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Constructing manifolds by homotopy equivalences
I. An obstruction to constructing PL-manifolds from homology manifolds


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CONSTRUCTING MANIFOLDS
BY HOMOTOPY EQUIVALENCES I.
AN OBSTRUCTION
TO CONSTRUCTING PL-MANIFOLDS
FROM HOMOLOGY MANIFOLDS

by Hajime SATO

0. Introduction.

A homology manifold can be given a canonical cell complex structure, where each cell is a contractible homology manifold. In this paper, given a homology manifold M, we aim at constructing a PL-manifold with a cell complex structure, where each cell is an acyclic PL-manifold, which is cellularly equivalent to the canonical cell complex structure of M. We obtain a theorem that, if the dimension $n$ of M is greater than 4 and if the boundary $\partial M$ is a PL-manifold or empty, there is a unique obstruction element in $H_{n-4}(M; \mathcal{H}^3)$, where $\mathcal{H}^3$ is the group of 3-dimensional PL-homology spheres modulo those which are the boundary of an acyclic PL-manifold. If the manifold is compact, the constructed PL-manifold is simple homotopy equivalent to M.

I have heard that similar results have been obtained independently and previously by M. Cohen and D. Sullivan, refer [1] and [9].

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1. Definition of homology manifold with boundary$^{(1)}$.

Let K be a locally finite simplicial complex and let $\sigma$ be a simplex of K. We define the subcomplexes of K as follows.

Let $M$ be a locally finite full simplicial complex of dimension $n$. We say that $M$ is a homology manifold of dimension $n$ if the following equivalent condition holds:

**Lemma 1.** — The followings are equivalent:

i) for any simplex $\sigma$ of dimension $p$,
\[ \tilde{H}_i(Lk(\sigma, M)) = \tilde{H}_i(S^{n-p-1}) \text{ or } 0. \]

ii) for any simplex $\sigma$ of dimension $p$,
\[ \tilde{H}_i(St(\sigma, M)/\partial St(\sigma, M)) = \tilde{H}_i(S^n) \text{ or } 0. \]

iii) for any point $x$ of $|M|$, where $|M|$ denotes the underlying topological space of $M$,
\[ H_i(|M|, |M| - x) = \tilde{H}_i(S^n) \text{ or } 0. \]

The definition is invariant by the PL-homeomorphism in the category of simplicial complexes.

**Lemma 2.** — For any $p$-simplex $\sigma$ of $M$, $Lk(\sigma, M)$ is a compact $(n - p - 1)$-dimensional homology manifold.

**Proof.** — It is compact because $M$ is locally finite. Let $\tau$ be a $q$-simplex of $Lk(\sigma, M)$. We have
\[ Lk(\tau, Lk(\sigma, M)) = Lk(\tau \sigma, M). \]
Hence $\tilde{H}_i(Lk(\tau, Lk(\sigma, M))) = \tilde{H}_i(S^{n-p-q-1})$ or $0$, which completes the proof.

Let us define the subset $\partial M$ of $M$ by
\[ \partial M = \{ \sigma \in M \mid \tilde{H}_i(\sigma, M) = 0 \} \]
We call it as the boundary of $M$. If $\partial M = \emptyset$, the manifold is classical.
and the following Poincaré duality is well known (see for example [7 ; (7,4)]).

**Lemma 3.** — Let $M$ be an orientable compact $n$-dimensional homology manifold without boundary. Let $A_1 \supset A_2$ be subcomplexes of $M$. Then we have the isomorphism

$$H^i(A_1, A_2) = H_{n-i}(|M| - |A_2|, |M| - |A_1|).$$

Using this we will prove the followings. By lemma 2, for $p$-simplex $\sigma$, $Lk(\sigma, M)$ is a homology manifold and we can define $\partial Lk(\sigma, M)$.

**Lemma 4.** — If $\partial Lk(\sigma, M) \neq \emptyset$, $Lk(\sigma, M)$ is acyclic and $\partial Lk(\sigma, M)$ is an $(n - p - 2)$-dimensional homology manifold such that

$$\widetilde{H}_i(\partial Lk(\sigma, M)) = \widetilde{H}_i(S^{n-p-2}).$$

**Proposition 5.** — If $\partial M \neq \emptyset$, $\partial M$ is a subcomplex and is an $(n - 1)$-dimensional homology manifold without boundary.

We prove that lemma 4 for $n = k$ implies proposition 5 for $n = k$ and proposition 5 for $n \leq k$ implies lemma 4 for $n = k + 1$. Since lemma 4 holds for $n = 1$, we can continue by induction.

Lemma $4_{n=k} \Rightarrow$ Proposition $5_{n=k}$. Let $\sigma$ be a $p$-simplex of $\partial M$ and let $\sigma_0 \subset \sigma$. Then we can write $\sigma = \sigma_0 \sigma_1$. We have

$$\widetilde{H}_* (Lk(\sigma_1, Lk(\sigma_0, M))) = \widetilde{H}_* (Lk(\sigma, M)) = 0,$$

which shows that $\sigma_1 \in \partial Lk(\sigma_0, M)$ and so $\partial Lk(\sigma_0, M) \neq \emptyset$. By the lemma 4, $Lk(\sigma_0, M)$ is acyclic and it follows that $\sigma_0 \in \partial M$. Hence $\partial M$ is a well-defined subcomplex of $M$. A $q$-simplex $\tau$ of $Lk(\sigma, M)$ is in $Lk(\sigma, \partial M)$ if and only if $\widetilde{H}_i(Lk(\tau \sigma, M)) = 0$. Since

$$Lk(\tau \sigma, M) = Lk(\tau, Lk(\sigma, M)),$$

it is equivalent to that $\tau$ belongs to $\partial Lk(\sigma, M)$. Hence the complex $Lk(\sigma, \partial M)$ coincides with $\partial Lk(\sigma, M)$. By lemma $4_k$, we have

$$\widetilde{H}_i(\partial Lk(\sigma, M)) = \widetilde{H}_i(S^{k-p-2}),$$

which shows that $\partial M$ is a $(k - 1)$-dimensional homology manifold without boundary.

Proposition $5_{n\leq k} \Rightarrow$ Lemma $4_{n=k+1}$. Let $M$ be a homology manifold of dimension $k + 1$. Let $\sigma$ be a $p$-simplex of $M$. By lemma 2,
$\text{Lk}(\sigma, M)$ is a homology manifold of dimension $k - p$. By proposition 5 for $n = k - p$, $\partial \text{Lk}(\sigma, M)$ is a $(k - p - 1)$-dimensional homology manifold without boundary if it is not empty. Let $2\text{Lk}(\sigma, M)$ be the double of $\text{Lk}(\sigma, M)$, i.e.,

$$2\text{Lk}(\sigma, M) = \text{Lk}(\sigma, M) \cup \text{Lk}(\sigma, M).$$

Let $\tau$ be a $q$-simplex of $2\text{Lk}(\sigma, M)$. If $\tau$ is not a simplex of $\partial \text{Lk}(\sigma, M)$, clearly,

$$\tilde{H}_i(\text{Lk}(\tau, 2\text{Lk}(\sigma, M))) = \tilde{H}_i(\text{Lk}(\tau, \text{Lk}(\sigma, M))) = \tilde{H}_i(S^{k-p-q-1}).$$

If $\tau$ is a simplex of $\partial \text{Lk}(\sigma, M)$, we have

$$\text{Lk}(\tau, 2\text{Lk}(\sigma, M)) = \text{Lk}(\tau, \text{Lk}(\sigma, M)) \cup \text{Lk}(\tau, \text{Lk}(\sigma, M)).$$

By definition $\tilde{H}_i(\text{Lk}(\tau, \text{Lk}(\sigma, M))) = 0$ and by the proposition 5 for $n = k - p - 1$, we have

$$\tilde{H}_i(\text{Lk}(\tau, \partial \text{Lk}(\sigma, M))) = \tilde{H}_i(S^{k-p-q-2}).$$

Hence in any case $\tilde{H}_i(\text{Lk}(\tau, 2\text{Lk}(\sigma, M))) = \tilde{H}_i(S^{k-p-q-1})$, which shows that $2\text{Lk}(\sigma, M)$ is a $(k - p)$-dimensional homology manifold without boundary. Applying lemma 3, we have

$$H^i(\text{Lk}(\sigma, M), \partial \text{Lk}(\sigma, M)) = H_{k-p-i}(| \text{Lk}(\sigma, M) | - | \partial \text{Lk}(\sigma, M) |).$$

Notice that for any homology manifold $M$, $H_i(|M| - |\partial M|) = H_i(M)$. Hence $H^i(\text{Lk}(\sigma, M), \partial \text{Lk}(\sigma, M)) = H_{k-p-i}(S^{k-p})$ or $H_{k-p-i}(pt.)$. But if it is isomorphic to $H_{k-p-i}(S^{k-p})$, we have

$$H^0(\text{Lk}(\sigma, M), \partial \text{Lk}(\sigma, M)) = Z,$$

which contradicts to the definition that $\tilde{H}_0(\text{Lk}(\sigma, M)) = 0$. Hence $\text{Lk}(\sigma, M)$ is acyclic and consequently $\tilde{H}_i(\partial \text{Lk}(\sigma, M)) = \tilde{H}_i(S^{k-p-1})$, which completes the proof.

2. Cell decomposition of a homology manifold.

We mean by a homology cell (resp. pseudo homology cell) of dimension $n$ or homology $n$-cell (resp. pseudo homology $n$-cell) a
compact contractible (resp. acyclic) homology manifold of dimension \( n \) with a boundary, the boundary being a homology sphere but not necessarily simply connected. A (pseudo) homology cell complex is a complex \( K \) with a locally finite family of (pseudo) homology cells \( C = \{ C_\alpha \} \), such that:

i) \( K = \bigcup C_\alpha \)

ii) \( C_\alpha , C_\beta \subseteq C \) implies \( \partial C_\alpha , C_\alpha \cap C_\beta \) are unions of cells in \( C \)

iii) If \( \alpha \neq \beta \), then \( \text{Int } C_\alpha \cap \text{Int } C_\beta = \emptyset \).

If a homology manifold \( M \) has a (pseudo) homology cell complex structure, we call it a (pseudo) cellular decomposition of \( M \). Two (pseudo) homology cell complexes \( K = \bigcup C_\alpha , K' = \bigcup C'_\alpha \) are isomorphic if there exists a bijection \( k : C \to C' \) such that both \( k \) and \( k^{-1} \) are incidence preserving. In such a case we say that they are cellularly equivalent.

Now we have the following:

**Proposition 1.** — If two finite homology cell complexes \( K, K' \) are cellularly equivalent, then they are simple homotopy equivalent.

We can define a simplicial map \( f : K \to K' \) inductively by the dimension of the cells. Hence it is sufficient to prove the following lemma.

**Lemma 2.** — Let \( A_j^i (j = 1, 2, \ldots, r) \) be subcomplex of simplicial complexes \( B^i \) for \( i = 1, 2 \) respectively such that \( B^i = \bigcup_j A_j^i \), and let \( f : B^1 \to B^2 \) be a simplicial map. For any subset \( s \) of \( \{ 1, 2, \ldots, r \} \), let \( A_s^i = \bigcap_{j \in s} A_j^i \) and let \( f_s \) be the restriction of \( f \) on \( A_s \). If \( f_s \) is a mapping from \( A_s^1 \) to \( A_s^2 \) which is a simple homotopy equivalence for any \( s \), then \( f \) itself is a simple homotopy equivalence.

**Proof.** — First suppose that \( r = 2 \). We have the exact sequence

\[
0 \to C_* (A_1^i) \to C_* (B^i) \to C_* (A_1^i \cap A_2^i) \to 0
\]

of the chain complexes. Let \( g : A_1^1/(A_1^1 \cap A_2^1) \to A_2^2/(A_1^2 \cap A_2^2) \) be the map induced by \( f \) and let us denote by \( w(\ ) \) the Whitehead torsion. Then by theorem 10 of [8], we have
\[ w(f) = w(f_{(1)}) + w(g) . \]

Remark here that \( f \) and \( g \) can easily be seen to be homotopy equivalences. Further we have the exact sequence

\[ 0 \to C_\ast(A^i_1 \cap A^i_2) \to C_\ast(A^i_2) \to C_\ast(A^i_1/(A^i_1 \cap A^i_2)) \to 0 \]

which shows that

\[ w(f_{(2)}) = w(f_{(1,2)}) + w(g) . \]

Since \( w(f_{(1)}) = w(f_{(2)}) = w(f_{(1,2)}) = 0 \), we have \( w(f) = 0 \). If \( r \geq 3 \), we can repeat this argument, which shows that \( f \) is a simple homotopy equivalence for any \( r \).

Now let \( \sigma \) be a simplex of a locally finite simplicial complex \( K \). We denote by \( b_\sigma \in K' \) its barycenter. We define dualcomplex \( D(\sigma) \) and its subcomplex \( \delta D(\sigma) \) which are subcomplexes of \( K' \) by

\[ D(\sigma) = D(\sigma, K) = \{ b_{\sigma_0} \ldots b_{\sigma_r} | \sigma_0 < \ldots < \sigma_r \in K \} \]

\[ \delta D(\sigma) = \delta D(\sigma, K) = \{ b_{\sigma_0} \ldots b_{\sigma_r} | \sigma_0 < \ldots < \sigma_r \in K \} \]

The followings are easy to see.

i) if \( \sigma < \sigma' \Rightarrow D(\sigma) \supset D(\sigma') \)

ii) \( D(\sigma) = b_\sigma * \delta D(\sigma) \)

iii) \( \delta D(\sigma) = \bigcup_{\tau > \sigma} D(\tau) \) where \( \tau > \sigma \) and \( \tau \neq \sigma \)

iv) \( \delta D(\sigma) \) is isomorphic to \( Lk(\sigma, K)' \).

Let \( M \) be a homology manifold. For each simplex

\[ \sigma = b_{\sigma_0} b_{\sigma_1} \ldots b_{\sigma_r} \]

of \( M' \), where \( \sigma_0^n < \sigma_1^{n_1} < \ldots < \sigma_r^{n_r} \) are a set of simplexes of \( M \), we have the dual cell \( D(\sigma, M') \). It is a compact homology manifold by lemma 2 of § 1. Further we have

\[ \delta D(\sigma, M') \cong Lk(\sigma, M) \]

\[ \cong Lk(\sigma, \sigma_r) * Lk(\sigma_r, M) \]

\[ \cong S^{n_r-r-1} * Lk(\sigma_r, M) \]

\[ \cong Lk(\sigma_r, M) \times D^{n_r-r-1} (Lk(\sigma_r, M) \ast (pt.)) \times S^{n_r-r-1} . \]
where \(\cong\) denotes that both sides are PL-homeomorphic and let
\[
d_{a}: \partial D(\sigma, M') \to Lk(\sigma, M) \times D^{n-r} \cup (Lk(\sigma, M) \ast (pt.)) \times S^{n-r-1}
\]
be the PL-homeomorphism, which we call the trivialization of \(\partial D(\sigma, M')\). If \(\sigma\) is not in \(\partial M\), \(\partial D(\sigma, M')\) is a homology manifold whose homology groups are isomorphic to those of \(S^{n-1}\), boundary being empty. If \(\sigma \in \partial M\), \(\partial D(\sigma, M')\) is an acyclic homology manifold with the boundary \(Lk(\sigma, \partial M'')\) which is PL-homeomorphic to \(\partial Lk(\sigma, M) \times D^{n-r} \cup (\partial Lk(\sigma, M) \ast (pt.)) \times S^{n-r-1}\). The union \(St(\sigma, \partial M'') \cup \delta(\sigma, M') = \partial D(\sigma, M')\) is a homology manifold without boundary whose homology groups are isomorphic to those of \(S^{n-1}\). Hence in any case \(D(\sigma, M')\) is a homology cell. The union \(\cup D(\sigma, M')\), \(\sigma\) moving all simplexes of \(M'\), gives the cellular decomposition of \(M\), which we call the canonical one.

We define the handle \(M_{i}\) of index \(i\) by the disjoint union
\[
M_{i} = \cup D(b_{o_{n-i}})
\]
where \(\sigma\) changes all \((n-i)\)-simplexes of \(M\). We have \(\partial D(b_{o}) = \cup D(\tau)\), where \(\sigma < \tau \in M'\), and it gives a cellular decomposition of \(M_{i}\). We can divide the boundary as \(\partial D(b_{o}) = LD(b_{o}) \cup HD(b_{o})\), which consists of unions of celles attached to the handles of lower indexes and higher indexes. We define them as
\[
LD(b_{o}) = \partial D(b_{o}) \cap \left( \bigcup_{j < i} M_{j} \right) \\
HD(b_{o}) = \partial D(b_{o}) \cap \left( \bigcup_{j > i} M_{j} \right) .
\]
Let \(\tau = b_{r_{0}} b_{r_{1}} \ldots b_{r_{r}} \neq \sigma\) be a simplex of \(M'\), where
\[
\tau_{0}^{m_{0}} < \tau_{1}^{m_{1}} \ldots < \tau_{r}^{m_{r}} \in M .
\]
Then \(D(\tau) \in LD(b_{o})\) if and only if \(\tau_{r} > \sigma\) and \(D(\tau) \in HD(b_{o})\) if and only if \(\tau_{0} < \sigma\). It is easy to see that
\[
LD(b_{o}) \cong Lk(\sigma, M) \times D^{n-i} \\
HD(b_{o}) \cong (Lk(\sigma, M) \ast (pt.)) \times S^{n-i-1} ,
\]
and these isomorphism together give the trivialization \(d_{b_{o}}\) of \(\partial D(b_{o})\).
Let $\Delta^{n-i}$ be the standard $(n-i)$-simplex and let
\[ \partial \Delta^{n-i} = S^{n-i-1} = \bigcup_{\alpha} C_{\alpha} \]
be the cell decomposition defined as above, which we call the standard decomposition of $S^{n-i-1}$. The decomposition
\[ \text{HD}(b_\sigma) = \bigcup D(\tau) . \]
is equal to the standard product decomposition
\[ \{ \text{Lk}(\sigma, M) \ast (pt.) \} \times \left( \bigcup_{\alpha} C_{\alpha} \right) . \]
All the cells of $\text{HD}(b_\sigma)$ which is not contained in $\text{LD}(b_\sigma) \cap \text{HD}(b_\sigma)$ is written as
\[ (\text{Lk}(\sigma, M) \ast (pt.)) \times C_{\alpha} . \]
Finally we define $M_{(i)}$ the subcomplex of $M$ composed of handles whose indexes are inferior or equal to $i$, that is,
\[ M_{(i)} = \bigcup_{j \leq i} M_j \subset M . \]
Then we have
\[ M_{(i)} = M_{(i-1)} \cup M_i \]
attached on $\bigcup_{\sigma} LD(b_\sigma)$, $\sigma$ being $(n-i)$-simplexes.

3. PL-homology spheres.

We call an $n$-dimensional homology manifold whose homology groups are isomorphic to those of $S^n$ a homology $n$-sphere or homology sphere of dimension $n$. If it is a PL-manifold, it is called a PL-homology $n$-sphere.

If dimension is smaller than 3, a homology sphere is the natural sphere. And so any 3-dimensional homology manifold is a PL-manifold. In order to study higher dimensional cases we define the group $\mathcal{H}^3$.

Let $X^3$ be the set of oriented 3-dimensional PL-homology spheres. Note that any homology sphere is orientable. We say that $H^3_1 \in X^3$ is equivalent to $H^3_2 \in X^3$ if $H^3_1 \# (-H^3_2)$ is the boundary of an acyclic PL-manifold, where $\#$ denotes the connected sum and
- $H^3_2$ is $H^3_2$ with the orientation inverted. Let $\mathcal{H}^3 = X^3/\sim$ be the set of equivalence classes. By the connected sum operation, $\mathcal{H}^3$ is an abelian group. Let $G$ be the binary dodecahedral group. The quotient space $S^3/G$ is a PL-homology sphere whose class in $\mathcal{H}^3$ is non-trivial.

On the contrary, for higher dimensions the following is known [2] [6] [4].

**Proposition 1** (Hsiang-Hsiang, Tamura, Kervaire). — Any PL-homology sphere is the boundary of a contractible PL-manifold, if the dimension is greater than 3.

We will prove the followings, where $x$ is a point in $S^i$, $i \geq 1$.

**Proposition 2.** — Let $H^3 \in X^3$, then $H^3 \times S^1$ is the boundary of a PL-manifold $K^5$ such that $H_*(K) \cong H_*(S^1)$ and the inclusion

$$j : S^1 \hookrightarrow \{x\} \times S^1 \hookrightarrow H^3 \times S^1 \hookrightarrow K$$

induce an isomorphism of the fundamental groups.

**Proposition 3.** — Let $H^3 \in X^3$ and let $i \geq 2$. Then $H^3 \times S^i$ is the boundary of a PL-manifold $K^{4+i}$ such that the inclusion

$$j : S^i \hookrightarrow \{x\} \times S^i \hookrightarrow H^3 \times S^i \hookrightarrow K$$

induces a homotopy equivalence.

**Proof of Proposition 2.** — Since any orientable closed 3-dimensional PL-manifold is a boundary of a 4-dimensional parallelizable PL-manifold (See by example [3]), we have a parallelizable PL-manifold $L^4$ such that $\partial L = H$. By doing surgery we can assume that $\pi_1(L) = 0$. By the Poincaré duality theorem, $H_2(L)$ is free abelian. Let $p : L \times S^1 \rightarrow S^1$ be the projection. Then it induces an isomorphism of the fundamental groups. Remark that if we have a manifold $K$ with boundary $H^3 \times S^1$ such that $H_2(K) \cong 0$ and the inclusion $j : S^1 \hookrightarrow K$ induces the isomorphism of the fundamental groups, then, by the Poincaré duality, we have $H_i(K) = 0$ for $i \geq 2$. Hence it is sufficient to kill $H_2(L \times S^1)$. Since $H_2(L)$ is free, so is $H_2(L \times S^1)$. We can follow the method of lemma 5.7 of Kervaire-Milnor [5]. Since $\pi_1(L) = 0$, the Hurewicz map of $L$, $\pi_2(L) \rightarrow H_2(L)$, is isomorphic,
and so is the Hurewicz map of \( L \times S^1 \)
\[
h : \pi_2(L \times S^1) \rightarrow H_2(L \times S^1) .
\]
Hence we can represent any element of \( H_2(L \times S^1) \) by an embedded sphere. In our case the boundary \( \partial(L \times S^1) \) is \( H^3 \times S^1 \) and it does not satisfy the hypothesis of that lemma. But since we have
\[
H_2(\partial(L \times S^1)) = 0 ,
\]
the result is the same.

Proof of Proposition 3. — Let \( K^5 \) be the 5-dimensional PL-manifold of proposition 2. Attach \( K \) with \( H^3 \times D^2 \) by the identity map on \( H^3 \times S^1 \). The constructed manifold \( W^5 \) is a simply connected PL-homology sphere, and by the generalized Poincaré conjecture, it is the natural sphere \( S^5 \). It shows that we can embed \( H^3 \) in \( S^5 \) with a trivial normal bundle. By composing with the natural embedding \( S^5 \hookrightarrow S^{4+i} \), we have an embedding of \( H^3 \) in \( S^{4+i} \) with the trivial normal bundle. The manifold \( N \) which is the complement of the open regular neighbourhood of \( H^3 \) in \( S^{4+i} \) has \( H^3 \times S^i \) as the boundary and the inclusion \( j : S^i \hookrightarrow N \) induces an isomorphism of homology groups, hence homotopy equivalence, which completes the proof.

4. An obstruction to constructing PL-manifold.

Let \( M \) be a homology manifold of dimension greater than 4. We assume that the boundary \( \partial M \) is a PL-manifold if it is not empty. As in § 2, it has the handle decomposition
\[
M = M_{(n)} = \bigcup_{0 \leq i \leq n} M_i
\]
which has also the canonical homology cell complex structure. We want to construct a PL-manifold with a pseudo homology cell complex structure which is cellulary equivalent to \( M \). Since \( M_{(3)} \) is a PL-manifold, a problem first arises when we attach handles of index 4.

Let \( \sigma \) be an \((n - 4)\)-simplex in the interior of \( M \). Then \( \text{LK}(\sigma, M) \) is a 3-dimensional PL-homology sphere. Connecting \( \sigma \) by a path from
a fixed base point of \( M \), we can give the orientation for the neighbourhood of \( \sigma \), and hence for \( \text{Lk}(\sigma, M) \).

Let \( \text{Lk}(\sigma, M) \) be the class in the group \( \mathcal{H}^3 \). To each \((n - 4)\)-simplex \( \sigma \) of \( M \), we define a function \( \lambda(M) : \{ (n - 4)\text{-simplex} \} \to \mathcal{H}^3 \) by

\[
\lambda(M)(\sigma) = \begin{cases} 
\{ \text{Lk}(\sigma, M) \} & \text{if } \sigma \in \text{Int. } M \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( \lambda(M) \) is an element of the chain group \( C_{n-4}(M, \mathcal{H}^3) \). The coefficient may be twisted if the manifold is not orientable.

**Lemma 1.** \( \lambda(M) \) is a cycle.

**Proof.** Let \( \mu \) be an \((n - 5)\)-simplex. In the homology 4-sphere \( \text{Lk}(\mu) \), the complex \( \bigcup \text{Lk}(\sigma_i) \ast (x_i) \), where \( x_i \) denotes the barycenter of the 1-simplex \( b_{\mu} b_{\sigma_i} \), and the sum extends to all the \((n - 4)\)-simplexes such that \( \sigma_i > \mu \), is a subcomplex whose complement in \( \text{Lk}(\mu) \) is a PL-manifold. So the connected-summed PL-manifold \( \Sigma \text{Lk}(\sigma_i) \) bounds an acyclic PL-manifold.

Hence \( \lambda(M) \) represents an element \( \{ \lambda(M) \} \) of \( H_{n-4}(M, \mathcal{H}^3) \). Now we have the theorem:

**Theorem.** Let \( M^n \) be a homology manifold with the dimension \( n > 4 \). Assume that \( \partial M \) is a PL-manifold if \( \partial M \neq \emptyset \). If the obstruction class

\[
\{ \lambda(M) \} \in H_{n-4}(M, \mathcal{H}^3)
\]

is zero, then there exists a PL-manifold \( N \) with a pseudo homology cell decomposition which is cellularly equivalent to \( M \).

**Proof.** Since \( \{ \lambda(M) \} = 0 \), there exists a correspondance \( g : \{ (n - 3)\text{-simplex} \} \to \mathcal{H}^3 \) such that

\[
\sum_{\tau_i > \sigma} g(\tau_i) = \{ \text{Lk}(\sigma, M) \} \in \mathcal{H}^3.
\]

We will inductively construct PL-manifolds \( N_p \) and \( N_{(p)} = \bigcup_{q \leq p} N_q \) with a pseudo homology cell decomposition \( N_p = \bigcup E_\alpha \) where all
pseudo cells are PL-manifolds such that $N_{(p)}$ is cellularly equivalent to $M_{(p)}$.

(a) $p \leq 2$. In this case, the manifolds $N_p$, $N_{(p)}$ and their cells are just equal to $M_p$, $M_{(p)}$ and their cells. That is, for any $j$-simplex $\sigma$, $j \geq n - 2$, we define the PL-manifolds as

$$E(\sigma) = D(\sigma)$$

$$N_p = \cup \{E(\sigma)\mid \dim \sigma = n - p\} = \cup \{D(\sigma)\mid \dim \sigma = n - p\} = M_p$$

For any simplex $\mu \in M'$ such that $\mu > b_a$, we put

$$E(\mu) = D(\mu).$$

Hence $\partial E(b_a) = \partial D(b_a) = \cup D(\mu) = \cup E(\mu)$, and $N_{(p)} = M_{(p)}$.

(b) $p = 3$. Let $\tau_i$ be an $(n - 3)$-simplex. Let $H^3_i$ be the 3-dimensional PL-homology sphere which represents $g(\tau_i)$ and let $K_i$ be the PL-manifold whose boundary is $H^3_i \times S^{n-4}$ such that the inclusion $j : S^{n-4} \hookrightarrow K_i$ induces the isomorphisms of the fundamental groups and the homology groups, whose existence is shown by propositions 2 and 3 of § 3. Let $D^3 \subset H^3_i$ be a disc. Then $D^3 \times S^{n-4} \subset \partial K_i$. We have the PL-homeomorphism $\partial D(b_{\tau_i}) = S^2 \times D^{n-3} \cup D^3 \times S^{n-4}$. We define the PL-manifolds $E(b_{\tau_i})$ and $N_3$ by

$$E(b_{\tau_i}) = D(b_{\tau_i}) \cup D^3 \times S^{n-4} K_i$$

$$N_3 = \cup_{\tau_i} E(b_{\tau_i})$$

where $D(b_{\tau_i})$ is attached to $K_i$ by the identity map on $D^3 \times S^{n-4}$. It is easy to see that $E(b_{\tau_i})$ is a homology cell. We will give the pseudo cell decomposition for $\partial E(b_{\tau_i})$. First we devide $\partial E(b_{\tau_i})$ as the union

$$\partial E(b_{\tau_i}) = LE(b_{\tau_i}) \cup HE(b_{\tau_i}),$$

where

$$LE(b_{\tau_i}) = \partial D(b_{\tau_i}) - D^3 \times D^{n-3}$$

$$HE(b_{\tau_i}) = \partial K_i - D^3 \times S^{n-4} = (H^3_i - D^3) \times S^{n-4}.$$
(H_i^3 - D^3) \times S^{n-4} = (H_i^3 - D^3) \times \left( \bigcup_{\alpha} C_{\alpha} \right) = \bigcup_{\alpha} (H_i^3 - D^3) \times C_{\alpha}, \]

where $S^{n-4} = \bigcup C_{\alpha}$ is the standard decomposition. These decompositions of $LE(b_{r_i})$ and $HE(b_{r_i})$ fit together on their intersection and give the decomposition of $\partial E(b_{r_i})$, which is clearly cellular equivalent to that of $\partial D(b_{r_i})$. For each simplex $\mu > b_{r_i}, \mu \in M'$, we denote by $E(\mu)$ the pseudo cell of $\partial E(b_{r_i})$ which corresponds by the equivalence to $D(\mu) \in \partial D(b_{r_i})$. We have $\partial E(b_{r_i}) = \bigcup E(\mu)$. We define $N_{(3)}$ by

$$N_{(3)} = N_{(2)} \cup N_{3}$$

attached by the identity on $LE(b_{r_i})$. $N_{(3)}$ is cellularly equivalent to $M_{(3)}$.

(c) $p = 4$. Let $\sigma$ be a $(n - 4)$-simplex. Let $\bigcup E(\mu) \subset \partial N_{(3)}$ be the union of pseudo cells such that $b_{\sigma} < \mu \in M', \mu \neq b_{\sigma}$. Then by the definition, it is PL-homeomorphic to the PL-manifold

$$(Lk(\sigma) \# \Sigma (-H_i^3)) \times D^{n-4}$$

where $H_i^3$ represents $g(\tau_i)$ and the sum extends to all $\tau_i > \sigma$.

Since $\{Lk(\sigma)\} = \Sigma g(\tau_i)$ in $\mathcal{E}^3$, the PL-homology 3-sphere

$$H_3^\sigma = Lk(\sigma) \# \Sigma (-H_i^3)$$

is the boundary of an acyclic PL-manifold $W_{\sigma}^4$. The union

$$W_{\sigma}^4 \times S^{n-5} \cup H_3^\sigma \times D^{n-4}$$

is a PL-homology $(n - 1)$-sphere. By the proposition 1 of § 3, it is the boundary of a contractible PL-manifold $Y_{\sigma}$. We define the PL-manifolds $E(b_{\sigma})$ and $N^4$ as

$$E(b_{\sigma}) = Y_{\sigma}$$
$$N^4 = \bigcup E(b_{\sigma}) .$$

Further we define $LE(b_{\sigma})$ and $HE(b_{\sigma})$ by

$$LE(b_{\sigma}) = H_3^\sigma \times D^{n-4}$$
$$HE(b_{\sigma}) = W_{\sigma}^4 \times S^{n-5} .$$

The pseudo cellular decomposition for $LE(b_{\sigma})$ is already defined and we give for $HE(b_{\sigma})$ by the product with the standard decomposition.
They give a pseudo cellular decomposition of
\[ \partial E(b_\sigma) = \text{LE}(b_\sigma) \cup \text{HE}(b_\sigma), \]
which is cellularly equivalent to that of \( \partial D(b_\sigma) \). For each simplex \( \mu > b_\sigma, \mu \in M' \), we define \( E(\mu) \) by the pseudo cell which corresponds to \( D(\mu) \) by this equivalence. We define \( N_{(4)} \) by \( N_{(3)} \cup N_4 \) attached by the identity of \( \text{LE}(b_\sigma) \), which is cellularly equivalent to \( M_{(4)}. \)

(d) \( p \geq 5 \). Let \( \sigma \) be a \( j \)-simplex \( j \leq n - 5 \). Let \( \cup E(\mu) \subset \partial N_{(n-j-1)} \) be the union of pseudo cells such that \( \mu > b_\sigma, \mu \neq b_\sigma \). Then by our definition, it is a PL-manifold
\[ H_{\sigma}^{p-1} \times D^{n-p} \]
where \( H_{\sigma}^{p-1} \) is a PL-homology \((p-1)\)-sphere, where \( p = n - j \). By the proposition 1 of § 3, \( H^{p-1} \) is the boundary of a contractible PL-manifold \( W_{\sigma}^p \). We define \( E(b_\sigma) \) by
\[ E(b_\sigma) = W_{\sigma}^p \times D^{n-p}. \]
The other definitions are just similar to the case when \( p = 4 \).

Continuing this process, we obtain a PL-manifold \( N = N_{(n)} \) which is cellularly equivalent to \( M = M_{(n)} \). Q.E.D.

5. Simple homotopy equivalence.

By the theorem of § 4, for the same \( M \), if the obstruction class is 0, we can construct a PL-manifold \( N \). In this section, we prove the following.

**Theorem.** — If \( M \) is compact, the constructed manifold \( N \) is simple homotopy equivalent to \( M \).

Let \( M^{(k)} \) denote the \( k \)-skelton of \( M \). Let \( L \) be a subcomplex of \( M^{(k)} \), we define the PL-submanifold \( N^{(L)} \) of \( N \) by
\[ N^{(L)} = \cup \{ E(b_\sigma) | \sigma \in L \}. \]
We put
\[ N^{(k)} = N^{(M^{(k)})} = \cup \{ E(b) | \sigma \in M^{(k)} \}. \]

By the induction of \( k \), we prove the stronger
**Lemma 1.** There exists a simple homotopy equivalence
\[ f : M^{(k)} \to N^{(k)} \]
such that, for any \((k + 1)\)-simplex \(\mu\), \(f(\partial \mu) \subset N^{(\partial \mu)}\) and
\[ f|_{\partial \mu} : \partial \mu \to N^{(\partial \mu)} \]
is a simple homotopy equivalence.

**Proof.** If \(k = 0\), it holds obviously. Now we will prove the lemma for \(k + 1\) assuming the lemma for \(k\). Let \(\mu\) be a \((k + 1)\)-simplex. Since the collar of \(\partial \mu\) is PL-homeomorphic to \(S^k \times I\), we can write
\[ \mu = S^k \times I \cup S^k \ast (b_{\mu}) \]
where \(S^k_0 = S^k \times \{0\} = \partial \mu\) and \(S^k_1 = S^k \times \{1\} = S^k \times I \cap S^k \ast (b_{\mu})\).
Recall that
\[ N^{(M^{(k)} \cup \mu)} = N^{(k)} \cup E(b_{\mu}) \]
\[ N^{(k)} \cap E(b_{\mu}) = N^{(\partial \mu)} \cap E(b_{\mu}) = HE(b_{\mu}) = W_{\mu}^{n-k-1} \times S^k \]
where \(W_{\mu}^{n-k-1}\) is an acyclic (or contractible) PL-manifold. Let \(x\) be a point in the interior of \(W_{\mu}\) and let \(d : S^k \to W_{\mu} \times S^k\) be the embedding defined by \(d(S^k) = \{x\} \times S^k\). We define a map
\[ \tilde{f} : S^k_0 \cup S^k_1 \to N^{(k)} \]
by
\[ \tilde{f} | S^k_0 = f \]
\[ \tilde{f} | S^k_1 = d \].

Since \(\tilde{f} | \partial M\) gives a simple homotopy equivalence \(\partial \mu \to N^{(\partial \mu)}\), \(N^{(\partial \mu)}\) is homotopy equivalent to \(S^k\), and so \(\tilde{f} | S^k_0\) and \(\tilde{f} | S^k_1\) are homotopic. Hence we can extend \(\tilde{f}\) on \(S^k \times I\). Further since \(E(b_{\mu})\) is contractible, we can extend \(\tilde{f}\) to a map from \(\mu = S^k \times I \cup S^k \ast (b_{\mu})\) to \(N^{(M^{(k)} \cup \mu)}\).

By the definition, \(f\) and \(\tilde{f}\) coincide on \(\partial \mu\), and so we have a map
\[ g = f \cup \tilde{f} : M^{(k)} \cup \mu \to N^{(M^{(k)} \cup \mu)} \].
Repeating this for all \((k + 1)\)-simplexes of \(M\), we obtain a map \(g : M^{(k+1)} \to N^{(k+1)}\). We have the exact sequences of chain groups,
\[ 0 \to C_* (M^{(k)}) \to C_* (M^{(k+1)}) \to \Sigma C_* (\mu/\partial \mu) \to 0 \]
\[ 0 \to C_* (N^{(k)}) \to C_* (N^{(k+1)}) \to \Sigma C_* (E(b_{\mu})/HE(b_{\mu})) \to 0 \],
where we regard them as \(\mathbb{Z} \pi_1 (M^{(k+1)}) = \mathbb{Z} \pi_1 (N^{(k+1)})\)-modules.
The map $g$ induces $f_\ast$ on the first elements and $id_\ast$ on the third elements. Since they are chain equivalences with trivial Whitehead torsion, so is $g_\ast$ by [8]. Hence $g$ is a simple homotopy equivalence. It is easy to see that, for any $(k + 2)$-simplex $\tau$, $g$ induce a simple homotopy equivalence

$$g|_{\partial \tau} : \partial \tau \to N(\partial \tau).$$

Q.E.D.

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