ANNALES DE L'INSTITUT FOURIER

BENT FUGLEDE

The quasi topology associated with a countably subadditive set function

Annales de l'institut Fourier, tome 21, nº 1 (1971), p. 123-169

<http://www.numdam.org/item?id=AIF_1971__21_1_123_0>

Ann. Inst. Fourier, Grenoble
21,1 (1971), 123-169

THE QUASI TOPOLOGY ASSOCIATED WITH A COUNTABLY SUBADDITIVE SET FUNCTION

by Bent FUGLEDE

Table of contents.

Introduction.

We present a rather general study of certain "quasi topological" concepts associated with a *capacity* C on a topological space X, that is, an increasing, countably subadditive set function, defined on the set $\mathfrak{P}(X)$ of all subsets of X, and such that $C(\emptyset) = 0$.

A set $A \subset X$ is called *quasi closed* if there exist closed sets F with $C(A \Delta F)$ as small as we please, Δ denoting symmetric difference. Any finite union or *countable* intersection of quasi closed sets is quasi closed. A key result (Theorem 2.7) asserts that if X has a countable base, and if C is sequentially order continuous from below, then every non-void class $\mathfrak{H}$ of quasi closed subsets of X which is stable under countable intersection has a quasi minimal element, that is, a set $H_0 \in \mathcal{H}$ such that $C(H_0 \backslash H) = 0$ for all $H \in \mathcal{H}$.

As corollaries we obtain the existence of a "*quasi closure*" of any set $A \subset X$, and a "*quasi support*" for any outer measure on X which does not charge the sets $E \subset X$ with $C(E) = 0$. Also there are similar results (§ 3.4) for quasi semicontinuous functions instead of quasi closed sets.

The method employed has been extracted from a proof given by Choquet [11] for a theorem of Getoor [18] concerning the fine topology in potential theory.

In § 4 we introduce a notion of compatibility between the quasi topology (associated with a given capacity C on the topological space X) and an actual new topology on X, termed the "fine" topology (though not always required to be finer than the original topology on X). For this compatibility it is required

1) that quasi closed sets be precisely the sets equivalent to finely closed sets,

2) that any set E whose points x are finely isolated in E and of capacity $C(\{x\}) = 0$ should be finely closed, and

3) that these sets E be precisely the sets with $C(E) = 0$.

After having derived a number of consequences of such a compatibility, we characterize the topologies (if any), compatible with the given quasi topology, by means of the so-called *base* operation (essentially that of forming the finely derived set), which is shown to determine a kind of "lifting" of the equivalence classes of quasi closed sets. The question remains open, however, whether such a lifting always exists under the present very general circumstances.

The final § 5 is intended as a general framework for the study of various capacities and fine topologies of potential theory, cf. Brelot [2], [6]. In addition to the capacity C is given a convex cone $\mathcal{U}$ of lower semicontinuous functions $u : X \longrightarrow [0, +\infty]$. Copying Cartan's definition from the classical case (where $\mathcal{U}$ is the cone of superharmonic functions), the fine topology is defined as the coarsest topology, finer than the original topology on X, such that every $u \in \mathcal{U}$ becomes continuous. Sufficient conditions are obtained under which this fine topology be compatible with the quasi topology associated with C. Applications, notably to the capacity and fine topology with respect to a kernel, are sketched (with reference to [15] and a forthcoming detailed exposition).

1. Capacity as a countably aubadditive set function.

We denote throughout by $\mathfrak{P}(X)$ the lattice of all subsets of a set X. In the present introductory section X denotes an arbitrary set (without topology), exept for § 1.5.

1.1. DEFINITION. – *By a capacity on* X *we understand, in this paper, an increasing, countably subadditive set function* C *on* $\mathfrak{P}(X)$ *into* $[0, +\infty]$ *such that* $C(\emptyset) = 0$. *Thus we should have, for all sets* $A_1, A_2, \ldots \subset X$:

$$(C_1) \qquad A_1 \subset A_2 \Longrightarrow C(A_1) \leqslant C(A_2) ,$$

$$(C_2) \qquad C\Big(\bigcup_{n\in N} A_n\Big) \leqslant \sum_{n\in N} C(A_n) .$$

$$(C_3) \qquad C(\emptyset) = 0 .$$

Throughout this paper C will denote a fixed capacity on X in the above sense.

1.2. A property $P[x]$ involving the generic point $x \in X$ is said to hold *quasi everywhere* (q.e.) in a set $A \subset X$ if

$$C(\{x \in A \mid \text{non } P[x]\}) = 0 .$$

In the case $A = X$ we may write simply q.e. (in place of q.e. in X).

1.3. A pre-order relation $\prec$ is defined on the lattice $[-\infty, +\infty]^X$ of all functions $X \longrightarrow [-\infty, +\infty]$ by

$$[f_1 \prec f_2] \Longleftrightarrow [f_1(x) \leqslant f_2(x) \text{ q.e.}] .$$

The associated equivalence relation (C-*equivalence*) is denoted by $\sim$:

$$[f_1 \sim f_2] \Longleftrightarrow [f_1(x) = f_2(x) \text{ q.e.}] .$$

Identifying sets $A \subset X$ with their indicater function, we may view

$\mathcal{R}(X)$ as a sublattice of $[-\infty, +\infty]^X$, and the relations $\prec$ and $\sim$ induce corresponding relations (of pre-order and equivalence, respectively) on $\mathcal{R}(X)$, for which we use the same notations. Thus we have for sets $A_1, A_2 \subset X$

$$[A_1 \prec A_2] \Longleftrightarrow [C(A_1 \backslash A_2) = 0] \ ,$$

and we read this as follows : A_1 is quasi contained in A_2 (or A_2 quasi contains A_1, or A_2 contains quasi every point of A_1). Similarly, $[A_1 \sim A_2] \Longleftrightarrow [C(A_1 \Delta A_2) = 0]$, Δ denoting symmetric difference.

1.4. *Quasi uniform convergence.* – Let Y denote a uniform space, e.g. $[-\infty, +\infty]$. We say that a sequence of functions $f_n : X \longrightarrow Y$ converges *quasi uniformly* to a function $f : X \longrightarrow Y$ if there corresponds to every $\varepsilon > 0$ a set $\omega \subset X$ with $C(\omega) < \varepsilon$ such that f_n converges uniformly to f on $X \backslash \omega$.

In the affirmative case we clearly have $f_n(x) \longrightarrow f(x)$ pointwise quasi everywhere. The quasi uniform convergence of $(f_n)_{n \in N}$ to f depends (like convergence q.e.) only on the equivalence classes of $f, f_1, f_2, \ldots$.

1.5. *Outer capacity.* – A capacity on a *topological* space X will be called an *outer capacity* if, for every set $A \subset X$,

$$(C_4) \qquad C(A) = \inf\{C(G) \mid G \text{ open }, G \supset A\} .$$

Note that, in case of an outer capacity C on a topological space X, the set ω occurring in the definition of quasi uniform convergence may be taken to be *open*. A similar remark applies to the set ω occurring in the definitions of certain "quasi topological" concepts in the subsequent sections.

1.6. *Remark.* – All the definitions and results of the present paper could be carried over with the appropriate modifications in the proofs to capacities with values in a certain type of complete lattice with a suitable notion of addition. This lattice should have the property that every family of elements of the lattice should contain a countable subfamily with the same infimum. Disregarding its largest element, the lattice should, moreover, be a semigroup under the addition. We shall not specify here the relevant order properties of this addition, but only mention one interesting example from potential theory of such a complete "lattice semigroup" which may replace $[0, +\infty]$, viz. the lattice of all hyperharmonic functions $\geqslant 0$ (on a harmonic space) with the natural pointwise notions of order and addition (cf. § 5.6, in particular note 11).

2. Quasi open, quasi closed and quasi compact sets.

Throughout the rest of this paper, X will always denote a *topological space.* In contexts involving compact subsets of X it is tacitly assumed that X is separated, i.e., a Hausdorff space. The closure of a set $A \subset X$ is denoted by $\bar{A}$. We continue the study of a capacity C on X in the sense of Definition 1.1.

2.1. DEFINITION. – *A set* $A \subset X$ *is called quasi open (quasi closed, quasi compact) with respect to* C *if*

$$\inf\{C(A \Delta E) \mid E \textit{ open (closed, compact)}\} = 0 \; .$$

Equivalently, there should exist sets $\omega \subset X$ with $C(\omega)$ as small as we please and such that $A \backslash \omega$ may be extended to an open (closed, compact) subset of X by adding points of ω. In other words (leaving out the quasi compact case), $A \backslash \omega$ should be open (resp. closed) relatively to $X \backslash \omega$.

Clearly, A is quasi open if and only if $X \backslash A$ is quasi closed. Each of the 3 notions depends only on the equivalence class of the set A.

2.2. *The case of an outer capacity.* – In that case the set ω in the above definitions may be taken to be *open.*

LEMMA. – *If* C *is an outer capacity then*

$$[A \textit{ quasi open}] \Longleftrightarrow [\inf\{C(E \backslash A) \mid E \textit{ open}, \; E \supset A\} = 0] \; , \qquad (1)$$

$$[A \textit{ quasi closed}] \Longleftrightarrow [\inf\{C(A \backslash E) \mid E \textit{ closed}, \; E \subset A\} = 0] \; , \qquad (2)$$

$$[A \textit{ quasi compact}] \Longleftrightarrow [\inf\{C(A \backslash E) \mid E \textit{ compact}, \; E \subset A\} = 0] . \qquad (3)$$

Proof. – The quasi open case is reduced to the quasi closed case by duality. If A is quasi closed (quasi compact) there exists an open set ω with $C(\omega)$ as small as we please such that $A \backslash \omega$ may be extended to a closed (compact) subset of X by adding points of ω. Since $X \backslash \omega$ is now closed, the set $E := A \backslash \omega$ is itself closed (compact), and $A \backslash E \subset \omega$ has arbitrary small capacity. ▯

2.3. LEMMA. – *Any countable union or finite intersection of quasi open sets is quasi open. Any countable intersection or finite union of quasi closed (quasi compact) sets is quasi closed (quasi compact).*

Proof. – Follows easily by use of the countable subadditivity of C under observation of the inclusions

$$(\cup A_n) \Delta (\cup E_n) \subset \cup (A_n \Delta E_n) \ ,$$

$$(\cap A_n) \Delta (\cap E_n) \subset \cup (A_n \Delta E_n) \ ,$$

valid for arbitrary subsets of X. ▯

Remark. – The intersection of a quasi compact set and a quasi closed set is quasi compact. In particular, any quasi closed subset of a quasi compact set is quasi compact.

2.4. *Quasi stable capacity.* – The capacity C on the topological space X is called *quasi stable* if every set $A \subset X$ is contained in some quasi closed set H such that $C(H) = C(A)$.

On account of the preceding lemma, C is quasi stable if and only if, for every $A \subset X$,

$$C(A) = \inf \{ C(H) \mid H \text{ quasi closed}, H \succ A \} \ . \tag{4}$$

2.5. THEOREM(1). – *Each of the following conditions is necessary and sufficient for a capacity* C *to be quasi stable :*

(1) This result is inspired by Choquet [10] who proved (using properties of the fine topology) that the classical newtonian outer capacity has the property a). Choquet also noted that a) implies b), and that one may add, in a), the further requirement that A_1 be "stable" in the sense that $C(\overline{A}_1) = C(A_1)$, and similarly that, in b), all the sets A_n may be required to be stable. He further established that, under certain additional assumptions, one may replace E by $\emptyset$ in b), but in that case the sets A_n cannot always be chosen to be stable sets, not even in case of newtonian outer capacity.

a) *Every set* $A \subset X$ *admits, for every* $\varepsilon > 0$, *a partition* $A = A_1 \cup E$ *such that*

$$C(\overline{A}_1) \leqslant C(A) \ , \ C(E) < \varepsilon \ .$$

b) *Every set* $A \subset X$ *admits, for every* $\varepsilon > 0$, *a partition*

$$A = \bigcup_{n \in N} A_n \cup E$$

such that

$$\sum_{n \in N} C(\overline{A}_n) \leqslant C(A) + \varepsilon \ , \ C(E) = 0 \ .$$

Proof. – It suffices to consider the case $C(A) < +\infty$. Suppose first that C is quasi stable according to the definition above. Choose closed sets $F_n \subset X$ so that $C(H \Delta F_n) < \varepsilon/2^n$, and put $F = \cap F_n$, $A_1 = A \cap F$, $E = A \backslash F$. Then $\overline{A}_1 \subset F \subset F_n$ for each n, and hence

$$C(\overline{A}_1) \leqslant C(F_n) \leqslant C(H) + \varepsilon/2^n \ ,$$

showing that $C(\overline{A}_1) \leqslant C(H) = C(A)$. On the other hand,

$$E \subset H \backslash F \subset \cup (H \backslash F_n) \ ,$$

and so $C(E) < \varepsilon$. This establishes a).

The implication a) $\Longrightarrow$ b) is obtained by repeated application of a) :

$$A = A_1 \cup E_1 \ , \ C(\overline{A}_1) \leqslant C(A) \ , \ C(E_1) < \varepsilon/2 \ ,$$
$$E_1 = A_2 \cup E_2 \ , \ C(\overline{A}_2) \leqslant C(E_1) \ , \ C(E_2) < \varepsilon/2^2 \ ,$$

etc. This leads to b) with $E = \bigcap_{n \in N} E_n$.

Finally, suppose b) holds for some set A. Writing $B_0 = \emptyset$ and

$$B_n = A_1 \cup \ldots \cup A_n \ , \qquad H = \bigcup_{n \in N} \overline{B}_n \ ,$$

we have $H \succ A$ because $C(E) = 0$ and

$$H \supset \bigcup_{n \in N} B_n = \bigcup_{n \in N} A_n = A \backslash E \ .$$

To show that H is quasi closed we consider, for $p = 0, 1, 2, \ldots$,

$$H \backslash \overline{B}_p = \bigcup_{n > p} (\overline{B}_n \backslash \overline{B}_p) \subset \bigcup_{n > p} \overline{B_n - B_p} \subset \bigcup_{n > p} (\overline{A}_{p+1} \cup \ldots \cup \overline{A}_n) = \bigcup_{n > p} \overline{A}_n \ ,$$

$$C(H\setminus\overline{B}_p) \leqslant \sum_{n>p} C(\overline{A}_n) \longrightarrow 0 \quad \text{as} \quad p \longrightarrow \infty$$

because $\sum_{n\in N} C(\overline{A}_n) \leqslant C(A) + \varepsilon < \infty$. Since $\overline{B}_p$ is a closed subset of H, we have proved that H is quasi closed. Taking instead $p = 0$, we find

$$C(H) \leqslant \sum_{n\in N} C(\overline{A}_n) \leqslant C(A) + \varepsilon .$$

Having thus established (4), we conclude that C is quasi stable. ▯

2.6. LEMMA. – *A quasi closed set* A_1 *is quasi contained in a quasi closed set* A_2 *if and only if*

$$C(A_1 \cap \omega) \leqslant C(A_2 \cap \omega) \quad \textit{for every open set } \omega .$$

Proof. – The necessity is obvious. As to the sufficiency, the stated inequality extends to quasi open sets ω. In fact, let G be open, and $C(G \Delta \omega) < \varepsilon$. Then

$$C(A_1 \cap \omega) \leqslant C(A_1 \cap G) + \varepsilon \leqslant C(A_2 \cap G) + \varepsilon \leqslant C(A_2 \cap \omega) + 2\varepsilon ,$$

and hence $C(A_1 \cap \omega) \leqslant C(A_2 \cap \omega)$. It remains only to apply this to $\omega = X\setminus A_2$. ▯

2.7. *Existence of quasi minimal sets.* – In compensation for the fact that the intersection of a non-countable family of quasi closed sets need not be quasi closed (even in the classical newtonian case), we have the following main result (cf. [15]).

THEOREM. – *Suppose that the topological space* X *has a countable base of open sets, and that the capacity* C *is sequentially order continuous from below in the sense that*

$$C\Big(\bigcup_{n\in N} A_n\Big) = \sup_{n\in N} C(A_n) \tag{5}$$

for any increasing sequence of sets $A_n \subset X$. *Then every non-void class* $\mathcal{H}$ *of quasi closed subsets of* X *which is stable under countable intersection has a quasi minimal element* H_0, *that is, a set* $H_0 \in \mathcal{H}$ *which is quasi contained in any other set* $H \in \mathcal{H}$.

If, in addition, $\mathcal{H}$ is saturated with respect to C-*equivalence, that is, if $\mathcal{H}$ is a union of equivalence classes of subsets of* X, *then the set of all quasi minimal elements of $\mathcal{H}$ is an equivalence class of subsets of* X.

There is of course a dual version of the theorem asserting the existence of a quasi maximal element of every class of quasi open sets which is stable under countable union.

Proof. – (inspired by Choquet [11]). Let $\{\omega_n\}_{n\in N}$ denote a countable base of open subsets of X. We may assume without loss of generality that any finite union of sets from this base belongs to the base.

For every set $A \subset X$ there exists a set $H \subset \mathcal{H}$ such that

$$C(A \cap H) \leqslant C(A \cap E) \qquad \text{for every } E \in \mathcal{H} .$$

In fact, we may take $H = \bigcap_{n\in N} E_n$, where the sets $E_n \in \mathcal{H}$ are so chosen that

$$\inf_{n\in N} C(A \cap E_n) = \inf_{E\in\mathcal{H}} C(A \cap E) .$$

Applying this construction to $A = \omega_n$, we obtain sets $H_n \in \mathcal{H}$ such that

$$C(\omega_n \cap H_n) \leqslant C(\omega_n \cap E)$$

for $n \in N$, $E \in \mathcal{H}$. It follows that $H_0 := \bigcap_{n\in N} H_n \in \mathcal{H}$, and that

$$C(\omega_n \cap H_0) \leqslant C(\omega_n \cap H_n) \leqslant C(\omega_n \cap E)$$

for $n \in N$, $E \in \mathcal{H}$. Since every open set $\omega \subset X$ is the union of an increasing sequence of sets from our base $\{\omega_n\}_{n\in N}$, and since C is supposed to be sequentially order continuous from below, it follows that

$$C(\omega \cap H_0) \leqslant C(\omega \cap E)$$

for every open set ω and every $E \in \mathcal{H}$. Since $\mathcal{H}$ consists of quasi closed sets, this means, according to the above lemma, that H_0 is quasi contained in each $E \in \mathcal{H}$. ‖

2.8. *Quasi closure. Quasi interior.* – By a *quasi closure* of a set $A \subset X$ we understand a quasi minimal element in the class of all quasi closed sets which quasi contain A.

THEOREM. – *Under the hypotheses of Theorem 2.7 every set* $A \subset X$ *has a quasi closure. The set of all quasi closures of* A *is an equivalence class depending only on the equivalence class of* A.

Proof. – Follows at once from the above theorem and the remark to it when applied to the class of all quasi closed sets quasi containing A. This class is, in fact, saturated and stable under countable intersection, and it depends only on the equivalence class of A. ▯

By duality we obtain similarly the existence of an equivalence class of *quasi interiors* of a set A, that is, quasi maximal elements in the saturated class of all quasi open sets quasi contained in A.

Remark. – Still under the hypotheses of Theorem 2.7, C is quasi stable if and only if $C(A^*) = C(A)$ for every set $A \subset X$ and every quasi closure A^* of A.

2.9. *Quasi support.* – As a further application of Theorem 2.7 we consider, in addition to the given capacity C, another such capacity μ on X (e.g. an outer measure on X). We suppose that

$$[C(E) = 0] \Longrightarrow [\mu(E) = 0]$$

for $E \subset X$. Denote by $\mathcal{H}(\mu)$ the class of all sets H which are quasi closed (with respect to C) and which carry μ in the sense that $\mu(X \backslash H) = 0$. Clearly $\mathcal{H}(\mu)$ is saturated and stable under countable intersection. Under the hypotheses of Theorem 2.7 there corresponds to μ an equivalence class of *quasi supports*, that is, quasi minimal elements in $\mathcal{H}(\mu)$. This result is the quasi topological analogue to Getoor's theorem [18] in the fine topology of potential theory (cf. Choquet [11], and Cor. 2 to Theorem 4.4 below).

2.10. *Capacity for decreasing sequences.* – Let X be a *Hausdorff* space, and denote by $\mathcal{K} = \mathcal{K}(X)$ the class of all compact subsets of X. Any *outer* capacity C on X is order continuous from above on $\mathcal{K}$, that is,

$$C\Big(\bigcap_{\alpha} K_\alpha\Big) = \inf C(K_\alpha) \tag{6}$$

for any downward directed family (K_α) of compact sets. (In fact, any open neighbourhood of $\cap K_\alpha$ contains some K_β). This property, when applied just to decreasing sequences of compact sets, carries over to the case of quasi compact sets :

THEOREM. – *Let* C *denote an outer capacity on a Hausdorff space* X. *For any decreasing sequence* $(H_n)_{n\in N}$ *of quasi compact sets* $H_n \subset X$ *we have*

$$C\left(\bigcap_{n\in N} H_n\right) = \inf_{n\in N} C(H_n) .$$

Proof(2). – Given $\varepsilon > 0$ choose compact sets $K_n \subset H_n$ so that $C(H_n\backslash K_n) < \varepsilon/2^n$ (cf. Lemma 2.2), and put

$$K'_n = \bigcap_{p=1}^{n} K_p , \qquad K = \bigcap_{n\in N} K'_n = \bigcap_{n\in N} K_n .$$

The sequence $(K'_n)_{n\in N}$ is decreasing, and $K \subset \bigcap_{n\in N} H_n$. Since $(H_n)_{n\in N}$ is decreasing, we have for every $n \in N$

$$H_n\backslash K'_n \subset \bigcup_{p=1}^{n} (H_p\backslash K_p) ,$$

and hence

$$C(H_n) \leqslant C(K'_n) + C(H_n\backslash K'_n) \leqslant C(K'_n) + \varepsilon .$$

It follows that

$$C\left(\bigcap_{n\in N} H_n\right) \geqslant C(K) = \inf_{n\in N} C(K'_n) \geqslant \inf_{n\in N} C(H_n) - \varepsilon . \ \blacksquare$$

Remark. – The property (6) remains valid for any downward directed family of *closed* sets contained in a quasi compact set (cf. the proof of [13, th. 7.2]). For a refinement see Cor. 4 b) to Theorem 4.4 below.

At this point we recall a capacitability theorem of Choquet [9] according to which every $\mathcal{K}$-Souslin set $A \subset X$ is $(C , \mathcal{K})$-capacitable in the sense that

$$C(A) = \sup\{C(K) \mid K \in \mathcal{K} , K \subset A\} ,$$

(2) This result was stated incorrectly in Fuglede [15, Lemma 2], where "quasi closed sets of finite capacity" should be read as "quasi compact sets" (as above).

provided that the outer capacity C has the property (5), § 2.7, of sequential order continuity from below (on arbitrary sets)([3]).

2.11. *Relation to capacities subadditive on compact sets.* – Consider an increasing mapping c of the class $\mathcal{K} = \mathcal{K}(X)$ of all compact subsets of a Hausdorff space X into $[-\infty, +\infty]$, and define increasing set functions c_*, $c^* : \mathfrak{P}(X) \longrightarrow [-\infty, +\infty]$ by

$$c_*(A) = \sup\{c(K) \mid K \text{ compact}, K \subset A\},$$
$$c^*(A) = \inf\{c_*(O) \mid O \text{ open}, \quad O \subset A\},$$

for arbitrary sets $A \subset X$. Clearly $c_*(A) \leqslant c^*(A)$ for every set A. We call a set A capacitable (with respect to c, or c-capacitable) if $c^*(A) = c_*(A)$, and we may then write $c(A)$ for this common value. This is justified since $c_*(K) = c(K)$ for every compact set. Every open set is capacitable. For any upward directed family of open sets O_α it follows from the Borel-Lebesgue theorem that

$$c(\cup O_\alpha) = \sup c(O_\alpha)\ . \tag{7}$$

If every compact set is capacitable, that is, if c is a capacity in the original sense of Choquet [8, § 15], then dually

$$c(\cap K_\alpha) = \inf c(K_\alpha) \tag{8}$$

for any downward directed family of compact sets K_α (cf. (6) above).

Suppose now, in addition, that $c(\emptyset) = 0$ and that c is subadditive on $\mathcal{K}$:

$$c(K_1 \cup K_2) \leqslant c(K_1) + c(K_2)$$

for arbitrary compact sets K_1, K_2. Then c_* is subadditive on open sets, and in fact countably subadditive on such sets in view of (7). Consequently c^* is *countably subadditive* on arbitrary sets. Note also that $c^*(\emptyset) = c_*(\emptyset) = c(\emptyset)$. Thus $c^* : \mathfrak{P}(X) \longrightarrow [0, +\infty]$ is an *outer capacity,* that is, c^* satisfies the requirements (C_1), (C_2), (C_3), (C_4) (§ 1.1, § 1.5). Moreover, a set $A \subset X$ is $(c^*, \mathcal{K})$-capacitable (§ 2.10) if and only if A is c-capacitable, $c^*(A) = c_*(A)$.

The capacities which have the form c^* with c as above are precisely those outer capacities C for which all open sets are $(C, \mathcal{K})$-

([3]) The same holds with $\mathcal{K}$ replaced by the class $\mathcal{K}^*$ of all quasi compact sets (use Theorem 3.7). Actually, this leads only apparently to an extension of the above result because every quasi compact set is equivalent to a set of class $\mathcal{K}_\sigma$, and hence every $\mathcal{K}^*$-Souslin set is equivalent to a $\mathcal{K}$-Souslin set.

capacitable. In particular, if every open subset of X is of class $\mathcal{K}_\sigma$ (e.g. if X is a locally compact space with a countable base of open sets) then every outer capacity C which is sequentially order continuous from below (cf. (5), § 2.7) has the form c^*.

3. Quasi continuity and quasi semicontinuity.

We consider again a capacity C on a topological space X (separated in contexts involving compact sets). We use the abbreviations l.s.c. (resp. u.s.c.) for lower (resp. upper) semicontinuous.

3.1. DEFINITION. – *A function $f : X \longrightarrow [-\infty, +\infty]$ is called quasi continuous (resp. quasi l.s.c., quasi u.s.c.) if there corresponds to every $\varepsilon > 0$ a subset ω of X with $C(\omega) < \varepsilon$ such that the restriction of f to $X \backslash \omega$ is continuous*(4) *(resp. l.s.c., u.s.c.). – Note that ω may be taken to be open if C is outer.*

It is evident that f is quasi continuous if and only if f is both quasi l.s.c. and quasi u.s.c. Moreover, f is quasi l.s.c. if and only if $-f$ is quasi u.s.c. Each of the 3 classes of functions is a sublattice of $[-\infty, +\infty]^X$, stable under addition (when defined) and under multiplication by constants $\geqslant 0$, and furthermore saturated with respect to the equivalence relation defined in § 1.3. The class of quasi l.s.c. (resp. quasi u.s.c.) functions is easily shown to be stable under *countable* supremum (resp. countable infimum).

3.2. THEOREM. – *Each of the 3 classes of functions (quasi continuous, quasi l.s.c., quasi u.s.c.) is closed under quasi uniform convergence of sequences.*

Proof. – Choose $\omega_0 \subset X$ with $C(\omega_0) < \varepsilon/2$ so that $f_n \longrightarrow f$ uniformly on $X \backslash \omega_0$. For every $n = 1, 2, \ldots$ choose $\omega_n \subset X$ with $C(\omega_n) < \varepsilon/2^{n+1}$ so that f_n is continuous, l.s.c., or u.s.c., respectively, relatively to $X \backslash \omega_n$. Then $\omega := \omega_0 \cup \omega_1 \cup \omega_2 \cup \ldots$ has $C(\omega) < \varepsilon$,

(4) Continuity is not understood to imply finiteness.

and the restriction of f to $X \setminus \omega$ is the uniform limit of the continuous, l.s.c., or u.s.c. restrictions of f_n to $X \setminus \omega$. ‖

Remark. – With the obvious extension of the notion of quasi continuity to mappings of X into a topological space Y, we note that the class of all quasi continuous mappings of X into Y is closed under quasi uniform convergence of sequences whenever this makes sense, that is, when Y is a uniform space.

3.3. LEMMA. – *In order that a function $f : X \longrightarrow [-\infty, +\infty]$ be quasi l.s.c., resp. quasi u.s.c., it is necessary and sufficient that the set*

$$\{x \in X \mid f(x) \leqslant t\}, \quad \text{resp. } \{x \in X \mid f(x) \geqslant t\},$$

be quasi closed for every real t.

In its equivalent dual form the condition states that the sets where $f(x) > t$, resp. $f(x) < t$, should be quasi open. Note that, on account of Lemma 2.3, it suffices to verify the condition in question for all t belonging to some dense set of reals.

Proof. – The necessity of the condition is obvious. As to the sufficiency, say in the quasi u.s.c. case, let $(t_n)_{n \in N}$ denote a dense sequence of reals such that $H_n := \{x \in X \mid f(x) \geqslant t_n\}$ is quasi closed for every $n \in N$. There exist sets $\omega_n \subset X$ with $C(\omega_n) < \varepsilon/2^n$ such that $H_n \setminus \omega_n$ is closed relatively to $X \setminus \omega_n$. For the set $\omega := \cup\, \omega_n$ we find $C(\omega) < \varepsilon$, and f is u.s.c. relatively to $X \setminus \omega$ because the sets $H_n \setminus \omega$ are closed relatively to $X \setminus \omega$. ‖

Remark. – Let Y denote a topological space with a countable base of open sets. In order that a mapping $f : X \longrightarrow Y$ be quasi continuous it is necessary and sufficient that $f^{-1}(\Omega)$ be quasi open for every open set $\Omega \subset Y$. It suffices to verify this condition for sets Ω from a base for Y. Proof as above.

3.4. THEOREM. – *Suppose that the topological space X has a countable base of open sets, and that the capacity C is sequentially order continuous from below (cf. Theorem 2.7.). Then every non-void set Φ of quasi u.s.c. functions on X which is stable under countable infimum has a quasi minimal element f_0, that is, a function $f_0 \in \Phi$ such that $f_0 \prec f$ for every $f \in \Phi$.*

If, in addition, Φ is saturated with respect to C-equivalence (§ 1.3), that is, if Φ is a union of equivalence classes of functions, then the quasi minimal elements of Φ constitute an equivalence class of functions.

There is a dual version of the theorem asserting the existence of a quasi maximal element of every set of quasi l.s.c. functions stable under countable supremum.

Proof. – We merely utilize the conclusion of the analogous Theorem 2.7 concerning quasi closed sets. For every $t \in [-\infty, +\infty]$ put for brevity

$$E_t^f = \{x \in X \mid f(x) \geqslant t\}$$

for functions $f : X \longrightarrow [-\infty, +\infty]$, and write

$$\mathcal{H}_t = \{E_t^f \mid f \in \Phi\} .$$

According to Lemma 3.3, $\mathcal{H}_t$ is a class of quasi closed sets. Clearly, $\mathcal{H}_t$ is stable under countable intersection. It follows from Theorem 2.7 that there exists a quasi minimal element H_t of $\mathcal{H}_t$. Denote by h_t a function from Φ such that $E_t^{h_t} = H_t$. Then the function f_0 defined by

$$f_0(x) = \inf\{h_r(x) \mid r \text{ rational}\}$$

belongs to Φ. For every $f \in \Phi$ and every t we have $E_t^f \in \mathcal{H}_t$ and hence $E_t^f \succ H_t$. Consequently,

$$\begin{aligned} E_t^f &= \cap\{E_r^f \mid r \text{ rational}, r < t\} \\ &\succ \cap\{H_r \mid r \text{ rational}, r < t\} \supset E_t^{f_0} . \end{aligned}$$

As to the last inclusion note that, for every $x \in E_t^{f_0}$ and every rational $r < t$, we have $h_r(x) \geqslant f_0(x) \geqslant t \geqslant r$, and hence $x \in E_r^{h_r} = H_r$. It now follows that $f \succ f_0$ because the set

$$\{x \in X \mid f(x) < f_0(x)\} = \bigcup_{t \text{ rat.}} \{x \in X \mid f(x) < t \leqslant f_0(x)\}$$

is a countable union of sets $E_t^{f_0} \backslash E_t^f$ of capacity 0. ▯

3.5. *Quasi u.s.c. envelope.* – For any function

$$f : X \longrightarrow [-\infty, +\infty]$$

let $\Phi(f)$ denote the class of all quasi u.s.c. functions

$$h : X \longrightarrow [-\infty, +\infty]$$

such that $h \succ f$, that is $h \geqslant f$ q.e. Then $\Phi(f)$ is saturated and stable under countable infimum, and it depends only on the equivalence class of f.

By a *quasi u.s.c. envelope* of f we understand a quasi minimal element of $\Phi(f)$. As a corollary of the above theorem we have the following result.

THEOREM. – *Under the hypotheses of Theorem 3.4 every function $f : X \longrightarrow [-\infty, +\infty]$ has a quasi u.s.c. envelope. The set of all such envelopes of f is an equivalence class depending only on the equivalence class of f.*

3.6. *Quasi limit at infinity.* – In view of certain applications elsewhere we introduce the following definition :

Let C denote an *outer capacity* on a Hausdorff space X. A function $f : X \longrightarrow [-\infty, +\infty]$ is said to have the quasi limit 0 at infinity if there are *open* sets $\omega \subset X$ with $C(\omega)$ as small as we please such that the restriction of f to $\complement\omega$ vanishes at infinity in the closed (hence locally compact) subspace $\complement\omega$ of X. (If X is compact, any function on X is said to have the quasi limit 0 at infinity).

The class of all functions $f : X \longrightarrow [-\infty, +\infty]$ having the quasi limit 0 at infinity is closed under quasi uniform convergence (cf. proof of Theorem 3.2).

LEMMA. – *In order that a function $f : X \longrightarrow [0, +\infty]$ be quasi u.s.c. and have the quasi limit 0 at infinity, it is necessary and sufficient that $\{x \in X \mid f(x) \geqslant t\}$ be quasi compact for every $t > 0$.*

Proof. – Necessity : Clearly there corresponds to any $\varepsilon > 0$ an open set $\omega \subset X$ such that the restriction of f to $\complement\omega$ is u.s.c. and vanishes at infinity. Writing $H := \{x \in X \mid f(x) \geqslant t\}$, we infer that $H\backslash\omega$ is compact. – Sufficiency : Suppose just that

$$H_n := \{x \in X \mid f(x) \geqslant t_n\}$$

is quasi compact for every t_n from some dense sequence of reals > 0. Proceeding as in the proof of the analogous Lemma 3.3, we find for any $\varepsilon > 0$ an open set $\omega = \cup\, \omega_n$ with $C(\omega) < \varepsilon$ such that f is u.s.c. and vanishes at infinity relatively to $\complement\,\omega$ because each of the sets $H_n \backslash \omega \subset H_n \backslash \omega_n$ is compact. ‖

4. Quasi topology and "fine" topology.

4.1. *The base of a set.* – Let X denote a set endowed with a topology. With a view at the applications below (where this topology is not the initially given topology on X), we denote the *closure* of a set A by $\widetilde{A}$. A set $E \subset X$ will be called *discrete* if every point of x is isolated in E (that is, if the topological subspace E is discrete).

Let there be given a certain subset X_0 of X. In view of the potential theoretic applications, (cf. §§ 5.6 and 5.7), the points of X_0 will be called the *polar* points of X. (In classical potential theory in R^n, the only non polar point is the point at infinity, which it is sometimes convenient to adjoin to R^n, $n \geqslant 3$).

DEFINITION. – *The base* $b(A)$ *of a set* $A \subset X$ *is the set of all points of* $\widetilde{A}$ *which are not both polar and isolated in* A.

Equivalently,

$$x \in b(A) \Longleftrightarrow \begin{cases} x \in (A\backslash\{x\})^{\sim} & \text{for } x \in X_0 \,, \\ x \in \widetilde{A} & \text{for } x \in \complement X_0 \,. \end{cases}$$

In the case $X_0 = X$ where all points of X are polar, $b(A)$ reduces to the derived set (the set of all "limit points" for A).

Immediate consequences. – For any set $A \subset X$ we have :

$$b(A) \subset \widetilde{A} \,. \tag{9}$$

$$b(A)\backslash A = \widetilde{A}\backslash A \,. \tag{10}$$

$$\widetilde{A} = A \cup b(A) \,. \tag{11}$$

$$[A \text{ closed}] \Longleftrightarrow [b(A) \subset A] \,. \tag{12}$$

$$b(A) = \widetilde{A}\backslash(A\backslash b(A)) \,. \tag{13}$$

Thus $A\backslash b(A)$ is the discrete set of all isolated, polar points of A. Conversely, any discrete set E of polar points, that is, any set E such that $E \cap b(E) = \emptyset$, has the form $E = A\backslash b(A)$ (e.g. with $A = E$).

The base mapping $b : \mathfrak{P}(X) \longrightarrow \mathfrak{P}(X)$ is additive (in particular increasing) :

$$b(A_1 \cup A_2) = b(A_1) \cup b(A_2) . \tag{14}$$

Let us now assume that every polar point forms a closed set : $\{x\}^{-} = \{x\}$ for any $x \in X_0$. Then it is an elementary fact from topology that A and $\widetilde{A}$ have the same isolated polar points and hence also the same base. Thus

$$b(\widetilde{A}) = b(A) , \quad \widetilde{A}\backslash b(\widetilde{A}) = A\backslash b(A) . \tag{15}$$

4.2. *The base of a function.* – The above notions extend to functions $f : X \longrightarrow [0 , + \infty]$ in the obvious way. The *u.s.c. envelope* $\widetilde{f}$ and the base $b(f)$ of f are defined as functions on X to $[0 , + \infty]$ by

$$\widetilde{f}(x) = \limsup_{y \to x} f(y) \tag{16}$$

$$b(f)(x) = \begin{cases} \limsup\limits_{y \to x ,\, y \neq x} f(y) & \text{for } x \in X_0 , \\ \widetilde{f}(x) & \text{for } x \in \complement X_0 . \end{cases} \tag{17}$$

It is understood that $b(f)(x) = 0$ for any isolated polar point $x \in X_0$.

The base mapping $b : [0 , + \infty]^X \longrightarrow [0 , + \infty]^X$ is clearly an extension of the base mapping of sets when these are identified with their indicator functions :

$$b(1_A) = 1_{b(A)} .$$

The relations (9)-(12) may be generalized to the case of functions $f : X \longrightarrow [0 , + \infty]$. Thus $b(f) \leqslant \widetilde{f}$, with equality at any point x such that $\widetilde{f}(x) > f(x)$. In other terms,

$$\widetilde{f} = f \vee b(f) ,$$

showing that f is u.s.c. (that is, $\widetilde{f} = f$) if and only if $b(f) \leqslant f$. Moreover,

$$b(f_1 \vee f_2) = b(f_1) \vee b(f_2) .$$

If every polar point of X forms a closed set, it is easily verified that $\tilde{f}$ and f have the same base :

$$b(\tilde{f}) = b(f) .$$

We now impose a much stronger condition.

THEOREM([5]). – *Suppose that any countable union of discrete sets of polar points is discrete and closed. Then*

a) $b(b(f)) = b(\tilde{f}) = b(f)$ *for any* $f : \mathrm{X} \longrightarrow [0, +\infty]$.

b) *Any convergent sequence of polar points is constant from a certain step.*

c) *Any compact subset of* X *contains at most a finite number of polar points.*

d) *If* X *is locally compact, the set* X_0 *of all polar points is discrete.*

e) *If* X *satisfies the first axiom of countability,* X_0 *is discrete.*

Proof. – Note first that any polar point x forms a discrete and hence closed set x . More generally, any countable set of polar points is discrete and closed. (The properties b)–e) depend only on this latter consequence of the hypothesis of the theorem).

Ad a). For $t > 0$ write

$$\mathrm{E}_t = \{x \in \mathrm{X} \mid b(f)(x) < t < \tilde{f}(x)\} .$$

Any point $x \in \mathrm{E}_t$ is polar, hence closed. It follows that x is isolated in $\{y \in \mathrm{X} \mid \tilde{f}(y) > t\}$, a fortiori in E_t . This shows that E_t is a discrete set of polar points, and consequently the union $\mathrm{E} = \cup\{\mathrm{E}_t \mid t \text{ rational}\}$ is a discrete and closed set of polar points, by our hypothesis. It is clear, however, that any two functions $f_1 , f_2 : \mathrm{X} \longrightarrow [0, +\infty]$ which differ only in some discrete and closed set $\mathrm{E} \subset \mathrm{X}_0$, have the *same base.* Since, in particular, $b(f) = \tilde{f}$ except in the above discrete

([5]) In classical potential theory the *fine topology,* introduced by Cartan [7], satisfies the hypothesis of this theorem. In the absence of non polar points, the base $b(\mathrm{A})$ coincides here with the finely derived set (the set of all fine limit points for A). It consists of all points of X at which A is not thin (effilé) in the sense of Brelot, cf. e.g. Brelot [1] or [6] (See also Theorem 5.2 below). – For an extensive and more general study of topologies such as the fine topology, see Doob [12].

and closed set $E \subset X_0$, we conclude that $b(b(f)) = b(\widetilde{f})$, which in turn equals $b(f)$ as remarked above.

Ad b). For any sequence (x_n) of polar points x_n converging to some point $x \in X$, the countable set of all $x_n \neq x$ is closed and hence finite since it does not contain the limit x.

Ad c). Any countable set $\subset K \cap X_0$ (K compact) is discrete and closed, hence compact and consequently finite.

Ad d). Any polar point $x \in X_0$ has a compact neighbourhood K. Since $K \cap X_0$ is finite, $\{x\}$ is open relatively to $K \cap X_0$ and hence also relatively to X_0 because $K \cap X_0$ is a neighbourhood of x in X_0.

Ad e). If X (or just X_0) satisfies the first axiom of countability, and if $x \in X_0$ were not isolated in X_0, then there would exist a sequence of points of X_0, different from x and converging to x, which contradicts b). ‖

4.3. *Topology compatible with the quasi topology.* – In the sequel X denotes a topological space and $C : \mathfrak{P}(X) \longrightarrow [0, +\infty]$ a *capacity* on X in the sense of Def. 1.1.

In addition to the given (or "initial") topology on X we shall consider another topology on X, termed the "*fine*" topology (although we do not require here that it be finer than the initial topology). Topological notions pertaining to this new topology on X will be distinguished by the qualification "fine(ly)". We denote by $\widetilde{A}$ the *fine* closure of a set $A \subset X$, and by $\overline{A}$ the closure of A in the initial topology on X.

We shall appeal to the content of §§ 4.1. and 4.2., taking from now on as *polar* points those points $x \in X$ for which $C(\{x\}) = 0$. Thus we put

$$X_0 = \{x \in X \mid C(\{x\}) = 0\} ,$$

and we define the *base* $b(A)$ of a set $A \subset X$ according to Def. 4.1 as the set of all points of $\widetilde{A}$ which are not both polar and finely isolated (in A). Similarly, the finely u.s.c. envelope $\widetilde{f}$ and the base $b(f)$ of a function $f : X \longrightarrow [0, +\infty]$ are given by (16), (17) :

$$\widetilde{f}(x) = \text{fine} \limsup_{y \to x} f(y) ,$$

$$b(f)(x) = \begin{cases} \text{fine} \limsup\limits_{y \to x,\, y \neq x} f(y) & \text{if } C(\{x\}) = 0\,, \\ \tilde{f}(x) & \text{if } C(\{x\}) > 0\,. \end{cases}$$

DEFINITION. – *A new topology (termed "fine") on* X *is said to be compatible with the quasi topology determined by the capacity* C *on* X *if the following two conditions are fulfilled :*

(T_1) *A set* $A \subset X$ *is quasi closed if and only if it is* C*-equivalent to some finely closed set.*

(T_2) *For any set* $E \subset X$ *we have the implications*

$$[E \cap b(E) = \emptyset] \Longrightarrow [C(E) = 0] \Longrightarrow [b(E) = \emptyset]\,.$$

The property (T_2) states that a set E has capacity 0 if and only if E is a finely discrete set of polar points (points of capacity 0), and further that any such set is finely closed. (It follows then that $b(E) = \emptyset$).

In particular, the hypothesis of Theorem 4.2 is a consequence of (T_2). – We mention in passing that the fine topology of Cartan in classical newtonian potential theory is compatible with the quasi topology determined by the newtonian capacity (see § 5.6 or § 5.7 below).

LEMMA. – (T_1) *implies the following statements :*

a) *A mapping* f *of* X *into a topological space* Y *with a countable base of open sets is quasi continuous if and only if* f *is finely continuous*(6) *relatively to some set* $X \setminus E$ *with* $C(E) = 0$.

b) *A function* $f : X \longrightarrow [-\infty, +\infty]$ *is quasi u.s.c. (resp. quasi l.s.c.) if and only if* f *is finely u.s.c. (resp. finely l.s.c.) relatively to some set* $X \setminus E$ *with* $C(E) = 0$.

Proof. – Ad a). Let (Ω_n) denote a countable base of open subsets of Y. If $f : X \longrightarrow Y$ is finely continuous relatively to $X \setminus E$, where $C(E) = 0$, then the sets $f^{-1}(\Omega_n) \setminus E$ are finely open relatively to $X \setminus E$, hence of the form $G_n \setminus E$ with G_n finely open. It follows from (T_1) applied to $X \setminus G_n$ that G_n is quasi open, and so is therefore

(6) The fine continuity refers to the (induced) fine topology on (subsets of) X, and the given topology on Y.

$f^{-1}(\Omega_n)$, being equivalent to G_n since $C(E) = 0$. Consequently f is quasi continuous by the remark to Lemma 3.3. – Conversely, suppose that $f : X \longrightarrow Y$ is quasi continuous. Then $f^{-1}(\Omega_n)$ is quasi open, hence equivalent to some finely open set G_n, again by complementation of (T_1). Writing $E_n := f^{-1}(\Omega_n) \Delta G_n$ and $E = \cup E_n$, we thus have $C(E_n) = 0$ and hence $C(E) = 0$. We show that the restriction of f to $X \backslash E$ is finely continuous at any point $x \in X \backslash E$. For any n such that $f(x) \in \Omega_n$ the set $f^{-1}(\Omega_n) \backslash E = G_n \backslash E$ contains x and is open in the induced fine topology on $X \backslash E$.

Ad b). Proceed as above, but with Ω_n replaced by $[-\infty, t_n[$ (in the case of upper semicontinuity), where t_n ranges over the rational reals, and apply Lemma 3.3. Next replace f by $-f$. ‖ Conversely, (T_1) is a particular case of b).

REMARK. – Since we have allowed infinite values for f in statement b), the fine semicontinuity of f relatively to some set C-equivalent to X amounts (even without the hypothesis (T_1)) to the apparently stronger property that f is *C-equivalent* to some finely (upper, resp. lower) semicontinuous function h on X (that is, $f = h$ quasi everywhere). In fact, we may take for h (say in the case of upper semicontinuity) the finely u.s.c. envelope of the function which equals f in $X \backslash E$ and $-\infty$ in E. – Note however that a *quasi continuous* function $f : X \longrightarrow [-\infty, +\infty]$ need not be equivalent to any finely continuous function. In the newtonian case this appears e.g. from the example $f(x) := \sin(1/|x|)$ for $x \neq 0$ (the value at $x = 0$ being immaterial).

THEOREM. – *Suppose that a new topology on* X *(called the "fine" topology) is compatible with the quasi topology determined by the capacity* C. *Then the following statements hold :*

a) *A mapping f of* X *into a topological space* Y *with a countable base of open sets is quasi continuous if and only if f is finely continuous quasi everywhere.*

b) *The following are equivalent for any function*

$$f : X \longrightarrow [-\infty, +\infty] :$$

1) *f is quasi u.s.c.,* 2) *f is finely u.s.c. quasi everywhere,* 3) *f is C-equivalent to some finely u.s.c. function,* 4) $f \sim \tilde{f}$.

c) *For any function* $f : X \longrightarrow [0, +\infty]$, *we have* $b(f) \sim \widetilde{f}$. *The base* $b(f)$ *is the smallest finely u.s.c. function majorizing* f *quasi everywhere. Hence* $b(f)$ *and* $\widetilde{f}$ *are both quasi u.s.c. envelopes of* f. *In particular,* $b(f) \sim \widetilde{f} \sim f$ *if* f *is quasi u.s.c.*

d) *For any functions* $f_1, f_2 : X \longrightarrow [0, +\infty]$ *we have the implications*

$$[f_1 \leqslant f_2 \text{ q.e.}] \Longrightarrow [b(f_1) \leqslant b(f_2)] ,$$
$$[f_1 = f_2 \text{ q.e.}] \Longrightarrow [b(f_1) = b(f_2)] .$$

The converse implications hold if f_2, *and in the latter case also* f_1, *is quasi u.s.c.*

Proof. – Ad a). According to the above lemma it remains only to establish the "only if" part. For this we merely use the "only if" part of (T_1) together with the implication

$$[C(E) = 0] \Longrightarrow [C(\widetilde{E}) = 0] ,$$

which is a consequence of (T_2) (and actually needed at this point, like (T_1), as shown by taking $f = 1_E$). Thus let $f : X \longrightarrow Y$ be quasi continuous. By the lemma, f is then finely continuous relatively to $X \backslash E$ for some set $E \subset X$ with $C(E) = 0$. It follows that f is actually finely continuous in the finely open set $X \backslash \widetilde{E}$, and hence quasi everywhere since $C(\widetilde{E}) = 0$.

Ad b). Without any hypothesis, 2) and 4) are equivalent and imply 3). According to the above lemma and remark we have equivalence between 1) and 3) under the sole hypothesis (T_1). Finally, 3) implies 2) by the above consequence of (T_2) (which again is also a necessary condition).

We proceed to establish the former part of c) and of d), now under full use of (T_2) as well as (T_1). According to the proof of Theorem 4.2 a), $b(f)$ equals $\widetilde{f}$ except in some finely discrete set E of polar points, that is, by (T_2), a set E such that $C(E) = 0$. It was also noted that changing a function f in a finely discrete and finely closed set E of polar points (that is, a set E such that $b(E) = \emptyset$) does not affect the base $b(f)$. In view of (T_2), this implies the former part of d).

As to the remaining assertions of c), let $h : X \longrightarrow [0, +\infty]$ be quasi u.s.c., and suppose that $h \geqslant f$ q.e. It follows from the former part of d) that $b(h) \geqslant b(f)$ everywhere. Since $h \sim \tilde{h} \sim b(h)$ by b) and the former part of c), we conclude that $h \geqslant b(f)$ q.e., and so $b(f)$ is a quasi u.s.c. envelope of f. Taking h to be even *finely* u.s.c., that is, $h \geqslant b(h)$, our argument shows moreover that $b(f)$ is the smallest finely u.s.c. function $\geqslant f$ q.e.

Finally, let us prove the latter part of d). Thus let $b(f_1) \leqslant b(f_2)$ everywhere (or just quasi everywhere), and suppose that f_2 is quasi u.s.c., hence $\tilde{f}_2 \sim f_2$. Then

$$f_1 \leqslant \tilde{f}_1 \sim b(f_1) \prec b(f_2) \leqslant \tilde{f}_2 \sim f_2 . \ \|$$

COROLLARY. – *(The Choquet Property, cf. Choquet* [10]*). In addition to the hypothesis of the above theorem, suppose that* C *is an outer capacity (§ 1.5). For any set* $A \subset X$ *and any* $\varepsilon > 0$ *there exists a closed set* $F \subset b(A)$ *such that* $C(A \backslash F) < \varepsilon$.

In fact, $b(A)$ is finely closed, hence quasi closed, by (T_1), and quasi contains A by Theorem 4.3 c). According to Lemma 2.2 there exists a closed set $F \subset b(A)$ such that $C(b(A) \backslash F) < \varepsilon$. It follows that $C(A \backslash F) < \varepsilon$ because $C(A \backslash b(A)) = 0$.

4.4. *The lattice of bases* (cf. Doob [12]). – We continue the study of a "fine" topology compatible with the quasi topology associated with a capacity C on X. According to § 4.2 the mapping $f \longrightarrow b(f)$ of $[0, +\infty]^X$ into itself has the following properties

$$b(f_1 \vee f_2) = b(f_1) \vee b(f_2) ,$$

$$\tilde{f} = f \vee b(f) ,$$

$$b(b(f)) = b(f) = b(\tilde{f}) .$$

According to Theorem 4.3, this base mapping induces a mapping with the same properties of the lattice of all *equivalence classes* of functions in $[0, +\infty]^X$, the image of such an equivalence class being the common base $b(f)$ of all f in the class. We denote by $\mathcal{B}$ the range of the base mapping $b : [0, +\infty]^X \longrightarrow [0, +\infty]^X$ (or of the induced mapping of equivalence classes in $[0, +\infty]^X$ into $[0, +\infty]^X$). Thus $\mathcal{B}$ consists of all *bases* in $[0, +\infty]^X$, a base being here defined as a "b-perfect function", that is a function $f : X \longrightarrow [0, +\infty]$ such that $b(f) = f$.

The restriction of the (induced) base mapping b to the lattice $\mathfrak{F}$ of all equivalence classes of *quasi u.s.c. functions*

$$f : \mathrm{X} \longrightarrow [0\,,+\infty]$$

carries any such equivalence class into the unique base in the class (because $b(f) \sim f$ when f is quasi u.s.c. by Theorem 4.3 c)). This mapping is an order isomorphism of $\mathfrak{F}$ onto $\mathcal{B}$ in view of d) of that theorem, being onto because any base is quasi u.s.c.

Via this order isomorphism $b : \mathfrak{F} \longrightarrow \mathcal{B}$ the results on $\mathfrak{K}$ obtained in §§ 3.4-3.5 carry over to the lattice $\mathcal{B}$ of all bases, and so do the results in §§ 2.7-2.10 concerning quasi closed sets. Thus we have

THEOREM. – *Suppose that* X *has a countable base (in the initial topology), and that the capacity* C *is sequentially order continuous from below (in the sense of (5), § 2.7). Further let there be given another topology (termed "fine") on X, compatible with the quasi topology associated with* C. *The set* $\mathcal{B}$ *of all bases* $f : \mathrm{X} \longrightarrow [0\,,+\infty]$ *(that is,* $b(f) = f$*) is then a complete lattice under the order relation* $\leqslant$ *(pointwise). The supremum in* $\mathcal{B}$ *of any finite family of bases is the pointwise supremum of these. The supremum, resp. infimum, in* $\mathcal{B}$ *of any family of bases is the base of the pointwise supremum, resp. infimum, and there is always a countable subfamily with the same infimum in* $\mathcal{B}$.

Proof. – (Cf. Doob [12]). The pointwise supremum $f_1 \vee f_2$ of two bases is a base because

$$b(f_1 \vee f_2) = b(f_1) \vee b(f_2) = f_1 \vee f_2\ ,$$

and hence is the supremum of $(f_1\,, f_2)$ in $\mathcal{B}$. Denoting by f the pointwise supremum, resp. infimum, of an arbitrary family (f_α) of bases, it is clear that $b(f)$ is a majorant, resp. a minorant, for (f_α) in $\mathcal{B}$ because the base mapping b is increasing. And for any majorant, resp. minorant, $h \in \mathcal{B}$ for (f_α) we have $h \geqslant f$, resp. $h \leqslant f$, and hence $h = b(h) \geqslant b(f)$, resp. $\leqslant b(f)$.

Every function φ representable as the pointwise infimum of some countable subfamily (f_{α_n}) of the given family of bases, or just of any family (f_α) of finely u.s.c. functions $f_\alpha \geqslant 0$, is finely u.s.c. and hence quasi u.s.c. The class Φ of all such countable infima φ is stable under countable pointwise infimum. According to Theorem 3.4, Φ has a quasi minimal element h, say

$$h(x) = \inf_n f_{\alpha_n}(x) \quad \text{for every} \quad x \in \mathrm{X} .$$

For any α we have $h \leqslant f_\alpha$ q.e., hence $b(h) \leqslant b(f_\alpha) \leqslant f_\alpha$ by Theorem 4.3 d). Thus $b(h) \leqslant f \leqslant h$, where f denotes the pointwise infimum of (f_α). In particular, $h = f$ q.e. (which leads to Cor. 3 below), and consequently $b(h) = b(f)$. ‖

COROLLARY 1. – *Under the hypotheses of Theorem 4.4, the collection of all bases (as sets)* $\mathrm{A} \subset \mathrm{X}$ *is a complete lattice under inclusion. The supremum of any finite family of such bases is the union thereof. The infimum of any family of bases is the base of their intersection, and there is always a countable subfamily with the same infimum.*

COROLLARY 2. – (cf. Getoor [18], Choquet [11]). *Let* μ :

$$\mathfrak{P}(\mathrm{X}) \longrightarrow [0\,, +\,\infty]$$

denote a further capacity on X, *and such that* $\mathrm{C(E)} = 0$ *implies* $\mu(\mathrm{E}) = 0$. *Under the hypotheses of Theorem 4.4 there exists a smallest finely closed set* A *such that* $\mu(\mathrm{X}\backslash\mathrm{A}) = 0$. *This smallest set is a base (called the fine support of* μ*).*

Proof. – For any finely closed set A carrying μ in the sense that $\mu(\mathrm{X}\backslash\mathrm{A})=0$, the base $b(\mathrm{A})$ likewise carries μ because $\mathrm{C}(\mathrm{A}\backslash b(\mathrm{A}))=0$ and hence

$$\mu(\mathrm{X}\backslash b(\mathrm{A})) \leqslant \mu(\mathrm{X}\backslash\mathrm{A}) + \mu(\mathrm{A}\backslash b(\mathrm{A}) = 0 .$$

It suffices, therefore, to find a smallest base in the class $\mathcal{B}(\mu)$ of all bases carrying μ, or equivalently to prove that

$$\mathrm{B} := \inf \mathcal{B}(\mu) = b(\cap\, \mathcal{B}(\mu))$$

carries μ. By the preceding corollary there exists a sequence of bases $\mathrm{A}_n \in \mathcal{B}(\mu)$ such that $\mathrm{B} = \inf_n \mathrm{A}_n = b\left(\bigcap_n \mathrm{A}_n\right)$. Being countably subadditive and carried by each A_n, μ is carried by $\mathrm{A} := \cap \mathrm{A}_n$, and consequently by $\mathrm{B} = b(\mathrm{A})$ as remarked above. ‖

COROLLARY 3. – *(The quasi Lindelöf principle, cf. Doob* [12]*). Under the hypotheses of Theorem 4.4, any family of finely open sets contains a countable subfamily whose union is* C*-equivalent to the union of the given family. – More generally, any family of finely l.s.c. functions on* X *to* $[-\infty, +\infty]$ *contains a countable subfamily whose pointwise supremum is equal quasi everywhere to that of the given family. This subsequence may be taken to be increasing if the given family is upward directed.*

Except for the last assertion, which is straightforward, this corollary was established in the latter part of the proof of Theorem 4.4 (in dual form : infimum of finely u.s.c. functions $\geqslant 0$, this latter restriction being immaterial).

COROLLARY 4. – (Cf. Brelot [4]). a) *Under the hypotheses of Theorem 4.4, we have*

$$C\left(\bigcup_{\alpha} A_{\alpha}\right) = \sup C(A_{\alpha})$$

for any upward directed family of finely open sets $A_{\alpha} \subset X$. b) *Under the additional hypotheses that* X *is a Hausdorff space (in the initial topology) and that* C *is an outer capacity, we have*

$$C\left(\bigcap_{\alpha} A_{\alpha}\right) = \inf_{\alpha} C(A_{\alpha})$$

for any downward directed family of finely closed sets A_{α} *contained in a quasi compact set.*

In view of the quasi Lindelöf principle above, these assertions reduce to the case of an increasing, resp. decreasing, sequence ; and here the result follows a) from the assumed sequential order continuity of C from below, and b) from Theorem 2.10 since the sets A_{α} are quasi compact by (T_1) together with the remark to Lemma 2.3.

4.5. While the above results on the fine topology were derived from corresponding, more elementary results concerning the quasi topology, the following is an example in the opposite direction :

THEOREM. – *In addition to the hypotheses of Theorem 4.4 suppose that the fine topology (compatible with the quasi topology) is uniformizable. Then every quasi u.s.c. function* $f : X \longrightarrow [-\infty, +\infty]$ *is representable as the pointwise infimum of a decreasing sequence of quasi continuous functions* f_n .

Proof. – Since the fine topology is uniformizable, any finely u.s.c. function $f : X \longrightarrow [-\infty, +\infty]$ is the pointwise infimum of its finely continuous majorants. By the quasi Lindelöf principle, f is equal q.e. to the pointwise infimum of some decreasing sequence of such majorants g_n. This latter assertion remains valid for any quasi u.s.c. function f on account of Remark 4.3. Thus there is a set $E \subset X$ with $C(E) = 0$ such that $f(x) = \inf_n g_n(x)$ for every $x \in X \backslash E$. The functions f_n defined by

$$f_n(x) = \begin{cases} f(x) & \text{for} \quad x \in E\,, \\ g_n(x) & \text{for} \quad x \in X \backslash E\,, \end{cases}$$

are quasi continuous by Lemma 4.3 (being equivalent to g_n), and $f(x) = \inf_n f_n(x)$ for every $x \in X$. ‖

4.6. *Reformulation of compatibility with the quasi topology in terms of the base mapping of quasi closed sets.* – We have found above that if a new topology (termed "fine") on X is compatible with the quasi topology determined by the given capacity C, then the associated base mapping b, now considered for sets, is a kind of "*lifting*" in the measure theoretic sense. It assigns to each equivalence class of *quasi closed sets* a unique representative for the class, and this assignment is additive (and hence increasing), and $b(\emptyset) = \emptyset$. – Conversely, we have the following result :

LEMMA. – *Suppose that one can select from each equivalence class of quasi closed sets a certain set $b(A)$ from the class, the same for all sets A in the class, in such a way that $b(\emptyset) = \emptyset$ and that $b(A_1 \cup A_2) = b(A_1) \cup b(A_2)$ for any two quasi closed sets A_1, A_2. Suppose further*(7) *that every set $A \subset X$ has an equivalence class of quasi closures A^*. The extension of the mapping b to arbitrary sets $A \subset X$ defined by*

$$b(A) = b(A^*)$$

is then an additive and idempotent mapping of $\mathfrak{P}(X)$ into $\mathfrak{P}(X)$. The

(7) Recall that this is the case if X has a countable base of open sets (in the initial topology) and if moreover C is sequentially order continuous from below (Theorem 2.8).

sets $A \subset X$ *such that* $b(A) \subset A$ *are the closed sets of a topology (termed "fine") on* X, *compatible with the quasi topology determined by the capacity* C *on* X. *For any set* $A \subset X$ *the set* $b(A) = b(A^*)$ *is the base of* A *relative to this topology in the sense of Def. 4.1, taking for* X_0 *the set of all points* $x \in X$ *such that* $C(\{x\}) = 0$.

Proof. – The equivalence class of quasi closures A^* of a set A depends only on the equivalence class of A. For any set $A \subset X$, $b(A) = b(A^*)$ is by definition equivalent to A^* and is hence itself a quasi closure of A. This implies that $b(b(A)) = b(A^*) = b(A)$. For any two sets A_1, $A_2 \subset X$ with quasi closures A_1^*, A_2^*, the union $A_1^* \cup A_2^*$ is clearly a quasi closure of $A_1 \cup A_2$, and

$$b(A_1 \cup A_2) = b(A_1^* \cup A_2^*) = b(A_1^*) \cup b(A_2^*) = b(A_1) \cup b(A_2) .$$

Thus the extended mapping $b : \mathfrak{P}(X) \longrightarrow \mathfrak{P}(X)$ is additive (in particular increasing) and idempotent. It is then clear that the sets $A \subset X$ such that $b(A) \subset A$ are indeed the closed sets in a certain topology on X which we now call the fine topology.

For any set $A \subset X$ the set $\widetilde{A} := A \cup b(A)$ is the fine closure of A. In fact, $\widetilde{A}$ is finely closed since

$$b(\widetilde{A}) = b(A) \cup b(b(A)) = b(A) \subset \widetilde{A} ;$$

and for any finely closed set $F \supset A$ we have $F \supset b(F) \supset b(A)$, hence $F \supset A \cup b(A) = \widetilde{A}$. Since $b(A)$ is a quasi closure of A, so is $\widetilde{A}$ (because $\widetilde{A} \backslash b(A) = A \backslash b(A)$, and $C(A \backslash b(A)) = 0$). This implies (T_1).

For any set $E \subset X$ such that $E \cap b(E) = \emptyset$ we have

$$C(E) = C(E \backslash b(E)) = 0 .$$

On the other hand, $C(E) = 0$, that is, $E \sim \emptyset$, implies $b(E) = b(\emptyset) = \emptyset$, and so we have established (T_2).

It remains to establish that $b(A)$ is the base of A in the sense of Def. 4.1. Suppose first that $x \in A \backslash X_0$. The quasi closure $b(A)$ of A quasi contains A and hence also $\{x\} \subset A$; and since $C(\{x\}) > 0$, $b(A)$ must actually contain x. Next suppose that, on the contrary, either $x \in \complement A$ or else $C(\{x\}) = 0$. In either case, $A \backslash \{x\}$ is equivalent to A, hence $b(A \backslash \{x\}) = b(A)$, and consequently

$$(A \backslash \{x\})^{\sim} = b(A \backslash \{x\}) \cup (A \backslash \{x\}) = b(A) \cup (A \backslash \{x\}) .$$

Since $x \notin A\backslash\{x\}$, we conclude (in the present case $x \notin A\backslash X_0$) that $x \in (A\backslash\{x\})^{\sim}$ if and only if $x \in b(A)$. ‖

4.7. The above lemma, combined with Theorem 4.3, exhibits a 1-1 correspondence between the topologies compatible with our quasi topology and the (additive) "liftings" for the lattice of all quasi closed sets. Those topologies which are finer than the initial topology on X correspond hereby to those liftings for which $b(A) \subset \overline{A}$.

It is an open problem whether such a lifting always exists, corresponding, say, to a given outer capacity C on a locally compact space X with a countable base of open sets, C being moreover sequentially order continuous from below.

In the final section of the present paper we shall consider a case (adapted to applications in potential theory) in which there exists an additive lifting b of quasi closed sets, or, equivalently, a topology compatible with the quasi topology. This new topology will be finer than the initial topology on X, and it will be completely regular (that is, separated and uniformizable) provided the initial topology is completely regular.

We close the present section by a discussion of the well known case of liftings associated with an outer Radon measure, in particular the case of Lebesgue measure.

4.8. *The case of an outer measure.* – Let μ denote a positive Radon measure on a locally compact space X with a countable base of open sets. The outer measure $\mu^* : \mathfrak{P}(X) \longrightarrow [0, +\infty]$ is an outer capacity on X, and sequentially order continuous from below. From Lusin's theorem (used, by Bourbaki, essentially as definition of measurability) follows that a function $f : X \longrightarrow [-\infty, +\infty]$ is quasi continuous with respect to μ^* if and only if f is μ-measurable. Any quasi semicontinuous function is quasi continuous, being equivalent to the limit of a sequence of semicontinuous, hence μ-measurable functions. In particular, *the quasi closed* (or quasi open) *sets are precisely the μ-measurable sets.* (The quasi compact sets are the μ-integrable sets, that is, the μ-measurable sets of finite outer measure).

Every set $A \subset X$ has an equivalence class of quasi closures, viz. those μ-measurable sets A^* for which $\mu^*(A\backslash A^*) = 0$ and $\mu_*(A^*\backslash A) = 0$, where μ_* denotes the inner measure (cf. § 2.11). If $\mu^*(A) < +\infty$, the latter condition may be replaced by $\mu(A^*) = \mu^*(A)$, a condition which is necessary in any case. In particular μ^* is *quasi stable* in the sense of § 2.4.

According to the existence theory for liftings on measure spaces (see Tulcea and Tulcea [22]) there always exists, in the present case, a lifting b assigning to each equivalence class of μ-measurable sets a unique representative of the class in such a way that $b(A) \subset \overline{A}$, $b(A_1 \cup A_2) = b(A_1) \cup b(A_2)$, $b(A_1 \cap A_2) = b(A_1) \cap b(A_2)$. (The latter "multiplicative" property is, however, irrelevant from our point of view). Consequently, Lemma 4.6 is applicable.

4.9. *Density topology and approximately continuous functions.* — In order to dispose of a *density theorem* let us restrict the attention to the case of *Lebesgue measure* μ on $X = \mathbf{R}^n$. The (outer) *upper density* of a set $A \subset \mathbf{R}^n$ at a point $x \in \mathbf{R}^n$ is defined by

$$\overline{d}(x\,,A) = \limsup \frac{\mu^*(A \cap Q)}{\mu(Q)}$$

as Q ranges over all open "intervals" in $\mathbf{R}^n$ containing x, the lim sup referring to the diameter of Q tending to 0. Here an open interval in $\mathbf{R}^n$ is understood as the product of n open intervals on $\mathbf{R}$. The *lower density* $\underline{d}(x\,,A)$ is defined similarly with lim inf in place of lim sup. Whenever $\overline{d}(x\,,A) = \underline{d}(x\,,A)$, the common value is called the *density* of A at x, and may be denoted by $d(x\,,A)$.

What we have defined here are, more precisely, the *strong* upper or lower density and the strong density, the qualification strong referring to the use of arbitrary intervals Q. If instead we allow *cubes* only (products of intervals on $\mathbf{R}$ of equal length) we obtain the *ordinary* upper and lower density, $\overline{d}_0(x\,,A)$ and $\underline{d}_0(x\,,A)$, and (when these are equal) the ordinary density $d_0(x\,,A)$.

The density theorem (cf. Saks [21]) asserts that $d(x\,,A)$ (hence $d_0(x\,,A)$) exists and equals 1 almost everywhere in A for any set $A \subset \mathbf{R}^n$, and that $d(x\,,A)$ (hence $d_0(x\,,A)$) exists and equals 0 almost everywhere in $\complement A$ if A is measurable.

We now define the (strong) *base* $b(A)$ of a set $A \subset \mathbf{R}^n$ by

$$b(A) = \{x \in \mathbf{R}^n \mid \overline{d}(x\,,A) > 0\}\,.$$

Clearly, $b(\mathrm{A}) \subset \bar{\mathrm{A}}$, and $b(\mathrm{A}_1 \cup \mathrm{A}_2) = b(\mathrm{A}_1) \cup b(\mathrm{A}_2)$ for any sets A, A_1, $\mathrm{A}_2 \subset \mathbf{R}^n$. Moreover, $b(\mathrm{A}) = b(\mathrm{A}^*)$ for any quasi closure A^* of A with respect to μ^*. (This follows from the identity

$$\mu^*(\mathrm{A} \cap \mathrm{Q}) = \mu(\mathrm{A}^* \cap \mathrm{Q}) \ ,$$

which in turn follows from $\mu_*(\mathrm{A}^* \backslash \mathrm{A}) = 0$). Thus Lemma 4.6 is again applicable. The corresponding topology on $\mathbf{R}^n$ (with the sets A such that $b(\mathrm{A}) \subset \mathrm{A}$ as closed sets) is called the (strong) *density topology.* It is strictly finer than the euclidean topology on $\mathbf{R}^n$. Since the density topology is compatible with the quasi topology associated with outer Lebesgue measure on $\mathbf{R}^n$, the closed sets in the density topology are quasi closed, that is (§ 4.8) measurable. The open sets in the density topology are accordingly those measurable sets $\mathrm{A} \subset \mathbf{R}^n$ for which $d(x, \mathrm{A}) = 1$ for every $x \in \mathrm{A}$.

It is also well known that the (strongly) *approximately continuous* (real valued) functions on $\mathbf{R}^n$ introduced by Denjoy (for $n = 1$) are precisely those functions which are continuous in the (strong) density topology. Referring to Saks [21, p. 131] we recall that a function f on $\mathbf{R}^n$ is called approximately continuous if for every $x_0 \in \mathbf{R}^n$ there exists a measurable set $\mathrm{A} \subset \mathbf{R}^n$ with $d(x_0, \mathrm{A}) = 1$ such that

$$f(x) \longrightarrow f(x_0)$$

as $x \longrightarrow x_0$ on A.

Replacing throughout the strong density $d(x, \mathrm{A})$, etc., by the ordinary density $d_0(x, \mathrm{A})$, we obtain the *ordinary base*

$$b_0(\mathrm{A}) = \{x \in \mathbf{R}^n \mid \bar{d}_0(x, \mathrm{A}) > 0\}$$

of sets $\mathrm{A} \subset \mathbf{R}^n$, and the corresponding *ordinary density topology* on $\mathbf{R}^n$, again compatible with the quasi topology associated with μ^*. All that was said above carries over mutatis mutandis. In particular, the continuous functions in the ordinary density topology are the "ordinary" approximately continuous functions.

The ordinary density topology on $\mathbf{R}^n$ is strictly finer than the strong density topology (except of course for $n = 1$ where $d_0 = d$). A rather deep result of Goffman, Neugebauer, and Nishiura [19] asserts that the *ordinary* density topology on $\mathbf{R}^n$ is *completely regular,* but not normal, whereas the strong density topology is not even regular for $n > 1$. It is also known (contained implicitly in Ridder [20]) that every connected open subset of $\mathbf{R}^n$ is connected in the ordinary density topology, hence also in the strong density topology. Of course, neither of the two density topologies is locally compact or first countable (follows from Theorem 4.2 above). – Finally, let us remark that Theorem 4.5, or rather the principal step in the proof of it, was established by Zink [23] in the case of the strong density topology.

5. The fine topology determined by a cone of lower semicontinuous functions.

In this section we consider, following Brelot [2], [6], a *convex cone* $\mathcal{U}$ of l.s.c. functions u defined on a topological space X and with values in $[0, +\infty]$. Such a cone $\mathcal{U}$ determines a new topology on X (finer than the given topology) as defined by Cartan [7] in case of classical potential theory, where $\mathcal{U}$ is the cone of superharmonic functions. Subsequently we shall obtain sufficient conditions for this new topology, the fine topology on X, to be compatible with the quasi topology associated with a given capacity C on X in the sense of Def. 4.3.

5.1. DEFINITION. – *The fine topology on* X, *determined by the cone* $\mathcal{U}$, *is the coarsest topology on* X, *finer than the initial topology, such that all the functions* $u \in \mathcal{U}$ *are continuous.*

THEOREM. – *If the initial topology on* X *is 1) separated, 2) regular, or 3) completely regular, then so is the fine topology. If* X *is locally compact in the initial topology, then* X *is a Baire space also in the fine topology.*

Proof. – The fine topology is defined as the coarsest topology on X such that the following mappings of the set X are continuous : The identity mapping of X into the topological space X with the given (initial) topology, and further each of the mappings $u \in \mathcal{U}$ of X into $[0, +\infty]$ (with the usual topology). In view of this, the former part of the theorem reduces to well known facts from general topology. And moreover, if X is locally compact (in the initial topology), then each point $x \in X$ has a fundamental system of fine neighbourhoods each of which is compact in the initial topology. From this latter property (together with the fact that, by hypothesis, X is separated in the initial topology) follows, however, easily that X is a Baire space in the fine topology (by use of the well known method of proof that any locally compact space is a Baire space(8). ▯

5.2. In addition to the cone $\mathfrak{U}$, let a subset X_0 of X be given. As in § § 4.1 and 4.2 the points of X_0 are called *polar* points(9). We then dispose of the concepts introduced there. In particular, the base of a set $A \subset X$, and more generally of a function $f : X \longrightarrow [0, +\infty]$ is defined by

$$x \in b(A) \Longleftrightarrow \begin{cases} x \in (A \backslash \{x\})^{\sim} & \text{if } x \in X_0 , \\ x \in \widetilde{A} & \text{if } x \in \complement X_0 , \end{cases}$$

$$b(f)(x) = \begin{cases} \text{fine} \limsup\limits_{y \to x ,\, y \neq x} f(y) & \text{if } x \in X_0 , \\ \text{fine} \limsup\limits_{y \to x} f(y) & \text{if } x \in \complement X_0 , \end{cases}$$

Here $\widetilde{A}$ denotes the fine closure of a set $A \subset X$. Similarly we denote by $\widetilde{f}$ the finely u.s.c. envelope of f, defined by $\widetilde{f}(x) = \text{fine} \limsup f(y)$ as $y \longrightarrow x$.

5.3. *The reduced functions* R_f *and* R_f^A. – Following Brelot [2], [6], we associate with any function $f : X \longrightarrow [0, +\infty]$ the reduced function $R_f : X \longrightarrow [0, +\infty]$ defined by

$$R_f(x) = \inf\{u(x) \mid u \in \mathfrak{U} , u \geqslant f\} ,$$

interpreted as $+\infty$ if no such u exists. And for any set $A \subset X$ we write $R_f^A = R_{f \cdot 1_A}$, that is

(8) The author owes this latter observation to C. Berg. Also the former part of the theorem is known (even in a slightly amplified form), see Brelot [6, ch. I,3].

(9) From § 5.5, X_0 will denote the set of all points x such that $C(\{x\}) = 0$, just as in § 4.3 – § 4.6. In most applications (cf. § 5.6 and § 5.7) the polar points are likewise those forming a *polar set*, that is a set E such that there exists $u \in \mathfrak{U}$ with $u = +\infty$ everywhere in E.

$$R_f^A(x) = \inf\{u(x) \mid u \in \mathcal{U}\ ,\ u \geqslant f \text{ in } A\}\ .$$

Clearly $R_f \geqslant f$. For every $x \in X$ the mapping $f \longrightarrow R_f(x)$ is increasing and sublinear.

Since the functions $u \in \mathcal{U}$ are finely continuous, R_f is finely u.s.c., and we have $R_{\tilde{f}} = R_f$. If f is finely l.s.c. then $R_f^{\tilde{A}} = R_f^A$ for any set A.

5.4. *Thinness of a set.* – According to Brelot [2], [6] a set $A \subset X$ is said to be *thin* (or effilé) at a point $x \in \complement A$ if there exists $u \in \mathcal{U}$ such that

$$u(x) < \liminf u(y) \quad \text{as} \quad y \longrightarrow x,\ y \in A\ .$$

Thus A is thin at $x \in \complement A$ if and only if there exists a neighbourhood V of x (in the initial topology on X) with the property that

$$R_1^{A \cap V}(x) < 1\ ,$$

or equivalently that there exists $u \in \mathcal{U}$ such that $u \geqslant 1$ on $A \cap V$, and $u(x) < 1$. In the affirmative case such a neighbourhood V can always be chosen from a given base of neighbourhoods of X.

A set $A \subset X$ is called thin at a point $x \in A$ if $A \setminus \{x\}$ is thin at x according to the above definition, and if moreover x is a *polar* point, that is $x \in X_0$.

For any set $A \subset X$ we denote by $e(A)$ the set of all points $x \in X$ at which A is thin.

THEOREM (Cartan). – *For any set* $A \subset X$ *the base of* A *consists of all points of* X *at which* A *is not thin :* $b(A) = \complement e(A)$. *In particular the fine neighbourhoods of a point* $x \in X$ *are precisely the complements of those sets* $A \subset X$ *which are thin at* x *and do not contain that point.*

Proof. – As to the latter assertion see Cartan [7, § 26] (formulated for the classical newtonian case), or Brelot [2, th. 4.5], [6, th. I.3]. As to the former assertion, let first $x \in \complement A$. Then $[x \in b(A)] \Longleftrightarrow [x \in \tilde{A}] \Longleftrightarrow [\complement A$ is not a fine neighbourhood of $x] \Longleftrightarrow$ [A is not thin at x]. Next let $x \in A$. If x is polar, then $[x \in b(A)] \Longleftrightarrow [x \in (A \setminus x\ )^{\sim} \Longleftrightarrow [A \setminus \{x\}$ not thin at $x] \Longleftrightarrow$ [A not thin at

x]. Finally a non polar point $x \in A$ is, by definition, in $b(A)$, but not in $e(A)$. ▯

5.5. *Sufficient conditions for compatibility between the fine topology and the quasi topology.* – In addition to the convex cone $\mathfrak{U}$ of l.s.c. functions $u : X \longrightarrow [0, +\infty]$ we now consider a *capacity* C on X (in the sense of Def. 1.1). As in § § 4.3 and 4.4, we take foı *polar* points the points $x \in X$ of capacity 0 :

$$x \in X_0 \Longleftrightarrow C(\{x\}) = 0 .$$

According to Def. 4.3, the fine topology on X, determined by the cone $\mathfrak{U}$, is compatible with the quasi topology determined by C if and only if the following 4 conditions are satisfied :

i) Every quasi closed set $A \subset X$ is equivalent to some finely closed set.

ii) Every set $E \subset X$ of capacity 0 is everywhere thin :

$$[C(E) = 0] \Longrightarrow [b(E) = \emptyset] .$$

iii) Every set $E \subset X$, thin at each of its points, has capacity 0 : $[E \cap b(E) = \emptyset] \Longrightarrow [C(E) = 0]$.

iv) Every finely closed set $A \subset X$ is quasi closed.

We shall now obtain sufficient conditions. For any function $f : X \longrightarrow [0, +\infty]$ denote by $\hat{f}$ the *l.s.c. envelope* of f in the initial topology (the largest l.s.c. function $\leqslant f$). In particular, $\hat{R}_f$ denotes the l.s.c. envelope of R_f, cf. § 5.3.

THEOREM. – *Suppose that* X *has a countable base of open sets (in the initial topology), and that* C *is sequentially order continuous from below in the sense of (5),* § 2.7. *The fine topology determined by the cone* $\mathfrak{U}$ *is compatible with the quasi topology determined by the capacity* C *provided that the following 4 axioms are fulfilled :*

I) C is finely stable : $C(\widetilde{A}) = C(A)$ for every $A \subset X$.

II) Every set $E \subset X$ of capacity 0 is finely closed :

$$[C(E) = 0] \Longrightarrow [b(E) \subset E] .$$

III) For any function $f : X \longrightarrow [0, +\infty]$, $\hat{R}_f = R_f$ quasi everywhere in X_0 .

IV) Every function $u \in \mathfrak{U}$ is quasi continuous.

Remarks. – 1) The axiom I is fulfilled, e.g., if $C(A)$ depends on $\{u \in \mathfrak{U} \mid u(x) \geqslant 1$ for every $x \in A\}$ only. (The relations $u \geqslant 1$ in A and $u \geqslant 1$ in $\widetilde{A}$ are, in fact, equivalent on account of the fine continuity of $u \in \mathfrak{U}$).

2) The axiom II is fulfilled, e.g., if $C(E) = 0$ implies $R_1^E < 1$ everywhere in $\complement E$. (For then $E = \{x \in X \mid R_1^E(x) \geqslant 1\}$ is finely closed because R_1^E is finely u.s.c.). In our applications, $C(E) = 0$ even implies $R_{+\infty}^E = 0$ in $\complement E$.

3) When applied to the pointwise infimum f of an arbitrary family (u_α) of functions $u_\alpha \in \mathfrak{U}$, the axiom III implies that

$$\left(\inf_\alpha u_\alpha\right)^{\widehat{}} = \inf_\alpha u_\alpha \quad \text{q.e. in } X_0 . \tag{19}$$

Conversely, this property implies III when applied to the family of all functions $u \in \mathfrak{U}$ such that $u \geqslant f$. When X has a countable base of open sets, it suffices to verify (19) for arbitrary *countable* families of functions from $\mathfrak{U}$. This follows from a topological lemma due to Choquet (cf. Brelot [1, p. 6]).

3′) For some applications it is important that the axiom III can be replaced by the following weaker axiom III′ without affecting the validity of the above theorem :

III′) There exists a constant $k \geqslant 1$ with the following 2 properties (α) and (β) :

(α) For any sequence (u_n) of functions $u_n \in \mathfrak{U}$

$$k \cdot \left(\inf_n u_n\right)^{\widehat{}} \geqslant \inf_n u_n \quad \text{q.e. in } X_0 .$$

(β) For any set $A \subset X$, thin at a polar point $x \in \complement A$, there exists $u \in \mathfrak{U}$ such that

$$k \cdot u(x) < \liminf u(y) \quad \text{as} \quad y \longrightarrow x,\ y \in A .$$

Note that, for $k = 1$, the property (β) becomes redundant by the definition of thinness (§ 5.4), while (α) reduces to the axiom III provided that X has a countable base (cf. the preceding remark). Actually, in our applications, (β) holds for any finite k, the thinness at a polar point being "strong" in the sense of Brelot [2, p. 79].

Proof of the above theorem in the sharper from indicated in the last remark. – We divide the proof into 4 steps, one for each of the properties i)–iv) to be established. In each step the hypotheses used are specified.

Ad i). *The axiom I alone implies i).* Suppose that $A \subset X$ is quasi closed. For any $\varepsilon > 0$ there exists a closed set $F \subset X$ such that $C(A \,\Delta\, F) < \varepsilon$. Since the fine topology determined by the cone $\mathfrak{U}$ is finer than the initial topology on X, F is finely closed. Writing $E := A \,\Delta\, F$, hence $A \subset F \cup E$, we obtain

$$\widetilde{A} \subset (F \cup E)^{\sim} \subset F \cup \widetilde{E} = A \cup \widetilde{E} \,,$$

$$C(\widetilde{A} \backslash A) \leqslant C(\widetilde{E}) = C(E) < \varepsilon \,,$$

and so A is equivalent to its fine closure $\widetilde{A}$.(10)

Ad ii). *The axiom II alone is equivalent to ii).* Suppose that $C(E) = 0$. Then every point x of E is polar (that is, $C(\{x\}) = 0$) and finely isolated because $E\backslash\{x\}$ is finely closed in view of II applied to the set $E\backslash\{x\}$. The converse is obvious.

Ad iii). When X has a countable base $(\omega_n)_{n \in N}$, *the axiom III′ implies iii).* Suppose that $E \subset X$ is thin at each of its points, that is, $E \cap b(E) = \emptyset$. In particular, every point of E is polar, that is, $E \subset X_0$. Writing $E_{n,p} = \{x \in E \cap \omega_n \mid (k + 1/p)\, R_1^{E \cap \omega_n \backslash \{x\}}(x) < 1\}$, we have $E = \cup E_{n,p}$ on account of (β) in the axiom III′ (cf. Remark 3′ above). In order to prove that $C(E) = 0$ it suffices, therefore, to prove that $C(E_{n,p}) = 0$ for any pair of natural numbers n, p. Let S denote a countable, dense subset of such a set $E_{n,p}$. For each $s \in S$ there exists $u_s \in \mathfrak{U}$ such that $(k + 1/p)\, u_s(s) < 1$, and $u_s \geqslant 1$ in $E \cap \omega_n \backslash \{s\}$, a fortiori in $E_{n,p} \backslash S$. Writing $f := \inf\{u_s \mid s \in S\}$ (pointwise), we thus have $(k + 1/p)\, f < 1$ in S, and $f \geqslant 1$ in $E_{n,p}\backslash S$. On account of (α) in the axiom III′, this latter inequality implies that $k\hat{f} \geqslant 1$ q.e. in $E_{n,p}\backslash S$. On the other hand, $(k + 1/p)\, \hat{f} \leqslant (k + 1/p)\, f < 1$ in S implies

(10) The fine stability of C (axiom I) alone implies, more generally that any quasi continuous function $f : X \longrightarrow Y$ is finely continuous quasi everywhere, see Brelot [6, th. IV, 3]. This should be compared with Theorem 4.3 a) above, where Y was supposed to have a countable base, but where the remaining hypotheses used were weaker, viz. condition i) above together with the implication $[C(E) = 0] \Longrightarrow [C(\widetilde{E}) = 0]$ which is likewise a consequence of the axiom I.

$(k + 1/p)\hat{f} \leqslant 1$ in the closure $\overline{S} \supset E_{n,p}$ because $\hat{f}$ is l.s.c. This leads to a contradiction unless $C(E_{n,p} \setminus S) = 0$ and hence $C(E_{n,p}) = 0$. We have used here that $C(S) = 0$ because S is a countable subset of E, and every point $s \in E$ is polar, $C(\{s\}) = 0$.

Ad iv). Suppose that X has a countable base, and that C is sequentially order continuous from below. *Then II,III and IV* (or just II, iii), IV) *together imply iv).* We begin by proving that R_f is quasi u.s.c. for every function $f : X \longrightarrow [0, +\infty]$. According to Lemma 3.3, this amounts to showing that the set

$$A := \{x \in X \mid R_f(x) \geqslant t\}$$

is quasi closed for every real t. Let $\mathcal{H}$ denote the class of all sets

$$H_u = \{x \in X \mid u(x) \geqslant t\},$$

where $u \in \mathcal{U}$ and $u \geqslant f$. According to the axiom IV these sets H_u are quasi closed, and it follows therefore from Theorem 2.7 that there exists a set $H \in \mathcal{H}_\delta$ (that is, the intersection of a sequence of sets of class $\mathcal{H}$) such that H is quasi contained in each set from $\mathcal{H}_\delta$, in particular in each set $H_u \in \mathcal{H}$: $C(H \setminus H_u) = 0$. In view of ii) (which was shown above to be equivalent to II), this implies that $b(H \setminus H_u) = \emptyset$, and hence $b(H) \subset b(H_u)$ by the additivity of the base operation. Since each H_u is finely closed, $b(H_u) \subset H_u$, and consequently

$$b(H) \subset \cap\{H_u \mid u \in \mathcal{U}\ ,\ u \geqslant f\} = A \subset H\ .$$

The set $E = H \setminus b(H)$ is thin at each of its points, hence $C(E) = 0$ according to iii), and we conclude that A is quasi closed, being equivalent to the quasi closed set H.

Finally we show that the quasi upper semicontinuity of R_f for all functions $f : X \longrightarrow [0, +\infty]$ (or just for indicator functions) implies iv) when X has a countable base (ω_n). Thus let A denote a finely closed set, that is, $b(A) \subset A$, or in other terms, A is thin at every point of $\complement A$. It follows that

$$\complement A = \bigcup_n (\omega_n \cap \{x \in X \mid R_1^{A \cap \omega_n}(x) < 1\}),$$

Being thus a countable union of quasi open sets, $\complement A$ is quasi open, that is, A is quasi closed. ‖

5.6. *Example.* – (Classical potential theory. Superharmonic functions). For X we take a connected open subset of $\mathbf{R}^n$, $n \geq 3$ (or more generally a Green space in the sense of Brelot and Choquet). For $\mathfrak{U}$ take the convex cone of all *superharmonic* functions $u : X \longrightarrow [0, +\infty]$. A set $E \subset X$ is called *polar* if it has one of the following properties, which are known to be equivalent for any choice of a function $f : X \longrightarrow]0, +\infty]$:

a) There exists $u \in \mathfrak{U}$ such that $u = +\infty$ everywhere in E.

b) $R_f^E(x) = 0$ for some, and hence for any $x \in \complement E$.

c) $\hat{R}_f^E(x) = 0$ for some, and hence for any $x \in X$.

Any countable union of polar sets is polar.

Adjoining to $\mathfrak{U}$ the constant function $+\infty$ we obtain the class $\mathfrak{U}_\infty = \mathfrak{U} \cup \{\infty\}$ of all *hyperharmonic* functions ≥ 0. With the natural (pointwise) order $\mathfrak{U}_\infty$ is known to be a complete lattice in which any family contains a countable subfamily with the same infimum. The infimum $\widehat{\inf_\alpha}\, u_\alpha$ of any family (u_α) in the lattice $\mathfrak{U}_\infty$ is the l.s.c. envelope (and also the base, cf. § 4.4) of the pointwise infimum $\inf_\alpha u_\alpha$, from which it differs at most on some polar set. For any function $f : X \longrightarrow [0, +\infty]$ we thus have $\hat{R}_f = R_f$ except in some polar set (contained in $\{x \in X \mid f(x) > 0\}$). It is easily shown that $R_f = f \vee \hat{R}_f$. Moreover,

$$R_f = \sup_n R_{f_n} \quad , \quad \hat{R}_f = \sup_n \hat{R}_{f_n}$$

for any increasing sequence of functions $f_n : X \longrightarrow [0, +\infty]$ with the pointwise supremum f.

Following Brelot [4], [5] we now consider the following set functions in which f denotes a fixed, strictly positive, finite and continuous (to simplify) function on X, x_0 a fixed point of X, and m a positive Radon measure ($\neq 0$) on X :

1) $C(A) = \hat{R}_f^A$ (11) .

2) $C(A) = R_f^A(x_0)$.

(11) Thus, in case 1, the set function C has its values in the lattice $\mathfrak{U}_\infty$ of hyperharmonic functions ≥ 0 on X (cf. § 1.6).

3) $C(A) = \int \hat{R}_f^A \, dm$.

4) $C(A)$ = the outer greenian capacity of A([12]).

Each of these set functions is a capacity in the sense of Def. 1.1, sequentially order continuous from below, finely stable ($C(\widetilde{A}) = C(A)$), vanishing precisely on the polar sets (not containing x_0 in case 2).

The important property discovered by Choquet [10] for the greenian capacity (cf. the corollary to Theorem 4.3 above) was extended by Brelot [5] to the remaining cases 1, 2, 3 which serve to replace the greenian capacity in the axiomatic theory of harmonic functions([13]). The Choquet property is the crucial point in establishing the compatibility (in the sense of Def. 4.3 above) between the fine topology (associated with the cone $\mathcal{U}$ of superharmonic functions $\geqslant 0$) and the quasi topology determined by any one of the above 4 capacities. The axioms I, II, III, IV of Theorem 5.5 are likewise satisfied here, but IV plays a secondary role in the present situation([14]).

In the cases 1, 2, and 4, C is an *outer* capacity (Def. 1.5), more precisely C equals the outer capacity c^* associated with the restriction c of C to compact sets (cf. § 2.11). The same holds in case 3 if m does not charge the polar sets and if $\int \hat{R}_f \, dm < +\infty$.

([12]) This capacity is the total mass of the positive measure λ on X whose greenian potential $G\lambda$ (defined in terms of the Green kernel G on X by

$$G\lambda(y) = \int G(x, y) \, d\lambda(x))$$

equals $\hat{R}_1^A$. (If the latter is not a potential we put $C(A) = +\infty$).

([13]) In case 3 it is assumed in [5] that m does not charge the polar sets. This restriction can be removed on account of a theory of finely harmonic functions which we shall develop elsewhere. – Note that, in case 2, the set $\{x_0\}$ has capacity > 0, and hence x_0 should be considered as a non polar point in the definition of the associated base operation and notion of thinness (§ 4.3), even if x_0 is polar in the usual sense.

([14]) In case 2 the relation $\hat{R}_f = R_f$ in the axiom III, though valid except in some polar set, need not hold quasi everywhere in all of X with respect to the present C because it may fail to hold at the point x_0 if this point is polar in the usual sense (cf. the preceding note). But it does hold q.e. in X_0 (as required in III) because $X_0 = \{x \in X \mid C(\{x\}) = 0\}$ does not contain x_0 in this case. In the cases 1,3, and 4 we have $\hat{R}_f = R_f$ q.e. in X.

If $\hat{R}_f$ is a *potential*, the whole space X is *quasi compact* (Def. 2.1), and so is therefore any quasi closed set (in particular any finely closed set)([15]).

If f is *superharmonic*, each of the four capacities is alternating of infinite order, in particular strongly subadditive (cf. Choquet [8]) :

$$C(A \cup B) + C(A \cap B) \leqslant C(A) + C(B) \ , \quad A, B \subset X \ .$$

With the fine topology determined by the cone $\mathcal{U}$ of all superharmonic functions $X \longrightarrow [0, +\infty]$, the space X is known to be a completely regular Baire space (follows from Theorem 5.1), but neither locally compact nor first countable (follows from Theorem 4.2). Furthermore, X is *connected* and *locally connected* in the fine topology. The complement of any polar set is finely connected ; in other words, any set $A \subset X$ whose fine boundary is polar, is either polar or the complement of a polar set. For these and other properties of connectivity of the fine topology in the present case see Fuglede [17].

In view of the compatibility between the fine topology and the quasi topology associated with any one of the above capacities, the latter result amounts to saying that the space X is *quasi connected* in the sense that any set $A \subset X$ which is both quasi open and quasi closed must be of capacity 0 or the complement of a set of capacity 0([16]).

Referring to Brelot [5] we observe that, in the cases 1, 2, and 3 with f finite and continuous, the Green space X may be replaced more generally by any harmonic space in the sense of Brelot [3], having a countable base of open sets, satisfying the axioms 1, 2, 3, and D, and further admitting a strictly positive potential. The Green capacity (case 4) may be generalized in another direction (at least for X without irregular boundary points), see Example 5.7 below.

([15]) In case 3 we suppose at this point that m has compact support (or just that $\int \hat{R}_f \, dm < +\infty$). In case 4 the Green space X should be without irregular boundary points.

([16]) In this respect the quasi topologies of potential theory are quite different from the quasi topology associated with an outer Radon measure μ^*, in which case the quasi open and the quasi closed sets are the *same*, viz., the μ-measurable sets, as noted in § 4.8. (We assume that the space X is locally compact with a countable base).

It can be shown, e.g. by application of the Wiener criterion, that the fine topology of classical potential theory, say in the case where X denotes an open subset of $\mathbf{R}^n$ (of non polar complement if $n = 2$), is strictly coarser than the *ordinary density topology* on $\mathbf{R}^n$ (§ 4.9), see [17]. This implies that every finely continuous function is "ordinary" approximately continuous and hence of Baire class 1 in the euclidean topology. It follows that the fine topology is *not normal*([17]).

5.7. *Example.* – (Potentials and capacity with respect to a kernel). Let X be a locally compact Hausdorff space, and G a *kernel* on X, that is, a l.s.c. mapping of $X \times X$ into $[0, +\infty]$. Denoting by $\mathfrak{M}^+$ the set of all positive Radon measures on X, and by $\mathfrak{M}^+_\infty$ the set of all bounded measures in $\mathfrak{M}^+$, we define the potential $G\mu$ of $\mu \in \mathfrak{M}^+$ and the adjoint potential $\check{G}\lambda$ (where $\check{G}(y, x) = G(x, y)$) of $\lambda \in \mathfrak{M}^+_\infty$ by

$$G\mu(x) = \int G(x, y)\, d\mu(y) \ , \quad \check{G}\lambda(y) = \int G(x, y)\, d\lambda(x) \ ,$$

for all $x \in X$, resp. $y \in X$. The G-*capacity* of a compact set $K \subset X$ is defined by

$$\begin{aligned} c(K) : &= \sup\{\mu(X) \mid \mu \in \mathfrak{M}^+ \ , \ G\mu \leqslant 1 \ , \ \mu(\complement K) = 0\} \\ &= \inf\{\lambda(X) \mid \lambda \in \mathfrak{M}^+_\infty \ , \ \check{G}\lambda \geqslant 1 \text{ in } K\} \ , \end{aligned}$$

the equality between the 2 expressions for $c(K)$ resulting from a well known extension of v. Neumann's minimax theorem. The associated inner, resp. outer, G-capacity of an arbitrary set $A \subset X$ is denoted by $c_*(A)$, resp. $c^*(A)$ (cf. § 2.11).

We now make the following *basic assumptions* on the kernel G :

1) $G(x, y)$ should for $x \neq y$ be finite and continuous.

2) $G(x, y)$ should tend to 0 as one of the variables x or y goes

([17]) Proceeding as in the analogous case of the ordinary density topology (Goffman, Neugebauer and Nishiura [19]), we merely choose 2 disjoint countable (hence polar, hence finely closed) sets A, $B \subset X$ each of which is dense in X in the euclidean topology. A finely continuous function $f : X \longrightarrow [0,1]$ with $f(A) = \{0\}$, $f(B) = \{1\}$ would have to be discontinuous everywhere in X in the euclidian topology, which is impossible since f is of Baire class 1. – It is, however, easily shown that any finely continuous function defined on a *base* can be extended to a finely continuous function in the whole space. As to the question of normality of the fine topology in more general cases than here, see a paper by C. Berg to appear in Bull. Sci. Math.

to infinity in X, uniformly with respect to the other variable on compact sets.

3) G and the adjoint kernel $\check{G}$ should satisfy the *continuity principle.*

Under these basic hypotheses $\check{G}\lambda$ is, for every $\lambda \in \mathfrak{M}_\infty^+$, quasi continuous and has the quasi limit 0 at infinity in X (cf. § 3.6). For this result, essentially due to Choquet, see Fuglede [13], where also the following results were obtained : For any set $A \subset X$,

$$c^*(A) = \inf\{\lambda(X) \mid \lambda \in \mathfrak{M}_\infty^+ , \check{G}\lambda \geqslant 1 \text{ q.e. in } A\} ,$$

the q.e. referring to c^*. If $c^*(A) < +\infty$, this infimum is attained. Furthermore, c^* is sequentially order continuous from below. The quasi compact sets are precisely the quasi closed sets of finite outer capacity.

In the sequel we suppose, moreover, that the locally compact space X has a *countable base* of open sets. It can then be shown that the restriction "q.e." in the above expression for $c^*(A)$ can be dropped for any set A at the expense that the infimum is no longer attained in general. The sets $E \subset X$ such that $c^*(E) = 0$ are precisely the polar sets, that is, the sets E such that there exists $\lambda \in \mathfrak{M}_\infty^+$ with $\check{G}\lambda = +\infty$ everywhere in E. And the polar points $x \in X$ are characterized by $G(x, x) = +\infty$.

For $\mathfrak{U}$ we now take the convex cone of all functions representable as the (adjoint) potential $\check{G}\lambda$ of a bounded measure $\lambda \in \mathfrak{M}_\infty^+$, and for C the outer G-capacity c^*. It is well known that $R_{+\infty}^E = 0$ in $\complement E$ for any polar set E, and that the thinness is always strong at any polar point (in the sense that (β), Remark 3′, § 5.5, holds for any finite k), see Brelot [2]. In view of Remarks 1 and 2, § 5.5, it now follows that the axioms I, II and IV are fulfilled under our basic hypotheses 1, 2, and 3 on G (provided that X has a countable base of open sets).

The remaining axiom III (resp. III′) will be fulfilled (even with X in place of X_0) if we further assume that G satisfies the *domination principle* (resp. the k-dilated domination principle for some constant $k \in [1, +\infty[$, which may then be used in (α) of III′), and if, in addition, G satisfies the k'-dilated maximum principle for some $k' \in [1, +\infty[$. In the former case (of the actual domination principle),

$\hat{R}_f$ is, for any function $f : X \longrightarrow [0, +\infty]$ such that $\hat{R}_{\check{f}} \not\equiv +\infty$, of class $\mathcal{U}$, and in fact the smallest among all potentials $\check{G}\lambda$ (with $\lambda \in \mathfrak{M}_\infty^+$) such that $\check{G}\lambda \geqslant f$ q.e.

These results (for $k = 1$) are due to Brelot [2, § 14] under more restrictive conditions than here, see also Fuglede [13] for the case $k \geqslant 1$. As a specific example recall that the (symmetric) *Riesz kernel* G_α of order α, $0 < \alpha < n$, on $X = \mathbf{R}^n$, defined by

$$G_\alpha(x, y) = |x - y|^{\alpha - n}$$

(interpreted as $+\infty$ for $x = y$), satisfies the domination principle and the maximum principle (and hence all the axioms I, II, III, IV) for $0 < \alpha \leqslant 2$, but only the dilated forms of these 2 principles (hence the axioms I, II, III', IV) if $2 < \alpha < n$.

Returning to the case of a general kernel G (satisfying our basic hypotheses 1, 2, 3 on a locally compact space X with a countable base), we finally remark that if we merely want to ensure that the property iii), § 5.5, holds – and hence that the fine topology determined by our cone $\mathcal{U}$ is compatible with the quasi topology determined by the outer G-capacity c^* – then it suffices to assume that quasi every point $x \in X$ has a compact neighbourhood V_x such that the restriction of G to $V_x \times V_x$ satisfies the k-dilated domination principle for some k which may depend on x,([18]) cf. Fuglede [13] and a forthcoming detailed exposition.

BIBLIOGRAPHY

[1] M. Brelot, Eléments de la théorie classique du potentiel. Les cours de Sorbonne. C.D.U., Paris (4e édit. 1969).

[2] M. Brelot, Introduction axiomatique de l'effilement, *Annali di Mat.* 57 (1962), 77-96.

([18]) Whether this rather weak, local form of the dilated domination principle is fulfilled by every kernel G satisfying our basic assumptions seems to be an open question (even in the case G symmetric and satisfying the maximum principle, and X compact and with a countable base).

[3] M. BRELOT, Axiomatique des fonctions harmoniques. Séminaire de mathématiques supérieures, Montréal (1966).

[4] M. BRELOT, Capacity and balayage for decreasing sets. Symposium on Statistics and Probability, Berkeley (1965).

[5] M. BRELOT, Recherches axiomatiques sur un théorème de Choquet concernant l'effilement, *Nagoya Math. Journal* 30 (1967), 39-46.

[6] M. BRELOT, On topologies and boundaries in potential theory. (Lectures in Bombay, 1966), Lecture Notes in Math., n° 175. Berlin (1970).

[7] H. CARTAN, Théorie générale du balayage en potentiel newtonien, *Ann. Univ. Grenoble* 22 (1946), 221-280.

[8] G. CHOQUET, Theory of capacities, *Ann. Inst. Fourier* 5 (1953), 131-295.

[9] G. CHOQUET, Forme abstraite du théorème de capacitabilité, *Ann. Inst. Fourier* 9 (1959), 83-89.

[10] G. CHOQUET, Sur les points d'effilement d'un ensemble, *Ann. Inst. Fourier* 9 (1959), 91-101.

[11] G. CHOQUET, Démonstration non probabiliste d'un théorème de Getoor, *Ann. Inst. Fourier* 15, fasc. 2 (1965), 409-414.

[12] J.L. DOOB, Applications to analysis of a topological definition of smallness of a set, *Bull. Amer. Math. Soc.* 72 (1966), 579-600.

[13] B. FUGLEDE, Le théorème du minimax et la théorie fine du potentiel, *Ann. Inst. Fourier* 15, fasc. 1 (1965), 65-88.

[14] B. FUGLEDE, Esquisse d'une théorie axiomatique de l'effilement et de la capacité, *C.R. Acad. Sci.* Paris 261, (1965), 3272-3274.

[15] B. FUGLEDE, Quasi topology and fine topology. Sém. Brelot-Choquet-Deny : Théorie du potentiel, 10 (1965-66), n° 12.

[16] B. FUGLEDE, Applications du théorème minimax à l'étude de diverses capacités, *C.R. Acad. Sci.* Paris 266 (1968), 921-923.

[17] B. FUGLEDE, Propriétés de connexion en topologie fine. Prépublication, Copenhague (1969).

[18] R.K. GETOOR, Additive functionals of a Markov process. (Lectures), Hamburg (1964).

[19] C. GOFFMAN, C.J. NEUGEBAUER and T. NISHIURA, Density topology and approximate continuity, *Duke Math. J.* 28 (1961), 497-505.

[20] J. RIDDER, Über approximativ stetigen Funktionen, *Fund. Math.* 13 (1929), 201-209.

[21] S. SAKS, Theory of the Integral, Warsaw (1937).

[22] A. Ionescu TULCEA and C. Ionescu TULCEA, Topics in the Theory of Lifting, Erg. Math. 48 (1969).

[23] R.E. ZINK, On semicontinuous functions and Baire functions, *Trans. Amer. Math. Soc.* 117 (1965), 1-9.

Manuscrit reçu le 1er juillet 1970

Bent FUGLEDE
Universitetets Matematiske Institut
Oersted Institutet, Universitetsparken 5
Copenhagen, Denmark