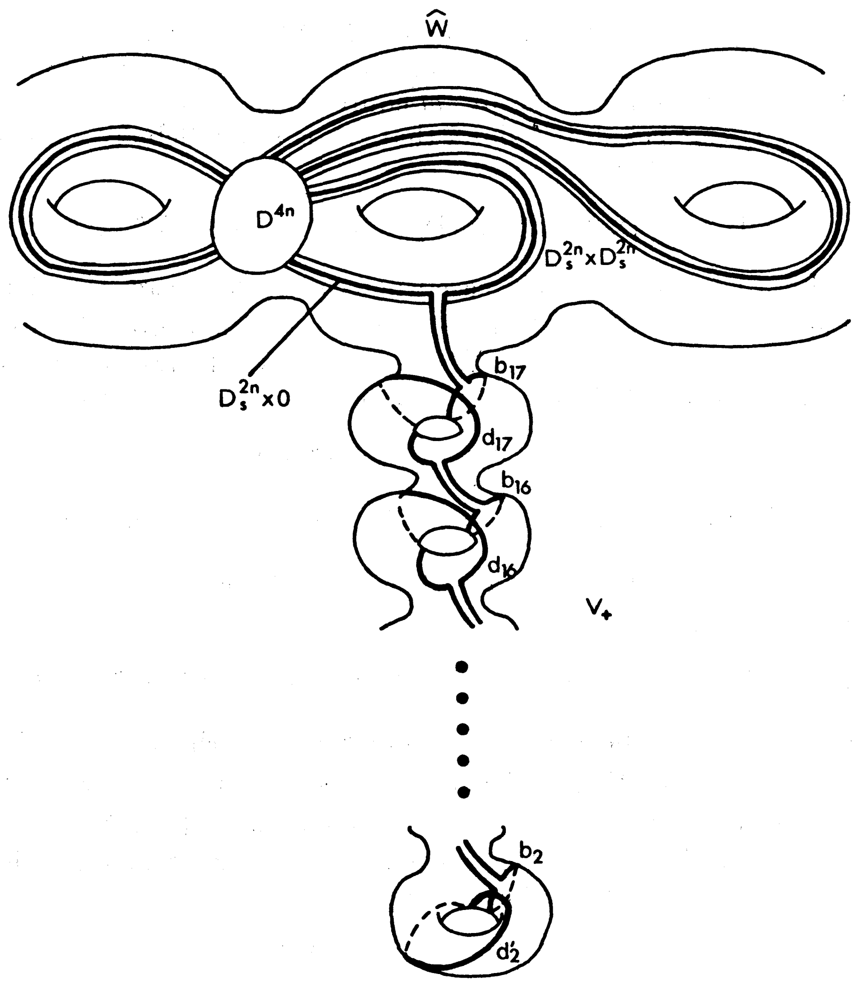

Fig. 2.

generator $\hat{F}$ such that $\partial \hat{F} = S^{4n-1}$. That is to say, we modified the axis ∂F to S^{4n-1} performing a surgery by making use of Theorem 1.

In case where m is odd, we can perform a surgery on the axis ∂F in a similar manner, by making use of Theorem 2, so that ∂F is modified to S^{2m-1}. This completes the proof.

As an immediate corollary of Theorems 8 and 5, we now have the following.

THEOREM 9. — *Every* $(m-1)$*-connected closed* $(2m+1)$*-dimensional differentiable manifold* $(m \geqslant 3)$ *has a codimension-one foliation.*

The existence of codimension-one foliations for stably parallelizable $(m-1)$-connected closed $(2m+1)$-dimensional differentiable manifolds is proved in [4], by using a classification theorem of such differentiable manifolds.

The following theorem is a direct consequence of Theorems 6 and 8.

THEOREM 10. — *Every* $(m-1)$*-connected closed* $(2m+1)$*-dimensional differentiable manifold* $(m \geqslant 3)$ *is obtained from a fibering over* S^1 *having an* $(m-1)$*-connected closed* $2m$*-dimensional differentiable manifold as fibre by performing a surgery on a cross-section.*

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