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CONSTRUCTING MANIFOLDS BY HOMOTOPY EQUIVALENCES I. AN OBSTRUCTION TO CONSTRUCTING PL-MANIFOLDS FROM HOMOLOGY MANIFOLDS

by Hajime SATO

0. Introduction.

A homology manifold can be given a canonical cell complex structure, where each cell is a contractible homology manifold. In this paper, given a homology manifold M, we aim at constructing a PL-manifold with a cell complex structure, where each cell is an acyclic PL-manifold, which is cellularly equivalent to the canonical cell complex structure of M. We obtain a theorem that, if the dimension n of M is greater than 4 and if the boundary ∂M is a PL-manifold or empty, there is a unique obstruction element in $H_{n-4}(M; \mathcal{H}^3)$, where $\mathcal{H}^3$ is the group of 3-dimensional PL-homology spheres modulo those which are the boundary of an acyclic PL-manifold. If the manifold is compact, the constructed PL-manifold is simple homotopy equivalent to M.

I have heard that similar results have been obtained independently and previously by M. Cohen and D. Sullivan, refer [1] and [9].

I would like to thank Professors V. Poénaru and F. Laudenbach for their kind support.

1. Definition of homology manifold with boundary([1]).

Let K be a locally finite simplicial complex and let σ be a simplex of K. We define the subcomplexes of K as follows.

([1]) We can refer the chapter 5 of the book : C.R.F. Maunder, "Algebraic topology", Van Nostrand, London (1970).

$$St(\sigma, K) = St(\sigma) = \{\tau \in K\,,\ \exists\,\alpha > \tau\,,\ \alpha > \sigma\}$$
$$\partial St(\sigma, K) = \partial St(\sigma) = \{\tau \in St(\sigma)\,,\ \tau \not> \sigma\}$$
$$Lk(\sigma, K) = Lk(\sigma) = \{\tau \in St(\sigma)\,,\ \tau \cap \sigma = \emptyset\}$$

We write by K', K'', the first and the second barycentric subdivisions of K.

Let M be a locally finite full simplicial complex of dimension n. We say that M is a homology manifold of dimension n if the following equivalent condition holds :

LEMMA 1. – *The followings are equivalent :*

i) *for any simplex σ of dimension p,*

$$\widetilde{H}_i(Lk(\sigma, M)) = \widetilde{H}_i(S^{n-p-1}) \quad \text{or} \quad 0.$$

ii) *for any simplex σ of dimension p,*

$$\widetilde{H}_i(St(\sigma, M)/\partial St(\sigma, M)) = \widetilde{H}_i(S^n) \quad \text{or} \quad 0.$$

iii) *for any point x of $|M|$, where $|M|$ denotes the underlying topological space of* M,

$$H_i(|M|, |M| - x) = \widetilde{H}_i(S^n) \quad \text{or} \quad 0.$$

The definition is invariant by the PL-homeomorphism in the category of simplicial complexes.

LEMMA 2. – *For any p-simplex σ of* M, $Lk(\sigma, M)$ *is a compact $(n-p-1)$-dimensional homology manifold.*

Proof. – It is compact because M is locally finite. Let τ be a q-simplex of $Lk(\sigma, M)$. We have

$$Lk(\tau, Lk(\sigma, M)) = Lk(\tau\sigma, M)\ .$$

Hence $\widetilde{H}_i(Lk(\tau, Lk(\sigma, M))) = \widetilde{H}_i(S^{n-p-q-1})$ or 0, which completes the proof.

Let us define the subset ∂M of M by

$$\partial M = \{\sigma \in M \mid \widetilde{H}_i(\sigma, M) = 0\}$$

We call it as the boundary of M. If $\partial M = \phi$, the manifold is classical and the following Poincaré duality is well known (see for example [7 ; (7,4)]).

LEMMA 3. – *Let* M *be an orientable compact n-dimensional homology manifold without boundary. Let* $A_1 \supset A_2$ *be subcomplexes of* M. *Then we have the isomorphism*

$$H^i(A_1, A_2) = H_{n-i}(|M| - |A_2|, |M| - |A_1|).$$

Using this we will prove the followings. By lemma 2, for p-simplex σ, $Lk(\sigma, M)$ is a homology manifold and we can define $\partial Lk(\sigma, M)$.

LEMMA 4. – *If* $\partial Lk(\sigma, M) \neq \emptyset$, $Lk(\sigma, M)$ *is acyclic and* $\partial Lk(\sigma, M)$ *is an* $(n - p - 2)$*-dimensional homology manifold such that*

$$\widetilde{H}_i(\partial Lk(\sigma, M)) = \widetilde{H}_i(S^{n-p-2}) .$$

PROPOSITION 5. – *If* $\partial M \neq \emptyset$, ∂M *is a subcomplex and is an* $(n - 1)$*-dimensional homology manifold without boundary.*

We prove that lemma 4 for $n = k$ implies proposition 5 for $n = k$ and proposition 5 for $n \leqslant k$ implies lemma 4 for $n = k + 1$. Since lemma 4 holds for $n = 1$, we can continue by induction.

Lemma $4_{n=k}$ $\Rightarrow$ Proposition $5_{n=k}$. Let σ be a p-simplex of ∂M and let $\sigma_0 < \sigma$. Then we can write $\sigma = \sigma_0 \sigma_1$. We have

$$\widetilde{H}_*(Lk(\sigma_1, Lk(\sigma_0, M))) = \widetilde{H}_*(Lk(\sigma, M)) = 0 ,$$

which shows that $\sigma_1 \in \partial Lk(\sigma_0, M)$ and so $\partial Lk(\sigma_0, M) \neq \emptyset$. By the lemma 4, $Lk(\sigma_0, M)$ is acyclic and it follows that $\sigma_0 \in \partial M$. Hence ∂M is a well-defined subcomplex of M. A q-simplex τ of $Lk(\sigma, M)$ is in $Lk(\sigma, \partial M)$ if and only if $\widetilde{H}_i(Lk(\tau\sigma, M)) = 0$. Since

$$Lk(\tau\sigma, M) = Lk(\tau, Lk(\sigma, M)) ,$$

it is equivalent to that τ belongs to $\partial Lk(\sigma, M)$. Hence the complex $Lk(\sigma, \partial M)$ coïncides with $\partial Lk(\sigma, M)$. By lemma 4_k, we have $\widetilde{H}_i(\partial Lk(\sigma, M)) = \widetilde{H}_i(S^{k-p-2})$, which shows that ∂M is a $(k - 1)$-dimensional homology manifold without boundary.

Proposition $5_{n \leqslant k}$ $\Rightarrow$ Lemma $4_{n=k+1}$. Let M be a homology manifold of dimension $k + 1$. Let σ be a p-simplex of M. By lemma 2,

$Lk(\sigma, M)$ is a homology manifold of dimension $k-p$. By proposition 5 for $n = k-p$, $\partial Lk(\sigma, M)$ is a $(k-p-1)$-dimensional homology manifold without boundary if it is not empty. Let $2Lk(\sigma, M)$ be the double of $Lk(\sigma, M)$, i.e.,

$$2Lk(\sigma, M) = Lk(\sigma, M) \bigcup_{\partial Lk(\sigma, M)} Lk(\sigma, M) .$$

Let τ be a q-simplex of $2Lk(\sigma, M)$. If τ is not a simplex of $\partial Lk(\sigma, M)$, clearly,

$$\widetilde{H}_i(Lk(\tau, 2Lk(\sigma, M))) = \widetilde{H}_i(Lk(\tau, Lk(\sigma, M))) = \widetilde{H}_i(S^{k-p-q-1})$$

If τ is a simplex of $\partial Lk(\sigma, M)$, we have

$$Lk(\tau, 2Lk(\sigma, M))$$
$$= Lk(\tau, Lk(\sigma, M)) \bigcup_{Lk(\tau, \partial Lk(\sigma, M))} Lk(\tau, Lk(\sigma, M)) .$$

By definition $\widetilde{H}_i(Lk(\tau, Lk(\sigma, M))) = 0$ and by the proposition 5 for $n = k-p-1$, we have

$$\widetilde{H}_i(Lk(\tau, \partial Lk(\sigma, M))) = \widetilde{H}_i(S^{k-p-q-2}) .$$

Hence in any case $\widetilde{H}_i(Lk(\tau, 2Lk(\sigma, M))) = \widetilde{H}_i(S^{k-p-q-1})$, which shows that $2Lk(\sigma, M)$ is a $(k-p)$-dimensional homology manifold without boundary. Applying lemma 3, we have

$$H^i(Lk(\sigma, M), \partial Lk(\sigma, M)) = H_{k-p-i}(|Lk(\sigma, M)| - |\partial Lk(\sigma, M)|) .$$

Notice that for any homology manifold M, $H_i(|M| - |\partial M|) = H_i(M)$. Hence $H^i(Lk(\sigma, M), \partial Lk(\sigma, M)) = H_{k-p-i}(S^{k-p})$ or $H_{k-p-i}(pt.)$. But if it is isomorphic to $H_{k-p-i}(S^{k-p})$, we have

$$H^0(Lk(\sigma, M), \partial Lk(\sigma, M)) = Z ,$$

which contradicts to the definition that $\widetilde{H}_0(Lk(\sigma, M)) = 0$. Hence $Lk(\sigma, M)$ is acyclic and consequently $\widetilde{H}_i(\partial Lk(\sigma, M)) = \widetilde{H}_i(S^{k-p-1})$, which completes the proof.

2. Cell decomposition of a homology manifold.

We mean by a homology cell (resp. pseudo homology cell) of dimension n or homology n-cell (resp. pseudo homology n-cell) a compact contractible (resp. acyclic) homology manifold of dimension n with a boundary, the boundary being a homology sphere but not necessarily simply connected. A (pseudo) homology cell complex is a complex K with a locally finite family of (pseudo) homology cells $C = \{C_\alpha\}$, such that :

i) $K = \cup C_\alpha$

ii) $C_\alpha, C_\beta \in C$ implies ∂C_α, $C_\alpha \cap C_\beta$ are unions of cells in C

iii) If $\alpha \neq \beta$, then $\mathrm{Int}\, C_\alpha \cap \mathrm{Int}\, C_\beta = \emptyset$.

If a homology manifold M has a (pseudo) homology cell complex structure, we call it a (pseudo) cellular decomposition of M. Two (pseudo) homology cell complexes $K = \cup C_\alpha$, $K' = \cup C'_\alpha$ are isomorphic if there exists a bijection $k : C \to C'$ such that both k and k^{-1} are incidence preserving. In such a case we say that they are cellularly equivalent.

Now we have the following :

PROPOSITION 1. – *If two finite homology cell complexes* K, K′ *are cellularly equivalent, then they are simple homotopy equivalent.*

We can define a simplicial map $f : K \to K'$ inductively by the dimension of the cells. Hence it is sufficient to prove the following lemma.

LEMMA 2. – *Let* $A_j^i (j = 1, 2, \ldots, r)$ *be subcomplex of simplicial complexes* B^i *for* $i = 1, 2$ *respectively such that* $B^i = \bigcup_j A_j^i$, *and let* $f : B^1 \to B^2$ *be a simplicial map. For any subset* s *of* $\{1, 2, \ldots, r\}$, *let* $A_s^i = \bigcap_{j \in s} A_j^i$ *and let* f_s be the restriction of f on A_s. *If* f_s *is a mapping from* A_s^1 *to* A_s^2 *which is a simple homotopy equivalence for any* s, *then* f *itself is a simple homotopy equivalence.*

Proof. – First suppose that $r = 2$. We have the exact sequence

$$0 \to C_*(A_1^i) \to C_*(B^i) \to C_*(A_1^i \cap A_2^i)) \to 0$$

of the chain complexes. Let $g : A_2^1/(A_1^1 \cap A_2^1) \to A_2^2/(A_1^2 \cap A_2^2)$ be the map induced by f and let us denote by $w(\ )$ the Whitehead torsion. Then by theorem 10 of [8], we have

$$w(f) = w(f_{\{1\}}) + w(g)\ .$$

Remark here that f and g can easily be seen to be homotopy equivalences. Further we have the exact sequence

$$0 \to C_*(A_1^i \cap A_2^i) \to C_*(A_2^i) \to C_*(A_2^i/(A_1^i \cap A_2^i)) \to 0$$

which shows that

$$w(f_{\{2\}}) = w(f_{\{1,2\}}) + w(g)\ .$$

Since $w(f_{\{1\}}) = w(f_{\{2\}}) = w(f_{\{1,2\}}) = 0$, we have $w(f) = 0$. If $r \geqslant 3$, we can repeat this argument, which shows that f is a simple homotopy equivalence for any r.

Now let σ be a simplex of a locally finite simplicial complex K. We denote by $b_\sigma \in K'$ its barycenter. We define dualcomplex $D(\sigma)$ and its subcomplex $\delta D(\sigma)$ which are subcomplexes of K' by

$$D(\sigma) = D(\sigma, K) = \{b_{\sigma_0} \dots b_{\sigma_r} \mid \sigma < \sigma_0 < \cdots < \sigma_r \in K\}$$

$$\delta D(\sigma) = \delta D(\sigma, K) = \{b_{\sigma_0} \dots b_{\sigma_r} \mid \sigma \lneqq \sigma_0 < \cdots < \sigma_r \in K\}$$

The followings are easy to see.

i) if $\sigma < \sigma' \Rightarrow D(\sigma) \supset D(\sigma')$

ii) $D(\sigma) = b_\sigma * \delta D(\sigma)$

iii) $\delta D(\sigma) = \bigcup_\tau D(\tau)$ where $\tau > \sigma$ and $\tau \neq \sigma$

iv) $\delta D(\sigma)$ is isomorphic to $Lk(\sigma, K)'$.

Let M be a homology manifold. For each simplex

$$\sigma = b_{\sigma_0} b_{\sigma_1} \dots b_{\sigma_r}$$

of M', where $\sigma_0^{n_0} < \sigma_1^{n_1} < \cdots < \sigma_r^{n_r}$ are a set of simplexes of M, we have the duall cell $D(\sigma, M')$. It is a compact homology manifold by lemma 2 of § 1. Further we have

$$\begin{aligned} \delta D(\sigma, M') &\cong Lk(\sigma, M) \\ &\cong Lk(\sigma, \sigma_r) * Lk(\sigma_r, M) \\ &\cong S^{n_r - r - 1} * Lk(\sigma_r, M) \\ &\cong Lk(\sigma_r, M) \times D^{n_r - r} \cup (Lk(\sigma_r, M) * (pt.)) \times S^{n_r - r - 1}\ , \end{aligned}$$

where $\cong$ denotes that both sides are PL-homeomorphic and let

$$d_\sigma : \delta D(\sigma, M') \to Lk(\sigma_r, M) \times D^{n_r - r} \cup (Lk(\sigma_r, M) * (pt.)) \times S^{n_r - r - 1}$$

be the PL-homeomorphism, which we call the trivialization of $\delta D(\sigma, M')$. If σ is not in ∂M, $\delta D(\sigma, M')$ is a homology manifold whose homology groups are isomorphic to those of S^{n-1}, boundary being empty. If $\sigma \in \partial M$, $\delta D(\sigma, M')$ is an acyclic homology manifold with the boundary $Lk(\sigma, \partial M'')$ which is PL-homeomorphic to $\partial Lk(\sigma_r, M) \times D^{n_r - r} \cup (\partial Lk(\sigma_r, M) * (pt.)) \times S^{n_r - r - 1}$. The union $St(\sigma, \partial M'') \cup \delta(\sigma, M') = \partial D(\sigma, M')$ is a homology manifold without boundary whose homology groups are isomorphic to those of S^{n-1}. Hence in any case $D(\sigma, M')$ is a homology cell. The union $\cup D(\sigma, M')$, σ moving all simplexes of M', gives the cellular decomposition of M, which we call the canonical one.

We define the handle M_i of index i by the disjoint union

$$M_i = \cup D(b_{\sigma^{n-i}})$$

where σ changes all $(n - i)$-simplexes of M. We have $\delta D(b_\sigma) = \cup D(\tau)$, where $\sigma < \tau \in M'$, and it gives a cellular decomposition of M_i. We can devide the boundary as $\delta D(b_\sigma) = LD(b_\sigma) \cup HD(b_\sigma)$, which consists of unions of celles attached to the handles of lower indexes and higher indexes. We define them as

$$LD(b_\sigma) = \delta D(b_\sigma) \cap \Big(\bigcup_{j<i} M_j\Big)$$
$$HD(b_\sigma) = \delta D(b_\sigma) \cap \Big(\bigcup_{j>i} M_j\Big) .$$

Let $\tau = b_{\tau_0} b_{\tau_1} \ldots b_{\tau_r} \neq \sigma$ be a simplex of M', where

$$\tau_0^{m_0} < \tau_1^{m_1} \cdots < \tau_r^{m_r} \in M .$$

Then $D(\tau) \in LD(b_\sigma)$ if and only if $\tau_r > \sigma$ and $D(\tau) \in HD(b_\sigma)$ if and only if $\tau_0 < \sigma$. It is easy to see that

$$LD(b_\sigma) \cong Lk(\sigma, M) \times D^{n-i}$$
$$HD(b_\sigma) \cong (Lk(\sigma, M) * (pt.)) \times S^{n-i-1} ,$$

and these isomorphism together give the trivialization d_{b_σ} of $\delta D(b_\sigma)$.

Let Δ^{n-i} be the standard $(n-i)$-simplex and let

$$\partial\Delta^{n-i} = S^{n-i-1} = \bigcup_{\alpha} C_\alpha$$

be the cell decomposition defined as above, which we call the standard decomposition of S^{n-i-1}. The decomposition

$$\mathrm{HD}(b_\sigma) = \cup\, \mathrm{D}(\tau)\ .$$

is equal to the standard product decomposition

$$\{\mathrm{L}k(\sigma, \mathrm{M}) * (pt.)\} \times \left(\bigcup_{\alpha} C_\alpha\right).$$

All the cells of $\mathrm{HD}(b_\sigma)$ which is not contained in $\mathrm{LD}(b_\sigma) \cap \mathrm{HD}(b_\sigma)$ is written as

$$(\mathrm{L}k(\sigma, \mathrm{M}) * (pt.)) \times C_\alpha\ .$$

Finally we define $M_{(i)}$ the subcomplex of M composed of handles whose indexes are inferior or equal to i, that is,

$$M_{(i)} = \bigcup_{j \leqslant i} M_j \subset M\ .$$

Then we have

$$M_{(i)} = M_{(i-1)} \cup M_i$$

attached on $\bigcup_{\sigma} \mathrm{LD}(b_\sigma)$, σ being $(n-i)$-simplexes.

3. PL-homology spheres.

We call an n-dimensional homology manifold whose homology groups are isomorphic to those of S^n a homology n-sphere or homology sphere of dimension n. If it is a PL-manifold, it is called a PL-homology n-sphere.

If dimension is smaller than 3, a homology sphere is the natural sphere. And so any 3-dimensional homology manifold is a PL-manifold. In order to study higher dimensional cases we define the group $\mathcal{H}^3$.

Let X^3 be the set of oriented 3-dimensional PL-homology spheres. Note that any homology sphere is orientable. We say that $H_1^3 \in X^3$ is equivalent to $H_2^3 \in X^3$ if $H_1^3 \# (-H_2^3)$ is the boundary of an acyclic PL-manifold, where # denotes the connected sum and

$-H_2^3$ is H_2^3 with the orientation inversed. Let $\mathcal{H}^3 = X^3/\sim$ be the set of equivalence classes. By the connected sum operation, $\mathcal{H}^3$ is an abelian group. Let G be the binary dodecahedral group. The quotient space S^3/G is a PL-homology sphere whose class in $\mathcal{H}^3$ is non trivial.

On the contrary, for higher dimensions the following is known [2] [6] [4].

PROPOSITION 1 (Hsiang-Hsiang, Tamura, Kervaire). – *Any* PL-*homology sphere is the boundary of a contractible* PL-*manifold, if the dimension is greater than 3.*

We will prove the followings, where x is a point in S^i, $i \geqslant 1$.

PROPOSITION 2. – *Let* $H^3 \in X^3$, *then* $H^3 \times S^1$ *is the boundary of a* PL-*manifold* K^5 *such that* $H_*(K) \cong H_*(S^1)$ *and the inclusion*

$$j : S^1 \hookrightarrow \{x\} \times S^1 \hookrightarrow H^3 \times S^1 \hookrightarrow K$$

induce an isomorphism of the fundamental groups.

PROPOSITION 3. – *Let* $H^3 \in X^3$ *and let* $i \geqslant 2$. *Then* $H^3 \times S^i$ *is the boundary of a* PL-*manifold* K^{4+i} *such that the inclusion*

$$j : S^i \hookrightarrow \{x\} \times S^i \hookrightarrow H^3 \times S^i \hookrightarrow K$$

induces a homotopy equivalence.

Proof of Proposition 2. – Since any orientable closed 3-dimensional PL-manifold is a boundary of a 4-dimensional parallelizable PL-manifold (See by example [3]), we have a parallelizable PL-manifold L^4 such that $\partial L = H$. By doing surgery we can assume that $\pi_1(L) = 0$. By the Poincaré duality theorem, $H_2(L)$ is free abelian. Let $p : L \times S^1 \to S^1$ be the projection. Then it induces an isomorphism of the fundamental groups. Remark that if we have a manifold K with boundary $H^3 \times S^1$ such that $H_2(K) \cong 0$ and the inclusion $j : S^1 \hookrightarrow K$ induces the isomorphism of the fundamental groups, then, by the Poincaré duality, we have $H_i(K) = 0$ for $i \geqslant 2$. Hence it is sufficient to kill $H_2(L \times S^1)$. Since $H_2(L)$ is free, so is $H_2(L \times S^1)$. We can follow the method of lemma 5.7 of Kervaire-Milnor [5]. Since $\pi_1(L) = 0$, the Hurewicz map of L, $\pi_2(L) \to H_2(L)$, is isomorphic, and so is the Hurewicz map of $L \times S^1$

$$h : \pi_2(L \times S^1) \to H_2(L \times S^1) .$$

Hence we can represent any element of $H_2(L \times S^1)$ by an embedded sphere. In our case the boundary $\partial(L \times S^1)$ is $H^3 \times S^1$ and it does not satisfy the hypothesis of that lemma. But since we have

$$H_2(\partial(L \times S^1)) = 0 ,$$

the result is the same.

Proof of Proposition 3. – Let K^5 be the 5-dimensional PL-manifold of proposition 2. Attach K with $H^3 \times D^2$ by the identity map on $H^3 \times S^1$. The constructed manifold W^5 is a simply connected PL-homology sphere, and by the generalized Poincaré conjecture, it is the natural sphere S^5. It shows that we can embed H^3 in S^5 with a trivial normal bundle. By composing with the natural embedding $S^5 \hookrightarrow S^{4+i}$, we have an embedding of H^3 in S^{4+i} with the trivial normal bundle. The manifold N which is the complement of the open regular neighbourhood of H^3 in S^{4+i} has $H^3 \times S^i$ as the boundary and the inclusion $j : S^i \hookrightarrow N$ induces an isomorphism of homology groups, hence homotopy equivalence, which completes the proof.

4. An obstruction to constructing PL-manifold.

Let M be a homology manifold of dimension greater than 4. We assume that the boundary ∂M is a PL-manifold if it is not empty. As in § 2, it has the handle decomposition

$$M = M_{(n)} = \bigcup_{0 \leqslant i \leqslant n} M_i$$

which has also the canonical homology cell complex structure. We want to construct a PL-manifold with a pseudo homology cell complex structure which is cellularly equivalent to M. Since $M_{(3)}$ is a PL-manifold, a problem first arises when we attach handles of index 4.

Let σ be an $(n-4)$-simplex in the interior of M. Then $Lk(\sigma, M)$ is a 3-dimensional PL-homology sphere. Connecting σ by a path from a fixed base point of M, we can give the orientation for the neighbourhood of σ, and hence for $Lk(\sigma, M)$.

Let $Lk(\sigma, M)$ be the class in the group $\mathcal{H}^3$. To each $(n-4)$-simplex σ of M, we define a function $\lambda(M) : \{(n-4)\text{-simplex}\} \to \mathcal{H}^3$ by

$$\lambda(M)(\sigma) = \begin{cases} \{Lk(\sigma, M)\} & \text{if } \quad \sigma \in \text{Int. } M \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda(M)$ is an element of the chain group $C_{n-4}(M, \mathcal{H}^3)$. The coefficient may be twisted if the manifold is not orientable.

LEMMA 1. – *$\lambda(M)$ is a cycle.*

Proof. – Let μ be an $(n-5)$-simplex. In the homology 4-sphere $Lk(\mu)$, the complex $\cup\, Lk(\sigma_i) * (x_i)$, where x_i denotes the barycenter of the 1-simplex $b_\mu b_{\sigma_i}$ and the sum extends to all the $(n-4)$-simplexes such that $\sigma_i > \mu$, is a subcomplex whose complement in $Lk(\mu)$ is a PL-manifold. So the connected-summed PL-manifold $\Sigma\, Lk(\sigma_i)$ bounds an acyclic PL-manifold.

Hence $\lambda(M)$ represents an element $\{\lambda(M)\}$ of $H_{n-4}(M, \mathcal{H}^3)$. Now we have the theorem :

THEOREM. – *Let M^n be a homology manifold with the dimension $n > 4$. Assume that ∂M is a PL-manifold if $\partial M \neq \emptyset$. If the obstruction class*

$$\{\lambda(M)\} \in H_{n-4}(M, \mathcal{H}^3)$$

is zero, then there exists a PL-manifold N with a pseudo homology cell decomposition which is cellularly equivalent to M.

Proof. – Since $\{\lambda(M)\} = 0$, there exists a correspondance

$$g : \{(n-3)\text{-simplex}\} \to \mathcal{H}^3$$

such that

$$\sum_{\tau_i > \sigma} g(\tau_i) = \{Lk(\sigma, M)\} \in \mathcal{H}^3 .$$

We will inductively construct PL-manifolds N_p and $N_{(p)} = \bigcup_{q \leqslant p} N_q$ with a pseudo homology cell decomposition $N_p = \cup\, E_\alpha$ where all pseudo cells are PL-manifolds such that $N_{(p)}$ is cellularly equivalent to $M_{(p)}$.

(a) $p \leqslant 2$. In this case, the manifolds N_p, $N_{(p)}$ and their cells are just equal to M_p, $M_{(p)}$ and their cells. That is, for any j-simplex σ, $j \geqslant n-2$, we define the PL-manifolds as

$$E(b_\sigma) = D(b_\sigma)$$

$$N_p = \cup \{E(b_\sigma) \mid \dim \sigma = n-p\} = \cup \{D(b_\sigma) \mid \dim \sigma = n-p\} = M_p$$

For any simplex $\mu \in M'$ such that $\mu > b_\sigma$, we put

$$E(\mu) = D(\mu) .$$

Hence $\partial E(b_\sigma) = \partial D(b_\sigma) = \cup D(\mu) = \cup E(\mu)$, and $N_{(p)} = M_{(p)}$.

(b) $p = 3$. Let τ_i be an $(n-3)$-simplex. Let H_i^3 be the 3-dimensional PL-homology sphere which represents $g(\tau_i)$ and let K_i be the PL-manifold whose boundary is $H_i^3 \times S^{n-4}$ such that the inclusion $j : S^{n-4} \hookrightarrow K_i$ induces the isomorphisms of the fundamental groups and the homology groups, whose existence is shown by propositions 2 and 3 of § 3. Let $D^3 \subset H_i^3$ be a disc. Then $D^3 \times S^{n-4} \subset \partial K_i$. We have the PL-homeomorphism $\partial D(b_{\tau_i}) = S^2 \times D^{n-3} \cup D^3 \times S^{n-4}$. We define the PL-manifolds $E(b_{\tau_i})$ and N_3 by

$$E(b_{\tau_i}) = D(b_{\tau_i}) \underset{D^3 \times S^{n-4}}{\cup} K_i$$

$$N_3 = \underset{i}{\cup} E(b_{\tau_i})$$

where $D(b_{\tau_i})$ is attaced to K_i by the identity map on $D^3 \times S^{n-4}$. It is easy to see that $E(b_{\tau_i})$ is a homology cell. We will give the pseudo cell decomposition for $\partial E(b_{\tau_i})$. First we devide $\partial E(b_{\tau_i})$ as the union $\partial E(b_{\tau_i}) = LE(b_{\tau_i}) \cup HE(b_{\tau_i})$, where

$$LE(b_{\tau_i}) = \partial D(b_{\tau_i}) - D^3 \times D^{n-3}$$

$$HE(b_{\tau_i}) = \partial K_i - D^3 \times S^{n-4} = (H_i^3 - D^3) \times S^{n-4} .$$

Since $LE(b_{\tau_i}) = LD(b_{\tau_i})$, we give the cell decomposition by that of $LD(b_{\tau_i})$. We give the pseudo cell decomposition in the interior of $HE(b_{\tau_i})$ as

$$(H_i^3 - D^3) \times S^{n-4} = (H_i^3 - D^3) \times \left(\bigcup_\alpha C_\alpha\right) = \bigcup_\alpha (H_i^3 - D^3) \times C_\alpha ,$$

where $S^{n-4} = \cup C_\alpha$ is the standard decomposition. These decompositions of $LE(b_{\tau_i})$ and $HE(b_{\tau_i})$ fit together on their intersection and give the decomposition of $\partial E(b_{\tau_i})$, which is clearly cellular equivalent to that of $\partial D(b_{\tau_i})$. For each simplex $\mu > b_{\tau_i}$, $\mu \in M'$, we denote by $E(\mu)$ the pseudo cell of $\partial E(b_{\tau_i})$ which corresponds by the equivalence to $D(\mu) \in \partial D(b_{\tau_i})$. We have $\partial E(b_{\tau_i}) = \cup E(\mu)$. We define $N_{(3)}$ by

$$N_{(3)} = N_{(2)} \cup N_3$$

attached by the identity on $LE(b_{\tau_i})$. $N_{(3)}$ is cellularly equivalent to $M_{(3)}$.

(c) $p = 4$. Let σ be a $(n-4)$-simplex. Let $\cup E(\mu) \subset \partial N_{(3)}$ be the union of pseudo cells such that $b_\sigma < \mu \in M'$, $\mu \neq b_\sigma$. Then by the definition, it is PL-homeomorphic to the PL-manifold

$$(Lk(\sigma) \# \Sigma(-H_i^3)) \times D^{n-4}$$

where H_i^3 represents $g(\tau_i)$ and the sum extends to all $\tau_i > \sigma$.

Since $\{Lk(\sigma)\} = \Sigma\, g(\tau_i)$ in $\mathcal{H}^3$, the PL-homology 3-sphere

$$H_\sigma^3 = Lk(\sigma) \# \Sigma(-H_i^3)$$

is the boundary of an acyclic PL-manifold W_σ^4. The union

$$W_\sigma^4 \times S^{n-5} \cup H_\sigma^3 \times D^{n-4}$$

is a PL-homology $(n-1)$-sphere. By the proposition 1 of § 3, it is the boundary of a contractible PL-manifold Y_σ. We define the PL-manifolds $E(b_\sigma)$ and N_4 as

$$E(b_\sigma) = Y_\sigma$$
$$N_4 = \cup E(b_\sigma) .$$

Further we define $LE(b_\sigma)$ and $HE(b_\sigma)$ by

$$LE(b_\sigma) = H_\sigma^3 \times D^{n-4}$$
$$HE(b_\sigma) = W_\sigma^4 \times S^{n-5} .$$

The pseudo cellular decomposition for $LE(b_\sigma)$ is already defined and we give for $HE(b_\sigma)$ by the product with the standard decomposition of S^{n-5}. They give a pseudo cellular decomposition of

$$\partial E(b_\sigma) = LE(b_\sigma) \cup HE(b_\sigma) ,$$

which is cellularly equivalent to that of $\partial D(b_\sigma)$. For each simplex $\mu > b_\sigma$, $\mu \in M'$, we define $E(\mu)$ by the pseudo cell which corresponds to $D(\mu)$ by this equivalence. We define $N_{(4)}$ by $N_{(3)} \cup N_4$ attached by the identity of $LE(b_\sigma)$, which is cellularly equivalent to $M_{(4)}$.

(d) $p \geqslant 5$. Let σ be a j-simplex $j \leqslant n-5$. Let $\cup E(\mu) \subset \partial N_{(n-j-1)}$ be the union of pseudo cells such that $\mu > b_\sigma$, $\mu \neq b_\sigma$. Then by our definition, it is a PL-manifold

$$H_\sigma^{p-1} \times D^{n-p}$$

where H_σ^{p-1} is a PL-homology $(p-1)$-sphere, where $p = n - j$. By the proposition 1 of § 3, H^{p-1} is the boundary of a contractible PL-manifold W_σ^p. We define $E(b_\sigma)$ by

$$E(b_\sigma) = W_\sigma^p \times D^{n-p} .$$

The other definitions are just similar to the case when $p = 4$.

Continuing this process, we obtain a PL-manifold $N = N_{(n)}$ which is cellularly equivalent to $M = M_{(n)}$. Q.E.D.

5. Simple homotopy equivalence.

By the theorem of § 4, for the same M, if the obstruction class is 0, we can construct a PL-manifold N. In this section, we prove the following.

THEOREM. – *If* M *is compact, the constructed manifold* N *is simple homotopy equivalent to* M.

Let $M^{(k)}$ denote the k-skelton of M. Let L be a subcomplex of $M^{(k)}$, we define the PL-submanifold $N^{(L)}$ of N by

$$N^{(L)} = \cup \{E(b_\sigma) \mid \sigma \in L\} .$$

We put

$$N^{(k)} = N^{(M^{(k)})} = \cup \{E(b) \mid \sigma \in M^{(k)}\} .$$

By the induction of k, we prove the stronger

LEMMA 1. – *There exists a simple homotopy equivalence*

$$f : M^{(k)} \to N^{(k)}$$

such that, for any $(k + 1)$*-simplex* μ, $f(\partial\mu) \subset N^{(\partial\mu)}$ *and*

$$f/\partial\mu : \partial\mu \to N^{(\partial\mu)}$$

is a simple homotopy equivalence.

Proof. – If $k = 0$, it holds obviously. Now we will prove the lemma for $k + 1$ assuming the lemma for k. Let μ be a $(k + 1)$-simplex. Since the collar of $\partial\mu$ is PL-homeomorphic to $S^k \times I$, we can write

$$\mu = S^k \times I \cup S^k * (b_\mu)$$

where $S_0^k = S^k \times \{0\} = \partial\mu$ and $S_1^k = S^k \times \{1\} = S^k \times I \cap S^k * (b_\mu)$. Recall that

$$N^{(M^{(k)} \cup \mu)} = N^{(k)} \cup E(b_\mu)$$
$$N^{(k)} \cap E(b_\mu) = N^{(\partial\mu)} \cap E(b_\mu) = HE(b_\mu) = W_\mu^{n-k-1} \times S^k$$

where W_μ^{n-k-1} is an acyclic (or contractible) PL-manifold. Let x be a point in the interior of W_μ and let $d : S^k \to W_\mu \times S^k$ be the embedding defined by $d(S^k) = \{x\} \times S^k$. We define a map

$$\tilde{f} : S_0^k \cup S_1^k \to N^{(k)}$$

by

$$\tilde{f} \mid S_0^k = f$$
$$\tilde{f} \mid S_1^k = d\ .$$

Since $\tilde{f} \mid \partial M$ gives a simple homotopy equivalence $\partial\mu \to N^{(\partial\mu)}$, $N^{(\partial\mu)}$ is homotopy equivalent to S^k, and so $\tilde{f} \mid S_0^k$ and $\tilde{f} \mid S_1^k$ are homotopic. Hence we can extend $\tilde{f}$ on $S^k \times I$. Further since $E(b_\mu)$ is contractible, we can extend $\tilde{f}$ to a map from $\mu = S^k \times I \cup S^k * (b_\mu)$ to $N^{(M^{(k)} \cup \mu)}$. By the definition, f and $\tilde{f}$ coïncide on $\partial\mu$, and so we have a map

$$g = f \cup \tilde{f} : M^{(k)} \cup \mu \to N^{(M^{(k)} \cup \mu)}\ .$$

Repeating this for all $(k + 1)$-simplexes of M, we obtain a map $g : M^{(k+1)} \to N^{(k+1)}$. We have the exact sequences of chain groups,

$$0 \to C_*(M^{(k)}) \to C_*(M^{(k+1)}) \to \Sigma\, C_*(\mu/\partial\mu) \to 0$$

$$0 \to C_*(N^{(k)}) \to C_*(N^{(k+1)}) \to \Sigma\, C_*(E(b_\mu)/HE(b_\mu)) \to 0\ ,$$

where we regard them as $Z\pi_1(M^{(k+1)}) = Z\pi_1(N^{(k+1)})$-modules.