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Some examples on quasi-barrelled spaces

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SOME EXAMPLES 
ON QUASI-BARRELLED SPACES (1) 

by Manuel VALDIVIA 

J. Dieudonné has proved in [2] the following theorem: 
a) Let $E$ be a bornological space. If $F$ is a subspace of $E$, of finite codimension, then $F$ is bornological.

We have given in [6] and [7], respectively, the following results:
b) Let $E$ be a quasi-barrelled space. If $F$ is a subspace of $E$, of finite codimension, then $F$ is quasi-barrelled.

c) Let $E$ be an ultrabornological space. If $F$ is a subspace of $E$, of infinite countable codimension, then $F$ is bornological.

The results a), b) and c) lead to the question if the results a) and b) will be true in the case of being $F$ a subspace of infinite countable codimension. In this paper we give an example of a bornological space $E$, which has a subspace $F$, of infinite countable codimension, such that $F$ is not quasi-barrelled.

In [8] we have proved the two following theorems:
d) Let $E$ be a $DF$-space. If $G$ is a subspace of $E$, of finite codimension, then $G$ is a $DF$-space.

e) Let $E$ be a sequentially complete $DF$-space. If $G$ is a subspace of $E$, of infinite countable codimension, then $G$ is a $DF$-space.

Another question is if the result d) is also true for subspaces of infinite countable codimension. Here we give an example of a quasi-barrelled $DF$-space, which has a subspace $G$, of infinite countable codimension, which is not a $DF$-space.

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N. Bourbaki, [1, p. 35], notices that it is not known if every bornological barrelled space is ultrabornological. In [9] we have proved that if $E$ is the topological product of an infinite family of bornological barrelled space, of non-zero dimension, there exists an infinite number of bornological barrelled subspaces of $E$, which are not ultrabornological. In this paper we give an example of a bornological barrelled space, which is not inductive limit of Baire spaces.

We use here vector spaces on the field $K$ of real or complex numbers. The topologies on these spaces are separated.

In [10] we have proved the following result:

f) Let $E$ be a barrelled space. If $\{E_n\}_{n=1}^{\infty}$ is an increasing sequence of subspaces of $E$, such that $\bigcup_{n=1}^{\infty} E_n = E$, then $E$ is the inductive limit of $\{E_n\}_{n=1}^{\infty}$.

**Theorem 1.** Let $E$ be the strict inductive limit of an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of metrizable locally convex spaces. Let $F$ be a sequentially dense subspace of $E$. If $E$ is barrelled, then $F$ is bornological.

**Proof.** Let $E_n$, $n = 1, 2, \ldots$, be the closure of $E_n$ in $E$. Obviously $E$ is the strict inductive limit of the sequence $\{E_n\}_{n=1}^{\infty}$. Let $F_n$ be the closure in $E$ of $F \cap E_n$, $n = 1, 2, \ldots$ If $x \in E$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points of $F$, which converges to $x$. Since the set of points of this sequence is bounded, there exists a positive integer $n_0$ such that $x_n \in E_{n_0}$, $n = 1, 2, \ldots$, and, therefore, $x \in F_{n_0}$. Hence $E = \bigcup_{n=1}^{\infty} F_n$. Since $E$ is barrelled, applying the result f), we obtain that $E$ is the strict inductive limit of the sequence $\{F_n\}_{n=1}^{\infty}$.

Given any Banach space $L$ and a linear and locally bounded mapping $u$ from $F$ into $L$, we must to prove that $u$ is continuous. Let $u_n$ be the restriction of $u$ to $F \cap E_n$. Since $F \cap E_n$ is a metrizable space and $u_n$ is locally bounded, $u_n$ is continuous. Let $v_n$ be the continuous extension of $u_n$ to $F_n$. Let $v$ be the linear mapping from $E$ into $L$, which coincides with $v_n$ in $F_n$, $n = 1, 2, \ldots$. Since $v_n$ is
equal to $v_{n+1}$ on $F \cap E_n$, then they are equal on $F_n$ and, therefore, $v$ is well defined. Since $E$ is the inductive limit of $\{F_n\}_{n=1}^{\infty}$ and since the restriction of $v$ to $F_n$ is continuous, $n = 1, 2, \ldots$, then $v$ is continuous. On other hand $u$ is the restriction of $v$ to $F$ and, therefore, $u$ is continuous. Q.E.D.

Example 1. — A. Grothendieck, [3], has given an example of a space $E$, which is strict inductive limit of an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of separable Frechet spaces, so that there exists in $E$ a non-closed subspace $G$, such that $G \cap E_n$ in closed, $n = 1, 2, \ldots$. In this example let $A_n^\ast$ be a countable set of $E_n^\ast$, dense in $E_n^\ast$. Let $P$ be the linear space generated by $\bigcup_{n=1}^{\infty} A_n^\ast$. Let $F$ be the linear hull of $P \cup G$. Since $P$ is sequentially dense in $E$, applying Theorem 1, it results that $F$ is a bornological space. Applying theorem $f$) it results that $G$ is not barrelled and since $G$ is quasicomplete, then $G$ is not quasi-barrelled. Since $P$ has a countable basis, $G$ is a subspace of $F$, of countable codimension, and by $a)$ the codimension of $G$ is infinite. Therefore, $F$ is a bornological space, which has a subspace $G$, of infinite countable codimension, so that $G$ is not quasi-barrelled.

Example 2. — G. Kothe, [4, p. 433-434] gives an example of a Montel $DF$-space, which has a closed subspace $L$, which is not a $DF$-space. In this example, let $\{B_n\}_{n=1}^{\infty}$ be a fundamental sequence of bounded sets. Since $E$ is a Montel $DF$-space, then $B_n$ is separable, $n = 1, 2, \ldots$. Let $A_n^\ast$ be a countable subset of $B_n^\ast$, dense in $B_n^\ast$, $n = 1, 2, \ldots$ Let $Q$ be the linear space generated by $\bigcup_{n=1}^{\infty} A_n^\ast$. Let $M$ be the linear hull of $Q \cup L$. Now, we shall prove that $M$ is quasi-barrelled. Indeed, given a closed, absolutely convex and bornivorous set $U$ in $M$, let $\bar{U}$ be its closure in $E$. If $x \in E$, there exists a positive integer $n_0$, such that $x \in B_{n_0}$ and, therefore, $x$ is in the closure of $A_{n_0}^\ast$. Hence, there exists a $\lambda \in K$, $\lambda > 0$, such that $\lambda \cdot x \in \bar{U}$, i.e. $\bar{U}$ is a barrel in $E$, and therefore,
U = U ∩ M is a neighborhood of the origin in M. Since Q has a countable basis, L is a subspace of M, of countable codimension, and by d), the codimension of L is infinite. The space M is, therefore, an example of quasi-barrelled space which has a subspace L, of infinite countable codimension, so that L is not a barrelled space.

We say that a subspace E of F is locally dense if, for every x ∈ F, there exists a sequence \( \{x_n\}_{n=1}^\infty \) of points of E, which converges to x in the Mackey sense. In [9] we have proved the following result:

g) Let F be a locally convex space. If E is a bornological locally dense subspace of F, then F is bornological.

**Theorem 2.** — Let E be a bornological barrelled space which has a family \( \{E_n\}_{n=1}^\infty \) of subspaces, which satisfy the following conditions:

I. \( \bigcup_{n=1}^\infty E_n = E. \)

II. For every positive integer n, there exists a topology \( \mathcal{V}_n \) on \( E_n \), finer than the initial one, so that \( E_n[\mathcal{V}_n] \) is a Frechet space.

III. — There exists in E a bounded set A, such that A ⊈ E_n, \( n = 1, 2, \ldots \)

Then there exists a bornological barrelled space F, which is not inductive limit of Baire spaces, so that E is a hyperplane of F.

**Proof.** — Let B be the closed, absolutely convex hull of A and let \( u \) be the canonical injection of \( E_B \) in E. If \( E_B \) is a Banach space, there exists, according to a theorem of Grothendieck, [4] or [5, p. 225], a positive integer \( n_1 \), such that \( u(E_B) = E_B \subseteq E_{n_1} \), hence A ⊈ E_{n_1}, which is in contradiction with the condition III. We take in \( E_B \) a Cauchy sequence \( \{x_n\}_{n=1}^\infty \) which is not convergent. Let \( \hat{B} \) be the closure of B in the completion \( \hat{E} \) of E. Since the topology of the Banach space \( \hat{E}_B \) induces in \( E_B \) a topology coarser than the initial one, \( \{x_n\}_{n=1}^\infty \) converges in \( \hat{E}_B \) to an element
x. Since the set \( M = \{x_1, x_2, \ldots\} \) is bounded in \( E_B \), there exists a \( \lambda \in K \), such that \( M \subseteq \lambda B \) and, therefore, if \( x \in E \), then \( x \in \lambda B \subseteq E_B \), hence, according to a result of N. Bourbaki, [5, p. 210-211], \( \{x_n\}_{n=1}^\infty \) converges to \( x \) in \( E_B \). This is a contradiction and, therefore, \( x \not\in E \). Let \( F \) be the space generated by \( E \cup \{x\} \), equipped with the topology induced by \( E \). Obviously \( F \) is a barrelled space and, according to result g), \( F \) is bornological.

Finally we need to prove that \( F \) is not inductive limit of Baire spaces. Suppose that there exists in \( F \) a family \( \{F_i : i \in I\} \) of subspaces, which union is \( F \), so that for every \( i \in I \), there exists a topology \( \mathcal{U}_i \) on \( F_i \), such that \( F_i[\mathcal{U}_i] \) is a Baire space and \( F \) is the locally convex hull of \( \{F_i[\mathcal{U}_i] : i \in I\} \). Since \( E \) is a dense hyperplane of \( F \), there exists an index \( i_0 \in I \), such that \( E \cap F_{i_0} \) is a dense hyperplane of \( F_{i_0}[\mathcal{U}_{i_0}] \). Let \( G \) be the vector space \( E \cap F_{i_0} \) with the topology induced by \( \mathcal{U}_{i_0} \) and let \( x_0 \) be an element of \( F_{i_0} \), which is not in \( G \). If \( \nu \) is the canonical injection of \( G \) in \( E \), \( \nu \) is continuous. Let \( G_n \) and \( H_n \) be the spaces \( G \cap \nu^{-1}(E_n) \) and that generated by \( (G \cap \nu^{-1}(E_n)) \cup \{x_0\} \), respectively, equipped with the topologies induced by \( \mathcal{U}_{i_0} \).

Obviously \( F_{i_0} = \bigcup_{n=1}^\infty H_n \) and, therefore, there exists a positive integer \( n_0 \) such that \( H_{n_0} \) is of the second category in \( F_{i_0}[\mathcal{U}_{i_0}] \). If \( \nu_{i_0} \) is the restriction of \( \nu \) to \( G_{n_0} \), the graph of \( \nu_{i_0} \) is closed in \( G_{n_0} \times E_{n_0}[\mathcal{U}_{n_0}] \) and, since \( G_{n_0} \) is barrelled and \( E_{n_0}[\mathcal{U}_{n_0}] \) is a Frechet space, \( \nu_{i_0} \) is continuous from \( G_{n_0} \) into \( E_{n_0}[\mathcal{U}_{n_0}] \). If \( \{y_m : m \in D\} \) is a net of elements of \( G_{n_0} \), which converges to \( y \in F_{i_0}[\mathcal{U}_{i_0}] \), then \( \{\nu_{i_0}(y_m) = y_m : m \in D\} \) is a Cauchy net in the Frechet space \( E_{n_0}[\mathcal{U}_{n_0}] \), which converges to \( z \), hence \( y = z \) and \( G_{n_0} \) is closed in \( F_{i_0}[\mathcal{U}_{i_0}] \). Also \( H_{n_0} \) is closed in \( F_{i_0}[\mathcal{U}_{i_0}] \) and since \( H_{n_0} \) is of the second category in \( F_{i_0}[\mathcal{U}_{i_0}] \), then \( H_{n_0} = F_{i_0}[\mathcal{U}_{i_0}] \) and, therefore, \( G_{n_0} = G \).

Finally, taking the net \( \{y_m : m \in D\} \) converging to \( x_0 \), it results that \( x_0 \in E \), which is not true. Hence \( F \) is not inductive limit of Baire spaces.

Example 3. — G. Kothe has given an example of a non-complete \((LB)\)-space, which is defined by a sequence \( \{E_n\}_{n=1}^\infty \).
of Banach spaces, so that there exists a bounded set $A$ in $E$, which is not subset of $E_n$, $n = 1, 2, \ldots$. This example, and our Theorem 2, assure the existence of bornological barrelled spaces which are not inductive limits of Baire spaces.

**BIBLIOGRAPHY**


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