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Annales de l'institut Fourier, tome 20, n° 2 (1970), p. 45-54

http://www.numdam.org/item?id=AIF_1970__20_2_45_0

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THEOREMS OF KREIN-MILMAN TYPE FOR CERTAIN CONVEX SETS OF FUNCTIONS AND OPERATORS

by Robert R. PHELPS

Let X be a compact Hausdorff space and E a real [or complex] locally convex Hausdorff vector space. Denote by $C(X, E)$ the real [or complex] linear space of all continuous functions from X to E , provided with the topology of uniform convergence. (Thus, a typical neighborhood of 0 has the form

$$\{f \in C(X, E) : \sup_x p(f(x)) \leq 1\}$$

where p is a continuous seminorm on E .)

For any subset A of E we let

$$C(X, A) = \{f \in C(X, E) : f(X) \subset A\}.$$

It follows that if B is a bounded closed convex subset of E , then $C(X, B)$ is a bounded closed convex subset of $C(X, E)$. We denote by $\text{ext } S$ the set of extreme points of a given convex set S ; it is readily verified that

$$C(X, \text{ext } B) \subset \text{ext } C(X, B).$$

[It is known [2, p. 755] that this inclusion can be proper, even for four dimensional E . There are also examples where $\text{ext } C(X, B)$ is empty for every X ; for instance, if $E = c_0$ in the norm topology and B is its unit ball.] The main purpose of this note is to exhibit conditions under which the set $C(X, B)$ will be the closed convex hull $\overline{\text{co}} C(X, \text{ext } B)$ of this subset of extreme points.

[Note that, even for one dimensional B , the set $C(X, B)$ need not be compact, so the Krein-Milman theorem does not apply.] Our main result was proved in two special cases in [6] (Theorems 2.1 and 4.1), where applications were made to various convex sets of bounded (or of compact) linear operators from a Banach space into $C(X)$. The more general result of the present note may be applied to analogous sets of *weakly* compact operators. We give one such application, as well as two results which were overlooked in [6].

As in [6], the problem is handled in two steps. First, we consider a condition (D) (below) on a pair of spaces (X, A) , with X compact Hausdorff and A bounded in E , which implies that

$$C(X, \overline{\text{co}} A) = \overline{\text{co}} C(X, A).$$

[This formulation was first considered by G. Seever [7].] We then apply this to bounded closed convex subsets B of E such that (with $A = \text{ext } B$), the pair (X, A) satisfies condition (D) and $B = \overline{\text{co}} A$.

DEFINITION. — *A pair of Hausdorff spaces (X, A) is said to satisfy condition (D) if the following holds for each $n > 0$:*

Given nonempty open sets $U_1, U_2, \dots, U_n$ in A and pairwise disjoint nonempty compact sets $K_1, K_2, \dots, K_n$ in X , there exists $f \in C(X, A)$ such that $f(K_i) \subset U_i$, $i = 1, 2, \dots, n$.

Condition (D) is a sort of density property for the subspace $C(X, A)$ in the space A^X of all functions from X to A . Indeed, condition (D) implies that $C(X, A)$ is dense in the pointwise topology on the space A^X , while density of $C(X, A)$ in the compact-open topology implies condition (D). As noted in [6], if X is a totally disconnected compact space, then (X, A) satisfies condition (D) for any A . On the other hand, if A is arcwise connected (or even only « almost arcwise connected » [6]), then (X, A) satisfies (D) for any compact X .

THEOREM 1. — *Let E be a real or complex locally convex Hausdorff vector space, X a compact Hausdorff space and A a bounded subset of E . If (X, A) satisfies condition (D), then*

$$C(X, \overline{\text{co}} A) = \overline{\text{co}} C(X, A).$$

The proof of the theorem depends on the following two technical lemmas.

LEMMA 1. — Suppose that L is a continuous linear functional on $C(X, E)$. Then there exists a continuous seminorm p on E and a regular Borel positive measure μ on X such that $\mu(X) \leq 1$ and

$$|L(f)| \leq \int_X p(f(x)) d\mu(x) \quad (\text{for each } f \in C(X, E)).$$

LEMMA 2. — Suppose that X and A are as described in the statement of the theorem. Given a continuous seminorm p on E , $\varepsilon > 0$, a regular Borel probability measure μ on X and $f \in C(X, \overline{\text{co}} A)$, there exist $g \in \overline{\text{co}} C(X, A)$ and a compact subset $K \subset X$ such that

$$p(g(x) - f(x)) < \varepsilon \quad \text{for } x \in K \quad \text{and} \quad \mu(X \setminus K) < \varepsilon.$$

Assuming that these lemmas have been proved, the theorem follows readily. Indeed, since $C(X, A) \subset C(X, \overline{\text{co}} A)$ and since the latter is closed and convex, we have

$$\overline{\text{co}} C(X, A) \subset C(X, \overline{\text{co}} A).$$

To show equality, it suffices to show that for each $\varepsilon > 0$, each $L \in C(X, E)^*$ and each $f \in C(X, \overline{\text{co}} A)$, there exists $g \in \overline{\text{co}} C(X, A)$ with

$$\operatorname{Re} L(g) > \operatorname{Re} L(f) - \varepsilon.$$

Choose p and μ according to Lemma 1, and let $M = \sup\{p(a) : a \in A\}$. Choose $K \subset X$ and $g \in \overline{\text{co}} C(X, A)$ according to Lemma 2, with ε replaced by $\varepsilon/2(M+1)$. It follows that

$$\operatorname{Re} L(f) - \operatorname{Re} L(g) \leq |L(f - g)| \leq \int_X p(f(x) - g(x)) d\mu(x).$$

The integral on the right is the sum of the integral over K and the integral over $X \setminus K$. From Lemma 2, the first summand is at most $\varepsilon/2(M+1)$, while the second is at most $M\varepsilon/2(M+1)$, hence the total is at most ε .

We now turn to the proof of Lemma 1. Since L is continuous on $C(X, E)$ it is bounded in absolute value by 1 on a

neighborhood of the form

$$\{f \in C(X, E) : p(f(x)) \leq 1, x \in X\},$$

where p is a continuous seminorm on E . Thus,

$$(*) \quad |L(f)| \leq \sup\{p(f(x)) : x \in X\}, \quad f \in C(X, E).$$

Let N denote the closed subspace $p^{-1}(0)$ and consider the space E/N , normed by the quotient norm $\|\cdots\|$ defined by p . Let φ denote the quotient map from E into $F = E/N$; the composition $f \rightarrow \varphi \circ f$ defines a linear mapping of $C(X, E)$ into $C(X, F)$ which satisfies

$$\|\varphi(f(x))\| = p(f(x))$$

for all $f \in C(X, E)$, $x \in X$. The space $C(X, F)$ has the norm

$$\|g\| = \sup\{\|g(x)\| : x \in X\}.$$

It follows from $(*)$ that the formula $J(\varphi \circ f) = L(f)$ defines a continuous linear functional J of norm at most 1 on the subspace $\varphi \circ C(X, E)$ of $C(X, F)$, and we can extend J to a functional of norm at most 1 on all of $C(X, F)$. At this point we could apply known results, which represent $C(X, F)^*$ in terms of dominated vector valued measures [4, p. 387], but we prefer to use the following direct (and simple) proof which was kindly furnished us by Dr. Erik Thomas. Let us define, for $h \in C(X)$, $h \geq 0$,

$$(**) \quad \mu(h) = \sup \{ |J(g)| : g \in C(X, F), \\ \|g(x)\| \leq h(x) \text{ for } x \text{ in } X \}$$

It is straightforward to verify that $\mu(h) < \infty$, that $\mu(\lambda h) = \lambda \mu(h)$ for $\lambda > 0$, and that $\mu(h_1 + h_2) \geq \mu(h_1) + \mu(h_2)$ if $h_1, h_2 \geq 0$ are in $C(X)$. The reverse inequality follows easily once we have the following fact: If $h = h_1 + h_2$ ($h_i \geq 0$) and $\|g(x)\| \leq h(x)$ for all x in X , then there exists g_1, g_2 in $C(X, F)$ such that $g = g_1 + g_2$ and $\|g_i(x)\| \leq h_i(x)$, $i = 1, 2$ and $x \in X$. Indeed, let $V = \{x \in X : \|g(x)\| > 0\}$ and for x in V let

$$\alpha_1(x) = \min(1, h_1(x)/\|g(x)\|), \quad \alpha_2(x) = 1 - \alpha_1(x).$$

If we define $g_i(x) = \alpha_i(x)g(x)$ for $x \in V$, $= 0$ for $x \in X \setminus V$,

then $g = g_1 + g_2$, $\|g_i(x)\| \leq h_i(x)$ and $\|g_i(x)\| \leq \|g(x)\|$ ($x \in X$, $i = 1, 2$). (The last inequality shows that each g_i is continuous.) Thus, μ is additive, non negative and positive homogeneous on the positive cone in $C(X)$, hence can be considered as an integral with respect to a finite positive regular Borel measure, say μ , on X . Furthermore, from $(**)$ it is obvious that $|J(g)| \leq \int \|g(x)\| d\mu(x)$ for all $g \in C(X, F)$ and that $\mu(1) = \|J\| \leq 1$. Finally, for $f \in C(X, E)$ we have $|L(f)| = |J(\varphi \circ f)| \leq \int \|\varphi(f(x))\| d\mu(x) = \int p(f(x)) d\mu(x)$, which completes the proof of Lemma 1.

We next give the proof of Lemma 2. For $x \in X$, let

$$V_x = \{y \in X : p(f(x) - f(y)) < \varepsilon/3\};$$

this is an open neighborhood of x , and we can choose $x_1, \dots, x_n$ such that the collection $\{V_{x_1}, \dots, V_{x_n}\}$ covers X , and such that no proper subcollection covers X . An easy induction argument, using the regularity of μ , shows that we can find another cover $\{V_1, \dots, V_n\}$ of open sets V_i such that $V_i \subset V_{x_i}$ and such that $\mu(D) < \varepsilon$, where $D = \cup \{V_i \cap V_j : i, j = 1, 2, \dots, n; i \neq j\}$. Let

$$K_i = V_i \setminus \cup \{V_j : j \neq i\} = X \setminus \cup \{V_j : j \neq i\}, i = 1, 2, \dots, n.$$

Then each K_i is compact, nonempty and $K_i \cap K_j$ is empty if $i \neq j$. Furthermore, if $K = \cup K_i$, then K is compact and $X \setminus K \subset D$, hence $\mu(X \setminus K) < \varepsilon$. Now, for each $i = 1, 2, \dots, n$ we have $f(x_i) \in \overline{\text{co}} A$, hence we can find $u_i \in \text{co } A$, with $p[u_i - f(x_i)] < \varepsilon/3$, of the following form:

$$u_i = \sum_{k=1}^{m_i} \lambda_{ik} a_{ik}, \quad \{a_{ik}\}_{k=1}^{m_i} \subset A, \quad \lambda_{ik} > 0, \quad \sum_{i=1}^{m_i} \lambda_{ik} = 1$$

where each λ_{ik} is a rational number, $k = 1, 2, \dots, m_i$. We can assume that the numbers λ_{ik} have a common denominator $Q > 0$, so by allowing repetitions of the points a_{ik} and by relabelling, we have

$$u_i = Q^{-1} \sum_{k=1}^Q b_{ik}, \quad \{b_{ik}\}_{k=1}^Q \subset A, \quad i = 1, 2, \dots, n.$$

By property (D), for each $k = 1, 2, \dots, Q$, we can choose

$g_k \in C(X, A)$ such that

$$g_k(K_i) \subset \{\nu \in E : p(\nu - b_{ik}) < \varepsilon/3\}.$$

Let $g = Q^{-1} \sum_{k=1}^Q g_k$, so that $g \in \text{co } C(X, A)$.

Suppose that $x \in K$; then $x \in K_i$ for some i and

$$\begin{aligned} p[g(x) - u_i] &= p[Q^{-1} \sum g_k(x) - Q^{-1} \sum b_{ik}] \\ &\leq Q^{-1} \sum p[g_k(x) - b_{ik}] < \varepsilon/3. \end{aligned}$$

Since $K_i \subset V_i \subset V_{x_i}$, we have $p[f(x) - f(x_i)] < \varepsilon/3$. Thus,

$$\begin{aligned} p[g(x) - f(x)] &\leq p[g(x) - u_i] + p[u_i - f(x_i)] \\ &\quad + p[f(x_i) - f(x)] < \varepsilon, \end{aligned}$$

which completes Lemma 2.

COROLLARY 1. — *Suppose that B is a bounded closed convex subset of the locally convex space E , and that X is a compact Hausdorff space. Let $A \subset \text{ext } B$. If $B = \overline{\text{co}} A$ and if (X, A) satisfies condition (D), then*

$$\overline{\text{co}} C(X, A) = C(X, B);$$

in particular, the latter set is the closed convex hull of its extreme points.

The hypothesis in Corollary 1 that $B = \overline{\text{co}} A$ is obviously a necessary one for the conclusion; indeed, since

$$C(X, A) \subset C(X, \overline{\text{co}} A)$$

and since the latter is closed and convex, it contains $\overline{\text{co}} C(X, A)$. Thus, if $C(X, B) = \overline{\text{co}} C(X, A)$, then $C(X, B) \subset C(X, \overline{\text{co}} A)$, whence $B = \overline{\text{co}} A$.

In general, condition (D) is not a necessary one for the validity of the equality $C(X, \overline{\text{co}} A) = \overline{\text{co}} C(X, A)$. Consider, for instance, $X = [0, 1]$, $E = \mathbb{C}$ (complex plane) and

$$A = \{z : |z| < 1/4\} \cup \{z : 3/4 < |z| < 1\}.$$

Then $\overline{\text{co}} A = \{z : |z| \leq 1\}$ is compact, and the above equality holds, but it is easily seen that $C(X, A)$ is not even pointwise dense in $A^\times$. If, however, A is the set of extreme points of $\overline{\text{co}} A$ — this is the situation we are mainly interested in — then there is a partial converse to Theorem 1.

THEOREM 2. — *If B is a compact convex subset of the locally convex space E and $A = \text{ext } B$ (so $B = \overline{\text{co}} A$), then the equality*

$$\overline{\text{co}} C(X, A) = C(X, \overline{\text{co}} A)$$

implies that $C(X, A)$ is pointwise dense in A^X .

We omit the proof, since it, closely parallels that of Theorem 3.1 of [6], in which E is a dual Banach space (in the weak* topology) and B is its unit ball. The same argument works in the general case, using the fact that each extreme point of B has a neighborhood base in B consisting of « slices » [3, p. 108].

We now consider some applications of the foregoing results to spaces of linear operators. Suppose that M is a real (resp. complex) Banach space and let $C(X)$ denote the real (resp. complex) continuous functions on the compact Hausdorff space X . The space $\mathcal{L}(M, C(X))$ (or simply $\mathcal{L}$) of all bounded linear operators from M into $C(X)$ is linearly isomorphic to the space $C(X, E)$, where $E = M_w^*$ is the space M^* in its weak* topology [5, p. 490]. The correspondence between an operator T in $\mathcal{L}$ and a function f in $C(X, E)$ is defined by

$$(Tm)(x) = \langle m, f(x) \rangle, \quad (x \in X, m \in M).$$

Moreover, $\|T\| = \sup \{ \|f(x)\| : x \in X \} = \|f\|$.

Thus, the unit ball $\mathcal{U}$ of $\mathcal{L}$ may be identified with the subset $C(X, U^*)$ of $C(X, E)$, where U^* is the unit ball of M^* . This correspondence was used in [6] to obtain various corollaries to Theorem 1, which was proved there for this particular choice of E . Similarly, the subspace

$$\mathcal{L}_c = \mathcal{L}_c(M, C(X))$$

of all compact operators in $\mathcal{L}$ can be identified with the subspace $C(X, M_n^*)$, of $C(X, E)$, where M_n^* is M^* in its norm topology [5], and Theorem 1 was also proved in [6] for this case. It is readily verified that the uniform topology on $C(X, E)$ carries over (under the correspondence indicated above) to the strong operator topology on $\mathcal{L}$, and that in $C(X, M_n^*)$ the uniform topology is the norm topology (norm defined as above) and this identifies on $\mathcal{L}_c$ with the norm (or « uniform operator ») topology. The fact that Theorem 1

was proved for arbitrary E allows us to consider the case where $E = M_w^*$, the space M^* in its weak (i.e. $\sigma(M^*, M^{**})$) topology. Under the above correspondence, $C(X, M_w^*)$ is exactly the space $\mathcal{L}_{wc} = \mathcal{L}_{wc}(M, C(X))$ of all weakly compact operators from M into $C(X)$. The topology induced on $\mathcal{L}_{wc}$ by the uniform topology on $C(X, M_w^*)$ is not one of the usual « operator » topologies, but is easily seen to be between the strong operator and norm topologies on $\mathcal{L}_{wc}$.

We will denote by $\mathcal{U}$, $\mathcal{U}_c$ and $\mathcal{U}_{wc}$ the unit ball of $\mathcal{L}$, $\mathcal{L}_c$ and $\mathcal{L}_{wc}$ respectively. These are, of course, the same as the sets $C(X, U^*)$, $C(X, U_n^*)$ and $C(X, U_w^*)$. An operator which corresponds to an element f of one of these sets such that $f(X) \subset \text{ext } U^*$ is called a *nice* (resp. nice compact, nice weakly compact) operator. They are of course, extreme points of the sets $\mathcal{U}$, $\mathcal{U}_c$ and $\mathcal{U}_{wc}$ respectively.

The next result is almost a direct application of Corollary 1 to the ball of weakly compact operators. The main point is to account for the difference between the two topologies involved.

PROPOSITION 1. — *Let M and $C(X)$ be as above, and let U^* be the unit ball of M^* . Suppose that there is a subset $A \subset \text{ext } U^*$ such that:*

- (i) *The pair (X, A_w) satisfies condition (D).*
- (ii) *U^* is the norm closed convex hull of A .*

Then the unit ball $\mathcal{U}_{wc}$ of $\mathcal{L}_{wc}$ is the strong operator closed convex hull of the nice weakly compact operators.

Proof. — Hypotheses (i) and (ii) allow us to apply Corollary 1 to obtain the equality $C(X, U_w^*) = \overline{\text{co}} C(X, A_w)$, where the closure is in the uniform topology of $C(X, M_w^*)$. Since $C(X, M_w^*) \subset C(X, M^{**})$, the uniform topology on the latter space induces a topology on $C(X, M_w^*)$ which is weaker than the original; we will call it the « strong » topology since it corresponds exactly to the strong operator topology on $\mathcal{L}_{wc}$. Thus, we want to show that $C(X, U_w^*)$ is the strong closed convex hull of $C(X, A_w)$, since the latter is clearly a subset of the nice weakly compact operators. But it is easily verified that (since U^* is weak* closed) $C(X, U_w^*)$ is strongly closed in $C(X, M_w^*)$, hence contains the strong closure of $\text{co } C(X, A_w)$, which in turn contains $\overline{\text{co}} C(X, A_w) = C(X, U_w^*)$.

The fact that in hypothesis (ii) above we used the norm closure instead of the weak closure (which Corollary 1 would have allowed) is no loss in generality, of course, since the set involved is convex.

Recall that a real or complex Banach space M is said to be *smooth* if for each point $x \in S(M) = \{x \in M: \|x\| = 1\}$ there exists a unique functional f_x in the unit sphere $S(M^*)$ of M^* such that $\operatorname{Re} f_x(x) = 1$. This is equivalent to Gateaux differentiability of the norm (at each nonzero point), and the functional f_x is the Gateaux differential of the norm at x .

THEOREM 3. — *Let M be a real or complex Banach space and X a compact Hausdorff space. In the real case, we assume that $\dim M > 1$.*

(a) *If M is smooth, then $\mathcal{U}$ is the strong operator closed convex hull of the nice operators.*

(b) *If the norm in M is Fréchet differentiable at each nonzero point, then $\mathcal{U}_{wc}$ [resp. $\mathcal{U}_c$] is the strong operator [resp. norm] closed convex hull of its nice operators.*

Proof. — (a) It is well known (and easily proved) that if M is smooth, then the map $x \rightarrow f_x$ defined above is continuous from $S(M)$ in its norm topology into $S(M^*)$ in its weak* topology. It is readily verified that U^* is the weak* closed convex hull of the image A of $S(M)$ under this map, and that $A \subset \operatorname{ext} U^*$. [In fact, A is known [1] to be norm dense in $S(M^*)$.] Since $S(M)$ is arcwise connected (in the real case this assertion obviously requires that $\dim M > 1$), the set A is arcwise connected in the weak* topology. Thus, (X, A) satisfies condition (D) so Corollary 1 yields the desired conclusion.

(b) The Fréchet differentiability of the norm in M implies that the derivative map $x \rightarrow f_x$ defined above is continuous from the norm topology on $S(M)$ into the norm topology on $S(M^*)$. With the same notation as in (a), the set A is norm arcwise connected and norm dense in $S(M^*)$, hence U^* is the norm closed convex hull of A and Proposition 1 [resp. Corollary 1] applies.

In the case when $M = C(X)$ for some compact Hausdorff space X , it is possible to obtain necessary and sufficient

conditions on X and Y that $\mathcal{U}_{wc} \subset \mathcal{L}_{wc}(C(X), C(Y))$ be the strong operator closed convex hull of the nice weakly compact operators. These conditions are the same as those in Theorem 4.6 of [6], and the methods for obtaining them are essentially the same. (We don't know, in this case, whether every extreme element of $\mathcal{U}_{wc}$ is a nice operator.) Similar results hold in the real case for the set of positive normalized weakly compact operators.

The following problem arises in the context of Corollary 1: Suppose that $C(X, B) = \overline{\text{co}} \text{ ext } C(X, B)$. Must $\text{ext } B$ be nonempty?

[*Note added in proof*: J. Lindenstrauss (private communication) has answered this question in the negative by showing that there exists a normed linear space E , a nonempty convex closed and bounded subset $B \subset E$ and a nonempty compact Hausdorff space X such that $\text{ext } B$ is empty, but $C(X, B) = \overline{\text{co}} \text{ ext } C(X, B)$.]

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Manuscrit reçu le 6 mars 1970.

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