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ON THE HAUSDORFF SUMMABILITY OF SERIES ASSOCIATED WITH A FOURRIER AND ITS ALLIED SERIES

by B. L. GUPTA

1. Let S_n be the n th partial sum of an infinite series $\sum_1^\infty a_n$ and let

$$t_n = \sum_{\nu=0}^n \binom{n}{\nu} (\Delta^{n-\nu} \mu_\nu) S_\nu. \quad (1.1)$$

Then the sequence $\{t_n\}$ is known as the Hausdorff means of sequence $\{S_n\}$, where $\{\mu_\nu\}$ is a sequence of real or complex numbers and the sequence $\{\Delta^p \mu_\nu\}$ denotes the differences of order p .

The series $\sum_1^\infty a_n$ is said to be summable by Hausdorff mean to the sum S , if $\lim t_n = S$, whenever $S_n \rightarrow S$. The necessary and sufficient condition for the Hausdorff summability to be conservative is that the sequence $\{\mu_n\}$ should be a sequence of moment constant, i.e. ;

$$\mu_n = \int_0^1 x^n d\chi(x), \quad n \geq 0 ;$$

where $\chi(x)$ is a real function of bounded variation in $0 \leq x \leq 1$. We may suppose without loss of generality that $\chi(0) = 0$, if also $\chi(1) = 1$ and $\chi(+0) = \chi(0) = 0$, so that $\chi(x)$ is continuous at the origin, then μ_n is a regular moment constant and the Hausdorff method i.e. (H, μ_n) is a regular method of summation [2].

If

$$\sum_{n=0}^\infty |t_n - t_{n-1}| < \infty, \quad (1.2)$$

then the series $\sum_1^\infty a_n$ is said to be absolutely summable (H, μ_n) or

summable $|H, \mu_n|$. It is also known that the Cesàro, Holder and Euler methods of summation are the particular cases of the above method.

2. Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue in $(-\pi, \pi)$. Let its Fourier series be

$$\frac{1}{2} a_0 + \sum_1^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t)$$

and its allied series is

$$\sum_1^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_1^{\infty} B_n(t).$$

We write

$$\varphi(t) = \frac{1}{2} \{f(\theta + t) + f(\theta - t)\},$$

$$\psi(t) = \frac{1}{2} \{f(\theta + t) - f(\theta - t)\}.$$

Let $g(x)$ be integrable L in $(0, 1)$, then for $\varepsilon > 0$

$$g_{\varepsilon}^{+}(x) = \frac{1}{\Gamma(\varepsilon)} \int_0^x (x-u)^{\varepsilon-1} g(u) du,$$

$$g_{\varepsilon}^{-}(x) = \frac{1}{\Gamma(\varepsilon)} \int_x^1 (u-x)^{\varepsilon-1} g(u) du.$$

Again, let

$$U_n(t) = \sum_{\nu=1}^n e^{i\nu t},$$

$$H(n, x, t) = E(n, x, t) + i F(n, x, t)$$

$$= \sum_{\nu=0}^n \nu^{\beta} \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu} e^{i\nu t}.$$

The object of this paper is to prove the following :

THEOREM 1. — *If*

$$\text{i) } \int_0^t |\varphi(u)| du = O(t)$$

ii) (H, μ_n) is conservative

and

$$\text{iii) } \left\{ \begin{array}{l} \text{either (a) } \chi(x) = g_{1+\beta+\varepsilon}^-(x) + c, \varepsilon > 0 ; \\ \text{or (b) } \chi(x) = g_{1+\beta+\varepsilon}^+(x) + c, \varepsilon > 0 ; \end{array} \right.$$

for some $g(x) \in L(0, 1)$;

then the series $\sum_{n=1}^{\infty} \frac{A_n(t)}{n^{1-\beta}}$, for $t > \beta \geq 0$ is summable $|H, \mu_n|$ at $t = \theta$, where c is an absolute constant.

THEOREM 2. — *If*

$$\text{i) } \int_0^t |\psi(u)| du = O(t)$$

ii) (H, μ_n) is conservative

and

$$\text{iii) } \left\{ \begin{array}{l} \text{either (a) } \chi(x) = g_{1+\beta+\varepsilon}^-(x) + c, \varepsilon > 0 ; \\ \text{or (b) } \chi(x) = g_{1+\beta+\varepsilon}^+(x) + c, \varepsilon > 0 ; \end{array} \right.$$

for some $g(x) \in L(0, 1)$,

then the series $\sum_{n=1}^{\infty} \frac{B_n(t)}{n^{1-\beta}}$, for $t > \beta \geq 0$ is summable $|H, \mu_n|$ at $t = \theta$, where c is an absolute constant.

It may also be remarked if

$$\chi(x) = 1 - (1 - x)^\delta, \quad \delta > 0 ;$$

the method (H, μ_n) reduces to the well known Cesàro method of summation of order δ .

Further if we choose β such that $\delta > \beta + \varepsilon$ then it can be proved that $\chi(x) - 1$ is the $(1 + \beta + \varepsilon)$ th backward integral of

$$- \frac{\Gamma(1 + \delta)}{\Gamma(\delta - \beta - \varepsilon)} (1 - x)^{\delta - \beta - \varepsilon - 1}$$

and $\chi(x)$ is also the $(\varepsilon + \beta + 1)$ th forward integral of

$$\frac{\delta}{\Gamma(1 - \beta - \varepsilon)} \left\{ x^{-(\beta + \varepsilon)} + (1 - \beta) \int_0^x (1 - v)^{\delta - 2} (x - v)^{-(\beta + \varepsilon)} dv \right\}.$$

Hence the method $|C, \delta|$ satisfies the hypothesis of our theorem 1 and 2 for $\varepsilon > 0$, $\delta > \beta \geq 0$ and the following theorems of Cheng [1] becomes the corollary of our theorems.

THEOREM. — *The series $\sum \frac{A_n(t)}{n^{1-\beta}}$ for $0 \leq \beta < 1$ is summable $|C, \delta|$ for $\delta > \beta$, at the point θ , whenever i) of theorem 1 holds and similarly the series $\sum_1 \frac{B_n(t)}{n^{1-\beta}}$, for $0 \leq \beta < 1$, is summable $|C, \delta|$, for $\delta > \beta$, at the point θ , whenever i) of theorem 2 holds.*

3. For the proof of the theorems, we require the following lemmas.

LEMMA 1. — *Uniformly in $0 < t \leq \pi$*

$$|U_n(t)| \leq \frac{k}{t}. \quad (3.1)$$

This can be easily proved.

LEMMA 2. — *If $g(x)$ and $h(x)$ be Lebesgue integrable in $(0, 1)$, then for $\varepsilon > 0$*

$$\int_0^1 g_\varepsilon^+(x) h(x) dx = \int_0^1 g(x) h_\varepsilon^-(x) dx. \quad (3.2)$$

This is known [3].

LEMMA 3. — *Uniformly in $0 \leq x \leq 1$*

$$\int_0^x H(n, v, t) dv = O\left(\frac{n^{\beta-1}}{t}\right) \quad (3.3)$$

LEMMA 4. — Let $\beta \geq 0$, $\varepsilon > 0$ and fixed, then for $\beta + \varepsilon < 1$

$$\int_0^x (x-u)^{\beta+\varepsilon-1} H(n, u, t) du = O\left(\frac{n^{-\varepsilon}}{t^{\beta+\varepsilon}}\right) \quad (3.4)$$

uniformly in $0 \leq x \leq 1$ and similarly

$$\int_x^1 (u-x)^{\beta+\varepsilon-1} \times H(n, u, t) du = O\left(\frac{n^{-\varepsilon}}{t^{\beta+\varepsilon}}\right). \quad (3.5)$$

The lemma 3 and 4 are due to Tripathy [4].

Proof of Theorem 1. — If t_n and u_n denote the Hausdorff means of $\sum \frac{A_n(\theta)}{n^{1-\beta}}$ and the sequence $\{n A_n(\theta)\}$ then for $n \geq 1$

$$u_n = n(t_n - t_{n-1}).$$

Hence, from (1.2) the series $\sum_{n=1}^{\infty} \frac{A_n(\theta)}{n^{1-\beta}}$ is summable $|H, \mu_n|$, if

$$I = \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{\nu=1}^n \binom{n}{\nu} (\Delta^{n-\nu} \mu_{\nu}) \nu^{\beta} A_{\nu}(\theta) \right| < \infty.$$

Since (H, μ_n) is conservative, we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^1 d\chi(x) \sum_{\nu=1}^n \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu} \nu^{\beta} A_{\nu}(\theta) \right| \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^1 d\chi(x) \sum_{\nu=1}^n \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu} \nu^{\beta} \int_0^{\pi} \varphi(t) \cos \nu t dt \right| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\frac{1}{n}} |\varphi(t)| \left| \int_0^1 d\chi(x) \sum_{\nu=1}^n \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu} \nu^{\beta} \cos \nu t \right| dt \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{n}}^{\pi} |\varphi(t)| \left| \int_0^1 d\chi(x) \sum_{\nu=1}^n \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu} \nu^{\beta} \cos \nu t \right| dt \\ &= I_1 + I_2, \text{ say.} \end{aligned}$$

Since

$$\begin{aligned} |H(n, x, t)| &\leq n^{\beta} \left| \sum_{\nu=0}^n \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu} \right| \\ &= n^{\beta} \end{aligned}$$

We have

$$\begin{aligned} I_1 &= O(1) \sum_{n=1}^{\infty} \frac{1}{n} n^{\beta} \int_0^{\frac{1}{n}} |\varphi(t)| dt \int_0^1 |d\chi(x)| \\ &= O(1) \sum_{n=1}^{\infty} \frac{1}{n^{1-\beta}} \cdot \frac{1}{n} \\ &= O(1) . \end{aligned}$$

Without loss of generality, we can suppose that $\beta + \varepsilon < 1$, if
a) $\chi(x) = g_{1+\beta+\varepsilon}(x) + c$, then

$$\begin{aligned} I_2 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{n}}^{\pi} |\varphi(t)| \cdot \left| \int_0^1 g_{\varepsilon+\beta}(x) E(n, x, t) dx \right| dt \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{n}}^{\pi} |\varphi(t)| \cdot \left| \int_0^1 g(x) E_{\beta+\varepsilon}^+(n, x, t) dx \right| dt . \end{aligned}$$

Since

$$\begin{aligned} E_{\beta+\varepsilon}^+(n, x, t) &= \frac{1}{\Gamma(\beta + \varepsilon)} \int_0^x (x - u)^{\beta + \varepsilon - 1} E(n, u, t) du \\ &= \frac{1}{\Gamma(\beta + \varepsilon)} \int_0^x (x - u)^{\beta + \varepsilon - 1} I_m H(n, u, t) du \\ &= O\left(\frac{1}{n^{\varepsilon} t^{\beta + \varepsilon}}\right) , \text{ by lemma-4.} \end{aligned}$$

Therefore

$$\begin{aligned} I_2 &\leq \int_0^1 |g(x)| dx \sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{n}}^{\pi} |\varphi(t)| \cdot O\left(\frac{1}{n^{\varepsilon} t^{\beta + \varepsilon}}\right) dt \\ &= \int_0^1 |g(x)| dx \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \int_{\frac{1}{n}}^{\pi} \frac{|\varphi(t)|}{t^{\beta + \varepsilon}} dt \\ &= \int_0^1 |g(x)| dx \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \{O(1) + O(n^{\beta + \varepsilon - 1})\} \\ &= O(1) \int_0^1 |g(x)| dx \\ &= O(1) . \end{aligned}$$

If b) $\chi(x) = g_{1+\beta+\varepsilon}^+(x) + c$, then proceeding in a similar way as in case a) and using estimate (4.5) of lemma 4, it can be proved that

$$I_2 = O(1) .$$

This completes the proof of theorem-1.

If we use the condition i) of Theorem-2 instead of the condition i) of theorem-1, we can prove that the series $\sum \frac{B_n(\theta)}{n^{1-\beta}}$ is summable $|H, \mu_n|$.

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BIBLIOGRAPHY

- [1] M.T. CHENG, Summability factors of Fourier series, *Duke Math. Jour.* 15 (1948), 17-27.
- [2] G.H. HARDY, Divergent series. Oxford 1949.
- [3] B. KUTTNER, On the "second theorem of consistency" for riesz summability (II), *Jour. Lond. Math. Soc.* 27 (1952), 207-217.
- [4] N. TRIPATHY, On the absolute Hausdorff summability of some series associated with Fourier series and its allied series, *Jour. Ind., Math. Soc.* 32 (1960) 141-154.

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