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THE GROWTH OF ENTIRE SOLUTIONS OF DIFFERENTIAL EQUATIONS OF FINITE AND INFINITE ORDER

by Lawrence GRUMAN

Let $f(z)$ be an entire function (of one or several variables) of finite order ρ. A proximate order $\rho(r)$ is a function which satisfies the conditions

$$\lim_{r\to\infty} \rho(r) = \rho \quad \text{and} \quad \lim_{r\to\infty} r\,\rho'(r)\ln r = 0\ . \tag{1}$$

The function $L(r) = r^{\rho(r)-\rho}$ satisfies

$$\lim_{r\to\infty} \frac{L(kr)}{L(r)} = 1 \quad \text{uniformly for} \quad 0 < a \leqslant k \leqslant b < \infty\ . \tag{2}$$

We assume in addition that $\lim\limits_{r\to\infty} L(r)$ exists (perhaps infinite). For every entire function of order ρ, there exists a proximate order $\rho(r)$ with respect to which $f(z)$ has normal type [5].

For a given proximate order $\rho(r)$, we define the functions

$$h_r^*(z) = \overline{\lim_{z'\to z}}\left[\overline{\lim_{r\to\infty}} \frac{\ln|f(rz')|}{r^{\rho(r)}}\right],\ r>0$$

$$\left(\text{resp. } h_c^*(z) = \overline{\lim_{z'\to z}}\left[\overline{\lim_{|u|\to\infty}} \frac{\ln|f(uz')|}{|u|^{\rho(r)}}\right],\ u\in \mathbf{C}\right).$$

If $f(z)$ is of normal type with respect to the proximate order $\rho(r)$, it follows from (2) that these functions are pluri-subharmonic and real positive homogeneous (resp. complex homogeneous) of order ρ [4]. The function $h_r^*(z)$ (resp. $h_c^*(z)$) is called the radial (resp. circular) indicator of growth function of $f(z)$.

A convex homogeneous function $g(z)$ is one which satisfies $g(z_1+z_2) \leqslant g(z_1)+g(z_2)$ and $g(tz) = tg(z)$, $t \geqslant 0$. To every convex homogeneous function $g(z)$, we associate the compact convex set $K_g = \{w : \mathrm{Re} < w, z > \leqslant g(z)\ \forall\, z \in \mathrm{C}^n\}$, and to every compact convex set K, we associate the convex homogeneous function

$$g_K(z) = \sup_{w \in K} \mathrm{Re} < w, z > ,$$

which is called the support function of K. If $\rho \equiv 1$, we define $h_K(z)$, the *convex indicator of growth function* of $f(z)$, to be the least convex homogeneous majorant of $h^*(z)$. It is evidently the support function of the closed convex hull of the set

$$\{w : \mathrm{Re} < w, z > \leqslant h^*(z)\ \forall\, z \in \mathrm{C}^n\} .$$

If the dimension $n = 1$, these two functions are the same [5].

In § 1, we investigate for the case $n = 1$ the relationship between the growth of the function $f(z)$ and that of solutions $u(z)$ of the differential equation $\mathrm{P(D)}\, u = f$ (where $\mathrm{D} = \dfrac{\partial}{\partial z}$ and P(D) is a differential polynomial).

Let $p(z)$ be a complex norm (i.e. $p(\lambda z) = |\lambda|\, p(z)$, $\lambda \in \mathrm{C}$), $\mathrm{B}^\rho_\mathrm{A}$ the space of functions which satisfy a majoration

$$|f(z)| \leqslant \mathrm{C_A} \exp\{(\mathrm{A}\, p(z))^\rho\}$$

and $\mathrm{E}^\rho_\mathrm{R} = \bigcap_{\mathrm{A}>\mathrm{R}} \mathrm{B}^\rho_\mathrm{A}$. In [8], A. Martineau introduced the notion of a constant coefficient differential operator as a convolution operator on the dual space $(\mathrm{E}^\rho_\mathrm{R})'$ of continuous linear functionals defined on $\mathrm{E}^\rho_\mathrm{R}$. We will take as our definition of such an operator the *transpose,* which is a linear operator on the space $\mathrm{E}^\rho_\mathrm{R}$ into itself. This category includes the usual constant coefficient differential operator as a special case. For $\rho \geqslant 1$, Martineau showed that for every such operator $\hat{\mu}$ on $\mathrm{E}^\rho_\mathrm{R}$ and every $f \in \mathrm{E}^\rho_\mathrm{R}$, there exists a solution $g \in \mathrm{E}^\rho_\mathrm{R}$ of the equation $\hat{\mu}(g) = f$.

In § 2, we extend this notion and this result to the case of $p(z)$ a pseudo-norm and $\rho(r)$ a proximate order ($\rho \neq 1$), including the important case of $\rho < 1$. In § 3, we extend this notion and result to the case $\rho = 1$ and $p(z)$ an arbitrary convex homogeneous function. In § 4, we extend this notion and result to those functions which satisfy a majoration of the type $\exp\{k(\ln r)^\rho\}$ for $\rho > 1$.

Remark. – The case of proximate orders for $\rho = 1$ is rendered much more difficult by the special role played by the exponentials. We do not treat this case.

1. Ordinary differential equations.

Let $f(z)$ be an entire function of a single variable and $h_r^*(z)$ its indicator function with respect to a proximate order $\rho(r)$. We will henceforth in this section use the notation $k_f(\theta) = h_r^*(e^{i\theta})$, which is the standard notation for $n = 1$. If $u(z)$ is a solution of the constant coefficient differential equation $\mathrm{P(D)}\,u = f$, then it is an easy consequence of Cauchy's theorem that $k_f(\theta) \leqslant k_u(\theta)$. We are interested in seeing if we can choose a solution such that equality holds (at least locally). We will need

LEMMA 1. – *The number of disjoint open intervals on which* $k_f(\theta)$ *can be negative is at most* $\sup\limits_{a<1} [2a\rho]$ *(where* [ ] *means "greatest integer in").*

Proof. – For $\theta_1 < \theta_2 < \theta_3$ and $\theta_3 - \theta_1 < \pi/\rho$, we have [5, p. 70]

$$k_f(\theta_1)\sin\rho(\theta_2 - \theta_3) + k_f(\theta_2)\sin\rho(\theta_3 - \theta_1) + \\ + k_f(\theta_3)\sin\rho(\theta_1 - \theta_2) \leqslant 0 \ .$$

Thus, any two disjoint intervals on which $k_f(\theta)$ is negative are separated by an interval of length at least π/ρ on which $k_f(\theta)$ is non-negative. Q.E.D.

THEOREM 1. – *Let* $f(z)$ *be an entire function with indicator* $k_f(\theta)$ *with respect to the proximate order* $\rho(r)$. *Then there exists a solution* $u(z)$ *of the differential equation* $\mathrm{P(D)}\,u = f$ *such that*

i) $k_u(\theta) = k_f(\theta)$ *for* $\rho \leqslant 1$.

ii) $k_f(\theta) \leqslant k_u(\theta) \leqslant k_f^+(\theta) = \max(k_f(\theta),\ 0)$ *for* $\rho > 1$ *and for any specific interval* (θ_1, θ_2) *on which* $k_f(\theta)$ *is negative, there exists a unique solution* u *with this property such that* $k_u(\theta) = k_f(\theta)$ *for* $\theta_1 \leqslant \theta \leqslant \theta_2$.

Proof. – It is enough to consider solutions of the equation $(D - a)u = f$ and then iterate the result. All such solutions are given by

$$u(z) = e^{az} \int_0^z f(\zeta)\, e^{-a\zeta}\, d\zeta + C e^{az} . \tag{3}$$

If for some open interval of θ, the function $f(z)\, e^{-az}$ has negative indicator (with respect to *any* proximate order), then

$$C = \int_0^\infty f(t\xi)\, e^{-at\xi}\, \xi dt \ , \ \xi = e^{i\theta} \ ,$$

defines a constant for all θ in this interval. If there is no such region, we choose $C = 0$. By Lemma 1, for $\rho \leqslant 1$, there is at most one such interval, but for $\rho > 1$ there may be more than one such interval and we may only be able to choose C to satisfy this relation in one of the intervals. (This explains the difference between i) and ii) above).

From (1), we have that

$$(r^{\rho(r)})' = \rho(r)\, r^{\rho(r)-1} + r^{\rho(r)}\, \rho'(r) \ln r \to \rho(r)\, r^{\rho(r)-1} . \tag{4}$$

Let us consider the case $\rho < 1$. For a given $\xi = e^{i\theta}$, let $b = k_f(\theta)$ and $s = \mathrm{Re}\, a\xi$. Then given $\varepsilon > 0$, we have $|f(t\xi)| \leqslant K \exp(b + \varepsilon)\, t^{\rho(t)}$.

i) If $s < 0$ and $b < 0$ and if $\varepsilon < -\dfrac{b}{2}$, then

$$|u(r\xi)| \leqslant K\, e^{sr} \int_0^r e^{(b+\varepsilon)\, t^{\rho(t)} - st}\, dt + |C|\, e^{sr}$$

$$\leqslant K_1'\, e^{sr} \int_{q_0}^r \left[(b+\varepsilon) \frac{d}{dt} (t^{\rho(t)}) - s \right] e^{(b+\varepsilon)t^{\rho(t)} - st}\, dt + |C|\, e^{sr} ,$$

where q_0 is chosen so large that $\left[(b + \varepsilon) \dfrac{d}{dt} (t^{\rho(t)}) - s \right]$ is bounded below and K_1' depends on q_0.

$$|u(r\xi)| \leqslant K_1' \left[e^{(b+\varepsilon)\, r^{\rho(r)}} - e^{sr} \cdot K_{q_0} \right] + |C|\, e^{sr}$$

$$\leqslant K_1''\, e^{(b+\varepsilon)\, r^{\rho(r)}} .$$

ii) If $s \geqslant 0$ and $b < 0$, then by the choice of C, we have

$$|u(r\xi)| \leqslant K\, e^{sr} \int_r^\infty e^{(b+\varepsilon)\, t^{\rho(t)}} \cdot e^{-st}\, dt$$

$$\leqslant K \int_r^\infty e^{(b+\varepsilon)\, t^{\rho(t)}}\, dt \leqslant K\, e^{(b+2\varepsilon) r^{\rho(t)}} \int_r^\infty e^{-\varepsilon\, t^{\rho(t)}}\, dt$$

$$\leqslant K_2'\, e^{(b+2\varepsilon)\, r^{\rho(r)}} ,$$

since by (4), $r^{\rho(r)}$ is increasing for sufficiently large r.

iii) If $s > 0$ and $b \geqslant 0$, then

$$|u(r\xi)| \leqslant K\, e^{sr} \int_r^\infty e^{(b+\varepsilon)\, t^{\rho(t)} - st}\, dt$$

$$\leqslant K_3'\, e^{sr} \int_r^\infty \left[(b+\varepsilon) \frac{d}{dt} (t^{\rho(t)}) - s \right] e^{(b+\varepsilon)\, t^{\rho(t)} - st}\, dt$$

$$\leqslant K_3''\, e^{(b+\varepsilon)\, r^{\rho(r)}} .$$

iv) If $s \leqslant 0$ and $b \geqslant 0$, then

$$|u(r\xi)| \leqslant K\, e^{sr} \int_0^r e^{(b+\varepsilon)\, t^{\rho(t)} - st}\, dt + |C|\, e^{sr}$$

$$\leqslant K_4'\, r\, e^{(b+\varepsilon)\, r^{\rho(r)}} .$$

The case $\rho \geqslant 1$ is treated similarly (for $\rho = 1$, we must make use of the assumption that $\lim_{r \to \infty} r^{\rho(r) - \rho}$ exists). For $\rho > 1$, if for some θ, $k_f(\theta) \neq k_u(\theta)$, then $u(z) = w(z) + Ce^{az}$, where $k_f(\theta) = k_w(\theta) < 0$, so $k_u(\theta) = 0$. Q.E.D.

Remark. – It follows from Theorem 6 below that if P(D) has a non-zero constant term, then for $\rho < 1$, the solution $u(z)$ in i) is unique.

The following example shows that it is not always possible to find a solution u of $P(D)\, u = f$ with the same indicator as f. Let $f(z) = e^{z^2}$ and let u be a solution of $Du = f$. The function $f(z)$ has two intervals on which its indicator is negative. If we integrate $f(z)$ along the positive imaginary axis, we obtain a constant different from that which we obtain by integrating along the negative imaginary axis.

There is even a more intimate connection between the growth of the function $f(z)$ and the solution $u(z)$ of $P(D)\, u = f$. If $f(z)$ grows regularly in a given direction, then so will $u(z)$. We introduce our criterion for regularity of growth.

Let E be a measurable set of positive real numbers and let $E^r = E \cap [0, r]$. A set is said to have upper relative measure U if $\overline{\lim_{r \to \infty}} \frac{\text{meas}(E^r)}{r} = U$. If $U = 0$, E is an E^0-set.

DEFINITION [5]. – *Let $f(z)$ be an entire function with indicator $k_f(\theta)$ with respect to a given proximate order $\rho(r)$; $f(z)$ is said to be of completely regular growth along the ray $re^{i\theta}$ if*

$$\lim_{r \to \infty} \frac{\ln |f(re^{i\theta})|}{r^{\rho(r)}} = k_f(\theta) ,$$

where r takes on all values except perhaps for some E^0-set.

Remark. – The property of being of completely regular growth is not invariant with respect to a change in proximate orders.

THEOREM 2. – *If $u(z)$ is a solution of $P(D)u = f$ for an entire function $f(z)$ and if $\rho(r)$ is a proximate order with respect to which both $k_f(\theta)$ and $k_u(\theta)$ are bounded, then if $f(z)$ is of completely regular growth along the ray $re^{i\theta}$, so is $u(z)$.*

Proof. – We consider a solution of $(D - a)u = f$. By Theorem 1, for given θ, there is an interval (θ_1, θ_2) containing θ such that $u = w + C e^{az}$ and w has the same indicator as f in the interval (θ_1, θ_2). Thus, if $k_u(\theta) \neq k_f(\theta)$, we have that $\lim_{r \to \infty} \frac{\ln |u(re^{i\theta})|}{r^{\rho(r)}}$ exists with no exceptional set. Hence, in the following, we assume that $k_u(\theta) = k_f(\theta)$. We assume without loss of generality that $\theta = 0$.

Let ε and η be given positive numbers. Then there exists a set E_1 of upper relative measure less than $\eta/4$ such that if $r \notin E_1$, the family of functions $k_{u,r}(\phi) = \frac{\ln |u(re^{i\phi})|}{r^{\rho(r)}}$ is equicontinuous [5, p. 96]. Thus, there is a $\delta > 0$ such that for $|\phi| < \delta$,

$$|k_{u,r}(\phi) - k_{u,r}(0)| < \frac{\varepsilon}{4} \text{ and } |k_u(\phi) - k_u(0)| < \frac{\varepsilon}{4} \text{ for } r \notin E_1 .$$

Since f is of completely regular growth along the positive real axis, given $\gamma > 0$ (depending eventually on η and ε), for r not in some E^0-set E_2,

$$-\frac{\gamma}{4} + k_f(0) \leqslant \frac{\ln |f(r)|}{r^{\rho(r)}} \leqslant k_f(0) + \frac{\gamma}{4} = k_u(0) + \frac{\gamma}{4}\,. \qquad (5)$$

We choose r so large that meas $(E_2^r) < \frac{\eta}{4} r$ and $\frac{\ln |u(re^{i\phi})|}{r^{\rho(r)}} \leqslant k_u(\phi) + \frac{\gamma}{4}$ [5, p. 71]. By Cauchy's formula,

$$f(r)\, e^{-ar} = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{u(\xi + r)}{\xi^2}\, e^{-a(\xi + r)}\, d\xi\,.$$

So by (5) for $r \notin E_2$ and r sufficiently large, there exists w with $|w - r| = 1$ such that, noting $\phi_w = \arg w$,

$$|a| + \ln |u(w)| > \left\{ k_f(0) - \frac{\gamma}{4} \right\} r^{\rho(r)} \geqslant \left\{ k_f(\phi_w) - \frac{\gamma}{2} \right\} |w|^{\rho(|w|)}$$

Let $R_m = \left(\frac{1+\eta}{1-\eta}\right)^m$. Then, as in the proof of Theorem 31 [5, p. 73], we can choose γ so small (depending on ε and η but independent of w since $k_u(\theta)$ is bounded) such that

$$\frac{\ln |u(r\,\phi_w)|}{r^{\rho(r)}} > k_u(\phi_w) - \frac{\varepsilon}{4}$$

except perhaps on a set of measure at most $\frac{\eta^2}{4} R_m$ for

$$(1 - 2\eta)\, R_m \leqslant r \leqslant (1 + 2\eta)\, R_m$$

(for $m \geqslant m_0$ so large that the above inequalities hold). Let

$$E_3 = [0\,, R_{m_0}] \cup \Big(\bigcup_{m \geqslant m_0} E_m \Big)\,.$$

Then

$$\frac{\text{meas}\,(E_3^r)}{r} \leqslant \frac{R_{m_0} + \sum_{i=m_0}^{m} \frac{\eta^2}{4} \dfrac{(R_{m_0} - R_m)}{1 - \dfrac{(1+\eta)}{(1-\eta)}}}{R_m(1-\eta)}$$

$$\leqslant 0(1) + \frac{\eta}{2}\left(1 - \frac{R_{m_0}}{R_m}\right) < \frac{\eta}{4}$$

for m sufficiently large. Let $E_\eta = E_1 \cup E_2 \cup E_3$. Then

$$\overline{\lim_{r\to\infty}} \frac{\operatorname{meas}(E^r_\eta)}{r} < \eta ,$$

and gathering together our inequalities, we have $|k_{u,r}(0) - k_u(0)| < \varepsilon$ for $r \notin E$. To see that this implies the theorem, we refer the reader to Theorem 1, part 3 [5, p. 141]. Q.E.D.

Remark. – The fact that a function is of completely regular growth in an interval has important consequences for the distribution of its zeros. This is fully discussed in [5].

2. Differential operators with constant coefficients.

Let $p_n(z)$ be a decreasing sequence of real valued functions and B_n the space of entire functions such that $|f(z)\exp\{-p_n(z)\}|$ goes to zero at infinity. This is a Banach space with norm

$$\|f\|_n = \sup_z |f(z)\exp\{-p_n(z)\}| .$$

We then set

$$E = \bigcap_n B_n , \tag{6}$$

which is a Fréchet space when we equip it with the projective limit topology. If B'_n is the dual space of B_n, E' that of E, then $E' = \bigcup_n B'_n$.

Let $p(z)$ be a complex pseudo-norm and $\rho(r)$ a proximate order. The space $E_p^{\rho(r)}$ will designate the space we get in (6) by setting $p_n(z) = \left\{ p(z) + \frac{1}{n}\|z\| \right\}^{\rho(r)}$ (where $r = \|z\|$, and we use the Euclidean norm). The space E^0 will be the space we get in (6) by setting $p_n(z) = \|z\|^{1/n}$ (the space of entire functions of zero order).

For a given proximate order $\rho(r)$, we have by (4) that $r^{\rho(r)}$ is increasing for sufficiently large r. For a given integer q, we define $\phi(q) = r_q$ to be the largest solution of $q = r^{\rho(r)}$. Then the type with respect to $\rho(r)$ of an entire function of one variable with coefficients c_q (in its Taylor series expansion at the origin) is given by the formula

$$(\sigma \rho e)^{1/\rho} = \overline{\lim_{q \to \infty}} \, (\phi(q) \, | c_q |^{1/q}) \quad [5, \text{ p. } 42] \, . \tag{7}$$

If $f \in E_p^{\rho(r)}$, we expand f at the origin in homogeneous polynomials $f(z) = \sum_q P_q(z)$. Let $A_q = \left(\frac{\phi(q)^\rho}{e\rho}\right)^{q/\rho}$. If we set

$$f_t(z) = \sum_q A_q P_q(z) \, ,$$

then $f_t(z)$ is a holomorphic function in the open set $D = \{z : p(z) < 1\}$, and when we equip the space $\mathcal{H}(D)$ of holomorphic functions defined on D with the topology of uniform convergence on compact subsets, the mapping $f \to f_t$ becomes an isomorphism of $E_p^{\rho(r)}$ onto $\mathcal{H}(D)$ (cf. [8], Prop. 4, p. 116 and [4]).

For $\mu \in (E_p^{\rho(r)})$, we define the linear functional μ_t on $\mathcal{H}(D)$ by $(f_t, \mu_t) = (f, \mu)$. This is an isomorphism of $(E_p^{\rho(r)})'$ onto $\mathcal{H}'(D)$, the space of continuous linear functionals on $\mathcal{H}(D)$. We say that a linear functional μ_t is carried by the compact convex set K if for every open neighborhood Ω of K, there exists a constant C_Ω such that $| \mu_t(f_t) | \leqslant C_\Omega \sup_\Omega |f_t|$. Every $\mu_t \in \mathcal{H}'(D)$ is carried by one of the sets $K_n = \left\{ z : p(z) + \frac{1}{n} \|z\| \leqslant 1 \right\}$.

We define the Fourier-Borel transform of the functional μ_t to be the entire function $\widetilde{\mu}_t(u) = \mu_t(\exp < z, u >)$. Then we have [3], [7].

Proposition 1. – *The functional μ_t is carried by the compact convex set* K *if and only if*

$$\widetilde{\mu}_t(u) \leqslant C_8 \exp(H_K(u) + \delta \|u\|) \quad \textit{for all} \quad \delta > 0 \, ,$$

where $H_K(u)$ is the support function of K.

Let $p'_n(u) = \sup_{z \in K_n} \operatorname{Re} < z, u >$. Then $p'_n(u)$ is a family of increasing complex norms, and since each $\mu_t \in \mathcal{H}'(D)$ is carried by some K_n, we have

$$\widetilde{\mu}_t(u) \leqslant C_n \exp H_{K_n}(u) \quad \text{for } n \text{ sufficiently large.}$$

Let α be a multi-index of positive numbers, $|\alpha| = \Sigma \, \alpha_i$ and

$z^\alpha = z^{\alpha_1} \dots z^{\alpha_n}$. Since the polynomials converge to $\exp < z , u >$ in $\mathcal{H}(D)$, we have

$$\mu_t(\exp < z , u >) = \mu_t \sum_q \sum_{|\alpha| = q} z^\alpha \frac{u^\alpha}{\alpha!} = \sum_q \sum_{|\alpha| = q} \mu_t(z^\alpha) \frac{u^\alpha}{\alpha!}$$

$$= \sum_q P_q^{\mu_t}(u)$$

and from (7) and Proposition 1, we have

$$\overline{\lim_{q \to \infty}} \left\{ \frac{q}{e} \, | P^{\mu_t}(u) |^{1/q} \right\} \leqslant p'_n(u)$$

for n sufficiently large. From the relation $\mu_t(z^\alpha) = \frac{1}{A_{|\alpha|}} \mu(z^\alpha)$, we see that $\mu \in (E_p^{\rho(r)})'$ (resp. $(E^0)'$) if and only if

$$\overline{\lim_{q \to \infty}} \left\{ \frac{q}{e} \left| \frac{1}{A_q} \sum_{|\alpha| = q} \mu(z^\alpha) \frac{u^\alpha}{\alpha!} \right|^{1/q} \right\} \leqslant p'_n(u) \tag{8}$$

for n sufficiently large (resp. for ρ sufficiently small).

For $\mu \in (E_p^{\rho(r)})'$ (resp. $(E^0)'$), we define its Fourier-Borel transform to be the *formal* power series

$$\widetilde{\mu}(u) = \mu(\exp < z , u >) = \sum_q \sum_{|\alpha| = q} \mu(z^\alpha) \frac{u^\alpha}{\alpha!} = \sum_q P_q^\mu(u).$$

If $\rho > 1$, we assume that the proximate order $\rho(r)$ satisfies :

i) $\rho(r) > 1$ for all r

ii) $\frac{d}{dr} (r^{\rho(r) - 1}) > 0$ for all r.

By (1), these properties hold eventually, so this is an inessential assumption. Then the equation $r = t^{\rho(t)-1}$ has a unique solution for all r. We define

$$\rho^*(r) = \frac{\rho(t)}{\rho(t) - 1}, \text{ where } t \text{ is this unique solution.}$$

It is an easy calculation to show that $\rho^*(r)$ satisfies the conditions (1) and so is a proximate order. For $\rho > 1$, we designate

$$F_{Ap'}^{\rho^*(r)} = \bigcup_n E_{Ap'_n}^{\rho^*(r)},$$

where $A = \dfrac{(\rho - 1)^{\frac{\rho-1}{\rho}}}{\rho}$

THEOREM 3. – *The mapping* $\mu \mapsto \widetilde{\mu}(u)$ *is a one-to-one linear mapping of* $(E_p^{\rho(r)})'$ *(resp.* $(E^0)'$*) onto*

i) $F_{Ap'}^{\rho^*(r)}$ *for* $\rho > 1$

ii) *the set* $Q_{p'}^{\rho(r)}$ *of formal power series at the origin which satisfy (8) for some n for* $\rho < 1$

iii) *the set* Q_0 *of formal power series at the origin which satisfy (8) for some* $\rho > 0$ *for* $(E^0)'$.

Proof. – We have that (8) holds for some n_0. Since

$$A_q^{1/q} = \frac{\phi(q)}{(e\rho)^{1/\rho}}, \quad \frac{q}{e}\,\frac{1}{A_q^{1/q}} = \frac{A\, r_q^{\rho(r_q)-1}}{(e\rho^*)^{1/\rho^*}}$$

(where $r_q = \phi(q)$). Let $r'_q = r_q^{\rho(r_q)-1}$. Then

$$(r'_q)^{\rho^*(r'_q)} = (r_q^{\rho(r_q)-1})^{\rho^*(r_q^{\rho(r_q)-1})}$$

$$= (r_q^{\rho(r_q)-1})^{\frac{\rho(r_q)}{\rho(r_q)-1}} = r_q^{\rho(r_q)} = q$$

so if $\phi'(q)$ is the unique solution of $(r'_q)^{\rho^*(r)} = q$, we have that $\frac{q}{e}\,\frac{1}{A^{1/q}} = A\,\frac{\phi'(q)}{(e\rho^*)^{1/\rho^*}}$ so the mapping is into. Since the calculations are all reversible, the mapping is also onto. This proves case i). Cases ii) and iii) follow directly from (8). Q.E.D.

Let $\mu \in (E_p^{\rho(r)})'$. Then for any other element ν, we define the convolution of ν with μ, $\mu * \nu = \tau$ by $(f(z), \mu * \nu) = (\mu_w f(z + w), \nu)$. This is defined at least on the polynomials, which are dense in $E_p^{\rho(r)}$. For $\rho > 1$, it is also defined on the exponentials [8]. We then have the relationship (for $\rho \neq 1$) $\widetilde{\tau}(u) = \widetilde{\mu}(u) \cdot \widetilde{\nu}(u)$, which, for the case $\rho < 1$, follows from

LEMMA 2. – *For* $\widetilde{\mu}(u)$, $\widetilde{\nu}(u) \in Q_{p'}^{\rho(r)}$ *(resp.* Q_0*), we have* $\widetilde{\tau}(u) = \widetilde{\mu}(u)\, \widetilde{\nu}(u) \in Q_{p'}^{\rho(r)}$ *(resp.* Q_0*) for* $\rho < 1$ *(i.e. these spaces are algebras).*

Proof. – We choose n_0 so large so that for $n \geqslant n_0$, (8) holds for both μ and ν. Consider such an n and let $\varepsilon > 0$ be given. Then there exist constants C_ε^μ and C_ε^ν such that

$$|P_q^\mu(u)| \leqslant C_\varepsilon^\mu [p'_n(u) + \varepsilon \,\|u\|]^q \left(\frac{\Phi(q)^\rho}{e\rho}\right)^{q/\rho} \left(\frac{e}{q}\right)^q$$

and

$$|P_q^\nu(u)| \leqslant C_\varepsilon^\nu [p'_n(u) + \varepsilon \,\|u\|]^q \left(\frac{\Phi(q)^\rho}{e\rho}\right)^{q/\rho} \left(\frac{e}{q}\right)^q .$$

Then

$$\begin{aligned} |P_q^\tau(u)| &= \Big|\sum_{m+n=q} P_m^\nu(u)\, P_n^\mu(u)\Big| \\ &\leqslant C_\varepsilon^\mu C_\varepsilon^\nu [p'(u) + \varepsilon\, \|u\|]^q \left(\frac{\Phi(q)^\rho}{e\rho}\right)\left(\frac{e}{q}\right)^q \\ &\qquad \sum_{m+n=q} \left[\frac{\Phi(m)^m\, \Phi(n)^n}{\Phi(m+n)^{m+n}}\right] \frac{(m+n)^{m+n}}{m^m n^n} . \end{aligned}$$

Let $r_q = \Phi(q)$. Then $\dfrac{q}{\Phi(q)} = r_q^{\rho(r_q)-1}$, and hence, since by (1), $r^{\rho(r)-1}$ is decreasing for r sufficiently large

$$\sum_{m+n=q} \frac{[r_{m+n}^{\rho(r_{m+n})-1}]^{m+n}}{[r_m^{\rho(r_m)-1}]^m\, [r_n^{\rho(r_n)-1}]^n} \leqslant K\, q \text{ for some constant } K .$$

Thus $|P_q^\tau(u)|$ satisfies (8). For Q_0, we choose ρ_0 so small that (8) holds for both μ and ν for $\rho < \rho_0$. The result then follows from the above calculations. Q.E.D.

Thus, by Theorem 3, for $\rho < 1$, the mapping $\nu \to \mu * \nu$ is a map of $(E_p^{\rho(r)})'$ (resp. $(E^0)'$) into $(E_p^{\rho(r)})'$ (resp. $(E_0)'$). If $\rho > 1$, this is only the case if $\widetilde{\mu}(u)$ is of minimal type with respect to the proximate order $\rho^*(r)$. Assuming μ to satisfy these conditions, we define $\check{\mu}$ to be the transpose of μ, $(\check{\mu}(f), \nu) = (f, \mu * \nu)$. We are interested in proving that the mapping $\check{\mu}(E_p^{\rho(r)})$ (resp. E^0) is onto (i.e. that there always exists a solution g such that $\check{\mu}(g) = f$). We will make use of [cf. 9, p. 85].

PROPOSITION 2. – *Let* E, F *be two Fréchet spaces,* α *a continuous linear map of* E *into* F. *The two following are equivalent*

i) α *is onto*

ii) ${}^t\alpha : \mathrm{F}' \to \mathrm{E}'$ *(the transpose map) is one-to-one and its image* ${}^t\alpha(\mathrm{F}')$ *is weakly closed in* E′.

We shall prove the closure of $\mu * \nu$ in the equivalent spaces as determined by Theorem 3, but first we must equip these spaces with topologies. For $\rho > 1$, we equip the space $\mathrm{F}^{\rho^*(r)}_{\mathrm{A}p'}$ with the topology of pointwise convergence. For $\rho < 1$, we equip $\mathrm{Q}^{\rho(r)}_{p'}$ (resp. Q_0) with the topology of convergence of Taylor's series coefficients. Each of these topologies is at least as weak as the weak topology.

We define a *differential operator with constant coefficients* (with respect to a given proximate order $\rho(r)$) to be

i) $\check{\mu}$ for $\mu \in (\mathrm{E}^{\rho(r)}_p)'$ for $\rho < 1$

ii) $\check{\mu}$ for $\mu \in (\mathrm{E}^0)'$

iii) $\check{\mu}$ for $\mu \in (\mathrm{E}^{\rho(r)}_p)'$ such that $\widetilde{\mu}(u)$ is of minimal type with respect to $\rho^*(r)$ for $\rho > 1$.

For $\rho > 1$, the mapping $\widetilde{\nu}(u) \to \widetilde{\mu}(u)\,\widetilde{\nu}(u)$ is closed in the topology we have chosen (the proof is carried out in [8] ; the modifications necessary to treat the case of proximate orders are obvious). Thus, we limit ourselves to the case $\rho < 1$ and E^0.

LEMMA 3. – *Let* $\mathrm{A}_n(u) = \dfrac{\mathrm{B}_{n+m}(u)}{\mathrm{C}_m(u)}$ *be a homogeneous polynomial which is the ratio of two homogeneous polynomials. Furthermore, assume that for some complex norm* $p_0(u)$ *that*

$$|\,\mathrm{B}_{n+m}(u)\,| \leqslant \mathrm{C}\,[p_0(u)]^{n+m} .$$

Then given $\delta > 0$, *there is a constant* K_δ *(depending only on* $\mathrm{C}_m(u)$ *and* δ*) such that* $|\,\mathrm{A}_n(u)\,| \leqslant \mathrm{C}\,\mathrm{K}_\delta\,[p_0(u)]^n\,(1+\delta)^{n+m}$.

Proof. – Let $\Omega = \{u : 1-\delta \leqslant p_0(u) \leqslant 1+\delta\}$. For every point u in Ω we find a polydisc (by making a non-singular linear change of variable if necessary) $\Delta(u\,;r^u)$ centered at u and lying in Ω such that $\mathrm{C}_m(u'_1, \ldots, u'_{n-1}, \xi_n) \neq 0$ for $|\,\xi_n - u_n\,| = r^u_n$ and

$$|u_i' - u_i| \leqslant r_i^u \,, i = 1, \ldots, n-1 \; [2] \,.$$

Let $\Omega' = \{u : p_0(u) = 1\}$. We now consider the polydisc $\Delta_u' = \Delta\left(u\,; \frac{r^u}{2}\right)$. Since Ω' is compact, it can be covered by a finite number of Δ'_{u^j}, $j = 1, \ldots, N$. The function $\dfrac{1}{C_m(u)}$ is bounded, say by $\dfrac{K_\delta}{2}$, on the compact set

$$K = \bigcup_j \{u' : u' \in \Delta_{u^j}\,, |u_i' - u_i| \leqslant r^{u_i^j}\,, i = 1, \ldots, n-1, |u_n' - u_n| = r^{u_n^j}\} \,.$$

Let the function A_n take its maximum on Ω' at the point u^0. Then $u^0 \in \Delta'_{u^j}$ for some j. By Cauchy's formula

$$|A_n(u^0)| = \left| \frac{1}{2\pi i} \int_{|\xi_n - u_n^j| = r_n^j} \frac{B_{n+m}(u_1^0, \ldots, u_{n-1}^0, \xi_n)\, d\xi_n}{C_m(u_1^0, \ldots, u_{n-1}^0, \xi_n)(\xi_n - u_n^0)} \right|$$
$$= K_\delta \, C \, p_0(u)(1+\delta)^{n+m} \,. \qquad \text{Q.E.D.}$$

THEOREM 4 (Division Theorem). – *Let* $H(u)$, $F(u) \in Q_p^{\rho(r)}$ *for* $\rho < 1$ (*resp.* Q_0) *with* $H(u) = F(u)\,G(u)$, *where* $G(u)$ *is a formal power series at the origin. Then* $G(u) \in Q_p^{\rho(r)}$ (*resp.* Q_0).

Proof. – Let $\varepsilon > 0$ be given and let

$$G(u) = \sum_q R_q(u) \,, \quad H(u) = \sum_q P_q(u) \,, \quad \text{and} \quad F(u) = \sum_q T_q(u) \,,$$

with s the smallest integer such that $T_s(u) \neq 0$. We choose n_0 so large that (8) holds for both $H(u)$ and $F(u)$ for $n \geqslant n_0$. Thus, there exist constants C_1 and C_2 such that

$$|P_q(u)| \leqslant C_1 [p_n'(u) + \varepsilon \|u\|]^q \left(\frac{\phi(q)^\rho}{e\rho}\right)^{q/\rho} \left(\frac{e}{q}\right)^q$$

and

$$|T_q(u)| \leqslant C_2 [p_n'(u) + \varepsilon \|u\|]^q \left(\frac{\phi(q)^\rho}{e\rho}\right)^{q/\rho} \left(\frac{e}{q}\right)^q \,.$$

We have

$$P_{q+s}(u) = \sum_{m+k=q} R_m(u)\, T_{k+s}(u)$$

or

$$R_q(u) = \frac{P_{q+s}(u) - \sum\limits_{\substack{m+k=q \\ m \neq q}} R_m(u)\, T_{k+s}(u)}{T_s(u)} .$$

We now show by induction that there exist constants K_q (with $K_{q-1} \leqslant K_q$) such that

$$|R_q(u)| \leqslant K_q [p_n'(u) + \varepsilon \, \|u\|]^q (1+\delta)^q \, q \left(\frac{\phi(q+s)^\rho}{e\rho}\right)^{\frac{\rho+s}{\rho}} \left(\frac{e}{q+s}\right)^{q+s} ,$$

where $K_q = K_{q-1}$ for q sufficiently large.

For $q = 0$, it follows from Lemma 3. We assume it true for $q \leqslant q_0 - 1$.

$$|R_{q_0}(u)| \leqslant \frac{|P_{q_0+s}(u)| + \sum\limits_{\substack{m+k=q_0 \\ m\neq q_0}} |R_m(u)\, T_{k+s}(u)|}{|T_s(u)|}$$

$$\leqslant K_\delta (1+\delta)^s [p_n'(u) + \varepsilon \, \|u\|]^{q_0} (1+\delta)^{q_0} \left(\frac{\phi(q_0+s)^\rho}{e\rho}\right)^{\frac{q_0+s}{\rho}} \left(\frac{e}{q+s}\right)^{q_0+s} \times$$

$$\times \left\{ C_1 + \sum_{\substack{m+k=q_0 \\ m\neq q_0}} K_{q-1} C_2 m \left[\frac{\phi(m)^m \, \phi(k+s)^{k+s}}{\phi(k+m+s)^{k+m+s}}\right] \frac{(m+k+s)^{m+k+s}}{m^m (k+s)^{k+s}} \right.$$

$$\leqslant \max [K_0 (1+\delta)^s C_1 , K_{q-1} C_2] [p_n'(u) + \varepsilon \, \|u\|]^{q_0} (1+\delta)^{q_0} \left(\frac{\phi(q+s)^\rho}{e\rho}\right)^{\frac{q_0+s}{\rho}} \left(\frac{e}{q+s}\right)^{q_0+s} \times$$

$$\times \left\{ 1 + \sum_{\substack{m+k=q_0 \\ m\neq q_0}} K_\delta (1+\delta)^s m \left[\frac{\phi(m)^m \, \phi(k+s)^{k+s}}{\phi(m+k+s)^{m+k+s}}\right] \frac{(m+k+s)^{m+k+s}}{(k+s)^{k+s} m^m} \right\} .$$

We assume that the function $r^{1-\rho(r)}$ is increasing. By (1), this holds eventually, so this is an inessential assumption

$$\left[\frac{\phi(m)^m\,\phi(k+s)^{k+s}}{\phi(q_0+s)^{q_0+s}}\right]\frac{(q_0+s)^{q_0+s}}{m^m(k+s)^{k+s}}=\frac{1}{\left[\dfrac{r_{q_0+s}^{1-\rho(r_{q_0+s})}}{r_m^{1-\rho(r_m)}}\right]^m\left[\dfrac{r_{q_0+s}^{1-\rho(r_{q_0+s})}}{r_{k+s}^{1-\rho(r_{k+s})}}\right]^{k+s}}.$$

Let us assume for the moment that $k+s\leqslant\frac{3}{4}(q_0+s)$. Then

$$\left[\frac{r_{q_0+s}^{1-\rho(r_{q_0+s})}}{r^{1-\rho(r_{k+s})}}\right]^{k+s}=\left[\frac{r_{q_0+s}^{\frac{1-\rho(r_{q_0+s})}{1-\rho}\cdot\frac{\rho}{2}}}{r_{k+s}^{\frac{1-\rho(r_{k+s})}{1-\rho}\cdot\frac{\rho}{2}}}\right]^{(k+s)\frac{2}{\rho}(1-\rho)}$$

Let $\psi(r)=r^{\frac{1-\rho(r)}{1-\rho}\cdot\frac{\rho}{2}}$. Then

$$\psi(r_{q_0+s})-\psi(r_{k+s})=\int_{r_{k+s}}^{r_{q_0+s}}\frac{d}{dr}\psi(r)\,dr\geqslant\int_{\frac{3}{4}(r_{q_0+s})}^{r_{q_0+s}}\frac{d}{dr}\psi(r)\,dr\geqslant$$

$$\geqslant\int_{\frac{3}{4}(r_{q_0+s})}^{r_{q_0+s}}\frac{d}{dr}r^{\frac{\rho(r_{q_0+s})}{4}}\,dr$$

for q_0+s sufficiently large, by (1). Thus

$$\psi(r_{q_0+s})-\psi(r_{k+s})\geqslant r_{q_0+s}^{\frac{\rho(r_{q_0+s})}{4}}\left[1-\left(\frac{3}{4}\right)^{1/4}\right]=\mathrm{T}(q_0+s)^{1/4}.$$

For $(k+s)\geqslant 12\,\frac{\rho}{2}\,\frac{1}{1-\rho}=\alpha$, we have

$$\left[\frac{r_{q_0+s}^{\frac{1-\rho(r_{q_0+s})}{1-\rho}\cdot\frac{\rho}{2}}}{r_{k+s}^{\frac{1-\rho(r_{k+s})}{1-\rho}\cdot\frac{\rho}{2}}}\right]^{(k+s)\frac{2}{\rho}(1-\rho)}\geqslant\left[1+\frac{\mathrm{T}(q_0+s)^{1/4}}{r_{k+s}^{\frac{1-\rho(r_{k+s})}{1-\rho}\cdot\frac{\rho}{2}}}\right]^{(k+s)\frac{2}{\rho}(1-\rho)}$$

$$\geqslant\left[1+\frac{(k+s)\,\mathrm{T}(q_0+s)^{1/4}}{r_{k+s}^{\frac{1-\rho(r_{k+s})}{1-\rho}\cdot\frac{\rho}{2}}}+\cdots+\mathrm{K}\,\mathrm{T}^{\gamma}(q_0+s)^{\gamma}+\cdots\right]^{\frac{2}{\rho}(1-\rho)},$$

where $\gamma \geqslant 3\,\dfrac{\rho}{2}\,\dfrac{1}{1-\rho}$, $\left(\text{since } r_{k+s}^{\frac{1-\rho(r_{k+s})}{1-\rho}\cdot\frac{\rho}{2}} = 0(k+s)^{1/2}\right)$

$$\geqslant \mathrm{T}'(q_0+s)^3 .$$

For $(k+s) \leqslant \alpha + 1$

$$\left[\frac{r_{q_0+s}^{1-\rho(r_{q_0+s})}}{r_{k+s}^{1-\rho(r_{k+s})}}\right]^{k+s} \geqslant \left[\frac{r_{q_0+s}^{1-\rho(r_{q_0+s})}}{\beta}\right]^{k+s} \geqslant (\alpha+1)^2\,\mathrm{K}_\delta\,(1+\delta)^s\,3 ,$$

$\left(\text{where } \beta = \max\limits_{(k+s)\leqslant\alpha} r_{k+s}^{1-\rho(r_{k+s})}\right)$ for q_0 sufficiently large. By symmetry, similar inequalities exist if we replace $(k+s)$ by m. We choose q_0 so large that $\dfrac{\mathrm{K}_\delta\,(1+\delta)^s}{\mathrm{T}'(q_0+s)^2} \leqslant \dfrac{1}{q_0}$. Thus

$$\left\{1 + \sum_{\substack{m+k=q_0\\ m\neq q_0}} \mathrm{K}_\delta\,(1+\delta)^s\, m \left[\frac{\phi(m)^m\,\phi(k+s)^{k+s}}{\phi(m+k+s)^{m+k+s}}\right] \frac{(m+k+s)^{m+k+s}}{m^m(k+s)^{k+s}}\right\}$$

$$\leqslant 1 + \frac{(q_0-1)}{3} + 2 \leqslant q_0$$

for q_0 sufficiently large, which establishes the induction.

Furthermore,

$$\left[\frac{\phi(q+s)}{q+s}\right]^{q+s} = \left[r_{q+s}^{1-\rho(r_{q+s})}\right]^{q+s} = \left[\frac{\phi(q)}{q}\right]^{q+s}\left[\frac{r_{q+s}^{1-\rho(r_{q+s})}}{r_q^{1-\rho(r_q)}}\right]^{q+s}$$

$$\leqslant (1+\delta)^{q+s}\left[\frac{\phi(q)}{q}\right]^{q+s+1}$$

for arbitrary $\delta > 0$ when q is sufficiently large. Thus

$$\overline{\lim_{q\to\infty}}\left\{\frac{q}{e}\left|\frac{1}{\mathrm{A}_q}\,\mathrm{R}_q(u)\right|^{1/q}\right\} \leqslant p_n'(u) ,$$

which proves the theorem. Q.E.D.

COROLLARY. – *Let* $F(u) = \sum_q T_q(u)$, $H(u) = \sum_q P_q(u)$ *be in* $Q_p^{\rho(r)}$ *(resp.* Q_0*) and assume* $T_0 \neq 0$. *Then there exists a unique* $G(u) \in Q_p^{\rho(r)}$ *(resp.* Q_0*) such that* $F(u)\,G(u) = H(u)$.

Proof. – It is well known that the set of formal power series with non-zero constant term forms a group under multiplication. By Theorem 4, $G(u) \in Q_p^{\rho(r)}$ (resp. Q_0). Q.E.D.

Combining Theorem 4 with Proposition 2, we obtain the following

THEOREM 5. – *Let* $\check{\mu}$ *be a differential operator with constant coefficients for some space* $E_p^{\rho(r)}$ *for a complex pseudo-norm* $p(z)$ *and a proximate order* $\rho(r)$ $(\rho \neq 1)$ *(resp.* E^0*). Then for* $f \in E_p^{\rho(r)}$ *(resp.* E^0*), there always exists* $g \in E_p^{\rho(r)}$ *(resp.* E^0*) such that* $\check{\mu}(g) = f$. *For* $\rho < 1$ *(resp.* E^0*), if* $\check{\mu}(1) \neq 0$, *the solution* g *is unique.*

Proof. – As a result of Theorem 4, the mapping $\nu \to \mu * \nu$ is one-to-one and closed. If $\widetilde{\mu}(u)$ has a non-zero constant term, then by the corollary to Theorem 4, this mapping is also onto, so its transpose $\check{\mu}$ is one-to-one. Q.E.D.

We now show that for $\rho < 1$, the uniqueness of the solution has important consequences for the circular indicator function. Instead of a complex pseudo-norm, we let $p_0(z)$ be any positive upper semi-continuous complex homogeneous function (i.e. $p_0(\lambda z) = |\lambda|\, p_0(z)$). We construct the space $E_{p_0}^{\rho(r)}$ as in (6).

LEMMA 4. – *Let* $p_0(z)$ *be a positive upper semi-continuous complex homogeneous function,* $\mathfrak{F} = \{p(z) : p(z)$ *a complex norm,* $p(z) \geqslant p_0(z)\}$. *Then* $p_0(z) = \inf_{p(z) \in \mathfrak{F}} \{p(z)\}$.

Proof. – Let $D = \{z : p_0(z) < 1\}$, $D_\varepsilon = \{z : p_0(z) + \varepsilon \|z\| < 1\}$, which are open. Consider a complex line (λz_0), $\lambda \in C$ (which we assume to be $(\lambda(z_1, 0, \ldots, 0))$), and let

$$D^{z_0} = D \cap (\lambda z_0)\ ,\ D_\varepsilon^{z_0} = D_\varepsilon \cap (\lambda z_0)\ .$$

This determines two concentric circles in the (λz_0) line. We choose a radius $r_{z_0} < \infty$ between the radii of these two concentric circles and ε_{z_0} so small that the convex set

$$K_{z_0} = \{ z : \|z_1\| < r_{z_0} , \sqrt{\sum_{i=2}^{n} |z_i|^2} < \varepsilon_{z_0} \} \subset D .$$

We define $p_{z_0}(z) = \inf_{\frac{1}{t} z \in K_{z_0}} t$, which is a complex norm. Since D_ε is a compact set, it can be covered by a finite number of the open sets K_{z_j}, $j = 1, \ldots, N$. Then $p_0(z) \leqslant \inf_j p_{z_j}(z) \leqslant p_0(t) + \varepsilon \|t\|$.

Q.E.D.

THEOREM 6. – *Let $\rho < 1$ and let f have circular indicator $h_c^*(z)$ with respect to $\rho(r)$. Let $\mu \in \bigcap_{A>0} (E_{A\|z\|}^{\rho(r)})'$ such that $\mu(1) \neq 0$. Then there is a unique solution g of the equation $\check{\mu}(x) = f$ such that, if $k_c^*(z)$ is the circular indicator of g with respect to $\rho(r)$, $k_c^*(z) \leqslant h_c^*(z)$.*

Proof. – Let $p_\alpha(z)$ be a family of norms such that

$$h_c^*(z)^{1/\rho} = \inf_\alpha p_\alpha(z) .$$

Then $\mu \in (E_{p_\alpha(z)}^{\rho(r)})'$ for every α, and by Theorem 5, there exists a unique solution g to the equation $\check{\mu}(g) = f$. We clearly have

$$k_c^*(z) \leqslant h_c^*(z) . \qquad \text{Q.E.D.}$$

In particular, if $P(D)$ is a differential polynomial with constant coefficients and non-zero constant term, then for $\rho < 1$, there is a unique solution g of the differential equation $P(D)\,g = f$ where g has the same circular indicator as f.

3. The case of $\rho = 1$ and convex functions.

Let h_k be a convex function, K the associated convex compact set. We make the space E_{h_k} of entire functions $F(u)$ whose convex indicator functions are less than or equal to h_k into a Frechet space as in (6) by choosing $p_n(z) = h_k(z) + \frac{1}{n} \|z\|$; $(E_{h_k})'$ is its dual space. We have the following characterization of $(E_{h_k})'$ [8].

PROPOSITION 3. – *The space* $(E_{h_k})'$ *is just the set of measures* m *for which there exists an* $\varepsilon > 0$ *such that* $m \cdot e^{h_k(z)+\varepsilon\|z\|}$ *is a bounded measure.*

We recall some of the basic notions that A. Martineau [8] used in defining the projective Laplace transformation of a function $f(z)$ of exponential type. Let V be an n-dimensional linear vector space, V′ its dual. Let P(V) be the projective space obtained from V by adding the points at infinity, P(V′) that obtained from V′ by adding the points at infinity. We write the coordinates of P(V) as (ζ_0, z), those of P(V′) as (ξ_0, ξ), and we let $\bar{\xi}$ be the hyperplane

$$\zeta_0 \cdot \xi_0 + \langle z , \xi \rangle = 0 .$$

We introduce the differential forms $\pi(z) = dz_1 \wedge \ldots \wedge dz_n$,

$$\theta(\xi) = \sum_{j=1}^{n} (-1)^j \, \xi_j \, d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_n$$

($d\xi_j$ omitted) and $\bar{\omega}(\xi, z) = \theta(\xi) \wedge \pi(z)$, which is defined in $V \times P(V')$.

Let Γ be the boundary of a strictly convex open set Ω and assume Γ regular and oriented by Stokes' formula $\int_{\partial\Omega} \pi = \int_{\Omega} d\pi$. To each point $z \in \Gamma$, we have the associated hyperplane $\bar{\xi}(z)$ through z tangent to Γ. This defines a manifold $\Sigma(\Gamma)$ in $V \times P(V')$.

For a compact convex set K, we designate by $\overset{*}{\complement} K$ the open subset of P(V′) formed of hyperplanes $\bar{\xi}$ such that $\bar{\xi} \cap K = \{\phi\}$.

PROPOSITION 4 [8]. – *Suppose* K *convex and compact. Let* ψ *be a function defined in* $\overset{*}{\complement} K$*, holomorphic there, and zero at the points at infinity* ($\xi_0 = 0$). *Let* $\bar{f} \in \mathcal{H}(K)$ *(functions holomorphic in a neighborhood of* K*) and* f *a representative of* $\bar{f}$ *in an open neighborhood* Ω *of* K*. Let* ω *be a strictly convex neighborhood of* K *with regular boundary included in* Ω*. Posing*

$$T_\psi(\bar{f}) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} f(z) \, \frac{\partial^{n-1}}{\partial \xi_0^{n-1}} \left(\frac{1}{\xi_0} \psi(\xi) \right) \bar{\omega}(z, \xi) \qquad (9)$$

we define a continuous linear functional on $\mathcal{H}(K)$ *which is independant of the choice of the representative* f *and of* ω.

Let $F(u)$ be an arbitrary element of E_{h_k}. We define the function

$$\mathcal{L}_F(\overline{\xi}) = \xi_0 \int_0^\infty F(-\xi t)\, e^{-\xi_0 t}\, dt\,.$$

This defines a function in $\overset{*}{C}K$ which is zero at the points at infinity $\xi_0 = 0$. The function $\mathcal{L}_F$ is called the projective Fourier-Borel transform of F. We then have

PROPOSITION 5 [8]. – *Let* $F(u) \in E_{h_k}$. *Then*

$$F(u) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} \exp < z\,,u > \frac{\partial^{n-1}}{\partial \xi_0^{n-1}} \left(\frac{\mathcal{L}_F(\xi)}{\xi_0}\right) \overline{\omega}(z\,,\xi), \quad (10)$$

where ω *is any strictly convex neighborhood of* K *with regular boundary.*

Let $\mu \in (E_{h_k})'$. We define the Fourier-Borel transform of μ to be $f_\mu(z) = \mu(\exp < z\,,u >)$, which, by Proposition 3, defines a function holomorphic in a neighborhood of K. For $\nu \in (E_{h_k})'$, we define the convolution of μ with ν as $(\nu * \mu)(F(u)) = \mu_u(\nu_v\, F(u+v))$. We refer the reader again to [8] to see that the convolution is well defined. We then have the relationship that $f_{\nu * \mu}(z) = f_\nu(z) \cdot f_\mu(z)$ where these functions are defined.

On the other hand, let $g(z)$ be a function holomorphic in a neighborhood of K. Then g defines a continuous linear operator S_g from E_{h_k} into E_{h_k} by

$$S_g(F(u)) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} g(z) \exp < z\,,u > \frac{\partial^{n-1}}{\partial \xi^{n-1}} \left(\frac{\mathcal{L}_F(\xi)}{\xi_0}\right) \overline{\omega}(z\,,\xi)\,,$$

where ω is a suitably small strictly convex regular neighborhood of K.

LEMMA 5. – *Let* $\psi_{z_0} = \mathcal{L}_{\exp < z_0, u >}$ *for* $z_0 \in K$. *Then the linear functional on* $\mathcal{H}(K)$ *determined by* ψ_{z_0}, $T_{\psi_{z_0}} = \delta(z_0)$, *the Dirac measure.*

Proof. – Let f be a representative of $\overline{f} \in \mathcal{H}(K)$ defined in some convex neighborhood ω of K. Since ω is a Runge domain, f can be uniformly approximated by polynomials in an open neighborhood of K, and since $z_i = \lim_{|\lambda| \to 0} \dfrac{e^{z_i \lambda} - 1}{\lambda}$, $\lambda \in \mathbf{C}$, f can be uniformly approximated by exponentials. But by (10), we have that $T_{\psi_{z_0}}$ is just $f(z_0)$ for the exponentials. It now follows from the uniform convergence in a neighborhood of K that $T_{\psi_{z_0}}(f) = f(z_0)$. Q.E.D.

LEMMA 6. – *Let $\nu \in (E_{h_k})'$. If f_ν is its Fourier-Borel transform, then the linear operator $Q_{f_\nu} : E_{h_k} \to E_{h_k}$ is just the transpose of the convolution $\nu * \mu$* (i.e. $(Q_{f_\nu}(F), \mu) = (F, \nu * \mu)$).

Proof. – By Proposition 3, we can represent μ by a measure m_μ such that $m_\mu e^{h_k(u) + \varepsilon \|u\|}$ is a bounded measure for ε sufficiently small. Then

$$\mu(F(u)) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} \mu(\exp < z , u >) \frac{\partial^{n-1}}{\partial \xi^{n-1}} \left(\frac{\mathcal{L}_F(\xi)}{\xi_0}\right) \bar{\omega}(z , \xi)$$

follows from Fubuni's theorem for ω a sufficiently small, strictly convex neighborhood of K. Thus, μ is completely determined by its values on a set of exponentials $\exp < z , u >$ defined for z in a neighborhood of K. We choose ω so small that f_ν is defined and bounded in ω. Then for $z_0 \in \omega$,

$$(Q_h(\exp < z_0 , u >), \mu) =$$

$$= \mu\left(\frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} \exp < z , u > f_\nu(z) \frac{\partial^{n-1}}{\partial \xi^{n-1}} \left(\frac{\psi_{z_0}(\xi)}{\xi_0}\right) \bar{\omega}(z , \xi)\right) =$$

$$= f_\nu(z_0)\, \mu(\exp < z_0 , u >) = f_\nu(z_0) f_\mu(z_0) ,$$

from which the lemma follows. Q.E.D.

For $\nu \in (E_{h_k})'$, we define the differential operator with constant coefficients $\check{\nu}$ on E_{h_k} to be the transpose of the convolution operation $\mu \to \nu * \mu$ on $(E_{h_k})'$.

THEOREM 7. – *Let* $\check{\nu}$ *be a differential operator with constant coefficients on* E_{h_k}. *Then*

(a) *for* $F \in E_{h_k}$, *there always exists* $G \in E_{h_k}$ *such that* $\check{\nu}(G) = F$,

(b) *if* f_ν *has no zeros in* K, *then* G *is unique*

(c) *the polynomial exponential solutions of* $\check{\nu}(x) = 0$ *are dense in the space of all solutions of this equation.*

Proof. – (a) The mapping $\mu \to f_\mu$ is a one-to-one linear mapping of $(E_{h_k})'$ onto $\mathcal{H}(K)$. We topologize $\mathcal{H}(K)$ with the topology of convergence of the Taylor series coefficients at each point of K. This is at least as weak as the equivalente on $\mathcal{H}(K)$ of the weak topology on $(E_{h_k})'$, since, for a multi-index α,

$$\mu(u^a \exp < z_0 , u >) = \frac{\partial^{|\alpha|} f_\mu(z_0)}{\partial z^\alpha}.$$

If $f_\nu \cdot f_{\mu_\gamma}$ is a filter converging to $g \in \mathcal{H}(K)$, then we must have $g = f_\gamma \cdot f_g$, since the Taylor series of g is divisible by that of f_ν at each point of K. Thus the mapping $f_\mu \to f_\nu \cdot f_\mu$ is one-to-one and closed, so $\mu \to \nu * \mu$ is also one-to-one and closed. By Proposition 2, its transpose is onto.

(b) If f_ν has no zeros in K, then $f_\mu \to f_\nu \cdot f_\mu$ is onto so $\mu \to \nu * \mu$ is onto and hence its transpose is one-to-one.

(c) See [8] and [6]. Q.E.D.

The following example, due to C.O. Kiselman, shows that in some sense the results of § 2 and § 3 are sharp. Let $P(D) = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$ and let $f(z) = \cos \sqrt{z_1 z_2}$, which is of exponential type. Let u be a solution of exponential type of $P(D)\,u = f$. Then

$$u(0,r) - u(-r,0) = \int_0^1 \frac{d}{dt} u(-r(1-t), tr)\,dt =$$

$$= r \int_0^1 \cos r \sqrt{-t(1-t)}\,dt = \frac{r}{2} \int_0^1 (e^{r\sqrt{t(1-t)}} + e^{-r\sqrt{t(1-t)}})\,dt \geqslant$$

$$\geqslant \frac{r}{2} \int_0^1 e^{r\sqrt{t(1-t)}}\,dt \geqslant \frac{r}{2\sqrt{2}}\, e^{\frac{r}{2\sqrt{2}}} .$$

But $h_c^*(z)$ the circular indicator of $f(z)$, is zero in both the complex line $(\lambda(0, z_2))$ and $(\lambda(z_1, 0))$, so that the circular indicator (and hence the radial indicator) of u is strictly greater than that of f.

4. Functions of slow growth.

In this section, we extend the notion of a differential operator with constant coefficients to entire functions which satisfy a majoration of the form

$$|f(z)| \leqslant C_k \exp(\ln [p(z)])^k \tag{11}$$

asymptotically for some $k > 1$ and some norm $p(z)$. These functions are known to have very even growth [1].

We define the *logarithmic order* ρ of such a function to be the infemum of all k for which (11) holds. We define the *logarithmic type* σ of f (with respect to a logarithmic order ρ) to be the infemum of all b such that

$$|f(z)| \leqslant C_b \exp b(\ln p(z))^\rho .$$

These values are clearly independent of the norm used to define them.

THEOREM 8. – *Let m be a multi-index of positive numbers $m = (m_1, \ldots, m_n)$, $|m| = \Sigma m_i$. Then the logarithmic order and logarithmic type of a function f are given by*

$$\frac{\rho}{\rho - 1} = \overline{\lim_{|m| \to \infty}} \frac{\ln \ln^+ \dfrac{1}{|c_m|}}{\ln n} \; \textit{and} \quad \left(\frac{\rho - 1}{\rho}\right) \left[\frac{1}{\sigma\rho}\right]^{\frac{1}{\rho - 1}} = \overline{\lim_{|m| \to \infty}} \frac{\ln \dfrac{1}{|c_m|}}{n^{\frac{\rho}{\rho - 1}}}$$

where $f(z) = \sum_m c_m z^m$ *and* $\ln^+ a = \sup(0, \ln a)$.

Remark. – We interpret this to mean $\rho = 1$ if the limit in (12) is infinite. In this case, if we have $\sigma < +\infty$, we have a polynomial. We do not consider this case but rather assume that if $\rho = 1$ that $\sigma = +\infty$.

Proof. – Let $b > 0$ and $k > 1$ be numbers such that

$$|f(z)| \leqslant C \exp b(\ln r)^k .$$

We assume without loss of generality that $r = \|z\|_1$, where $\|z\|_1 = \max_i |z_i|$. By applying Cauchy's formula to the distinguished boundary of the polydisc of radius r, we get

$$|c_n| \leqslant C \exp\{b(\ln r)^k - |m| \ln r\} .$$

This function takes on its maximum (for $k > 1$) when $\ln r = \dfrac{|m|^{\frac{1}{k-1}}}{kb}$ and equals $\exp\left\{\left(\frac{1}{kb}\right)^{\frac{1}{k-1}} \left(\frac{1}{k} - 1\right) |m|^{\frac{k}{k-1}}\right\}$, which establishes the theorem in one direction.

On the other hand, if $|c_m| \leqslant K \exp\left\{\left(\frac{1}{kb}\right)^{\frac{1}{k-1}} \left(\frac{1}{k} - 1\right) |m|^{\frac{k}{k-1}}\right\}$,

$$|f(z)| \leqslant \sum_m K |m|^n \exp\left\{\left(\frac{1}{kb}\right)^{\frac{1}{k-1}} \left(\frac{1}{k} - 1\right) |m|^{\frac{k}{k-1}} + |m| \ln r\right\}$$

on the distinguished boundary of the polydisc of radius r. The function $\left(\frac{1}{kb}\right)^{\frac{1}{k-1}} \left(\frac{1}{k} - 1\right) x^{\frac{k}{k-1}} + x \ln r$ takes on its maximum for

$$x = \{(kb)^{\frac{1}{k-1}} \ln r\}^{k-1}$$

and equals $\exp b(\ln r)^k$.

Let $M_0 = [\{(kb)^{\frac{1}{k-1}} \ln r\}^{k-1}]$ and

$$M'_0 = \left[\left\{\frac{1}{2} \frac{k}{(k-1)} (kb)^{\frac{1}{k-1}} \ln r\right\}^{k-1}\right]$$

("greatest integer in"). Then

$$|f(z)| \leqslant K'(\ln r)^{2n(k-1)} \exp b(\ln r)^k +$$

$$+ \sum_{|m| = M'_0 + 1}^{\infty} r^{|m|} \exp\left\{\left(\frac{1}{kb}\right)^{\frac{1}{k-1}} \left(\frac{1}{k} - 1\right) |m|^{\frac{k}{k-1}}\right\}.$$

But

$$\sum_{|m|=M_0'+1}^{\infty} r^{|m|} \exp\left\{\left(\frac{1}{kb}\right)^{\frac{1}{k-1}} \left(\frac{1}{k}-1\right) |m|^{\frac{k}{k-1}}\right\} \leqslant$$

$$\leqslant \sum_{|m|=M_0'+1}^{\infty} \exp\left\{\left(\frac{1}{kb}\right)^{\frac{1}{k-1}} \left(\left(\frac{1}{k}-1\right) |m|^{\frac{k}{k-1}} + |m|(M_0+1)^{\frac{1}{k-1}}\right)\right\}$$

and this last series is bounded independently of M_0' since

$$\left(\frac{1}{k}-1\right) |m|^{\frac{k}{k-1}} + |m|(M_0+1)^{\frac{1}{k-1}} =$$

$$= |m|\left(\frac{1}{k}-1\right)\left(|m|^{\frac{1}{k-1}} - \frac{(k-1)}{k}(M_0+1)^{\frac{1}{k-1}}\right) < |m|\left(\frac{1}{k}-1\right) T$$

for some $T > 0$. Q.E.D.

We let $E_{\sigma,\rho}$ be the Fréchet space that we get by taking

$$p_n = \left(\sigma + \frac{1}{n}\right) (\ln r)^{\rho}$$

in (6), E_1 that which we get by taking $p_n = (\ln r)^{\left(1+\frac{1}{n}\right)}$, and we designate their duals by $(E_{\sigma,\rho})'$ and $(E_1)'$.

LEMMA 7. – *A linear functional μ on $E_{\sigma,\rho}$ (resp. E_1) is in $(E_{\sigma,\rho})'$ (resp. $(E_1)'$) if and only if*

$$|\mu(z^m)| \leqslant K_\varepsilon \exp\left[\frac{1}{(\sigma+\varepsilon)\rho}\right]^{\frac{\rho}{\rho-1}} \left[1-\frac{1}{\rho}\right] |m|^{\frac{\rho}{\rho-1}} \qquad (13)$$

(resp.

$$|\mu(z^m)| \leqslant K_\varepsilon \exp\left[\frac{1}{1+\varepsilon}\right]^{\frac{1+\varepsilon}{\varepsilon}} \left[\frac{\varepsilon}{1+\varepsilon}\right] |m|^{\frac{1+\varepsilon}{\varepsilon}}) \qquad (14)$$

for some $\varepsilon > 0$.

Proof. – It follows from the proof of Theorem 8 that the Taylor series of an element in $E_{\sigma,\rho}$ (resp. E_1) converges to the function in this space (cf. [8]). Thus, if μ is a continuous linear functional, it follows that (13) (resp. (14)) holds.

On the other hand, if (13) (resp. (14)) holds, it follows from the estimates of Theorem 8 that μ is a continuous linear functional on $E_{\sigma,\rho}$ (resp. E_1). Q.E.D.

For $\mu \in (E_{\sigma,\rho})'$ (resp. $(E_1)'$), we define its Fourier-Borel transform $\widetilde{\mu}(u) = \mu(\exp < z , u >) = \Sigma\, \mu(z^m) \dfrac{u^m}{m\,!}$, in the sense of a formal power series at the origin. We topologize this space with the topology of convergence of coefficients. Let $Q_{\sigma,\rho}$ (resp. Q_1) be the space of formal power series whose coefficients satisfy (13) (resp. (14)) above.

For $\nu , \mu \in (E_{r,\rho})'$ (resp. $(E_1)'$), we define the convolution of μ with ν, $\nu * \mu$ to be

$$\nu * \mu(f(u)) = \mu(\nu_v(f(u + v))) \ .$$

A differential operator with constant coefficients on $E_{\sigma,\rho}$ (resp. E_1) is defined as the transpose of this convolution operation. We then have the following

THEOREM 9. – *Let $\check{\nu}$ be a differential operator with constant coefficients on the space $E_{\sigma,\rho}$ (resp. E_1). Then for $f \in E_{\sigma,\rho}$ (resp. E_1) there is always a solution $g \in E_{\sigma,\rho}$ (resp. E_1) of the equation $\check{\nu}(g) = f$. If $\check{\nu}(1) = 0$, then g is unique.*

The proof is the same as that of Theorem 6, with some alterations in the calculations of Theorem 5 to prove that the operation of convolution is closed. The details are left to the interested reader.

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