ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 22, nº 3 (1972), p. 131-140

<http://www.numdam.org/item?id=AIF_1972__22_3_131_0>

Ann. Inst. Fourier, Grenoble
22,3 (1972) 131-140

DECOMPOSITION OF GROUP-VALUED ADDITIVE SET FUNCTIONS

by Tim TRAYNOR

Introduction.

This article is concerned with the decomposition of additive set functions with values in a topological group. The main object is to obtain Lebesgue-type decomposition theorems for such functions. However, since the method has some other interesting applications, we have decided to present the general theory :

Given a finitely additive set function m, defined on a ring H of sets, with values in a commutative Hausdorff topological group, and given an ideal K of H, we study representations of m as the sum of two other additive set functions, one essentially supported on K, the other vanishing on K. We find that such a decomposition must be unique and exists under mild conditions. The main condition, s-boundedness, was introduced by C.E. Rickart [1] to obtain a similar decomposition, with respect to σ-ideals, of additive set functions taking values in a normed space. For complete spaces, this condition is equivalent to the monotone convergence condition used by M. Sion [2] in his extension theorem. In this form, s-boundedness allows a direct construction of the decomposition, by taking certain limits.

In the second section, we apply the general theorem to obtain two Lebesgue-type decompositions, one for the null set notion of absolute continuity, the other to (the group-valued version of) the "ε, δ" notion of absolute continuity.

We close with some other applications and an indication of the corresponding theory for outer measures.

Notation

In all that follows,

N is the set of non-negative integers ;

$A\backslash B = \{s \in A : s \notin B\}$;

$K_\sigma = \{\bigcup_i A_i : A$ is a sequence in $K\}$

The symbol Y denotes a commutative Hausdorff topological group, written additively, and $\mathcal{V}$ is a base for the neighborhoods of 0 in Y, consisting of symmetric sets. We recall that Y is metrizable if and only if $\mathcal{V}$ can be chosen to be countable. We suppose that

H is a ring of subsets of a space S, and

m is finitely additive on H to Y.

1. General decomposition theorems.

Let K be an ideal of H. We are concerned here with the problem of representing m as the sum of two additive functions, one vanishing on K, the other nearly supported on K (in a sense to be made precise in 1.3). For this purpose, we will impose on m the following condition of C.E. Rickart [1] :

1.1. DEFINITION. – *m is s-bounded on* H *iff for every disjoint sequence* A *in* H, $m(A_i) \to 0$.

Remarks. – If H is a σ-ring on which m is σ-additive, then m is s-bounded. If Y is a locally convex space, a finitely additive s-bounded function is bounded. (See Rickart's proof for normed spaces, [1, p. 655])(1).

It is easy to prove the following lemma, which gives the form of s-boundedness which best suits our purposes.

1.2. LEMMA. – *m is s-bounded iff for every increasing or decreasing sequence* A *in* H, *$m(A_i)$ is Cauchy.*

(1) This result is actually true for any commutative topological group in which singleton sets are bounded.

1.3. DEFINITIONS

1) m is nearly supported on K iff for every V in $\mathcal{V}$, there exists E in K such that $m(A) \in V$, whenever $S \setminus E \supset A \in H$.

2) m vanishes on K iff $m(E) = 0$, for all E in K.

1.4. DEFINITIONS. – We put

$$m_K(A) = \lim_E m(A \cap E) \quad \textit{and} \quad m_K^\circ(A) = \lim_E m(A \setminus E),$$

as E *runs over* K *directed upward by inclusion, provided these limits exist.*

The following result indicates that the desired decomposition must be given by 1.4. (and is therefore unique).

1.5. LEMMA. – *If m is the sum of two finitely additive functions m_1 and m_2 on* H *to* Y *such that m_1 is nearly supported on* K *and m_2 vanishes on* K, *then $m_1 = m_K$ and $m_2 = m_K^\circ$ on* H.

Proof. – Suppose m, m_1 and m_2 are as described. Given V in $\mathcal{V}$, choose E_0 in K such that for all A in H, $m_1(A \setminus E_0) \in V$. Then, for all E in K containing E_0 and each A in H,

$$\begin{aligned} m(A \cap E) &= m_1(A \cap E) + m_2(A \cap E) \\ &= m_1(A \cap E) \\ &= m_1(A) - m_1(A \setminus E) \\ &\in m_1(A) + V, \end{aligned}$$

and

$$\begin{aligned} m(A \setminus E) &= m_1(A \setminus E) + m_2(A \setminus E) \\ &= m_1(A \setminus E) + m_2(A) \\ &\in m_2(A) + V. \end{aligned}$$

Hence $\lim_E m(A \cap E) = m_1(A)$ and $\lim_E m(A \setminus E) = m_2(A)$, as required.

The following is the key result, giving the properties of m_K and m_K°

1.6. THEOREM. – *Suppose that* m *is finitely additive on* H *to* Y. *If* K *is an ideal of* H *and one of* m_K *and* m_K° *is defined on* H *to* Y, *then so is the other, and* $m = m_K + m_K^\circ$ *on* H. *Both are then finitely additive on* H ; *σ-additive, if* m *is σ-additive* ; *s-bounded, if* m *is s-bounded. Moreover,* m_K° *vanishes on* K *and, if* $m_K(S)$ *also exists,* $m_K|H$ *is nearly supported on* K.

Suppose Y is metrizable and $m_K(A)$ exists for all A in $H \cup \{S\}$. If either

(a) K is a σ-ideal (so that $K_\sigma = K$)

or (b) H is a σ-ring on which m is σ-additive,

then there exists an E in K_σ such that

$$m_K(A) = m(A \cap E) \quad \text{and} \quad m_K^\circ(A) = m(A \backslash E)\ , \quad \text{for all } A \text{ in } H,$$

and

$$m_K(S) = m(E).$$

Proof. – For all A in H, $m(A) = m(A \cap E) + m(A \backslash E)$, for all E in K. Thus, since subtraction is continuous,

$$m(A) = \lim_E m(A \cap E) + \lim_E m(A \backslash E),$$

whenever one of the two limits exist.

Finite additivity is a consequence of continuity of addition.

Let A be in H. Since $m_K(A)$ exists, given any V in $\mathcal{V}$, there exists E_0 in K such that $m(A \cap E) - m(A \cap E_0) \in V$, whenever $E_0 \subset E \in K$. Then, for all B in H contained in A, we have

$$E_0 \subset (B \cap E) \cup E_0 \in K,$$

and calculation shows that $m(B \cap E) - m(B \cap E_0) \in V$. Thus , $\lim_E m(B \cap E)$ exists uniformly for $A \supset B \in H$.

Now, suppose that m is σ-additive on H and B is an increasing sequence in H with union A belonging to H. Then, we use the uniform convergence just proved to interchange limits :

$$m_K(A) = \lim_E m(A \cap E) = \lim_E \lim_i m(B_i \cap E) = \lim_i \lim_E m(B_i \cap E)$$
$$= \lim_i m_K(B_i).$$

Thus, m_K is σ-additive on H. Since $m_K^\circ = m - m_K$ on H, it too is σ-additive on H.

Suppose that m is s-bounded, let A be a disjoint sequence in H, and let V in $\mathcal{V}$ be given. Using the definition of m_K, for each i in N, choose E_i in K, contained in A_i with $m_K(A_i) - m(E_i) \in V$. Since m is s-bounded, $m(E_i) \to 0$, so eventually $m_K(A_i) \in V + V$. This shows that m_K (and hence m_K°) is s-bounded on H.

It is evident that m_K° vanishes on K. If, in addition, $m_K(S)$ exists, we repeat an argument given above to prove that $\lim_E m(A \cap E)$ exists uniformly for A in H $\cup$ {S}. Thus, we may choose an E in K such that, for all A in H, $m_K(A) \in m(A \cap E) + V$, so $m_K | H$ is nearly supported on K.

To prove the remarks about the metrizable case, let V_i, $i \in N$, form a base for the neighborhoods of 0 in Y. Again using uniform convergence, we choose, for each i in N, E'_i in K such that $E'_{i+1} \supset E'_i$ and $m_K(A) - m(A \cap E'_i) \in V_i$, for all A in H $\cup$ {S}. Then, for all A in H $\cup$ {S}, $m_K(A) = \lim_i m(A \cap E'_i)$. Let E be the union of the E'_i and let A be in H $\cup$ {S}. Then,

(a) if K is a σ-ideal, we have $m_K(A \backslash E) = \lim_i m(A \backslash E \cap E'_i) = 0$ so $m_K(A) = m_K(A \cap E) = m(A \cap E)$;

(b) if K is not necessarily a σ-ideal, but H is a σ-ring on which m is σ-additive, then $m_K(A) = \lim_i m(A \cap E'_i) = m(A \cap E)$. Thus, in either case, $m_K(A) = m(A \cap E)$, for all A in H $\cup$ {S}. Also, for A in H, $m_K^\circ(A) = m(A) - m_K(A) = m(A \backslash E)$. This completes the proof.

The essential work of establishing the decomposition theorem is now done. We need only introduce hypotheses ensuring that m_K and m_K° exist.

1.7. THEOREM. – *Let m be a finitely additive s-bounded function on the ring* H *with values in* Y. *If* K *is an ideal of* H *and* $m[K]$ *has complete closure* (1), *then m is the sum of unique finitely additive functions* m_1 *and* m_2 *on* H *to* Y *such that* m_1 *is nearly supported on* K *and* m_2 *vanishes on* K. *The* m_i *are s-bounded ; σ-additive if m is σ-additive.*

(1) In case Y is a quasi-complete locally convex space, this condition is automatically satisfied, since an s-bounded additive function is bounded.

In case Y is metrizable and either

(a) K is a σ-ideal (so that $K_\sigma = K$), or

(b) H is a σ-ring on which m is σ-additive,

the completeness condition may be dropped. Then, there exists an E in K_σ such that $m_1(A) = m(A \cap E)$ and $m_2(A) = m(A \setminus E)$, for all A in H.

Proof. – we need only prove that, under the completeness condition, or if Y is metrizable and (a) or (b) holds, $m_K(A)$ exists in Y, for all A in $H \cup \{S\}$. The other statements follow immediately from 1.5 and 1.6.

Now, for all A in $H \cup \{S\}$, lemma 1.2 implies that for every increasing sequence E in K, $m(A \cap E_i)$ forms a Cauchy sequence. Hence, the elements $m(A \cap E)$, E in K, form a Cauchy net, (Sion (2, lemma 2.5)). Under the completeness condition, this net converges.

If Y is metrizable, but not necessarily complete, we think of Y embedded in its completion $\bar{Y}$. Then, $m_K(A)$ exists in $\bar{Y}$, for all A in $H \cup \{S\}$. But then, if (a) or (b) holds, there exists, by 1.6, E in K_σ such that, for all A in $H \cup \{S\}$, $m_K(A) = m(A \cap E)$, which is already in Y.

2. Lebesgue decomposition.

In this section, we apply the general theory to obtain two Lebesgue-type decomposition theorems. In addition to the previous notation, Z is another commutative topological group, and n is an additive function on H to Z.

2.1. DEFINITIONS.

1) N is n-null iff $N \in H$ and $n(A) = 0$, whenever $N \supset A \in H$; $\mathcal{N}_n = \{N : N \text{ is n-null}\}$.

2) m is n-continuous iff $m(N) = 0$, for all N in $\mathcal{N}_n$.

3) m is nearly n-singular iff for each V in $\mathcal{V}$, there exists N in $\mathcal{N}_n$ with $m(A \setminus N) \in V$, for all A in H.

4) m is n-singular iff there exists N in $\mathcal{N}_n$ with $m(A \setminus N) = 0$, for all A in H.

The Lebesgue decomposition theorem for this type of n-continuity is the following :

2.2. THEOREM. – *Let m be finitely additive and s-bounded on* H *to* Y *and n be finitely additive on* H *to* Z. *If the range of m has complete closure, then there exist unique finitely additive functions* m_n *and* m'_n *on* H *to* Y *such that* $m = m_n + m'_n$, m_n is *n-continuous, and* m'_n *is nearly n-singular. Both are s-bounded functions ; σ-additive, if m is σ-additive.*

Suppose, in addition, that Y *is metrizable and* H *is a* σ-ring.

(a) *If n is σ-additive, then* m'_n *is n-singular.*

(b) *If m is σ-additive, then there exists a countable union* E *of n-null sets with* $m'_n(A\backslash E) = 0$, *for all* A *in* H.

In these two cases, the completeness condition may be dropped.

Proof. – Let $K = \mathfrak{N}_n$. Then K is an ideal in H. If H is a σ-ring on which K is σ-additive, then K is a σ-ideal. The theorem now follows from the definitions and theorem 1.7.

The following alternative notion of absolute continuity is often used for additive functions.

2.3. DEFINITIONS.

1) *m is topologically n-continuous iff for every* V *in* $\mathcal{V}$, *there exists a neighborhood* W *of* 0 *in* Z *such that* $m(A) \in V$, *whenever* $A \in H$ *and* $n(E) \in W$ *for all* E *in* H *contained in* A.

2) *For* A *in* H, m_A *denotes the restriction of m to* A :

$$m_A(E) = m(A \cap E), \quad \textit{for all} \quad E \textit{ in } H.$$

3) *We say that m is nowhere topologically n-continuous iff the only members* A *of* H *for which* m_A *is topologically n-continuous are the m-null sets.*

Remark. – In general, the two types of n-continuity are distinct. However, if H is a σ-ring, *n* and *m* are σ-additive and Z is metrizable, the two notions coincide [4].

The following is the Lebesgue decomposition theorem for topological n-continuity.

2.4. THEOREM. – *Let m and n be finitely additive on* H *to* Y *and* Z, *respectively. If m is s-bounded and the range of m has complete closure, then there exist unique finitely additive functions* m_c *and* m'_c *such that* $m = m_c + m'_c$, m_c *is topologically n-continuous, and* m'_c *is nowhere topologically n-continuous. Both are s-bounded ; they are σ-additive if m is σ-additive.*

If Y *is metrizable,* H *is σ*-ring, *and one of m and n is σ-additive , the completeness condition may be dropped. In this event, there is an* E *in* H *such that* $m_c = m(\mathrm{A} \cap \mathrm{E})$ *and* $m'_c(\mathrm{A}) = m(\mathrm{A} \backslash \mathrm{E})$, *for all* A *in* H.

Proof. – Let $\mathrm{K} = \{\mathrm{E} \in \mathrm{H} : m_{\mathrm{E}}$ is topologically n-continuous$\}$. Then, K is an ideal of H. Using 1.7, let m_c be the part of m nearly supported on K and let m'_c be the part of m which vanishes on K. To show that m_c is topologically n-continuous, given V in $\mathcal{V}$, let E in K be so large that, for all A in H, $m_c(\mathrm{A}\backslash\mathrm{E}) \in \mathrm{V}$. Since $\mathrm{E} \in \mathrm{K}$, there exists a neighborhood W of 0 in Z such that, whenever $\mathrm{A} \in \mathrm{H}$ and $n(\mathrm{B}) \in \mathrm{W}$ for all B in H contained in A, we have $m_{\mathrm{E}}(\mathrm{A}) \in \mathrm{V}$. Then, for such an A,

$$\begin{aligned} m_c(\mathrm{A}) &= m_c(\mathrm{E} \cap \mathrm{A}) + m_c(\mathrm{A}\backslash\mathrm{E}) \\ &= m(\mathrm{E} \cap \mathrm{A}) + m_c(\mathrm{A}\backslash\mathrm{E}) \\ &\in \mathrm{V} + \mathrm{V}. \end{aligned}$$

To show that m'_c is nowhere topologically continuous, let $\mathrm{A} \in \mathrm{H}$ and suppose $(m'_c)_{\mathrm{A}}$ is topologically n-continuous. Then A is in K, since $m_{\mathrm{A}} = (m_c)_{\mathrm{A}} + (m'_c)_{\mathrm{A}}$. But m'_c vanishes on K so A is m'_c-null.

The other properties are direct consequences of 1.7.

3. Other applications and remarks.

In this section we wish to indicate briefly some other applications, of the decomposition theorems. The symbols m, H, Y, and $\mathcal{V}$ have their usual meanings.

3.1. *Atomic functions.* – An element A of H is called an m-atom iff for every B in H either $A \cap B$ or $A \setminus B$ is m-null. The function m is called nearly atomic iff, for every V in $\mathcal{V}$, there is a finite union E of m-atoms such that $m(A \setminus E) \in V$, for all A in H ; it is called atomless iff every m-atom is m-null.

Applying 1.7, taking for K the collection of finite unions of m-atoms, we can decompose an s-bounded additive function m into atomless and nearly atomic parts, provided its range has complete closure. In the metrizable situation, if m is σ-additive on H and H is σ-ring, completeness isn't needed, and the decomposition takes the form $m(A) = m(A \cap E) + m(A \setminus E)$, where E is a countable union of m-atoms.

If H is a σ-ring, there is a less interesting theorem, taking for K the collection of countable unions of m-atoms. In this case, we obtain no information from the individual atoms about the values of the part supported on K, (unless, of course, m is σ-additive, in which case the decomposition is the same as that above).

3.2. *σ-additive functions.* – Theorem 1.7 may be used to obtain a representation of m as the sum of a σ-additive function and a function which is "nowhere σ-additive" in the sense that, for no non-null E in H, is its restriction to E σ-additive. However, there is a much stronger result, using a variation of the Carathéodory process [5].

3.3. *Capacitability and outer regularity.* – In the situation of the decomposition theorems, if m has certain approximation properties, say by compact sets from within or by open sets from without, these are retained by each of the two parts. This is because of the uniform convergence property mentioned in the proof of 1.6.

3.4. *Outer measures.* – Again because of uniform convergence properties, if one begins a G-outer measure μ (see Sion [3]) and an ideal in the μ-measurable sets, one may obtain a representation of μ as the sum of two G-outer measures - one vanishing on the ideal, the other nearly supported on it.

We leave the details to the reader.

BIBLIOGRAPHIE

[1] C.E. RICKART, Decomposition of additive set functions, *Duke Math. Jour.*, 10 (1943), 653-665.

[2] M. SION, Outer measures with values in a topological group, *Proc. Lond. Math. Soc.* (3), 19 (1969), 89-106.

[3] M. SION, Group-valued outer measures, International Congress of Mathematicians, Nice, 1970.

[4] T. TRAYNOR, Absolute continuity for group-valued measures (to appear), *Can. Math. Bull.*, 1973.

[5] T. TRAYNOR, A general Hewitt-Yosida Decomposition, (to appear). *Can. Jour. Math.*

Manuscrit reçu le 26 mai 1971
accepté par G. CHOQUET

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