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SOME REMARKS ON CONVOLUTION EQUATIONS

by C.A. BERENSTEIN and M.A. DOSTAL

Introduction.

Let $\mathcal{D}(\Omega)$ ($\mathcal{E}'(\Omega)$) be the space of all Schwartz testfunctions (distributions) with compact support in an open convex subset Ω of $\mathbf{R}^n$ (1) ; both spaces are considered in their respective topologies (cf. [18]). Starting from an idea due to B. Malgrange [15], we studied in [3], [5], [9] a special description of these topological vector spaces by means of Fourier transform. The main objective of the present note is to derive by this method a formula for the supporting function $h_{ss.\Phi}$ of the set cv. sing supp Φ , $\Phi \in \mathcal{E}'$. Although the expression for $h_{ss.\Phi}$, at which we arrive in Section 2, is again a formula of Paley-Wiener type, it is of different kind than the known one (cf. [12], [13], [7]). Some applications of this result to convolution equations are discussed in Section 3. Section 1 contains several auxiliary statements some of which seem to be of independent interest. Part of the material presented in this article was announced in our note [4].

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(1) Further notation : $\mathcal{E}' = \mathcal{E}'(\mathbf{R}^n)$; $\mathbf{S}^{n-1} = \{\alpha \in \mathbf{R}^n : \|\alpha\| = 1\}$; for $x, y \in \mathbf{R}^n$, $< x , y > = \sum_{i=1}^{n} x_i y_i$ $\mathbf{S}_+^{n-1}$ ($\mathbf{S}_-^{n-1}$) denotes the open upper (lower) hemisphere on $\mathbf{S}^{n-1}$; sing supp Φ is the singular support of a distribution Φ (cf. § 1 below) ; $h_A(\eta)$ is the supporting function of the set A, i.e. $h_A(\eta) = \sup_A < x , \eta >$, $\eta \in \mathbf{R}^n$; cv. A is the convex hull of A, hence $h_{\text{cv.A}} \equiv h_A$; ∂A denotes the boundary of A. If $\Phi \in \mathcal{E}'$, we shall write $h_{s,\Phi}$ ($h_{ss.\Phi}$ resp.) instead of $h_{\text{supp}\,\Phi}$ ($h_{\text{sing supp}\,\Phi}$ resp.).

1. Auxiliary statements.

We start with several facts about convex functions, which will be needed in the sequel.(1)

Given an open convex set $\Omega \neq \emptyset$ in $\mathbf{R}^n$, $\mathfrak{L}(\Omega)$ will denote the class of all convex functions p defined on Ω and such that

$$\lim_{x \to \partial\Omega} \frac{p(x)}{\|x\|} = \infty. \quad (^2) \tag{1}$$

If K is a compact subset of $\mathbf{R}^n$, let $\mathfrak{L}(\mathrm{K}) = \bigcup_{\Omega \supset \mathrm{K}} \mathfrak{L}(\Omega)$. Obviously $\mathfrak{L}(\mathrm{K}) = \mathfrak{L}(\mathrm{cv.K})$. When $\mathrm{K} = \emptyset$ ($\mathrm{K} = \operatorname{supp} \Phi$, $\Phi \in \mathcal{E}'$, resp.) we shall write $\mathfrak{L}$ ($\mathfrak{L}(\Phi)$, resp.) instead of $\mathfrak{L}(\mathrm{K})$. Furthermore, for $p \in \mathfrak{L}$, Ω_p denotes the domain of definition of p,

$$\mathrm{N}_p = \min_{\Omega_p} p(x), \quad \mathrm{H}_p(s) = \{x : p(x) \leqslant \mathrm{N}_p + s\}$$

for $s \geqslant 0$, and $h_p(s\,;\alpha) = h_{\mathrm{H}_p(s)}(\alpha)$. In a certain sense, the mapping $s \mapsto h_p(s\,;\alpha)$ represents a substitute for the inverse function of p.

The first lemma is an immediate corollary of the Hahn-Banach theorem in its geometric form (cf. [6]).

LEMMA 1. – *For any fixed* $p \in \mathfrak{L}$, $s \geqslant 0$ *and* $\alpha \in \mathrm{S}^{n-1}$, *let* $x^* = x^*(s, \alpha) \in \partial\mathrm{H}_p(s)$ *be an arbitrary point such that*

$$\langle x^*, \alpha\rangle = h_p(s\,;\alpha).$$

If $\pi(x^*, s, \alpha)$ *denotes the linear variety*

$$\{(x, y) \in \mathbf{R}^{n+1} : \langle x - x^*, \alpha\rangle = 0,\ y = \mathrm{N}_p + s\},$$

then there exists a hyperplane $\widetilde{\pi}(p, x^*, \alpha)$ *which is tangent to* $\partial[p]$ (cf.(2)) *at the point* $(x^*, \mathrm{N}_p + s)$ *and contains* $\pi(x^*, s, \alpha)$.

For each $p \in \mathfrak{L}(\Omega)$, the so-called *normal mapping* ν_p of the convex surface $\partial[p]$ can be defined as follows (cf. [1], [2]). For $x \in \Omega$,

(1) Cf. [9] where similar questions are discussed.

(2) The convexity of p means that the set $[p] \overset{\mathrm{def}}{=} \{(x, y) \in \mathbf{R}^{n+1} : x \in \Omega,\ y \geqslant p(x)\}$ is convex and p is continuous.

let $\widetilde{\pi}(p, x)$ be any tangent hyperplane of $\partial[p]$ passing through the point $(x, p(x))$. If $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \in S^n$ is the corresponding outer normal vector, then $\alpha \in S^n_-$. (1). Hence, for some positive λ, $\lambda\, \alpha_{n+1} = -1$. Set $\nu_p(x) = \nu_{p,\widetilde{\pi}}(x) = (\lambda\alpha_1, \ldots, \lambda\alpha_n)$. If x is fixed, $\mathfrak{N}_p(x)$ will denote the normal cone of p at x, i.e. the set of all possible vectors $\nu_p(x)$. Each $\mathfrak{N}_p(x)$ is a compact convex subset of $\mathbf{R}^n$ (cf. [2]). The mapping $x \mapsto \nu_p(x)$, which in general is multivalued, is called the normal mapping of the surface $\partial[p]$. If the function p has total differential at the point x, then obviously the set $\mathfrak{N}(x)$ contains only one vector, namely $\nu_p(x) = \left(\frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n}\right)$. Hence, by a well-known theorem of K. Reidemeister [16], the normal mapping ν_p is univalent almost everywhere in Ω. Furthermore, it is not difficult to see that if, for a fixed $\alpha \in S^{n-1}$ and $s \geqslant 0$, $x^*(s, \alpha)$ denotes any point of the sort appearing in Lemma 1, then $\|\nu_p(x^*(s, \alpha))\| \nearrow \infty$ when $s \to \infty$. Moreover, an easy compactness argument shows that for each $p \in \mathfrak{P}$, the function

$$C_p(s) = \inf\{\|\nu_p(x)\| : \nu_p(x) \in \mathfrak{N}_p(x),\ x \in \partial H_p(s)\}$$

is such that

$$C_p(s) \nearrow \infty \qquad \text{for } s \to \infty. \tag{2}$$

It is not difficult to construct functions p for which $C_p(s)$ grows arbitrarily fast ; this can be done by a procedure called in [9] the convex interpolation. Here we state only a special case which will be used in the sequel.

LEMMA 2. — *Given Ω as above and constants* N *real and* $\delta > 0$, *there exists a function* $p \in \mathfrak{P}(\Omega)$ *such that* $N_p = N$ *and*

$$C_p(s) \geqslant \frac{s}{\delta} \qquad (s \geqslant 1). \tag{3}$$

This lemma has an interesting consequence which will be used in Section 2. Let p be the function of Lemma 2 and $x_1 = x^*(s_1, \alpha)$ as above ($s_1 > 1$). By Lemma 1, there exists a vector $\nu_p(x_1) \in \mathfrak{N}_p(x_1)$ such that $\nu_p(x_1) = \|\nu_p(x_1)\| . \alpha$. Hence, for any $x_2 \in \Omega$ and $s_2 = p(x_2)$, the convexity of p implies

$$<(x_2 - x_1, p(x_2) - p(x_1)),\ (\nu_p(x_1), -1)> \leqslant 0. \tag{4}$$

(1) Cf. (1), p. 55.

whence

$$C_p(s_1) \leqslant \frac{s_2 - s_1}{< x_2 - x_1 \, , \alpha >} \tag{5}$$

for $< x_2 - x_1 \, , \alpha > > 0$. In particular, if x_2 is such that

$$x_2 = x^*(s_2 \, , \alpha) \notin \mathrm{H}_p(s_1)$$

then Lemma 2 combined with (5) shows that

$$s_1(h_p(s_2 \, , \alpha) - h_p(s_1 \, , \alpha)) \leqslant \delta(s_2 - s_1) \quad (\alpha \in \mathrm{S}^{n-1} \, ; s_2 > s_1 > 1) \tag{6}$$

Remark 1. – We have just proved the existence of functions p with property (6). However much more can be said. If Ω is bounded, then for any $p \in \mathfrak{L}(\Omega)$ and $\delta > 0$, there is a constant σ such that the inequality (6) holds for all α and $s_2 > s_1 \geqslant \sigma$. For Ω unbounded, the same statement remains valid, if one assumes that each p grows at infinity faster than indicated in (1). It suffices to suppose, for instance, that

$$\lim_{x \to \partial\Omega} \frac{\log p(x)}{\| x \|} = \infty. \tag{1*}$$

The proofs of these statements are omitted since they are rather tedious. However, it will be clear from the context that by using these statements one could simplify the construction in the next section.

Finally, the convex interpolation mentioned above easily yields the following lemma about extensions of convex functions :

LEMMA 3. – *Assume that $\Omega' \subset \Omega$. If p' is a function in $\mathfrak{L}(\Omega')$ satisfying condition* (6), *then for each $s_0 > 0$, there exists a function $p \in \mathfrak{L}(\Omega)$ also satisfying* (6) *and such that $p = p'$ on $\mathrm{H}_{s_0}(p')$ and $p \leqslant p'$ everywhere.*

Let $\mathcal{C}$ be the class of all functions λ which are concave, increasing, continuously differentiable on $[0 \, , \infty)$ and such that

$$0 < 2(s + 1)\, \lambda'(s) \leqslant 1 \quad \text{for all} \quad s \geqslant 0.$$

LEMMA 4.[1] – *Let λ be a function in $\mathcal{C}$ and p its inverse. Then for any $a > 0$ and $b \geqslant 1$,*

$$\max\left(\sum_{s=0}^{\infty} e^{a\lambda(s)-bs}\ ,\ \sum_{s=0}^{\infty} e^{as-bp(s)}\right) \tag{7}$$

$$\leqslant \left(3+\frac{a}{2b}\right) e^{a\lambda\left(\frac{a}{b}\right)}.\ ^{(2)}$$

Proof. – Set $f(s) = \exp\,[a\lambda(s/b) - s]$ and estimate the first sum in (7). The Euler-Maclaurin formula implies

$$2\sum_{s=0}^{\infty} f(bs) \leqslant f(0) + \left(\frac{a}{b}+1\right)\int_0^{\infty} f(s)\,ds. \tag{8}$$

The integration by parts in the integral $\mathfrak{I}_a = \int_a^{\infty} f(s)\,ds$ gives $\mathfrak{I}_a \leqslant 2f(a)$. This inequality combined with (8) and the obvious estimate,

$$\int_0^a f(s)\,ds \leqslant \exp\left[a\lambda\left(\frac{a}{b}\right)\right],$$

yields (7). The estimate of the second sum in (7) is similar.

In the next section we shall study the singular support of a distribution Φ. Let us recall that the set sing supp Φ is defined as the complement of the largest open set on which Φ is equal (in the sense of distributions) to a C^{∞}-function.[3] The Fourier transform of a a distribution $\Phi \in \mathcal{E}'$ is defined here as the entire function

$$\hat{\Phi}(\zeta) = \Phi(e^{-i\langle x,\zeta\rangle}) \qquad (\zeta = \xi + i\eta \in \mathbf{C}^n)$$

Set $\omega(\xi) = \log\,(2 + |\xi|)$, and for t real and $\alpha \in S^{n-1}$,

$$\Gamma(t\,,\alpha) = \{\zeta \in \mathbf{C}^n\ :\ \zeta = \zeta(\xi) = \xi + i\alpha t\omega(\xi)\}\ .$$

By virtue of the Paley-Wiener theorem and the formulae of Plancherel and Cauchy-Poincaré, one has

(1) This lemma is a slightly stronger version of a similar statement used in [5], [9].

(2) Actually, in the second sum the index s runs only over the integers $\geqslant \lambda(0)$.

(3) We write $\Phi \in C^{(j)}(\mathcal{U})$ ($j = 0\,,1\,,\ldots,\infty\,;\mathcal{U}$ open) if Φ is equal in $\mathcal{U}$ to a function which is continuously differentiable in $\mathcal{U}$ up to the order j. Similarly, $\Phi \in C^{(j)}(x_0)$ if $\Phi \in C^{(j)}(\mathcal{U})$ for some open neighborhood $\mathcal{U}$ of x_0.

$$\Phi(\varphi) = \frac{1}{(2\pi)^n} \int_{\Gamma(t,\alpha)} \hat{\Phi}(\zeta)\, \hat{\varphi}(-\zeta)\, d\zeta\,, \tag{9}$$

where the integral $\int_{\Gamma(t,\alpha)} f(\zeta)\, d\zeta$ is understood to be

$$\int_{\mathbf{R}^n} f(\zeta(\xi)) \frac{\partial(\zeta_1, \ldots, \zeta_n)}{\partial(\xi_1, \ldots, \xi_n)} d\xi_1, \ldots, d\xi_n\,. \tag{10}$$

Then, for some positive constant C,

$$\left| \frac{\partial(\zeta_1, \ldots, \zeta_n)}{\partial(\xi_1, \ldots, \xi_n)} \right| \leqslant \mathrm{C}(1+t)^n. \tag{11}$$

Using this estimate, equation (9) and the Fubini theorem, we arrive at the following standard lemma :

LEMMA 5. – *Given a distribution $\Phi \in \mathcal{E}'$ and an integer $j \geqslant 0$, assume that there exist t real, $\alpha \in \mathrm{S}^{n-1}$ and an open set* U *in* $\mathbf{R}^n$ *such that for any multiindex γ, $|\gamma| \leqslant j$, the integral*

$$\mathfrak{I}_\gamma(x) = \int_{\Gamma(t,\alpha)} \zeta^\gamma\, e^{i\langle x,\zeta\rangle}\, \hat{\Phi}(\zeta)\, d\zeta \tag{12}$$

converges absolutely and locally uniformly in U. *Then* $\Phi \in \mathrm{C}^{(j)}(\mathrm{U})$.

The order of a distribution Φ ($\Phi \in \mathcal{E}'$) is defined here as the infimum $\mathrm{M}(\Phi)$ of all numbers M such that, for some $\mathrm{C} > 0$ and $\mathrm{R} > 0$,

$$|\hat{\Phi}(\zeta)| \leqslant \mathrm{C}e^{\mathrm{M}\omega(\xi) + \mathrm{R}|\eta|}$$

By the Paley-Wiener theorem (or by the previous lemma), sing supp $\Phi = \emptyset$ (i.e. $\Phi \in \mathrm{C}_0^\infty(\mathbf{R}^n)$) if and only if $\mathrm{M}(\Phi) = -\infty$.

Let $\Delta(\zeta\,; r)$ be the polydisk $\{z \in \mathbf{C}^n : \max_j |\zeta_j - z_j| \leqslant r\}$ and $\dot{\Delta}(\zeta\,; r)$ its distinguished boundary. If g is analytic in $\Delta(\zeta\,; r)$, set

$$\left\{ \begin{aligned} |g(\zeta)|_r &= \max_{z \in \Delta(\zeta;r)} |g(z)|\,; \\ [g(\zeta)]_r &= \frac{1}{(2\pi r)^n} \oint_{\dot{\Delta}(\zeta;r)} |g(z)|\, |dz|\,. \end{aligned} \right. \tag{13}$$

LEMMA 6. – *Given $\varepsilon > 0$ and g analytic in $\Delta(z_0\,; 2\varepsilon)$, then for each $k = 0, 1, \ldots,$*

$$|g^{(k)}(z_0)|_\varepsilon \leqslant \frac{k!\,(k+2)}{2\pi\varepsilon^{k+1}}\,[g(z_0)]_{2\varepsilon}\,. \tag{14}$$

The proof is an easy application of the Cauchy formula. One can clearly assume $n = 1$. Fix $r \in (0\,,\varepsilon]$ and ζ such that $|\zeta - z_0| = r$. Then the Cauchy formula and the monotonicity of $G(r) = [g(z_0)]_r$ yield

$$\frac{1}{k!}\int_0^\varepsilon r^{k+1}|g^{(k)}(\zeta)|\,dr \leqslant \int_0^\varepsilon [g(\zeta)]_r\; r\,dr$$

$$\leqslant \iint_{\Delta(z_0\,;\,2\varepsilon)} |g(w)|\,|dw| \leqslant \int_0^{2\varepsilon} [g(z_0)]_{2\varepsilon}\; r\,dr\,,$$

and the estimate (14) easily follows.

2. Formula for $h_{ss.\Phi}$.

Let $\mathcal{R}_0(\Omega) = \{p \in \mathcal{R}(\Omega) : N_p = 0\}$ and $\mathcal{R}_0 = \bigcup\limits_{\Omega} \mathcal{R}_0(\Omega)$. For each $p \in \mathcal{R}_0(\Omega)$, $\mu > 0$ and any convergent series $\sum\limits_{s=1}^{\infty} \varepsilon_s$ such that $0 < \varepsilon_s \leqslant 1$, the series

$$k(\zeta) = k(p\,;\{\varepsilon_s\}\,;\mu\,;\zeta) = \sum_{s=1}^{\infty} \varepsilon_s \exp\,[h_p(s\,,\eta) - (\mu + s)\,\omega(\xi)] \tag{15}$$

is obviously locally uniformly convergent in $\mathbf{C}^n$ and hence represents there a positive continuous function. Let $\mathcal{K}(\Omega)$ be the family of all such functions k and $\mathcal{A}(\mathcal{K}(\Omega))$ the space of all entire functions f satisfying the condition

$$\|f\|_k = \sup_{\zeta\in\mathbf{C}^n} \frac{|f(\zeta)|}{k(\zeta)} < \infty \qquad (\forall k \in \mathcal{K}(\Omega)). \tag{16}$$

The natural locally convex topology on $\mathcal{A}(\mathcal{K}(\Omega))$ is the one generated by all the norms $\|\cdot\|_k$. It was shown in [3], [9] that the topological vector spaces $\mathcal{A}(\mathcal{K}(\Omega))$ and $\mathcal{D}(\Omega)$ are isomorphic, the corresponding isomorphism $\mathcal{F} : \mathcal{D}(\Omega) \to \mathcal{A}(\mathcal{K}(\Omega))$ being the Fourier transform, $\mathcal{F}(\varphi) = \hat{\varphi}$. Furthermore, it was proved in [4], [5] that by a suitable extension of the "Taylor"-type series (15) to the "Laurent"-type series $k(p\,;\{\varepsilon_s\}\,;\zeta) = \sum_{s=-\infty}^{\infty} \ldots$, one obtains a similar description of the space $\mathcal{E}'(\Omega)$ (1).

Let $\Phi \in \mathcal{E}'(\Omega)$ be fixed. Lemma 4 and the Paley-Wiener theorem imply that for each $k \in \mathcal{K}(\Omega)$ there are constants N$'$ real and $C > 0$ such that

$$|\hat{\Phi}(\zeta)| \leqslant C e^{N'\omega(\xi)} k(\zeta). \tag{17}$$

Let $N(p\,,\Phi) = 1 + \inf(N' - \mu)$ where the infimum is taken over all numbers $N' - \mu$ which appear in inequalities (17) with the k's of the form $k = k(p\,;\ldots)$. Hence, for some $C_1 > 0$ and a sequence $\{\varepsilon_s\}$,

$$|\hat{\Phi}(\zeta)| \leqslant C_1 e^{N(p,\Phi)} \sum_{s=0}^{\infty} \varepsilon_s \exp[h_p(s\,,\eta) - s\omega(\xi)]. \tag{17*}$$

It follows from Lemma 4 that

$$N(p\,,\Phi) \geqslant M(\Phi). \tag{18}$$

In the sequel we shall need a decomposition of the class $\mathcal{L}_0(\Omega)$ into subclasses of functions with the same growth (in the sense of condition (6)). Let δ be a positive number and $\sigma(\alpha)$ an arbitrary function of $\alpha \in S^{n-1}$, $\sigma(\alpha) \geqslant 1$. Then $\mathcal{L}_0(\Omega\,;\sigma\,;\delta)$ denotes the family of all functions $p \in \mathcal{L}_0(\Omega)$ satisfying inequality (6) for all α and $s_2 > s_1 \geqslant \sigma(\alpha)$. As was shown in Sect. 1, $\mathcal{L}_0(\Omega\,;1\,;\delta) \neq \emptyset$; therefore, $\mathcal{L}_0(\Omega\,;\sigma\,;\delta) \neq \emptyset$ for all δ and σ as above. For each σ fixed, there is an obvious decomposition of $\mathcal{L}_0(\Omega)$,

$$\begin{cases} \mathcal{L}_0(\Omega) = \bigcup\limits_{\delta} \mathcal{L}_0(\Omega\,;\sigma\,;\delta)\ ; \\ \mathcal{L}_0(\Omega\,;\sigma\,;\delta_1) \subset \mathcal{L}_0(\Omega\,;\sigma\,;\delta_2) \quad \text{for} \quad \delta_1 < \delta_2 . \end{cases} \tag{19}$$

Remark 2. – If we limit ourselves to those p's which satisfy condition (1*) instead of (1), then by Remark 1 (cf. Sect. 1), for each fixed $\delta > 0$, there is a decomposition of $\mathcal{L}_0(\Omega)$ given by the formula

(1) Actually, in [4], [5] this was shown only for $\Omega = \mathbf{R}^n$. However, the extension to general Ω is straightforward.

$$\mathcal{P}_0(\Omega) = \bigcup_\sigma \mathcal{P}_0(\Omega ; \sigma ; \delta). \tag{19*}$$

Moreover, in (19*) we can limit ourselves to constant functions σ.

Let $\mathcal{L}_\Phi(\alpha) = \mathcal{L}(\Phi ; \Omega ; \sigma ; \alpha)$ be the function

$$\mathcal{L}_\Phi(\alpha) = \lim_{\delta \to 0} \lim_{s \to \infty} \inf_p \{h_p(s + \mathrm{N}(p, \Phi) ; \alpha) : p \in \mathcal{P}_0(\Omega ; \sigma ; \delta)\}. \tag{20}$$

Setting

$$\mathcal{P}_0(\Phi ; \sigma ; \delta) = \bigcup_{\operatorname{supp}\Phi \subset \Omega} \mathcal{P}_0(\Omega ; \sigma ; \delta),$$

one could similarly define a function $\mathcal{L}^*_\Phi(\alpha)$ given by the same formula (20), but with $\mathcal{P}_0(\Omega ; \sigma ; \delta)$ replaced by the class $\mathcal{P}_0(\Phi ; \sigma ; \delta)$. Obviously, for each $\Omega \supset \operatorname{supp} \Phi$,

$$\mathcal{L}^*_\Phi \leqslant \mathcal{L}_\Phi, \tag{21}$$

and both limits in (20) can be replaced by the suprema. Thus, for instance, we can write

$$\mathcal{L}^*_\Phi(\alpha) = \sup_\delta \sup_s \inf_p \{h_p(\ldots) : p \in \mathcal{P}_0(\Phi ; \sigma ; \delta)\} \tag{20*}$$

The interest in functions given by formulae (20) or (20*) is justified by the fact that the only other possible choice of the order of the three limits in (20*) has an interesting geometric interpretation :

PROPOSITION 1. – *For every $\Phi \in \mathcal{E}'$ and arbitrary σ,*

$$\sup_\delta \inf_p \sup_s h_p(\ldots ; \alpha) \equiv h_{s.\Phi}(\alpha) \quad (p \in \mathcal{P}(\Phi ; \sigma ; \delta)\} , \tag{22}$$

Proof. – As can be easily seen from the Paley-Wiener theorem, for each fixed δ,

$$h_{s.\Phi}(\alpha) = \inf_p \sup_s \{h_p(s ; \alpha) : p \in \mathcal{P}_0(\Phi ; \sigma ; \delta)\}, \tag{23}$$

and formula (22) follows.

Remark 3. – Actually, formula (23) leads to a simpler expression for $h_{s,\Phi}$,

$$h_{s.\Phi}(\alpha) = \inf_p \sup_s \{h_p(s\,;\alpha) : p \in \mathfrak{P}(\Phi)\}. \tag{24}$$

Moreover, by the above proposition,

$$\mathcal{L}^*_\Phi \leqslant h_{s.\Phi}\,. \tag{25}$$

In view of the last proposition, it is natural to consider the function $\mathcal{L}^*_\Phi$ defined by (20*) and to ask whether the function $\mathcal{L}^*$ (or $\mathcal{L}$) has a similar geometric interpretation. The answer is remarkably simple :

THEOREM 1. – *For every distribution $\Phi \in \mathcal{E}'$, an arbitrary convex region Ω containing the support of Φ, and any function $\sigma(\alpha) \geqslant 1$,*

$$\mathcal{L}(\Phi\,;\Omega\,;\sigma\,;\alpha) = \mathcal{L}^*(\Phi\,;\sigma\,;\alpha) = h_{ss.\Phi}(\alpha) \qquad (\forall\alpha). \tag{26}$$

Proof. – First we shall prove the inequality

$$\mathcal{L}^*(\Phi\,;\sigma\,;\alpha) \geqslant h_{ss.\Phi}(\alpha). \tag{27}$$

By (25) we can obviously assume that sing supp $\Phi \neq \emptyset$; and, that $\alpha \in S^{n-1}$ is fixed so that

$$\mathcal{L}^*(\Phi\,;\sigma\,;\alpha) < h_{s.\Phi}(\alpha).$$

(On the other hand, the case $\mathcal{L}^*(\Phi\,;\sigma\,;\alpha) = -\infty$ is not excluded). Let A be any number $> -\infty$ such that

$$\mathcal{L}^*(\Phi\,;\sigma\,;\alpha) \leqslant A < h_{s.\Phi}(\alpha).$$

Choose $\varepsilon > 0$, $\varepsilon < h_{s.\Phi}(\alpha) - A$. In order to prove (27), it suffices to show that $\Phi \in C^{(j)}(x_0)$ (1) for all $j = 0, 1, \ldots$ and all x_0 in the set

$$A + 4\varepsilon < \langle x_0, \alpha\rangle < h_{s.\Phi}(\alpha) + \varepsilon. \tag{28}$$

From now on, A, ε, j and x_0 are fixed. Since sing supp $\Phi \neq \emptyset$, $M(\Phi) > -\infty$ (cf. Sect. 1). Let $\delta < \varepsilon/2$ and

$$s'_0 > \max\{\sigma(\alpha)\,,\ |M(\Phi)| + 1\,,\ j + n + 1\}.$$

By (20*) and (28) there exists $p \in \mathfrak{P}_0(\Phi\,;\sigma\,;\delta)$ such that, for $N = N(p\,,\Phi)$,

$$h_p(s'_0 + N\,;\alpha) < \langle x_0, \alpha\rangle - 3\varepsilon\,.$$

(1) Cf. (3), p. 59.

Fix p and δ. Then the second inequality in (28) combined with (22) shows that, for some integer $s_0 \geqslant s_0'$,

$$h_p(s_0 + N\,;\alpha) < \langle x_0\,,\alpha\rangle - 3\varepsilon \leqslant h_p(s_0 + N + 1\,;\alpha). \tag{29}$$

If we choose $s_1 \geqslant s_0 + 1$ so that

$$h_p(s_1 - 1 + N\,;\alpha) \leqslant \langle x_0\,,\alpha\rangle - 2\varepsilon < h_p(s_1 + N\,;\alpha), \tag{30}$$

then

$$s_0 + N > 1\ ;\ s_1 > 3s_0 + 2N > s_0 + 1. \tag{31}$$

Indeed, $s_0 + N \geqslant s_0' + N > |M(\Phi)| + 1 + N > 1$; on the other hand, by (29), (30) and (6),

$$\varepsilon < h_p(s_1 + N) - h_p(s_0 + N) < \frac{\varepsilon(s_1 - s_0)}{2(s_0 + N)},$$

which gives (31). Similarly, one obtains the inequality

$$h_p(s_1 + N\,;\alpha) < \langle x_0\,,\alpha\rangle - \varepsilon. \tag{32}$$

For each $s = 0\,,1\,,\ldots,$ let $x^* = x^*(s\,,\alpha)$ be as in Lemma 1. Then there exists a hyperplane $\tilde{\pi}$ which is tangent to $\partial[p]$ at the point $(x^*(s_1 + N\,;\alpha),\ p(x^*(s_1 + N\,;\alpha)))$ and such that, for some $t > 0$, $t\alpha = \nu_{p,\tilde{\pi}}(x^*(s_1 + N\,;\alpha))$. It will be shown below that, for

$$U = \{x \in \mathbf{R}^n\ :\ \langle x - x_0\,,\alpha\rangle > -\varepsilon\}$$

and the variety $\Gamma(t\,,\alpha)$, the hypotheses of Lemma 5 are satisfied. This will complete the proof of (12).

Let γ be any multiindex of length $|\gamma| \leqslant j$. We claim that the integral $\mathfrak{J}_\gamma(x)$ in (12) is uniformly and absolutely convergent in U. Since by (12), (11) and (17*),

$$|\mathfrak{J}_\gamma(x)| \leqslant C \sum_{s=0}^{\infty} \varepsilon_s \int_{\mathbf{R}^n} (2 + |\xi|)^{E_s} d\xi, \tag{33}$$

where

$$E_s(x) = N + j - s + t\,[h_p(s\,;\alpha) - \langle x\,,\alpha\rangle] \qquad (s \geqslant 0), \tag{34}$$

it suffices to prove that, for instance,

$$E_s(x) \leqslant -n - 1. \tag{35}$$

Consider first $s = 0, 1, \ldots, s_1 - 1 + \mathrm{N}$. Then the inequality (4), applied to $x_1 = x^*(s_1 + \mathrm{N}\,; \alpha)$, $x_2 = x^*(s_0 + \mathrm{N}\,; \alpha)$, $\nu_p(x_1) = t\alpha$, and the inequalities (29), (30) and (31) give

$$t \geqslant \frac{s_1 - s_0}{h_p(s_1 + \mathrm{N}\,; \alpha) - h_p(s_0 + \mathrm{N}\,; \alpha)} > \frac{2(s_0 + \mathrm{N})}{\varepsilon}. \tag{36}$$

Hence by (30) and (31), the function $\mathrm{E}_s(x)$ satisfies in U the estimate

$$\mathrm{E}_s(x) < \mathrm{N} + j - s - t\varepsilon < \mathrm{N} + j - s - 2(s_0 + \mathrm{N}) < -n - 1. \tag{37}$$

Now let $s \geqslant s_1 + \mathrm{N}$. Then, for $x \in \mathrm{U}$, inequalities (4) and (32) yield

$$\left.\begin{aligned} \mathrm{E}_s(x) &\leqslant \mathrm{N} + j - s + t\,[h_p(s\,; \alpha) - h_p(s_1 + \mathrm{N}\,; \alpha)] \\ &\quad + t\,[h_p(s_1 + \mathrm{N}) - \langle x\,, \alpha\rangle] < j - s_1 \\ &\quad < -n - 1. \end{aligned}\right\} \tag{38}$$

Thus, inequality (27) is established.

By inequality (21), it remains to prove that, for an arbitrary $\Omega \supset \operatorname{supp} \Phi$,

$$\mathcal{L}(\Phi\,; \Omega\,; \sigma\,; \alpha) \leqslant h_{ss.\Phi}(\alpha). \tag{39}$$

We can assume $\mathcal{L}(\Phi\,; \Omega\,; \sigma\,; \alpha) > -\infty$. Let $\varepsilon > 0$ be fixed. Then one has $\Phi = \Phi_1 + \Phi_2$, where $\Phi_2 \in \mathrm{C}_0^\infty(\mathbf{R}^n)$ and $\Phi_1 \in \mathcal{E}'$ is such that

$$h_{s.\Phi_1}(\alpha) < \varepsilon + h_{ss.\Phi}(\alpha). \tag{40}$$

Let Ω' be the open ε-neighborhood of the set cv. sing supp Φ. Choose $p' \in \mathcal{R}_0(\Omega'\,; 1\,; \delta)$. Then by (40),

$$|\hat{\Phi}_1(\zeta)| \leqslant \mathrm{C}' e^{\mathrm{N}'\omega(\xi)}\, k(p'\,; \{\varepsilon_s\}\,; 1\,; \zeta)\ (^1) \tag{41}$$

where $\mathrm{N}' = \mathrm{N}(p'\,; \Phi)$. Taking an arbitrary $s_0' \geqslant |\mathrm{N}'|$ and setting $s_0 = s_0' + \mathrm{N}'$ in Lemma 3, we obtain a function

$$p \in \mathcal{R}_0(\Omega\,; 1\,; \delta) \subset \mathcal{R}_0(\Omega\,; \sigma\,; \delta).$$

However the Paley-Wiener theorem for C_0^∞-functions shows that for

(1) The Paley-Wiener theorem implies that if p' is chosen so that supp $\Phi_1 \subset$ int $\mathrm{H}_{p'}(1)$, then $\mathrm{N}' \leqslant \mathrm{M}(\Phi_1) + 1 = \mathrm{M}(\Phi) + 1$.

any real M and the same sequence $\{\varepsilon_s\}$ as in (41), there is a $C_M > 0$ such that

$$|\hat{\Phi}_2(\zeta)| \leqslant C_M e^{M\omega(\xi)} k(p ; \{\varepsilon_s\} ; 1 ; \zeta) . \tag{42}$$

Adding the last two inequalities (with $M = N(p' ; \Phi)$ in (42)) we get

$$|\hat{\Phi}(\zeta)| \leqslant C'' e^{N(p' ; \Phi)} k(\ldots ; \zeta). \tag{43}$$

Hence $N = N(p ; \Phi) \leqslant N'$ and

$$h_p(s_0' + N) \leqslant h_p(s_0' + N') = h_{p'}(s_0) \leqslant h_{ss.\Phi} + \varepsilon ,$$

which proves (39).

Remark 4. – It follows from Theorem 1 that σ plays only an auxiliary role in the formulae (20) and (20*). However, Theorem 1 also has the following consequence. It can be shown (cf. [4], Prop. 3 ([1])) that, as in the case of the space $\mathcal{D}(\Omega)$ (cf. (16)), one can describe the topology of the space $\mathcal{E}'(\Omega)$ by means of all inequalities (17) with $p \in \mathfrak{L}_0(\Omega)$. But then the equation $\mathcal{L}(\Phi ; \Omega ; \alpha) \equiv h_{ss.\Phi}(\alpha)$ means that by looking at the inequalities (17) for $p \in \mathfrak{L}_0(\Omega)$ only ([2]) (i.e. at all the seminorms of Φ in the topology of $\mathcal{E}'(\Omega)$), one can determine the set cv. sing supp Φ. This obviously fails for the set cv. supp Φ. In this case we have to use inequalities (17) for all $p \in \hat{\mathfrak{L}}_0(\Phi)$. (In other words, the set cv. supp Φ cannot be described by the function which, in accordance with the above notation, could be denoted by $\mathcal{L}(\Phi ; \Omega ; \sigma ; \alpha)$.) This shows that the topology of $\mathcal{E}'(\Omega)$ is more closely related to singular supports than to supports.

3. Some applications.

Consider the convolution equation

$$\Phi * \varphi = \psi , \tag{44}$$

where Φ and ψ are given distributions, $\Phi \in \mathcal{E}'$, and write

([1]) The formulation of Proposition 3 in [4] is not precise and must be slightly changed. We shall return to these questions elsewhere.

([2]) and not necessarily for all $p \in \mathfrak{L}_0(\Phi)$.

$$\Theta(\Phi\,;C\,;A\,;\zeta) = C\,\exp\,[A\omega(\xi) + h_{s.\Phi}(\eta)]. \tag{45}$$

PROPOSITION 2. – *Given a distribution* $\Phi \in \mathcal{E}'$, *the following conditions are equivalent :*

(i) *there exist constants* $r > 0$, $C > 0$ *and* A *real such that for every entire function f,*

$$\Theta(\Phi\,;C\,;A\,;\zeta)\,|f(\zeta)| \leqslant |\hat{\Phi}(\zeta)f(\zeta)|_r \qquad (\forall\zeta \in \mathbf{C}^n)\ ; \tag{46}$$

(ii) *condition* (i) *with* $f \equiv 1$ *only;*

(iii) *condition* (i) *with* $|\ldots|_r$ *replaced in* (46) *by* $[\ldots]_r$;

(iv) *condition* (iii) *with* $f \equiv 1$ *only.*

Proof. – Let ζ be fixed. It suffices to show that (ii) ⇒ (i) ⇒ (iii).

Consider the inner function $B(z)$ corresponding to the function $\hat{\Phi}(z)$ and the polydisk $\Delta(\zeta\,;r)$ (cf. [17]). Then $\hat{\Phi}(z) = Q(z)\,B(z)$ where both Q and B are analytic in $\Delta(\zeta\,;r)$, $Q(z) \neq 0$ in $\Delta(\zeta\,;r)$ and $|B(z)| = 1$ on $\Delta(\zeta\,;r)$. The mean value property of harmonic functions combined with condition (ii) gives

$$|Q(\zeta)| = |Q(\zeta)|_r = |\hat{\Phi}(\zeta)|_r \geqslant \Theta(\cdots;\zeta).$$

Hence, if f is an arbitrary entire function, then

$$|f(\zeta)|\;\Theta(\ldots;\zeta) \leqslant |f(\zeta)\,Q(\zeta)| \leqslant |f(\zeta)\,Q(\zeta)|_r = |f(\zeta)\,\Phi(\zeta)|_r\,,$$

and the inequality (46) is established. The rest follows easily from Lemma 6.

DEFINITION. – *A distribution* Φ *with compact support is said to be of class* $\mathcal{R}$, *provided* Φ *satisfies one of the conditions* (i) – (iv) *in Proposition 2.* $\hat{\mathcal{R}}$ *will denote the class* $\{\hat{\Phi} : \Phi \in \mathcal{R}\}$.

The simplest examples of distributions of class $\mathcal{R}$ are linear partial differential operators with constant coefficients ; in other words, all polynomials are in the class $\hat{\mathcal{R}}$. This was proved by Malgrange in [14], and a similar statement appears in Ehrenpreis [10]. Actually, in this case one can show that inequality (46) is satisfied for each positive r (1). It is natural to ask for other examples of such distributions.

(1) Distributions with this property will be called of class $\mathcal{R}_0$.

PROPOSITION 3. – *Let* $P(\zeta)$ *be an exponential polynomial, i.e.*

$$P(\zeta) = \sum_{k=1}^{m} P_k(\zeta)\, e^{<\alpha_k, \zeta>},$$

where the P_k*'s are polynomials and* $\alpha_k = (\alpha_k^1, \ldots, \alpha_k^n) \in C^n$ *are the so-called frequencies of* P. *Set* $h_P(\zeta) = \max_k \operatorname{Re} <\alpha_k, \zeta>$. *Then, for each* $\varepsilon > 0$, *there exists a constant* $C = C(\varepsilon, P)$ *such that, if* f *is an analytic function in the polydisk* $\Delta(\zeta; \varepsilon)$, *then*

$$|f(\zeta)|\, e^{h_P(\zeta)} \leqslant C[f(\zeta)\, P(\zeta)]_\varepsilon. \tag{47}$$

Remark 5. – In particular, if all frequencies are pure imaginary, i.e. $\alpha_k = -ia_k$, $a_k \in \mathbf{R}^n$, then P is the Fourier transform of a distribution T with finite support and $h_P(\zeta) = h_{s.T}(\eta)$. In this case, Proposition 3 implies that $T \in \mathcal{R}_0$

Proof of Proposition 3 : We can assume that

$$h_P(\zeta) = \operatorname{Re} <\alpha_1, \zeta>.$$

There exists a differential operator $Q_1(D)$ with constant coefficients such that

$$Q_1(D)\, P(\zeta) = C_1 e^{<\alpha_1, \zeta>} \qquad (C_1 \neq 0).$$

($Q_1(x_1, \ldots, x_n)$ can be defined as a certain product of factors $(x_j - \alpha_k^j)$; hence C_1 depends on the coefficients of the polynomial P_1 and the frequencies of P). The Cauchy formula then yields

$$|C_1 e^{<\alpha_1, \zeta>}| \leqslant C_2(\varepsilon, P)\, [P(\zeta)]_\varepsilon.$$

As in the proof of Proposition 2 we shall consider a function $R(z)$ which is analytic and $\neq 0$ in $\Delta(\zeta; \varepsilon)$ and such that $|R(z)| = |P(z)|$ on $\dot{\Delta}(\zeta; \varepsilon)$. (For $n = 1$, it suffices to divide P by the corresponding Blaschke product ; for $n > 1$, one has to use inner functions, cf. Rudin [17]). Then,

$$|R(\zeta)| = [R(\zeta)]_\varepsilon \geqslant C_3 e^{h_P(\zeta)}.$$

Applying the Cauchy formula to $R(\zeta)\, f(\zeta)$ completes the proof.

Remark 6. – By virtue of this Proposition, the characteristic function of each bounded parallelepiped in $\mathbf{R}^n$ is in the class $\mathcal{R}$;

but then it is natural to pose the following problem : Given an arbitrary compact convex set K in $\mathbf{R}^n$, is its characteristic function χ_K in the class $\mathcal{R}$ (or even $\mathcal{R}_0$) ? We can prove it only when K is a polyhedron. (*)

Remark 7. – If we replace in (47) the exponent $h_P(\zeta)$ by $-Ch_P(\zeta)$, $C > 0$, we obtain a weaker inequality which was proved by Ehrenpreis in [11] by using mean periodicity. This inequality is still sufficient for the proof of the division problem for exponential polynomials P in the spaces $\mathcal{D}'_F(\mathbf{R}^n)$, $\mathcal{D}(\mathbf{R}^n)$, etc. (cf. [10], [11]). However, if we want to prove that, e.g., for each $\psi \in \mathcal{D}'(\Omega)$, equation (44) (with $\Phi = P$) has a solution $\varphi \in \mathcal{D}'(\Omega')$ where Ω' – cv. supp $P = \Omega$, then we need in (47) h_P with the "+" sign. More generally, one can show that for distributions of class $\mathcal{R}$ the solution of this division problem is surprisingly simple (cf. [9], Th. 2). To some extent this explains the interest in such distributions. One can expect that these distributions will have some further interesting properties. We shall mention only two of them.

PROPOSITION 4. – *For each* $\Phi \in \mathcal{R}$,

$$\text{cv. supp } \Phi = \text{cv. sing supp } \Phi. \tag{48}$$

Proof. – Inequality (46) with $f \equiv 1$ and estimate (17) with arbitrary $p \in \mathfrak{L}_0(\Phi\,;1\,;\delta)$ imply by Lemma 4, (¹)

$$\exp[h_{s.\Phi}(\eta)] \leqslant C'(1 + |\eta|)\exp[(N_p - A)\omega + h_p(|\eta|/\omega\,;\eta)]. \tag{49}$$

Fix $\varepsilon \in (0, 1)$, $\alpha \in S^{n-1}$ and consider only those $\zeta \in \mathbf{C}^n$ for which $\eta/|\eta| = \alpha$. Then for each $\delta \leqslant \delta_\varepsilon$ and $s \geqslant s_\varepsilon$ there is a function $q = q_s \in \mathfrak{L}_0(\Phi\,;1\,;\delta)$ such that

$$\left\{\begin{array}{ll} h_q(s + N_q\,;\alpha) < \varepsilon + h_{ss.\Phi}(\alpha) & \\ N_q \leqslant M(\Phi) + 1 & \text{(cf. (¹), p. 66).}\end{array}\right. \tag{50}$$

(¹) $N_p = N(p\,;\Phi)$.

(*) *Added in proofs* : If K is a sphere, or more generally, an ellipsoid, then $\chi_K \notin \mathcal{R}$ (cf. our forthcoming paper, On convolution equations I).

Taking logarithms on both sides of (49) and dividing by $|\eta|$ we get

$$h_{s.\Phi}(\alpha) \leqslant [\log C'_q + (N_q - A)\omega + \log(1 + |\eta|)]\, |\eta|^{-1} + h_q(|\eta|/\omega \,;\alpha). \tag{51}$$

Setting $|\eta| = (s + N_q)\omega$ and letting $s \to \infty$, inequalities (50) and (51) imply

$$h_{s.\Phi}(\alpha) \leqslant h_{ss.\Phi}(\alpha) + \varepsilon\,,$$

which proves the proposition.

THEOREM 2. – (The Titchmarsh-Lions formula for singular supports). *For each $\Phi \in \mathcal{R}$ and $\Psi \in \mathcal{E}'$,*

$$\text{cv. sing supp } (\Phi * \Psi) = \text{cv. sing supp } \Phi + \text{cv. sing supp } \Psi. \tag{52}$$

Proof. – It suffices to show that

$$\text{cv. sing supp } (\Phi * \Psi) \supset \text{cv. sing supp } \Phi + \text{cv. sing supp } \Psi \tag{53}$$

By Proposition 4, one has to prove,

$$h_{ss.\Phi*\Psi}(\alpha) - h_{s.\Phi}(\alpha) \geqslant h_{ss.\Psi}(\alpha). \tag{54}$$

Let α be fixed. If ε is any positive number, then by Theorem 1, for all δ small and s large, there is a function q satisfying (50) with Φ replaced by $\Phi * \Psi$. Hence by Lemma 4 and inequality (17) applied to $\Phi * \Psi$ and $k = k(q\,;\ldots)$, we obtain

$$|\hat{\Phi}(\zeta)\,\hat{\Psi}(\zeta)|_r \leqslant C(1 + |\eta|)\exp[N_q\omega + h_q(|\eta|/\omega\,;\eta)]. \tag{55}$$

By (46), this estimate gives

$$\log|\hat{\Psi}(\zeta)| \leqslant C' + (N_q - A)\omega + \log(1 + |\eta|) + h_q(|\eta|/\omega\,;\eta) - h_{s.\Phi}(\eta)\,. \tag{56}$$

Taking as before $\eta = (s + N_q)\omega\alpha$, we obtain by (50),

$$\varlimsup_{s\to\infty}\ \varlimsup_{|\xi|\to\infty} \frac{\log|\hat{\Psi}(\zeta)|}{|\eta|} \leqslant \varepsilon + h_{ss.\Phi*\Psi}(\alpha) - h_{s.\Phi}(\alpha).$$

Since the double limit in (57) is the well-known formula for the function $h_{ss.\Psi}$ (cf. [12], [7], [8]), inequality (54) follows.

Remark 8. – Another condition on Φ, which is sufficient for the validity of the equality (52) with any $\Psi \in \mathcal{E}'$, appears in [8]. This condition imposes certain restriction on the regularity of growth of $\hat{\Phi}$ at infinity. It would be interesting to compare the two conditions. Yet another approach to the problem of singular supports of convolutions is due to Hörmander [13]. Hörmander uses the technique of plurisubharmonic functions. Besides having important consequences, this method is interesting, because as above, the set cv. sing supp Φ is described in terms of a whole family $\mathcal{H}(\Phi)$ of functions. However, in concrete cases it is not easy to describe such families $\mathcal{H}(\Phi)$. Nevertheless, there seem to be some common features of Hörmander's description of cv. sing supp Φ and the formula (26) of the present article. Thus, for instance, the concluding statement in [13], which also concerns the equality (52), requires that $\mathcal{H}(\Phi)$ consist of a single function, namely, $h_{s.\Phi}$. (Incidentally, this condition is satisfied for distributions with finite support and it also implies (48)). Hörmander's result resembles Theorem 2 and Proposition 4 above. However, we were unable to find the exact relationship between Hörmander's condition on Φ and Φ being of class $\mathcal{R}$.

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