

# ANNALES DE L'INSTITUT FOURIER

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*Annales de l'institut Fourier*, tome 21, n° 4 (1971), p. 175-177

[<http://www.numdam.org/item?id=AIF\\_1971\\_\\_21\\_4\\_175\\_0>](http://www.numdam.org/item?id=AIF_1971__21_4_175_0)

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## ANOTHER CHARACTERIZATION OF ABSOLUTE STABILITY

by Roger C. McCANN

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It is well known that absolute stability of a compact subset  $M$  of a locally compact metric space can be characterized by the presence of a fundamental system of absolutely stable neighborhoods, and also by the existence of a continuous Liapunov function  $\varphi$  defined on some neighborhood of  $M = \varphi^{-1}(0)$ , [1]. In a more general setting it has been shown that a set  $M$  is closed and absolutely stable if and only if  $M = \bigcap \varphi_i^{-1}(0)$  for suitable Liapunov functions  $\varphi_i$ , [2]. This paper presents a more elementary description of absolute stability in terms of positively invariant neighborhoods only.

Throughout this paper  $\mathbb{R}$  and  $\mathbb{R}^+$  will denote the reals and the non-negative reals respectively. A rational number  $r$  is called dyadic iff there are integers  $n$  and  $j$  such that  $n \geq 0$ ,  $1 \leq j < 2^n$ , and  $r = j/2^n$ .

A dynamical system on a topological space  $X$  is a mapping  $\pi$  of  $X \times \mathbb{R}$  into  $X$  satisfying the following axioms (where  $x\pi t = \pi(x, t)$ ):

- (1)  $x\pi 0 = x$  for  $x \in X$ .
- (2)  $(x\pi t)\pi s = x\pi(t + s)$  for  $x \in X$  and  $t, s \in \mathbb{R}$ .
- (3)  $\pi$  is continuous in the product topology.

If  $A \subset X$  and  $B \subset \mathbb{R}$ , then  $A\pi B$  will denote the set  $\{x\pi t : x \in A, t \in B\}$ . A subset  $A$  of  $X$  is called positively invariant if and only if  $A\pi \mathbb{R}^+ = A$ .

A mapping  $\varphi: X \rightarrow \mathbb{R}^+$  is called a Liapunov function (relative to  $\pi$ ) if and only if  $\varphi$  is continuous and  $\varphi(x\pi t) \leq \varphi(x)$  for all  $x \in X$  and  $t \in \mathbb{R}^+$ .

Absolute stability is defined in terms of a prolongation ([1], [2]) and, in [1], is characterized in a special setting by the following theorem.

**THEOREM A.** — *Let  $M$  be a compact subset of a locally compact metric space. Then the following are equivalent:*

- (a) *There is a Liapunov function  $\nu$  with  $\nu^{-1}(0) = M$ .*
- (b)  *$M$  possesses a fundamental system of absolutely stable neighborhoods.*
- (c)  *$M$  is absolutely stable.*

In [2], absolutely stable sets, in a more general setting, are characterized by Liapunov functions.

**THEOREM B.** — *Let  $M$  be a subset of a space  $X$  which is Hausdorff paracompact, and locally compact. Then  $M$  is closed and absolutely stable if and only if  $M = \bigcap \nu_i^{-1}(0)$  for suitable Liapunov functions  $\nu_i: X \rightarrow [0, 1]$ .*

In order to obtain our result we will need the following result [2, Corollary 18].

**THEOREM C.** — *In a locally compact metric space  $X$ , the closed absolutely stable sets are precisely the zero-sets of Liapunov functions mapping  $X$  into  $[0, 1]$ .*

**THEOREM.** — *Let  $M$  be a closed subset of a locally compact metric space  $X$ . Then  $M$  is absolutely stable if and only if  $M$  possesses a family  $\mathcal{F}$  of neighborhoods satisfying*

- (i) *If  $U \in \mathcal{F}$ , then  $U$  is open and positively invariant.*
- (ii)  *$\bigcap \mathcal{F} = M$ .*
- (iii) *If  $U \in \mathcal{F}$ , then there is a  $V \in \mathcal{F}$  such that  $\overline{V} \subset U$ .*
- (iv) *If  $U, V \in \mathcal{F}$  are such that  $\overline{U} \subset V$ , then there is a  $W \in \mathcal{F}$  such that  $\overline{U} \subset W \subset \overline{W} \subset V$ .*

*Proof.* — *If.* Let  $U \in \mathcal{F}$ . For each dyadic rational  $r$  we construct a set  $U(r) \subset U$  such that  $U(r) \in \mathcal{F}$  and  $\overline{U}(r) \subset U(s)$  if  $r < s$ . Then we construct a Liapunov function  $\nu_U: X \rightarrow [0, 1]$  and show that  $M = \bigcap \{\nu_U^{-1}(0): U \in \mathcal{F}\}$ . The result will then follow from Theorem B. First obtain from  $\mathcal{F}$  a system

of neighborhoods  $U\left(\frac{1}{2^n}\right)$ ,  $n$  a non-negative integer, such that  $U(1) = U$  and  $U\left(\frac{1}{2^{n+1}}\right) \subset U\left(\frac{1}{2^n}\right)$ . This is clearly possible by (iii). Using (iv) this system of neighborhoods can be extended to one with the desired properties. For example, we choose  $U\left(\frac{3}{4}\right)$  to be any member  $W$  of  $\mathcal{F}$  such that  $\overline{U}\left(\frac{1}{2}\right) \subset W \subset \overline{W} \subset U(1)$ . Now define  $\nu_U: X \rightarrow \mathbb{R}^+$  by  $\nu_U(x) = 1$  if  $x \notin U = U(1)$  and  $\nu_U(x) = \inf \{\nu: x \in U(r)\}$  if  $x \in U$ . If  $x \in U(r)$  and  $t \in \mathbb{R}^+$ , then  $x\pi t \in U(r)$  since  $U(r)$  is positively invariant. Therefore

$$\nu_U(x) = \inf \{r: x \in U(r)\} \geq \inf \{r: x\pi t \in U(r)\} = \nu_U(x\pi t).$$

The continuity of  $\nu_U$  is proved as in the proof of Urysohn's lemma. Thus for each  $U \in \mathcal{F}$  we have constructed a continuous Liapunov function  $\nu_U$  such that  $M \subset \nu_U^{-1}(0) \subset U$ . By (ii),  $\cap \nu_U^{-1}(0) = M$ .

*Only if.* — Let  $M$  be absolutely stable. Then by theorem C,  $M = \nu_U^{-1}(0)$  for some Liapunov function  $\nu$ . Let  $\mathcal{F}$  consist of all sets of the form  $\{x: \nu(x) < r\}$  where  $r \in (0, 1)$ . Evidently  $\mathcal{F}$  satisfies conditions (i)-(iv).

*Remark.* — In the « If » part of the proof we only need that  $X$  is Hausdorff, paracompact, and locally compact.

The author wishes to thank Professor Otomar Hájek for several helpful conversations during the preparation of this paper.

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Manuscrit reçu le 20 décembre 1970.

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