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PRINCIPAL SERIES REPRESENTATIONS OF IWAHORI–HECKE ALGEBRAS FOR KAC–MOODY GROUPS OVER LOCAL FIELDS

by Auguste HÉBERT (*)

Abstract. — Recently, Iwahori–Hecke algebras were associated with Kac–Moody groups over non-Archimedean local fields. We introduce principal series representations for these algebras. We study these representations and partially generalize irreducibility criteria of Kato and Matsumoto.

Résumé. — Des algèbres d'Iwahori–Hecke ont récemment été associées aux groupes de Kac–Moody sur les corps locaux non-archimédiens. Nous introduisons les représentations de la série principale pour ces algèbres. Nous étudions ces représentations et généralisons partiellement les critères d'irréductibilité de Kato et de Matsumoto.

1. Introduction

1.1. The reductive case

Let G be a split reductive group over a non-Archimedean local field $\mathcal{K}$. Let T be a maximal split torus of G and Y be the cocharacter lattice of (G, T). Let B be a Borel subgroup of G containing T. Let $T_{\mathbb{C}} = \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathbb{C}^*)$. Then τ can be extended to a character $\tau : B \to \mathbb{C}^*$. If $\tau \in T_{\mathbb{C}}$, the principal series representation $I(\tau)$ of G is the induction of $\tau\delta^{1/2}$ from B to G, where $\delta : B \to \mathbb{R}_+^*$ is the modulus character of B. More explicitly, this is the space of locally constant functions $f : G \to \mathbb{C}$ such that $f(bg) = \tau\delta^{1/2}(b)f(g)$ for every $g \in G$ and $b \in B$. Then G acts on $I(\tau)$ by right translation.

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To each open compact subgroup K of G is associated the Hecke algebra $\mathcal{H}_K$. This is the algebra of functions from G to $\mathbb{C}$ which have compact support and are K-bi-invariant. There exists a strong link between the smooth representations of G and the representations of the Hecke algebras of G. Let K_I be the Iwahori subgroup of G. Then the Hecke algebra $\mathcal{H}_\mathbb{C}$ associated with K_I is called the Iwahori–Hecke algebra of G and plays an important role in the representation theory of G.

The algebra $\mathcal{H}_\mathbb{C}$ acts on $I_{\tau,G} := I(\tau)^{K_I}$ by the formula

$$\phi.f = \int_G \phi(g)g.f\mathrm{d}\mu(g), \quad \forall\, (\phi, f) \in \mathcal{H}_\mathbb{C} \times I(\tau)^{K_I},$$

where μ is a Haar measure on G. This formula can actually be rewritten as

$$\phi.f = \mu(K_I) \sum_{g\in G/K_I} \phi(g)g.f, \quad \forall\, (\phi, f) \in \mathcal{H}_\mathbb{C} \times I(\tau)^{K_I}. \tag{1.1}$$

Then $I(\tau)$ is irreducible as a representation of G if and only $I_{\tau,G}$ is irreducible as a representation of $\mathcal{H}_\mathbb{C}$.

Let W^v be the vectorial Weyl group of (G, T). By the Bernstein–Lusztig relations, $\mathcal{H}_\mathbb{C}$ admits a basis $(Z^\lambda H_w)_{\lambda\in Y, w\in W^v}$ such that $\bigoplus_{\lambda\in Y} \mathbb{C}Z^\lambda$ is a subalgebra of $\mathcal{H}_\mathbb{C}$ isomorphic to the group algebra $\mathbb{C}[Y]$ of Y. We identify $\bigoplus_{\lambda\in Y} \mathbb{C}Z^\lambda$ and $\mathbb{C}[Y]$. We regard τ as an algebra morphism $\tau : \mathbb{C}[Y] \to \mathbb{C}$. Then $I_{\tau,G}$ is isomorphic to the induced representation $I_\tau = \mathrm{Ind}_{\mathbb{C}[Y]}^{\mathcal{H}_\mathbb{C}}(\tau)$ and we refer to [37, Section 3.2] for a survey on this subject.

Matsumoto and Kato gave criteria for the irreducibility of I_τ. The group W^v acts on Y and thus it acts on $T_\mathbb{C}$. If $\tau \in T_\mathbb{C}$, we denote by W_τ the stabilizer of τ in W^v. Let $\Phi^\vee$ be the coroot lattice of G. Let q be the residue cardinal of $\mathcal{K}$. Let $W_{(\tau)}$ be the subgroup of W_τ generated by the reflections $r_{\alpha^\vee}$, for $\alpha^\vee \in \Phi^\vee$ such that $\tau(\alpha^\vee) = 1$. Then Kato proved the following theorem (see [20, Theorem 2.4]):

Theorem 1.1. — *Let $\tau \in T_\mathbb{C}$. Then I_τ is irreducible if and only if it satisfies the following conditions:*

(1) $W_\tau = W_{(\tau)}$,
(2) *for all* $\alpha^\vee \in \Phi^\vee$, $\tau(\alpha^\vee) \neq q$.

When τ is *regular*, that is when $W_\tau = \{1\}$, condition (1) is satisfied and this is a result by Matsumoto (see [25, Théorème 4.3.5]).

1.2. The Kac–Moody case

Let G be a split Kac–Moody group over a non-Archimedean local field $\mathcal{K}$. We do not know which topology on G could replace the usual topology on reductive groups over $\mathcal{K}$. There is up to now no definition of smoothness for the representations of G. However one can define certain Hecke algebras in this framework. In [5] and [6], Braverman, Kazhdan and Patnaik defined the spherical Hecke algebra and the Iwahori–Hecke $\mathcal{H}_{\mathbb{C}}$ of G when G is affine. In [12] and [2], Bardy-Panse, Gaussent and Rousseau generalized these constructions to the case where G is a general Kac–Moody group. They achieved this construction by using masures (also known as hovels), which are analogous to Bruhat–Tits buildings (see [11]). Together with Abdellatif, we attached Hecke algebras to subgroups slightly more general than the Iwahori subgroup (see [1]).

Let B be a positive Borel subgroup of G and T be a maximal split torus of G contained in B. Let Y be the cocharacter lattice of G, W^v be the Weyl group of G and Y^{++} be the set of dominant cocharacters of Y. The Bruhat decomposition does not hold on G: if G is not reductive,

$$G^+ := \bigsqcup_{\lambda\in Y^{++}} K_I \lambda K_I \subsetneq G.$$

The set G^+ is a sub-semi-group of G. Then $\mathcal{H}_{\mathbb{C}}$ is defined to be the set of functions from $K_I\backslash G^+/K_I$ to $\mathbb{C}$ which have finite support. The Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{C}}$ of G admits a Bernstein–Lusztig presentation but it is no longer indexed by Y. Let $Y^+ = \bigcup_{w\in W^v} w.Y^{++} \subset Y$. Then Y^+ is the *integral Tits cone* and we have $Y^+ = Y$ if and only G is reductive. The *Bernstein–Lusztig–Hecke algebra of* G is the space ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}} = \bigoplus_{w\in W^v} \mathbb{C}[Y]H_w$ subject to some relations (see Section 2.3). Then $\mathcal{H}_{\mathbb{C}}$ is isomorphic to $\bigoplus_{w\in W^v} \mathbb{C}[Y^+]H_w$.

Let $B^+ = B\cap G^+$. Let $T_{\mathbb{C}}^+ = \mathrm{Hom}_{\mathrm{Mon}}(Y^+,\mathbb{C})\setminus\{0\}$ and $T_{\mathbb{C}} = \mathrm{Hom}_{\mathrm{Gr}}(Y,\mathbb{C}^*)$. Let $\epsilon\in\{+,\emptyset\}$. If $\tau^\epsilon\in T_{\mathbb{C}}^\epsilon$ we define the space $\widehat{I(\tau^\epsilon)^\epsilon}$ of functions f from G^ϵ to $\mathbb{C}$ such that for every $g\in G^\epsilon$ and $b\in B^\epsilon$, $f(bg) = \tau\delta^{1/2}(b)f(g)$. As we do not know which condition could replace "locally constant", we do not impose any regularity condition on the functions of $\widehat{I(\tau^\epsilon)^\epsilon}$. Then G^ϵ acts by right translation on $\widehat{I(\tau^\epsilon)^\epsilon}$. Let $I_{\tau^\epsilon,G^\epsilon}$ be the subspace of $\widehat{I(\tau^\epsilon)^\epsilon}$ of functions which are invariant under the action of K_I and whose support satisfy some finiteness conditions (see 6.2.1). Inspired by formula (1.1), we define an action of $\mathcal{H}_{\mathbb{C}}$ on $I_{\tau^\epsilon,G^\epsilon}$ by

$$\phi.f = \sum_{g\in G/K_I} \phi(g)g.f,\quad \forall\,(\phi,f)\in\mathcal{H}_{\mathbb{C}}\times I_{\tau^\epsilon,G^\epsilon}.$$

As often in the Kac–Moody theory, the fact that this formula is well-defined is not obvious. We prove some finiteness results on G to prove that the formula only involves finite sums and that $\phi.f$ is an element of $I_{\tau^\epsilon,G^\epsilon}$ (see Definition/Proposition 6.12).

We regard τ^ϵ as an algebra morphism $\mathbb{C}[Y^\epsilon] \to \mathbb{C}$. Let $I^\epsilon_{\tau^\epsilon}$ be the representation of ${}^{\mathrm{BL}}\mathcal{H}^\epsilon_{\mathbb{C}}$ (where ${}^{\mathrm{BL}}\mathcal{H}^+_{\mathbb{C}} = \mathcal{H}_{\mathbb{C}}$) defined by induction of τ^ϵ from $\mathbb{C}[Y^\epsilon]$ to ${}^{\mathrm{BL}}\mathcal{H}^\epsilon_{\mathbb{C}}$.

We prove the following proposition, which seems to indicate that the representations of $\mathcal{H}_{\mathbb{C}}$ correspond to representations of G^+ and that the representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$ correspond to representations of G:

Proposition 1.2 (see Proposition 6.28). — *Let $\tau^+ \in T^+_{\mathbb{C}}$.*

(1) *Suppose that τ^+ is not the restriction to Y^+ of an element of $T_{\mathbb{C}}$. For every $f \in \widehat{I(\tau^+)} \setminus \{0\}$, for every G-module M, the restriction of M to G^+ is not isomorphic to $G^+.f$. For every $x \in I^+_{\tau^+} \setminus \{0\}$, for every ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$-module M, the restriction of M to $\mathcal{H}_{\mathbb{C}}$ is not isomorphic to $\mathcal{H}_{\mathbb{C}}.x$.*

(2) *Suppose that τ^+ is the restriction to Y^+ of a (necessarily unique) element τ of $T_{\mathbb{C}}$. Every element f^+ of $\widehat{I(\tau^+)^+}$ can be extended uniquely to an element f of $\widehat{I(\tau)}$. Then $f^+ \mapsto f$ is an isomorphism of G^+-modules. The action of $\mathcal{H}_{\mathbb{C}}$ on $I^+_{\tau^+}$ extends uniquely to an action of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$ on $I^+_{\tau^+}$. Then $I^+_{\tau^+}$ is naturally isomorphic to I_τ as a ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$-module.*

Note that the existence of elements of $T^+_{\mathbb{C}}$ which do not extend to elements of $T_{\mathbb{C}}$ depends on G. We prove that in some cases (for example when G is affine or associated with a size 2 Kac–Moody matrix) every element of $T^+_{\mathbb{C}}$ is the restriction of an element of $T_{\mathbb{C}}$. We also prove that for some size 3 Kac–Moody matrices, there exists $\tau \in T^+_{\mathbb{C}}$ which is not the restriction of an element of $T_{\mathbb{C}}$ (see Lemma 6.20 and Lemma 6.24).

We then restrict our study to the elements τ^+ of $T^+_{\mathbb{C}}$ which are the restriction of an element τ of $T_{\mathbb{C}}$. We prove that $I^+_{\tau^+}$ is irreducible if and only if I_τ is (see Proposition 2.12). We then study the irreducibility of I_τ. We prove the following theorem, generalizing Matsumoto's irreducibility criterion (see Corollary 4.10):

Theorem 1.3. — *Let τ be a regular character. Then I_τ is irreducible if and only if for all $\alpha^\vee \in \Phi^\vee$,*

$$\tau(\alpha^\vee) \neq q.$$

We also generalize one implication of Kato's criterion (see Lemma 4.5 and Proposition 4.17). Let $W_{(\tau)}$ be the subgroup of W_τ generated by the reflections $r_{\alpha^\vee}$, for $\alpha^\vee \in \Phi^\vee$ such that $\tau(\alpha^\vee) = 1$.

Theorem 1.4. — *Let $\tau \in T_{\mathbb{C}}$. Assume that I_τ is irreducible. Then:*

(1) $W_\tau = W_{(\tau)}$,
(2) *for all* $\alpha^\vee \in \Phi^\vee$, $\tau(\alpha^\vee) \neq q$.

We then obtain Kato's criterion when the Kac–Moody group G is associated with a size 2 Kac–Moody matrix (see Theorem 5.35):

Theorem 1.5. — *Assume that G is associated with a size 2 Kac–Moody matrix. Let $\tau \in T_{\mathbb{C}}$. Then I_τ is irreducible if and only if it satisfies the following conditions:*

(1) $W_\tau = W_{(\tau)}$,
(2) *for all* $\alpha^\vee \in \Phi^\vee$, $\tau(\alpha^\vee) \neq q$.

In order to prove these theorems, we first establish the following irreducibility criterion. For $\tau \in T_{\mathbb{C}}$ set $I_\tau(\tau) = \{x \in I_\tau \mid \theta.x = \tau(\theta).x,\ \forall\ \theta \in \mathbb{C}[Y]\}$. Then:

Theorem 1.6 (see Theorem 4.8). — *I_τ is irreducible if and only if:*

- $\tau(\alpha^\vee) \neq q$ *for all* $\alpha^\vee \in \Phi^\vee$
- $\dim I_\tau(\tau) = 1$.

Remark 1.7. — Suppose that G is an affine Kac–Moody group. Then by [2, 7], some extension $\widetilde{{}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}}$ of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$ contains the double affine Hecke algebra introduced in [8]. It would therefore be interesting to find a link between the representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$ and those of this algebra.

Framework

Actually, following [2] we study Iwahori–Hecke algebras associated with abstract masures. In particular our results also apply when G is an almost-split Kac–Moody group over a non-Archimedean local field. The definition of $W_{(\tau)}$ and the statements given in this introduction are not necessarily valid in this case and we refer to Proposition 4.17, Theorem 5.35 and Theorem 4.8 for statements valid in this frameworks.

Organization of the paper

The paper is organized as follows. In a first part (Sections 2 to 5) we consider “abstract” Iwahori–Hecke algebras. We define them using the Bernstein–Lusztig presentation and they are a priori not associated with a group. The techniques used are mainly algebraic, based on the Bernstein–Lusztig relations. In a second part (Section 6), we introduce Kac–Moody groups, masures and Iwahori–Hecke algebras associated with groups, and we associate some principal series representations to these groups. The techniques involved are mainly building theoretic.

In Section 2, we recall the definition of the Iwahori–Hecke algebras and of the Bernstein–Lusztig–Hecke algebras, introduce principal series representations and define an algebra ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ containing ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$, where $\mathcal{F}$ is the field of coefficients of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$.

In Section 3, we study the $\mathcal{F}[Y]$-module I_τ and we study the intertwining operators from I_τ to $I_{\tau'}$, for $\tau, \tau' \in T_{\mathcal{F}}$.

In Section 4, we establish Theorem 1.6. We then apply it to obtain Theorem 1.3 and Theorem 1.4.

In Section 5 we consider the weight vectors of I_τ and use them to prove Kato’s irreducibility criterion for size 2 Kac–Moody matrices.

In Section 6, we introduce Kac–Moody groups over local fields, masures, and Iwahori–Hecke algebras of these groups. We introduce some principal series representations of these groups, study them and relate them to the principal series representations studied in the previous sections.

There is an index of notations at the end of the paper.

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2. Bernstein–Lusztig presentation of Iwahori–Hecke algebras

Let G be a Kac–Moody group over a non-Archimedean local field. Then Gaussent and Rousseau constructed a space $\mathcal{I}$, called a masure on which G acts, generalizing the construction of the Bruhat–Tits buildings (see [11], [35] and [36]). Rousseau then gave in [34] an axiomatic definition of masures inspired by the axiomatic definition of Bruhat–Tits buildings. We call a masure satisfying these axioms an abstract masure. It is a priori not associated with any group.

In [2], Bardy-Panse, Gaussent and Rousseau attached an Iwahori–Hecke algebra $\mathcal{H}_\mathcal{R}$ to each abstract masure satisfying certain conditions and to each ring $\mathcal{R}$. The algebra $\mathcal{H}_\mathcal{R}$ is an algebra of functions defined on some pairs of chambers of the masure, equipped with a convolution product. Then they prove that under some additional hypothesis on the ring $\mathcal{R}$ (which are satisfied by $\mathbb{R}$ and $\mathbb{C}$), $\mathcal{H}_\mathcal{R}$ admits a Bernstein–Lusztig presentation. In this section, we will only introduce the Bernstein–Lusztig presentation of $\mathcal{H}_\mathcal{R}$ and we do not introduce masures (we introduce them in Section 6). We however introduce the standard apartment of a masure. We restrict our study to the case where $\mathcal{R} = \mathcal{F}$ is a field.

2.1. Standard apartment of a masure

2.1.1. Root generating system

A *Kac–Moody matrix* (or generalized Cartan matrix) is a square matrix $A = (a_{i,j})_{i,j\in I}$ indexed by a finite set I, with integral coefficients, and such that :

(i) $\forall\, i \in I,\ a_{i,i} = 2$;
(ii) $\forall\, (i,j) \in I^2, (i \neq j) \Rightarrow (a_{i,j} \leqslant 0)$;
(iii) $\forall\, (i,j) \in I^2,\ (a_{i,j} = 0) \Leftrightarrow (a_{j,i} = 0)$.

A *root generating system* is a 5-tuple $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac–Moody matrix A indexed by the finite set I, of two dual free $\mathbb{Z}$-modules X and Y of finite rank, and of a free family $(\alpha_i)_{i\in I}$ (respectively $(\alpha_i^\vee)_{i\in I}$) of elements in X (resp. Y) called *simple roots* (resp. *simple coroots*) that satisfy $a_{i,j} = \alpha_j(\alpha_i^\vee)$ for all i, j in I. Elements of X (respectively of Y) are called *characters* (resp. *cocharacters*).

Fix such a root generating system $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ and set $\mathbb{A} := Y \otimes \mathbb{R}$. Each element of X induces a linear form on $\mathbb{A}$, hence X

can be seen as a subset of the dual $\mathbb{A}^*$. In particular, the α_i's (with $i \in I$) will be seen as linear forms on $\mathbb{A}$. This allows us to define, for any $i \in I$, an involution r_i of $\mathbb{A}$ by setting $r_i(v) := v - \alpha_i(v)\alpha_i^\vee$ for any $v \in \mathbb{A}$. Let $\mathscr{S} = \{r_i \,|\, i \in I\}$ be the (finite) set of *simple reflections*. One defines the *Weyl group of* $\mathcal{S}$ as the subgroup W^v of $\mathrm{GL}(\mathbb{A})$ generated by $\mathscr{S}$. The pair $(W^v, \mathscr{S})$ is a Coxeter system, hence we can consider the length $\ell(w)$ with respect to $\mathscr{S}$ of any element w of W^v. If $s \in \mathscr{S}$, $s = r_i$ for some unique $i \in I$. We set $\alpha_s = \alpha_i$ and $\alpha_s^\vee = \alpha_i^\vee$.

The following formula defines an action of the Weyl group W^v on $\mathbb{A}^*$:

$$\forall\, x \in \mathbb{A}, w \in W^v, \alpha \in \mathbb{A}^*,\ (w.\alpha)(x) := \alpha(w^{-1}.x).$$

Let $\Phi := \{w.\alpha_i \,|\, (w,i) \in W^v \times I\}$ (resp. $\Phi^\vee = \{w.\alpha_i^\vee \,|\, (w,i) \in W^v \times I\}$) be the set of *real roots* (resp. *real coroots*): then Φ (resp. $\Phi^\vee$) is a subset of the *root lattice* $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ (resp. *coroot lattice* $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$). By [22, 1.2.2(2)], one has $\mathbb{R}\alpha^\vee \cap \Phi^\vee = \{\pm\alpha^\vee\}$ and $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ for all $\alpha^\vee \in \Phi^\vee$ and $\alpha \in \Phi$.

2.1.2. Fundamental chamber, Tits cone and vectorial faces

As in the reductive case, define the *fundamental chamber* as $C_f^v := \{v \in \mathbb{A} \,|\, \forall\, s \in \mathscr{S}, \alpha_s(v) > 0\}$.

Let $\mathcal{T} := \bigcup_{w\in W^v} w.\overline{C_f^v}$ be the *Tits cone*. This is a convex cone (see [22, 1.4]).

For $J \subset \mathscr{S}$, set $F^v(J) = \{x \in \mathbb{A} \,|\, \alpha_j(x) = 0, \forall\, j \in J \text{ and } \alpha_j(x) > 0, \forall\, j \in \mathscr{S} \setminus J\}$. A *positive vectorial face* (resp. *negative*) is a set of the form $w.F^v(J)$ $(-w.F^v(J))$ for some $w \in W^v$ and $J \subset \mathscr{S}$. Then by [30, 5.1 Théorème (ii)], the family of positive vectorial faces of $\mathbb{A}$ is a partition of $\mathcal{T}$ and the stabilizer of $F^v(J)$ is $W_J = \langle J \rangle$.

One sets $Y^{++} = Y \cap \overline{C_f^v}$ and $Y^+ = Y \cap \mathcal{T}$.

Remark 2.1. — By [18, Section 4.9] and [18, Section 5.8] the following conditions are equivalent:

(1) the Kac–Moody matrix A is of finite type (i.e. is a Cartan matrix),
(2) $\mathbb{A} = \mathcal{T}$
(3) W^v is finite.

2.2. Recollections on Coxeter groups

2.2.1. Bruhat order

Let $(W_0, \mathscr{S}_0)$ be a Coxeter system. We equip it with the Bruhat order $\leqslant_{W_0}$ (see [3, Definition 2.1.1]). We have the following characterization (see [3, Corollary 2.2.3]): let $u, w \in W_0$. Then $u \leqslant_{W_0} w$ if and only if every reduced expression for w has a subword that is a reduced expression for u. By [3, Proposition 2.2.9], $(W_0, \leqslant_{W_0})$ is a *directed poset*, i.e. for every finite set $E \subset W_0$, there exists $w \in W_0$ such that $v \leqslant_{W_0} w$ for all $v \in E$.

We write $\leqslant$ instead of $\leqslant_{W^v}$. For $u, v \in W^v$, we denote by $[u, v]$, $[u, v)$, $\ldots$ the sets $\{w \in W^v \mid u \leqslant w \leqslant v\}$, $\{w \in W^v \mid u \leqslant w < v\}$, $\ldots$.

2.2.2. Reflections and coroots

Let $\mathscr{R} = \{wsw^{-1} \mid w \in W^v, s \in \mathscr{S}\}$ be the set of *reflections* of W^v. Let $r \in \mathscr{R}$. Write $r = wsw^{-1}$, where $w \in W^v$, $s \in \mathscr{S}$ and $ws > w$ (which is possible because if $ws < w$, then $r = (ws)s(ws)^{-1}$). Then one sets $\alpha_r = w.\alpha_s \in \Phi_+$ (resp. $\alpha_r^\vee = w.\alpha_s^\vee \in \Phi_+^\vee$). This is well-defined by the lemma below.

Lemma 2.2. — *Let $w, w' \in W^v$ and $s, s' \in \mathscr{S}$ be such that $wsw^{-1} = w's'w'^{-1}$ and $ws > w$, $w's' > w'$. Then $w.\alpha_s = w'.\alpha_{s'} \in \Phi_+$ and $w.\alpha_s^\vee = w'.\alpha_{s'}^\vee \in \Phi_+^\vee$.*

Proof. — One has $r(x) = x - w.\alpha_s(x)w.\alpha_s^\vee = x - w'.\alpha_{s'}(x)w'.\alpha_{s'}^\vee$ for all $x \in \mathbb{A}$ and thus $w.\alpha_s \in \mathbb{R}^* w'.\alpha_{s'}$ and $w.\alpha_s^\vee \in \mathbb{R}^* w'.\alpha_{s'}^\vee$. As Φ and $\Phi^\vee$ are reduced, $w.\alpha_s = \pm w'.\alpha_{s'}$ and $w.\alpha_s^\vee = \pm w'.\alpha_s^\vee$. By [22, Lemma 1.3.13], $w.\alpha_s, w'.\alpha_{s'} \in \Phi_+$ and $w.\alpha_s^\vee, w'.\alpha_{s'}^\vee \in \Phi_+^\vee$, which proves the lemma. □

Lemma 2.3. — *Let $r, r' \in \mathscr{R}$ and $w \in W^v$ be such that $w.\alpha_r = \alpha_{r'}$ or $w.\alpha_r^\vee = \alpha_{r'}^\vee$. Then $wrw^{-1} = r'$.*

Proof. — Write $r = vsv^{-1}$ and $r' = v's'v'^{-1}$ for $s, s' \in \mathscr{S}$ and $v, v' \in W^v$. Then $v'^{-1}wv.\alpha_s = \alpha_{s'}$. Thus by [22, Theorem 1.3.11 (b5)],

$$v'^{-1}wvsv^{-1}w^{-1}v' = s'$$

and hence $wrw^{-1} = r'$. □

Let $r \in \mathscr{R}$. Then for all $x \in \mathbb{A}$, one has:

$$r(x) = x - \alpha_r(x)\alpha_r^\vee.$$

Let $\alpha^\vee \in \Phi^\vee$. One sets $r_{\alpha^\vee} = wsw^{-1}$ where $(w, s) \in W^v \times \mathscr{S}$ is such that $\alpha^\vee = w.\alpha_s^\vee$. This is well-defined, by Lemma 2.3. Thus $\alpha^\vee \mapsto r_{\alpha^\vee}$ and $r \mapsto \alpha_r^\vee$ induce bijections $\Phi_+^\vee \to \mathscr{R}$ and $\mathscr{R} \to \Phi_+^\vee$. If $r \in \mathscr{R}$, $r = wsw^{-1}$, one sets $\sigma_r = \sigma_s$, which is well-defined by assumption on the σ_t, $t \in \mathscr{S}$ (see Section 2.3).

For $w \in W^v$, set $N_{\Phi^\vee}(w) = \{\alpha^\vee \in \Phi_+^\vee \mid w.\alpha^\vee \in \Phi_-^\vee\}$.

Lemma 2.4 ([22, Lemma 1.3.14]). — *Let $w \in W^v$. Then $|N_{\Phi^\vee}(w)| = \ell(w)$ and if $w = s_1 \dots s_r$ is a reduced expression, then*

$$N_{\Phi^\vee}(w) = \{\alpha_{s_r}^\vee, s_r.\alpha_{s_{r-1}}^\vee, \dots, s_r \dots s_2.\alpha_{s_1}^\vee\}.$$

2.2.3. Reflections subgroups of a Coxeter group

If W_0 is a Coxeter group, a *Coxeter generating set* is a set $\mathscr{S}_0$ such that $(W_0, \mathscr{S}_0)$ is a Coxeter system. Let $(W_0, \mathscr{S}_0)$ be a Coxeter system and $\mathscr{R}_0 = \{w.s.w^{-1} \mid w \in W_0, s \in \mathscr{S}_0\}$ be its set of reflections. A *reflection subgroup of* W_0 is a group of the form $W_1 = \langle \mathscr{R}_1 \rangle$ for some $\mathscr{R}_1 \subset \mathscr{R}_0$. For $w \in W_0$, set $N_{\mathscr{R}_0}(w) = \{r \in \mathscr{R}_0 \mid rw^{-1} < w^{-1}\}$. By [9, 3.3] or [10, 1], if $\mathscr{S}(W_1) = \{r \in \mathscr{R}_0 \mid N_{\mathscr{R}_0}(r) \cap W_1 = \{r\}\}$, then $(W_1, \mathscr{S}(W_1))$ is a Coxeter system.

Let $(W_0, \mathscr{S}_0)$ be a Coxeter system. The *rank of* $(W_0, \mathscr{S}_0)$ is $|\mathscr{S}_0|$.

Remark 2.5.

(1) The rank of a Coxeter group is not well-defined. For example, by [26, 3], if $k \in \mathbb{Z}_{\geqslant 1}$ and $n = 4(2k+1)$ then the dihedral group of order n admits Coxeter generating sets of order 2 and 3. However by [27], all the Coxeter generating sets of the infinite dihedral group have cardinal 2.

(2) Using [4, IV 1.8 Proposition 7] we can prove that if $(W_0, \mathscr{S}_0)$ is a Coxeter system of infinite rank, then every Coxeter generating set of W_0 is infinite.

(3) Reflection subgroups of finite rank Coxeter groups are not necessarily of finite rank. Indeed, let W_0 be the Coxeter group generated by the involutions s_1, s_2, s_3, with $s_i s_j$ of infinite order when $i \neq j \in [\![1, 3]\!]$. Let $W_0' = \langle s_1, s_2 \rangle \subset W_0$ and $\mathscr{R}_1 = \{w s_3 w^{-1} \mid w \in W_0'\} \subset \mathscr{R}_0$. Then $W_1 = \langle \mathscr{R}_1 \rangle$ has infinite rank. Indeed, let $\psi : W_0 \to W_0'$ be the group morphism defined by $\psi_{|W_0'} = \mathrm{Id}_{W_0'}$ and $\psi(s_3) = 1$. Then $\mathscr{R}_1 \subset \ker \psi$. Thus s_3 appears in the reduced writing of every non-trivial element of W_1. By [3, Corollary 1.4.4] if $r \in \mathscr{R}_1$, then the unique element of $N_{\mathscr{R}_0}(r)$ containing an s_3 in its reduced writing is r. Thus $\mathscr{S}(W_1) \supset \mathscr{R}_1$ is infinite.

2.3. Iwahori–Hecke algebras

In this subsection, we give the definition of the Iwahori–Hecke algebra via its Bernstein–Lusztig presentation, as done in [2, Section 6.6].

Let $\mathcal{R}_1 = \mathbb{Z}[(\sigma_s)_{s\in\mathscr{S}}, (\sigma'_s)_{s\in\mathscr{S}}]$, where $(\sigma_s)_{s\in\mathscr{S}}, (\sigma'_s)_{s\in\mathscr{S}}$ are two families of indeterminates satisfying the following relations:

- if $\alpha_s(Y) = \mathbb{Z}$, then $\sigma_s = \sigma'_s$;
- if $s, t \in \mathscr{S}$ are such that the order of st is finite and odd (i.e. if $\alpha_s(\alpha_t^\vee) = \alpha_t(\alpha_s^\vee) = -1$), then $\sigma_s = \sigma_t = \sigma'_s = \sigma'_t$.

To define the Iwahori–Hecke algebra $\mathcal{H}_{\mathcal{R}_1}$ associated with $\mathbb{A}$ and $(\sigma_s, \sigma'_s)_{s\in\mathscr{S}}$, we first introduce the Bernstein–Lusztig–Hecke algebra. Let ${}^{BL}\mathcal{H}_{\mathcal{R}_1}$ be the free $\mathcal{R}_1$-vector space with basis $(Z^\lambda H_w)_{\lambda\in Y, w\in W^v}$. For short, one sets $H_w = Z^0 H_w$ for $w \in W^v$ and $Z^\lambda = Z^\lambda H_1$ for $\lambda \in Y$. The *Bernstein–Lusztig–Hecke algebra* ${}^{BL}\mathcal{H}_{\mathcal{R}_1}$ is the module ${}^{BL}\mathcal{H}_{\mathcal{R}_1}$ equipped with the unique product $*$ that turns it into an associative algebra and satisfies the following relations (known as the *Bernstein–Lusztig relations*):

(BL1) $\forall\, (\lambda, w) \in Y \times W^v$, $Z^\lambda * H_w = Z^\lambda H_w$;

(BL2) $\forall\, s \in \mathscr{S}, \forall\, w \in W^v$,

$$H_s * H_w = \begin{cases} H_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\ (\sigma_s - \sigma_s^{-1}) H_w + H_{sw} & \text{if } \ell(sw) = \ell(w) - 1 \end{cases}$$

(BL3) $\forall\, (\lambda, \mu) \in Y^2$, $Z^\lambda * Z^\mu = Z^{\lambda+\mu}$;

(BL4) $\forall\, \lambda \in Y$, $\forall\, i \in I$, $H_s * Z^\lambda - Z^{s.\lambda} * H_s = Q_s(Z)(Z^\lambda - Z^{s.\lambda})$, where $Q_s(Z) = \frac{(\sigma_s - \sigma_s^{-1}) + (\sigma'_s - \sigma_s'^{-1}) Z^{-\alpha_s^\vee}}{1 - Z^{-2\alpha_s^\vee}}$.

The existence and uniqueness of such a product $*$ comes from [2, Theorem 6.2].

Definition 2.6. — *Let $\mathcal{F}$ be a field of characteristic* 0 *and $f : \mathcal{R}_1 \to \mathcal{F}$ be a ring morphism such that $f(\sigma_s)$ and $f(\sigma'_s)$ are invertible in $\mathcal{F}$ for all $s \in \mathscr{S}$. Then the* Bernstein–Lusztig–Hecke algebra of $(\mathbb{A}, (\sigma_s)_{s\in\mathscr{S}}, (\sigma'_s)_{s\in\mathscr{S}})$ *over $\mathcal{F}$ is the algebra ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} = {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{R}_1} \otimes_{\mathcal{R}_1} \mathcal{F}$. Following* [2, Section 6.6], *the* Iwahori–Hecke algebra $\mathcal{H}_{\mathcal{F}}$ *associated with $\mathcal{S}$ and $(\sigma_s, \sigma'_s)_{s\in\mathscr{S}}$ is now defined as the $\mathcal{F}$-subalgebra of ${}^{BL}\mathcal{H}_{\mathcal{F}}$ spanned by $(Z^\lambda H_w)_{\lambda\in Y^+, w\in W^v}$ (recall that $Y^+ = Y \cap \mathcal{T}$ with $\mathcal{T}$ being the Tits cone). Note that for G reductive, we recover the usual Iwahori–Hecke algebra of G, since $Y \cap \mathcal{T} = Y$.*

In certain proofs, when $\mathcal{F} = \mathbb{C}$, we will make additional assumptions on the σ_s and σ'_s, $s \in \mathscr{S}$. To avoid these assumptions, we can assume that $\sigma_s, \sigma'_s \in \mathbb{C}$ and $|\sigma_s| > 1, |\sigma'_s| > 1$ for all $s \in \mathscr{S}$.

Remark 2.7.

(1) Let $s \in \mathscr{S}$. Then if $\sigma_s = \sigma'_s$, $Q_s(Z) = \frac{(\sigma_s - \sigma_s^{-1})}{1 - Z^{-\alpha_s^\vee}}$.
(2) Let $s \in \mathscr{S}$ and $\lambda \in Y$. Then $Q_s(Z)(Z^\lambda - Z^{s.\lambda}) \in \mathcal{F}[Y]$. Indeed, $Q_s(Z)(Z^\lambda - Z^{s.\lambda}) = Q_s(Z).Z^\lambda(1 - Z^{-\alpha_s(\lambda)\alpha_s^\vee})$. Assume that $\sigma_s = \sigma'_s$. Then

$$\frac{1 - Z^{-\alpha_s(\lambda)\alpha_s^\vee}}{1 - Z^{-\alpha_s^\vee}} = \begin{cases} \sum_{j=0}^{\alpha_s(\lambda)-1} Z^{-j\alpha_s^\vee} & \text{if } \alpha_s(\lambda) \geqslant 0 \\ -Z^{\alpha_s^\vee} \sum_{j=0}^{-\alpha_s(\lambda)-1} Z^{j\alpha_s^\vee} & \text{if } \alpha_s(\lambda) \leqslant 0, \end{cases}$$

and thus $Q_s(Z)(Z^\lambda - Z^{s.\lambda}) \in \mathcal{F}[Y]$. Assume $\sigma'_s \neq \sigma_s$. Then $\alpha_s(Y) = 2\mathbb{Z}$ and a similar computation enables to conclude.
(3) From (BL4) we deduce that for all $s \in \mathscr{S}$, $\lambda \in Y$,

$$Z^\lambda * H_s - H_s * Z^{s.\lambda} = Q_s(Z)(Z^\lambda - Z^{s.\lambda}).$$

(4) When G is a split Kac–Moody group over a non-Archimedean local field $\mathcal{K}$ with residue cardinal q, we can choose $\mathcal{F}$ to be a field containing $\mathbb{Z}[\sqrt{q}^{\pm 1}]$ and take $f(\sigma_s) = f(\sigma'_s) = \sqrt{q}$ for all $s \in \mathscr{S}$.
(5) By (BL4), the family $(H_w * Z^\lambda)_{w \in W^v, \lambda \in Y}$ is also a basis of ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$.
(6) Let $w \in W^v$ and $w = s_1 \dots s_k$, with $k \in \mathbb{Z}_{\geqslant 0}$ and $s_1, \dots, s_k \in \mathscr{S}$ be a reduced expression of w. We set $\sigma_w = \sigma_{s_1} \dots \sigma_{s_k}$. This is well-defined, independently of the choice of a reduced expression of w by the conditions imposed on the σ_s and by [3, Theorem 3.3.1 (ii)].

We equip $\mathcal{F}[Y]$ with an action of W^v. For $\theta = \sum_{\lambda \in Y} a_\lambda Z^\lambda \in \mathcal{F}[Y]$ and $w \in W^v$, set $\theta^w := \sum_{\lambda \in Y} a_\lambda Z^{w.\lambda}$.

Lemma 2.8. — *Let $\theta \in \mathcal{F}[Y]$ and $w \in W^v$. Then $\theta * H_w - H_w * \theta^{w^{-1}} \in {}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}^{<w} := \bigoplus_{v<w} H_v \mathcal{F}[Y]$. In particular, ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}^{\leqslant w} := \bigoplus_{v \leqslant w} H_v \mathbb{C}[Y]$ is a left finitely generated $\mathcal{F}[Y]$-submodule of ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$.*

Proof. — We do it by induction on $\ell(w)$. Let $\theta \in \mathcal{F}[Y]$ and $w \in W^v$ be such that $u := \theta H_w - H_w \theta^{w^{-1}} \in {}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})^{<w}$. Let $s \in \mathscr{S}$ and assume that $\ell(ws) = \ell(w) + 1$. Then by (BL4):

$$\theta * H_{ws} = (H_w \theta^{w^{-1}} + u) * H_s = H_{ws}\theta^{sw^{-1}} + aH_w + uH_s,$$

for some $a \in \mathcal{F}$. Moreover, by [22, Corollary 1.3.19] and (BL2), $u * H_s \in {}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})^{<ws}$ and the lemma follows. □

Definition 2.9. — *Let $\mathcal{H}_{\mathcal{F},W^v} = \bigoplus_{w \in W^v} \mathcal{F} H_w \subset \mathcal{H}_\mathcal{F}$. Then $\mathcal{H}_{\mathcal{F},W^v}$ is a subalgebra of $\mathcal{H}_\mathcal{F}$. This is the Hecke algebra of the Coxeter group $(W^v, \mathscr{S})$.*

2.4. Principal series representations

In this subsection, we introduce the principal series representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$.

We now fix $(\mathbb{A}, (\sigma_s)_{s\in\mathscr{S}}, (\sigma'_s)_{s\in\mathscr{S}})$ as in Section 2.3 and a field $\mathcal{F}$ as in Definition 2.6. Let $\mathcal{H}_{\mathcal{F}}$ and ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ be the Iwahori–Hecke and the Bernstein–Lusztig Hecke algebras of $(\mathbb{A}, (\sigma_s)_{s\in\mathscr{S}}, (\sigma'_s)_{s\in\mathscr{S}})$ over $\mathcal{F}$.

Let $T_{\mathcal{F}} = \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathcal{F}^\times)$ be the group of homomorphisms from Y to $\mathcal{F}^*$. Let $\tau \in T_{\mathcal{F}}$. Then τ induces an algebra morphism $\tau : \mathcal{F}[Y] \to \mathcal{F}$ by the formula $\tau(\sum_{y\in Y} a_y e^y) = \sum_{y\in Y} a_y \tau(y)$, for $\sum a_y e^y \in \mathcal{F}[Y]$. This equips $\mathcal{F}$ with the structure of an $\mathcal{F}[Y]$-module.

Let $I_\tau = \mathrm{Ind}_{\mathcal{F}[Y]}^{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}}(\tau) = {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} \otimes_{\mathcal{F}[Y]} \mathcal{F}$. For example if $\lambda \in Y$, $w \in W^v$ and $s \in \mathscr{S}$, one has:

$$\begin{aligned} Z^\lambda.1 \otimes_\tau 1 = \tau(\lambda) 1 \otimes_\tau 1, \quad H_w * Z^\lambda \otimes_\tau 1 = \tau(\lambda) H_w \otimes_\tau 1 \quad \text{and} \\ Z^\lambda.H_s \otimes_\tau 1 = H_s * Z^{s.\lambda} \otimes_\tau 1 + Q_s(Z)(Z^\lambda - Z^{s.\lambda}) \otimes_\tau 1 \\ = \tau(s.\lambda) H_i \otimes_\tau 1 + \tau\big(Q_s(Z)(Z^\lambda - Z^{s.\lambda})\big) \otimes_\tau 1. \end{aligned}$$

Let $h \in I_\tau$. Write $h = \sum_{\lambda\in Y, w\in W^v} h_{w,\lambda} H_w Z^\lambda \otimes_\tau c_{w,\lambda}$, where $(h_{w,\lambda})$, $(c_{w,\lambda}) \in \mathcal{F}^{(W^v\times Y)}$, which is possible by Remark 2.7. Thus

$$h = \sum_{\lambda\in Y, w\in W^v} h_{w,\lambda} c_{w,\lambda} \tau(\lambda) H_w \otimes_\tau 1 = \left(\sum_{\lambda\in Y, w\in W^v} h_{w,\lambda} c_{w,\lambda} \tau(\lambda) H_w\right) 1 \otimes_\tau 1.$$

Thus I_τ is a principal ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module and $(H_w \otimes_\tau 1)_{w\in W^v}$ is a basis of I_τ. Moreover $I_\tau = \mathcal{H}_{W^v,\mathcal{F}}.1 \otimes_\tau 1$ (see Definition 2.9 for the definition of $\mathcal{H}_{W^v,\mathcal{F}}$).

The definition of principal series representations of $\mathcal{H}_{\mathcal{F}}$ is very similar: we replace $T_{\mathcal{F}}$ by $T_{\mathcal{F}}^+ = \mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathbb{C}) \setminus \{0\}$ and $\mathcal{F}[Y]$ by $\mathcal{F}[Y^+]$ in the definition above. If $\tau \in T_{\mathcal{F}}^+$, we denote by $I_{\tau^+}^+$ the principal series representation of $\mathcal{H}_{\mathcal{F}}$ associated with τ^+.

Remark 2.10. — Let $\tau \in T_{\mathcal{F}}$. By Lemma 2.8, $I_\tau^{\leqslant w}$ and $I_\tau^{\ngeq w} := \bigoplus_{v\in W^v | v \ngeq w} \mathcal{F} H_v \otimes_\tau 1$ are $\mathcal{F}[Y]$-submodules of I_τ. In particular $\mathcal{F}[Y].x$ is finite dimensional for all $x \in I_\tau$.

LEMMA 2.11. — *Let $\tau \in T_{\mathcal{F}}$. Let $M \subset I_\tau$ be a finite dimensional $\mathcal{F}[Y^+]$-submodule of I_τ. Then M is an $\mathcal{F}[Y]$-submodule of I_τ.*

Proof. — Let $\lambda \in Y^+$. Let $\phi_\lambda : M \to M$ be defined by $\phi_\lambda(x) = Z^\lambda.x$, for all $m \in M$. Let $x \in \ker(\phi_\lambda)$. Then $Z^{-\lambda}.Z^\lambda.x = 0 = x$ and thus ϕ_λ

is an isomorphism. Moreover, $\phi_\lambda^{-1}(x) = Z^{-\lambda}.x$ for all $x \in M$ and thus $Z^{-\lambda}.x \in M$, for all $x \in M$. As $Y^+ - Y^+ = Y$, we deduce the lemma. $\square$

Proposition 2.12. — *Let $\tau \in T_{\mathcal{F}}$ and $M \subset I_\tau$. Then M is an $\mathcal{H}_{\mathcal{F}}$-submodule of I_τ if and only if M is an ${}^{BL}\mathcal{H}_{\mathcal{F}}$-submodule of I_τ. In particular, I_τ is irreducible as a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module if and only if I_τ is irreducible as an $\mathcal{H}_{\mathcal{F}}$-module.*

Proof. — Let $M \subset I_\tau$ be a $\mathcal{H}_{\mathcal{F}}$-submodule. Then M is an $\mathcal{F}[Y^+]$ submodule of I_τ. Let $x \in M$. Then by Remark 2.10, $\mathcal{F}[Y^+].x \subset \mathcal{F}[Y].x$ is finite dimensional. Thus $M = \sum_{x\in M} \mathcal{F}[Y^+].x$ and by Lemma 2.11, M is an $\mathcal{F}[Y]$-submodule of I_τ. As ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ is generated as an algebra by $\mathcal{H}_{\mathcal{F}}$ and $\mathcal{F}[Y]$, we deduce the proposition. $\square$

2.5. The algebra ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}(T_{\mathcal{F}})$

In this subsection, we introduce an algebra ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ containing ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$. This algebra will enable us to regard the elements of I_τ as specializations at τ of certain elements of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$. When $\mathcal{F} = \mathbb{C}$, this will enable us to make $\tau \in T_{\mathbb{C}}$ vary and to use density arguments and basic algebraic geometry to study the I_τ.

2.5.1. Description of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$

Let ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ be the right $\mathcal{F}(Y)$ vector space $\bigoplus_{w\in W^v} H_w\mathcal{F}(Y)$. We equip $\mathcal{F}(Y)$ with an action of W^v. For $\theta = \frac{\sum_{\lambda\in Y} a_\lambda Z^\lambda}{\sum_{\lambda\in Y} b_\lambda Z^\lambda} \in \mathcal{F}(Y)$ and $w \in W^v$, set $\theta^w := \frac{\sum_{\lambda\in Y} a_\lambda Z^{w.\lambda}}{\sum_{\lambda\in Y} b_\lambda Z^{w.\lambda}}$.

Proposition 2.13. — *There exists a unique multiplication $*$ on ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ which equips ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ with the structure of an associative algebra and such that:*

- *$\mathcal{F}(Y)$ embeds into ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ as an algebra,*
- (BL2) *is satisfied,*
- *the following relation is satisfied:*

(BL4′) *for all $\theta \in \mathcal{F}(Y)$ and $s \in \mathscr{S}$, $\theta * H_s - H_s * \theta^s = Q_s(Z)(\theta - \theta^s)$.*

The proof of this proposition is postponed to 2.5.2.

We regard the elements of $\mathcal{F}[Y]$ as polynomial functions on $T_{\mathcal{F}}$ by setting:

$$\tau\left(\sum_{\lambda\in Y} a_\lambda Z^\lambda\right)=\sum_{\lambda\in Y} a_\lambda \tau(\lambda),$$

for all $(a_\lambda)\in\mathcal{F}^{(Y)}$. The ring $\mathcal{F}[Y]$ is a unique factorization domain. Let $\theta\in\mathcal{F}(Y)$ and $(f,g)\in\mathcal{F}[Y]\times\mathcal{F}[Y]^*$ be such that $\theta=\frac{f}{g}$ and f and g are coprime. Set $\mathcal{D}(\theta)=\{\tau\in T_{\mathcal{F}}\mid \theta(g)\neq 0\}$. Then we regard θ as a map from $\mathcal{D}(\theta)$ to $\mathcal{F}$ by setting $\theta(\tau)=\frac{f(\tau)}{g(\tau)}$ for all $\tau\in\mathcal{D}(\theta)$.

For $w\in W^v$, let $\pi^H_w : {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})\to\mathcal{F}(Y)$ be defined by $\pi_w^H(\sum_{v\in W^v} H_v\theta_v)=\theta_w$, for $(\theta_v)\in(\mathcal{H}_{W^v,\mathcal{F}})^{W^v}$ with finite support. If $\tau\in T_{\mathcal{F}}$, let $\mathcal{F}(Y)_\tau=\{\frac{f}{g}\mid f,g\in\mathcal{F}[Y] \text{ and } g(\tau)\neq 0\}\subset\mathcal{F}(Y)$. Let ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau=\bigoplus_{w\in W^v} H_w\mathcal{F}(Y)_\tau\subset {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$. This is a not a subalgebra of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ (consider for example $\frac{1}{Z^\lambda-1}*H_s=H_s*\frac{1}{Z^{s.\lambda}-1}+\dots$ for some well chosen $\lambda\in Y$, $s\in\mathscr{S}$ and $\tau\in T_{\mathbb{C}}$). It is however an $\mathcal{H}_{W^v,\mathcal{F}}-\mathcal{F}(Y)_\tau$ bimodule. For $\tau\in T_{\mathcal{F}}$, we define $\mathrm{ev}_\tau : {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau\to\mathcal{H}_{W^v,\mathcal{F}}$ by $\mathrm{ev}_\tau(h)=h(\tau)=\sum_{w\in W^v} H_w\theta_w(\tau)$ if $h=\sum_{w\in W^v} H_w\theta_w\in\mathcal{H}(Y)_\tau$. This is a morphism of $\mathcal{H}_{W^v,\mathcal{F}}-\mathcal{F}(Y)_\tau$-bimodules.

2.5.2. Construction of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$

We now prove the existence of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$. For this we use the theory of Asano and Ore of rings of fractions: ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ will be the ring ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}*(\mathcal{F}[Y]\setminus\{0\})^{-1}$.

Let $V={}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}\otimes_{\mathcal{F}[Y]}\mathcal{F}(Y)\supset {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$, where ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ is equipped with its structure of a right $\mathcal{F}[Y]$-module. As a right $\mathcal{F}(Y)$-vector space, $V=\bigoplus_{w\in W^v} H_w\mathcal{F}(Y)$. The left action of $\mathcal{F}[Y]$ on ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ extends to an action of $\mathcal{F}[Y]$ on V by setting $\theta.\sum_{w\in W^v} H_w f_w=\sum_{w\in W^v}(\theta.H_w)f_w$, for $\theta\in\mathcal{F}[Y]$ and $(f_w)\in\mathcal{F}(Y)^{W^v}$ with finite support. This equips V with the structure of an $(\mathcal{F}[Y]-\mathcal{F}(Y))$-bimodule.

Lemma 2.14. — *The left action of $\mathcal{F}[Y]$ on V extends uniquely to a left action of $\mathcal{F}(Y)$ on V. This equips V with the structure of an $(\mathcal{F}(Y)$-$\mathcal{F}(Y))$-bimodule.*

Proof. — Let $w\in W^v$ and $P\in\mathcal{F}[Y]\setminus\{0\}$. Let $V^{\leqslant w}=\bigoplus_{v\in[1,w]} H_v\mathcal{F}(Y)$. By Lemma 2.8, the map $m_P : V^{\leqslant w}\to V^{\leqslant w}$ defined by $m_P(h)=P.h$ is well-defined. Thus the left action of $\mathcal{F}[Y]$ on $V^{\leqslant w}$ induces a ring morphism $\phi_w : \mathcal{F}[Y]\to\mathrm{End}_{v.s}(V^{\leqslant w})$, where $\mathrm{End}_{v.s}(V^{\leqslant w})$ is the space of endomorphisms of the $\mathcal{F}(Y)$-vector space $V^{\leqslant w}$.

Let us prove that $\phi_w(P)$ is injective. Let $h \in V^{\leqslant w}$. Write $h = \sum_{v\in[1,w]} H_v\theta_v$, with $\theta_v \in \mathcal{F}(Y)$ for all $v \in [1,w]$. Suppose that $h \neq 0$. Let $v \in [1,w]$ be such that $\theta_v \neq 0$ and such that v is maximal for this property for the Bruhat order. By Lemma 2.8, $P * h \neq 0$ and thus $\phi_w(P)$ is injective. As $V^{\leqslant w}$ is finite dimensional over $\mathcal{F}(Y)$, we deduce that $\phi_w(P)$ is invertible for all $P \in \mathcal{F}[Y]$. Thus ϕ_w extends uniquely to a ring morphism $\widetilde{\phi_w} : \mathcal{F}(Y) \to V^{\leqslant w}$. As $(W^v, \leqslant)$ is a directed poset, there exists an increasing sequence $(w_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ (for the Bruhat order) such that $\bigcup_{n\in\mathbb{Z}_{\geqslant 0}}[1,w_n] = W^v$. Let $m,n \in \mathbb{Z}_{\geqslant 0}$ be such that $m \leqslant n$. Let $P \in \mathcal{F}[Y]$ and $f^{(m)} = \widetilde{\phi_{w_m}}(P)$ and $f^{(n)} = \widetilde{\phi_{w_n}}(P)$. Then $f^{(n)}_{|V^{\leqslant w_m}} = f^{(m)}$ and thus for all $\theta \in \mathcal{F}(Y)$ and $x \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$, $\theta.x := \widetilde{\phi}_{w_k}(\theta)(x)$ is well-defined, independently of $k \in \mathbb{Z}_{\geqslant 0}$ such that $x \in V^{\leqslant w_k}$. This defines an action of $\mathcal{F}(Y)$ on V.

Let $h \in V$, $\theta \in \mathcal{F}(Y)$ and $P \in \mathcal{F}[Y] \setminus \{0\}$. Let $x = \frac{1}{P}.h$. Then as V is an $(\mathcal{F}[Y]$-$\mathcal{F}(Y))$-bimodule, $(P * x) * \theta = h * \theta = P * (x * \theta)$ and thus $x * \theta = \frac{1}{P} * (h * \theta) = (\frac{1}{P} * h) * \theta$. Thus V is an $(\mathcal{F}(Y) - \mathcal{F}(Y))$-bimodule. $\square$

Lemma 2.15. — *The set $\mathcal{F}[Y] \subset {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ satisfies the right Ore condition: for all $P \in \mathcal{F}[Y] \setminus \{0\}$ and $h \in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} \setminus \{0\}$, $P * {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} \cap h * \mathcal{F}[Y] \neq \{0\}$.*

Proof. — Let $P \in \mathcal{F}[Y] \setminus \{0\}$ and $h \in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} \setminus \{0\}$. Then by definition, $P * (\frac{1}{P} * h) = h \in V$. Moreover, $V = \bigoplus_{w\in W^v} H_w\mathcal{F}(Y)$ and thus there exists $\theta \in \mathcal{F}[Y] \setminus \{0\}$ such that $\frac{1}{P} * h * \theta \in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} \setminus \{0\}$. Then $P * \frac{1}{P} * h * \theta = h * \theta \in P * {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}} \cap h * \mathcal{F}[Y]$, which proves the lemma. $\square$

Definition 2.16. — *Let R be a ring and r in R. Then r is said to be* regular *if for all $r' \in R \setminus \{0\}$, $rr' \neq 0$ and $r'r \neq 0$.*

Let R be a ring and $X \subset R$ a multiplicative set of regular elements. A right ring of fractions for R with respect to X *is any overring $S \supset R$ such that:*

- *Every element of X is invertible in S.*
- *Every element of S can be expressed in the form ax^{-1} for some $a \in R$ and $x \in X$.*

We can now prove Proposition 2.13. The uniqueness of such a product follows from (BL4′). By Lemma 2.8, the elements of $\mathcal{F}[Y] \setminus \{0\}$ are regular. By Lemma 2.15 and [13, Theorem 6.2], there exists a right ring of fractions ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ for ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ with respect to $\mathcal{F}[Y] \setminus \{0\}$. Then ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ is an algebra over $\mathcal{F}$ and as a vector space, ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}}) = \bigoplus_{w\in W^v}(H_w\mathcal{F}[Y])(\mathcal{F}[Y] \setminus \{0\})^{-1} = \bigoplus_{w\in W^v} H_w\mathcal{F}(Y)$.

Let $(f,g) \in \mathcal{F}[Y] \times (\mathcal{F}[Y] \setminus \{0\})$. Then it is easy to check that $g * \big(H_s * \frac{1}{g^s} + Q_s(Z)\big)(\frac{1}{g} - \frac{1}{g^s})\big) = H_s$ and thus $\frac{1}{g} * H_s = (H_s * \frac{1}{g^s} + Q_s(Z)(\frac{1}{g} - \frac{1}{g^s})$. Let $f \in \mathcal{F}[Y]$. A straightforward computation yields the formula $\frac{f}{g} * H_s = H_s * (\frac{f}{g})^s + Q_s(Z)(\frac{f}{g} - (\frac{f}{g})^s)$ which finishes the proof of Proposition 2.13.

Remark 2.17.

- Inspired by the proof of [2, Theorem 6.2] we could try to define $*$ on V as follows. Let $\theta_1, \theta_2 \in \mathcal{F}[Y]$ and $w_1, w_2 \in W^v$. Write $\theta_1 * H_{w_2} = \sum_{w \in W^v} H_w \theta_w$, with $(\theta_w) \in \mathcal{F}(Y)^{(W^v)}$. Then $(H_{w_1} * \theta_1) * (H_{w_2} * \theta_2) = \sum_{w \in W}(H_{w_1} * H_w) * (\theta_2 \theta_w)$. However it is not clear a priori that the so defined law is associative.
- Suppose that $\mathcal{H}_\mathcal{F}$ is the Iwahori–Hecke algebra associated with some masure defined in [2, Definition 2.5]. Using the same procedure as above (by taking $S = \{Y^\lambda | \lambda \in Y^+\}$), we can construct the algebra ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$ from the algebra $\mathcal{H}_\mathcal{F}$. In this particular case, this gives an alternative proof of [2, Theorem 6.2].

3. Weight decompositions and intertwining operators

Let $\tau \in T_\mathcal{F}$. In this section, we study the structure of I_τ as a $\mathcal{F}[Y]$-module and the set $\mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}-\mathrm{mod}}(I_\tau, I_{\tau'})$ for $\tau' \in T_\mathcal{F}$.

In Section 3.1, we study the weights of I_τ and decompose every ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$-submodule of I_τ as a sum of generalized weight spaces (see Lemma 3.2).

In Section 3.2, we relate intertwining operators and weight spaces. We then prove the existence of nontrivial intertwining operators $I_\tau \to I_{w.\tau}$ for all $w \in W^v$.

In Section 3.3, we prove that when W^v is infinite, then every nontrivial submodule of I_τ is infinite dimensional. We deduce that contrary to the reductive case, there exist irreducible representations of ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$ which does not embed in any I_τ.

3.1. Generalized weight spaces of I_τ

Let $\tau \in T_\mathcal{F}$. Let $x \in I_\tau$. Write $x = \sum_{w \in W^v} x_w H_w \otimes_\tau 1$, with $(x_w) \in \mathcal{F}^{(W^v)}$. Set $\mathrm{supp}(x) = \{w \in W^v \,|\, x_w \neq 0\}$. Equip W^v with the Bruhat order. If E is a finite subset of W^v, $\max(E)$ is the set of elements of E that are maximal for the Bruhat order. Let R be a binary relation on W^v (for example $R=$"$\leqslant$", $R=$"$\ngeqslant$", ...) and $w\in W^v$. One sets

$$I_\tau^{Rw}=\bigoplus_{v\in W^v\,|\,vRw}\mathcal{F}H_v\otimes_\tau 1,\quad \mathcal{H}_{W^v,\mathcal{F}}{}^{Rw}=\bigoplus_{vRw}\mathcal{F}H_v,$$

$$^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})^{Rw}=\bigoplus_{vRw}H_v\mathcal{F}(Y)$$

and

$$^{\mathrm{BL}}\mathcal{H}_\mathcal{F}^{Rw}={}^{\mathrm{BL}}\mathcal{H}(T_\mathcal{F})^{Rw}\cap{}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}=\bigoplus_{vRw}H_v\mathcal{F}[Y].$$

Let V be a vector space over $\mathcal{F}$ and $E\subset\mathrm{End}(V)$. For $\tau\in\mathcal{F}^E$ set $V(\tau)=\{v\in V\,|\,e.v=\tau(e).v,\forall\ e\in E\}$ and $V(\tau,\mathrm{gen})=\{v\in V\,|\,\exists\ k\in\mathbb{Z}_{\geqslant 0}\,|\,(e-\tau(e)\,\mathrm{Id})^k.v=0,\ \forall\ e\in E\}$. Let $\mathrm{Wt}(E)=\{\tau\in\mathcal{F}^E\,|\,V(\tau)\neq\{0\}\}$.

The following lemma is well known.

Lemma 3.1. — *Let V be a finite dimensional vector space over $\mathcal{F}$. Let $E\subset\mathrm{End}(V)$ be a subset such that for all $e,e'\in E$,*

(1) *e is triangularizable*
(2) *$ee'=e'e$.*

Then $V=\bigoplus_{\tau\in\mathrm{Wt}(E)}V(\tau,\mathrm{gen})$ and in particular $\mathrm{Wt}(E)\neq\emptyset$.

For $\tau\in T_\mathcal{F}$, set $W_\tau=\{w\in W^v\,|\,w.\tau=\tau\}$.

Let M be a $^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$-module. For $\tau\in T_\mathcal{F}$, set

$$M(\tau)=\{m\in M\,|\,P.m=\tau(P).m,\ \forall\ P\in\mathcal{F}[Y]\}$$

and

$$M(\tau,\mathrm{gen})=\{m\in M\,|\,\exists\ k\in\mathbb{Z}_{\geqslant 0}\,|\,\forall\ P\in\mathcal{F}[Y],(P-\tau(P))^k.m=0\}\supset M(\tau).$$

Let

$$\mathrm{Wt}(M)=\{\tau\in T_\mathcal{F}\,|\,M(\tau)\neq\{0\}\}$$

and

$$\mathrm{Wt}(M,\mathrm{gen})=\{\tau\in T_\mathcal{F}\,|\,M(\tau,\mathrm{gen})\neq\{0\}\}.$$

Lemma 3.2.

(1) *Let $\tau,\tau'\in T_\mathcal{F}$. Let $x\in I_\tau(\tau',\mathrm{gen})$. Then if $x\neq 0$,*

$$\max\mathrm{supp}(x)\subset\{w\in W^v\,|\,w.\tau=\tau'\}.$$

In particular, if $I_\tau(\tau',\mathrm{gen})\neq\{0\}$, then $\tau'\in W^v.\tau$ and thus

$$\mathrm{Wt}(I_\tau)\subset W^v.\tau.$$

(2) *Let* $\tau \in T_{\mathcal{F}}$. *Let* $M \subset I_\tau$ *be a* $\mathcal{F}[Y]$*-submodule of* I_τ. *Then* $\mathrm{Wt}(M) = \mathrm{Wt}(M, \mathrm{gen}) \subset W^v.\tau$ *and* $M = \bigoplus_{\chi \in \mathrm{Wt}(M)} M(\chi, \mathrm{gen})$. *In particular,* $\mathrm{Wt}(M) \neq \emptyset$.

Proof.

(1). — Let $x \in I_\tau(\tau', \mathrm{gen}) \setminus \{0\}$. Let $w \in \max\mathrm{supp}(x)$. Write $x = a_w H_w \otimes_\tau 1 + y$, where $a_w \in \mathcal{F} \setminus \{0\}$ and $y \in I_\tau^{\not\geqslant w}$. Then by Lemma 2.8,

$$\begin{aligned} Z^\lambda.x = a_w H_w Z^{w^{-1}.\lambda} \otimes_\tau 1 + y' &= \tau(w^{-1}.\lambda) a_w H_w \otimes_\tau 1 + y' \\ &= \tau'(\lambda) a_w H_w \otimes_\tau 1 + \tau'(\lambda) y, \end{aligned}$$

where $y' \in I_\tau^{\not\geqslant w}$. Therefore $w.\tau = \tau'$.

(2). — Let $w \in W^v$. Let $P \in \mathcal{F}[Y]$ and $m_P : I_\tau^{\leqslant w} \to I_\tau^{\leqslant w}$ be defined by $m_P(x) = P.x$ for all $x \in I_\tau^{\leqslant w}$. Then by Lemma 2.8, $(m_P - w.\tau(P)\,\mathrm{Id})(I_\tau^{\leqslant w}) \subset I_\tau^{< w}$. By induction on $\ell(w)$ we deduce that m_P is triangularizable on $I_\tau^{\leqslant w}$ and $\mathrm{Wt}(I_\tau^{\leqslant w}) \subset [1, w].\tau \subset W^v.\tau$.

Let $x \in M$ and $M_x = \mathcal{F}[Y].x$. By the fact that $(W^v, \leqslant)$ is a directed poset and by Lemma 2.8, there exists $w \in W^v$ such that $M_x \subset I_\tau^{\leqslant w}$. Therefore, for all $P \in \mathcal{F}[Y]$, $m_P : M_x \to M_x$ is triangularizable. Thus by Lemma 3.1,

$$\mathcal{F}[Y].x = \bigoplus_{\chi \in \mathrm{Wt}(M_x, \mathrm{gen})} M_x(\chi, \mathrm{gen}) = \bigoplus_{\chi \in W^v.\tau} M_x(\chi, \mathrm{gen}).$$

Consequently, $M = \sum_{x \in M} M_x = \bigoplus_{\chi \in \mathrm{Wt}(M, \mathrm{gen})} M(\chi, \mathrm{gen})$ and $\mathrm{Wt}(M) \subset \bigcup_{w \in W^v} \mathrm{Wt}(I_\tau^{\leqslant w}) \subset W^v.\tau$.

Let $\chi \in \mathrm{Wt}(M, \mathrm{gen})$. Let $x \in M(\chi, \mathrm{gen}) \setminus \{0\}$ and $N = \mathcal{F}[Y].x$. Then by Lemma 2.8, N is a finite dimensional submodule of I_τ. By Lemma 3.1, $\mathrm{Wt}(N) \neq \emptyset$. As $\mathrm{Wt}(N) \subset \{\chi\}$, $\chi \in \mathrm{Wt}(M)$. Thus $\mathrm{Wt}(M, \mathrm{gen}) \subset \mathrm{Wt}(M)$ and as the other inclusion is clear, we get the lemma. □

Proposition 3.3 (see [25, 4.3.3 Théorème (iii)]). — *Let* $\tau, \tau' \in T_{\mathcal{F}}$ *and* M *(resp.* M'*) be a* ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$*-submodule of* I_τ *(resp.* $I_{\tau'}$*). Assume that* $\mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(M, M') \setminus \{0\}$. *Then* $\tau' \in W^v.\tau$.

Proof. — Let $f \in \mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}}(M, M') \setminus \{0\}$. Then by Lemma 3.2 (2), there exists $w \in W^v/W_\tau$ such that $f\big(M(w.\tau, \mathrm{gen})\big) \neq \{0\}$. Then $w.\tau \in \mathrm{Wt}(I_{\tau'})$ and by Lemma 3.2 (1) the proposition follows. □

An element $\tau \in T_{\mathcal{F}}$ is said to be *regular* if $w.\tau \neq \tau$ for all $w \in W^v \setminus \{1\}$. We denote by $T_{\mathcal{F}}^{\mathrm{reg}}$ the set of regular elements of $T_{\mathcal{F}}$.

Proposition 3.4 (see [20, Proposition 1.17]). — *Let* $\tau \in T_{\mathcal{F}}$.

(1) *There exists a basis* $(\xi_w)_{w\in W^v}$ *of* I_τ *such that for all* $w\in W^v$*:*
   - $\xi_w \in I_\tau^{\leqslant w}$ *and* $\pi^H_w(\xi_w)=1$
   - $\xi_w\in I_\tau(w.\tau,\mathrm{gen})$.

   Moreover, if $w\in W^v$ *is minimal for* $\leqslant$ *among* $\{v\in W^v | v.\tau = w.\tau\}$*, then* $\xi_w\in I_\tau(w.\tau)$*. In particular,* $\mathrm{Wt}(I_\tau)=W^v.\tau$*.*

(2) *If* τ *is regular, then* $I_\tau(w.\tau,\mathrm{gen})=I_\tau(w.\tau)$ *is one dimensional for all* $w\in W^v$ *and* $I_\tau=\bigoplus_{w\in W^v} I_\tau(w.\tau)$*.*

Proof.

(1). — Let $w\in W^v$. Then by Lemma 2.8, Lemma 3.1 and Lemma 3.2,

$$I_\tau^{\leqslant w}=\bigoplus_{\overline{v}\in W^v/W_\tau} I_\tau^{\leqslant w}(\overline{v}.\tau,\mathrm{gen}).$$

Write $H_w\otimes_\tau 1=\sum_{\overline{v}\in W^v/W_\tau} x_{\overline{v}}$, where $x_{\overline{v}}\in I_\tau^{\leqslant w}(v.\tau,\mathrm{gen})$ for all $\overline{v}\in W^v/W_\tau$. Let $\overline{v}\in W^v/W_\tau$ be such that $\pi^H_w(x_{\overline{v}})\neq 0$. Then $\max\mathrm{supp}(x_{\overline{v}})=\{w\}$ and by Lemma 3.2, $w.\tau=\overline{v}.\tau$. Set $\xi_w=\frac{1}{\pi^H_w(x_{\overline{v}})}x_{\overline{v}}$. Then $(\xi_u)_{u\in W^v}$ is a basis of I_τ and has the desired properties. Let $w\in W^v$ be minimal for $\leqslant$ among $\{v\in W^v|v.\tau=w.\tau\}$. Let $\lambda\in Y$. Then by Lemma 2.8, $(Z^\lambda-w.\tau(\lambda).\xi_w)\in I_\tau(w.\tau,\mathrm{gen})\cap I_\tau^{<w}$. By Lemma 3.2, we deduce that $(Z^\lambda-w.\tau(\lambda)).\xi_w=0$ and thus that $\xi_w\in I_\tau(w.\tau)$. Thus $w.\tau\in\mathrm{Wt}(I_\tau)$ and by Lemma 3.2, $\mathrm{Wt}(I_\tau)=I_\tau$.

(2). — Suppose that τ is regular. Let $w\in W^v$, $\lambda\in Y$ and $x\in I_\tau(\tau,\mathrm{gen})$. Then by Lemma 3.2 (1), $x-\pi^H_w(x)\xi_w\in I_\tau(\tau,\mathrm{gen})\cap I_\tau^{<w}=\{0\}$. By (1), $\xi_w\in I_\tau(w.\tau)$ and thus $I_\tau(\tau)=I_\tau(\tau,\mathrm{gen})$ is one dimensional. By Lemma 3.2, we deduce that $I_\tau=\bigoplus_{w\in W^v} I_\tau(w.\tau)$. □

3.2. Intertwining operators and weight spaces

In this subsection, we relate intertwining operators and weight spaces and study some consequences. Let $\tau\in T_{\mathcal{F}}$. Using Section 3.1, we prove the existence of nonzero morphisms $I_\tau\to I_{w.\tau}$ for all $w\in W^v$. We will give a more precise construction of such morphisms in Section 4.4.

Let M be a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module and $\tau\in T_{\mathcal{F}}$. For $x\in M(\tau)$ define $\Upsilon_x: I_\tau\to M$ by $\Upsilon_x(u.1\otimes_\tau 1)=u.x$, for all $u\in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$. Then Υ_x is well-defined. Indeed, let $u\in {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ be such that $u.1\otimes_\tau 1=0$. Then $u\in\mathcal{F}[Y]$ and $\tau(u)=0$. Therefore $u.x=0$ and hence Υ_x is well-defined. The following lemma is then easy to prove.

Lemma 3.5 (Frobenius reciprocity, see [20, Proposition 1.10]). — *Let* M *be a* ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$*-module,* $\tau\in T_{\mathcal{F}}$ *and* $x\in M(\tau)$*. Then the map* $\Upsilon: M(\tau)\to$

$\mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}-mod}(I_\tau, M)$ *mapping each* $x \in M(\tau)$ *to* Υ_x *is a vector space isomorphism and* $\Upsilon^{-1}(f) = f(1 \otimes_\tau 1)$ *for all* $f \in \mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}-mod}(I_\tau, M)$.

Proposition 3.6 (see [25, (4.1.10)]). — *Let* M *be a* ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$*-module such that there exists* $\xi \in M$ *satisfying:*

(1) *there exists* $\tau \in T_{\mathcal{F}}$ *such that* $\xi \in M(\tau)$,
(2) $M = {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}.\xi$.

Then there exists a surjective morphism $\phi : \mathcal{I}_\tau \twoheadrightarrow M$ *of* ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$*-modules.*

Proof. — One can take $\phi = \Upsilon_\xi$, where Υ is as in Lemma 3.5. □

Proposition 3.7 (see [25, Théorème 4.2.4]). — *Let* M *be an irreducible representation of* ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ *containing a finite dimensional* $\mathcal{F}[Y]$*-submodule* $M' \neq \{0\}$. *Then there exists* $\tau \in T_{\mathcal{F}}$ *such that there exists a surjective morphism of* ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$*-modules* $\phi : I_\tau \twoheadrightarrow M$.

Proof. — By Lemma 3.1, there exists $\xi \in M' \setminus \{0\}$ such that $Z^\mu.\xi \in \mathcal{F}.\xi$ for all $\mu \in Y$. Let $\tau \in T_{\mathcal{F}}$ be such that $\xi \in M(\tau)$. Then we conclude with Proposition 3.6. □

Remark 3.8. — Let $\mathcal{Z}({}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}})$ be the center of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$. When W^v is finite, it is well known that ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ is a finitely generated $\mathcal{Z}({}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}})$ module and thus every irreducible representation of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ is finite dimensional. Assume that W^v is infinite. Using the same reasoning as in [1, Remark 4.32] we can prove that ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ is not a finitely generated $\mathcal{Z}({}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}})$-module. As we shall see (see Remark 4.11), when $\mathcal{F} = \mathbb{C}$, there exist irreducible infinite dimensional representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$. However we do not know if there exist an irreducible representation V of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ such that for all $x \in V \setminus \{0\}$, $\mathcal{F}[Y].x$ is infinite dimensional or equivalently, a representation which is not a quotient of a principal series representation.

Proposition 3.9 (see [20, (1.21)]). — *Let* $\tau \in T_{\mathcal{F}}$ *and* $w \in W^v$. *Then*

$$\mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_\tau, I_{w.\tau}) \neq \{0\}.$$

Proof. — By Proposition 3.4 $w.\tau \in \mathrm{Wt}(I_\tau)$ and we conclude with Lemma 3.5. □

3.3. Nontrivial submodules of I_τ are infinite dimensional

In this subsection, we prove that when W^v is infinite, then every submodule of I_τ is infinite dimensional. We then deduce that there can exist an irreducible representation of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$ such that V does not embed in any I_τ, for $\tau \in T_{\mathbb{C}}$.

Lemma 3.10. — *Assume that W^v is infinite. Let $w \in W^v$. Then there exists $s \in \mathscr{S}$ such that $sw > w$.*

Proof. — Let $D_L(w) = \{s \in \mathscr{S} \mid sw < w\}$. By the proof of [3, Lemma 3.2.3], $\mathscr{S} \not\subseteq D_L(w)$, which proves the lemma. □

Proposition 3.11 (compare [25, 4.2.4]). — *Let $\tau \in T_{\mathcal{F}}$. Let $M \subset I_\tau$ be a nonzero $\mathcal{H}_{W^v,\mathcal{F}}$-submodule. Then the dimension of M is infinite. In particular, if V is a finite dimensional irreducible representation of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$, then $\mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(V, I_\tau) = \{0\}$ for all $\tau \in T_{\mathcal{F}}$.*

Proof. — Let $m \in M \setminus \{0\}$. Let $\ell(m) = \max\{\ell(v) \mid v \in \mathrm{supp}(m)\}$. Let $w \in \mathrm{supp}(m)$ be such that $\ell(w) = \ell(m)$. By Lemma 3.10 there exists $(s_n) \in \mathscr{S}^{\mathbb{Z}_{\geqslant 1}}$ such that if $w_1 = w$ and $w_{n+1} = s_n w_n$ for all $n \in \mathbb{Z}_{\geqslant 1}$, one has $\ell(w_{n+1}) = \ell(w_n) + 1$ for all $n \in \mathbb{Z}_{\geqslant 1}$. Let $m_1 = m$ and $m_{n+1} = H_{s_n}.m_n$ for all $n \in \mathbb{Z}_{\geqslant 1}$. Then for all $n \in \mathbb{Z}_{\geqslant 1}$, $w_n \in \max\big(\mathrm{supp}(m_n)\big)$, which proves that M is infinite dimensional. □

As we shall see in Appendix A, there can exist finite dimensional representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$.

4. Study of the irreducibility of I_τ

In this section, we study the irreducibility of I_τ.

In Section 4.1, we describe certain intertwining operators between I_τ and $I_{s.\tau}$, for $s \in \mathscr{S}$ and $\tau \in T_{\mathcal{F}}$. For this, we introduce elements $F_s \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ such that $F_s(\chi) \otimes_\chi 1 \in I_\chi(s.\chi)$ for all $\chi \in \mathcal{T}_{\mathcal{F}}$ for which this is well-defined.

In Section 4.2, we establish that the condition (2) appearing in Theorems 1.1, 1.3 and 1.4 is a necessary condition for the irreducibility of I_τ. This conditions comes from the fact that when I_τ is irreducible, certain intertwinners have to be isomorphisms.

In Section 4.3, we prove an irreducibility criterion for I_τ involving the dimension of $I_\tau(\tau)$ and the values of τ (see Theorem 4.8). We then deduce Matsumoto criterion.

In Section 4.4 we introduce and study, for every $w \in W^v$, an element $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ such that $F_w(\chi) \otimes_\chi 1 \in I_\chi(w.\chi)$ for every $\chi \in T_{\mathbb{C}}$ for which this is well-defined.

In Section 4.5 we prove one implication of Kato's criterion (see Proposition 4.17).

The definition we gave for I_τ is different from the definition of Matsumoto (see [25, (4.1.5)]). It seems to be well known that these definitions are equivalent. We justify this equivalence in Section 4.6. We also explain why it seems difficult to adapt Kato's proof in our framework.

4.1. Intertwining operators associated with simple reflections

Let $s \in \mathscr{S}$. In this subsection we define and study an element $F_s \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ such that $F_s(\chi) \otimes_\chi 1 \in I_\chi(s.\chi)$ for all χ such that $F_s(\chi)$ is well-defined.

Let $s \in \mathscr{S}$ and $T_s = \sigma_s H_s$. Let $w \in W^v$ and $w = s_1 \dots s_k$ be a reduced writing. Set $T_w = T_{s_1} \dots T_{s_k}$. This is independent of the choice of the reduced writing by [2, 6.5.2].

Set $B_s = T_s - \sigma_s^2 \in \mathcal{H}_{W^v,\mathcal{F}}$. One has $B_s^2 = -(1+\sigma_s^2)B_s$. Let $\zeta_s = -\sigma_s Q_s(Z) + \sigma_s^2 \in \mathcal{F}(Y) \subset {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$. When $\sigma_s = \sigma_s' = \sqrt{q}$ for all $s \in \mathscr{S}$, we have $\zeta_s = \frac{1-qZ^{-\alpha_s^\vee}}{1-Z^{-\alpha_s^\vee}} \in \mathcal{F}(Y)$. Let $F_s = B_s + \zeta_s \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$.

Let $\alpha^\vee \in \Phi^\vee$. Write $\alpha^\vee = w.\alpha_s^\vee$ for $w \in W^v$ and $s \in \mathscr{S}$. We set $\zeta_{\alpha^\vee} = (\zeta_s)^w$.

Let $\alpha^\vee \in \Phi^\vee$. Write $\alpha = w.\alpha_s^\vee$, with $w \in W^v$ and $s \in \mathscr{S}$. We set $\sigma_{\alpha^\vee} = \sigma_s$ and $\sigma'_{\alpha^\vee} = w.\sigma'_s$. This is well-defined by Lemma 2.4 and by the relations on the σ_t, $t \in \mathscr{S}$ (see Section 2.3).

The ring $\mathcal{F}[Y]$ is a unique factorization domain. For $\alpha^\vee$, write $\zeta_{\alpha^\vee} = \frac{\zeta_{\alpha^\vee}^{\mathrm{num}}}{\zeta_{\alpha^\vee}^{\mathrm{den}}}$ where $\zeta_{\alpha^\vee}^{\mathrm{num}}, \zeta_{\alpha^\vee}^{\mathrm{den}} \in \mathcal{F}[Y]$ are pairwise coprime. For example if $\alpha^\vee \in \Phi^\vee$ is such that $\sigma_{\alpha^\vee} = \sigma'_{\alpha^\vee}$ we can take $\zeta_{\alpha^\vee}^{\mathrm{den}} = 1 - Z^{-\alpha^\vee}$ and in any case we will choose $\zeta_{\alpha^\vee}^{\mathrm{den}}$ among $\{1 - Z^{-\alpha^\vee}, 1 + Z^{-\alpha^\vee}, 1 - Z^{-2\alpha^\vee}\}$.

Remark 4.1. — Let $\tau \in T_{\mathcal{F}}$ and $r = r_{\alpha^\vee} \in \mathscr{R}$. Suppose that $r.\tau \neq \tau$. Then $\zeta_{\alpha^\vee}^{\mathrm{den}}(\tau) \neq 0$. Indeed, let $\lambda \in Y$ be such that $\tau(r.\lambda) \neq \tau(\lambda)$. Then $\tau(r.\lambda - \lambda) = \tau(\alpha_r^\vee)^{\alpha_r(\lambda)} \neq 1$. Suppose $\sigma_{\alpha^\vee} = \sigma'_{\alpha^\vee}$, then $\zeta_{\alpha^\vee}^{\mathrm{den}} = 1 - Z^{-\alpha_r^\vee}$ and thus $\tau(\zeta_{\alpha^\vee}^{\mathrm{den}}) \neq 0$. Suppose $\sigma_r = \sigma'_r$. Then $\alpha_r(\lambda) \in 2\mathbb{Z}$ thus $\tau(\alpha_r^\vee) \notin \{-1, 1\}$ and hence $\tau(\zeta_{\alpha^\vee}^{\mathrm{den}}) \neq 0$.

Lemma 4.2. — *Let $s \in \mathscr{S}$ and $\theta \in \mathcal{F}(Y)$. Then*

$$\theta * F_s = F_s * \theta^s.$$

In particular, for all $\tau \in T_{\mathcal{F}}$ such that $\tau(\zeta_s^{\mathrm{den}}) \neq 0$, $F_s(\tau) \otimes_\tau 1 \in I_\tau(s.\tau)$ and $F_s(\tau) \otimes_{s.\tau} 1 \in I_{s.\tau}(\tau)$.

Proof. — Let $\lambda \in Y$. Then

$$\begin{aligned} Z^\lambda * B_s - B_s * Z^{s.\lambda} &= \sigma_s(Z^\lambda * H_s - H_s * Z^{s.\lambda}) + \sigma_s^2(Z^{s.\lambda} - Z^\lambda) \\ &= -\sigma_s Q_s(Z)(Z^{s\lambda} - Z^\lambda) + \sigma_s^2(Z^{s.\lambda} - Z^\lambda) \\ &= \zeta_s(Z^{s.\lambda} - Z^\lambda). \end{aligned}$$

Thus $Z^\lambda * F_s = Z^\lambda * (B_s + \zeta_s) = F_s * Z^{s.\lambda}$ and hence $\theta * F_s = F_s * \theta^s$ for all $\theta \in \mathcal{F}[Y]$.

Let $\theta \in \mathcal{F}[Y] \setminus \{0\}$. Then $\theta * (F_s * \frac{1}{\theta^s}) = F_s$ and thus $\frac{1}{\theta} * F_s = F_s * \frac{1}{\theta^s}$. Lemma follows. □

Lemma 4.3. — *Let $s \in \mathscr{S}$. Then $F_s^2 = \zeta_s \zeta_s^s \in \mathcal{F}(Y) \subset {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$.*

Proof. — By Lemma 4.2, one has:

$$\begin{aligned} F_s^2 &= (B_s + \zeta_s) * F_s \\ &= B_s * F_s + F_s * \zeta_s^s \\ &= B_s^2 + B_s \zeta_s + B_s \zeta_s^s + \zeta_s \zeta_s^s \\ &= B_s(-1 - \sigma_s^2 + \zeta_s + \zeta_s^s) + \zeta_s \zeta_s^s \\ &= \zeta_s \zeta_s^s. \end{aligned}$$
□

4.2. A necessary condition for irreducibility

In this subsection, we establish that the condition (2) appearing in Theorems 1.1, 1.3 and 1.4 is a necessary condition for the irreducibility of I_τ.

Recall the definition of Υ from Section 3.2.

Lemma 4.4. — *Let $\tau \in T_{\mathcal{F}}$ and $s \in \mathscr{S}$ be such that $\tau(\zeta_s^{\mathrm{den}})\tau((\zeta_s^{\mathrm{den}})^s) \neq 0$. Let $\phi(\tau, s.\tau) = \Upsilon_{F_s(\tau) \otimes_{s.\tau} 1} : I_\tau \to I_{s.\tau}$ and $\phi(s.\tau, \tau) = \Upsilon_{F_s(\tau)\otimes_\tau 1} : I_{s.\tau} \to I_\tau$. Then*

$$\phi(s.\tau, \tau) \circ \phi(\tau, s.\tau) = \tau(\zeta_s \zeta_s^s)\,\mathrm{Id}_{I_\tau} \ \text{ and } \phi(\tau, s.\tau) \circ \phi(s.\tau, \tau) = \tau(\zeta_s \zeta_s^s)\,\mathrm{Id}_{I_{s.\tau}}.$$

Proof. — By Lemma 4.2 and Lemma 3.5, $\phi(s.\tau, \tau)$ and $\phi(\tau, s.\tau)$ are well-defined. Let $f = \phi(s.\tau, \tau) \circ \phi(\tau, s.\tau) \in \mathrm{End}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_\tau)$. Then by Lemma 4.2 and Lemma 4.3:

$$\begin{aligned} f(1 \otimes_\tau 1) &= \phi(s.\tau, \tau)\big(F_s(\tau) \otimes_{s.\tau} 1\big) \\ &= F_s(\tau).\phi(s.\tau, \tau)\big(1 \otimes_{s.\tau} 1\big) \\ &= F_s(\tau)^2 \otimes_\tau 1 \\ &= \tau(\zeta_s \zeta_s^s) \otimes_\tau 1. \end{aligned}$$

By symmetry, we get the lemma. □

Let $\mathcal{U}_{\mathcal{F}}$ be the set of $\tau \in T_{\mathcal{F}}$ such that for all $\alpha^\vee \in \Phi^\vee$, $\tau(\zeta_{\alpha^\vee}^{\mathrm{num}}) \neq 0$. When $\sigma_s = \sigma_s' = \sqrt{q}$ for all $s \in \mathscr{S}$, then $\mathcal{U}_{\mathcal{F}} = \{\tau \in T_{\mathcal{F}} \mid \tau(\alpha^\vee) \neq q,\ \forall\, \alpha^\vee \in \Phi^\vee\}$.

We assume that for all $s \in \mathscr{S}$, $\sigma_s' \notin \{\sigma_s^{-1}, -\sigma_s, -\sigma_s^{-1}\}$. Under this condition, if $\alpha^\vee \in \Phi^\vee$ and $\tau \in T_{\mathcal{F}}$ are such that $\tau(\zeta_{\alpha^\vee}^{\mathrm{den}}) = 0$, then $\tau(\zeta_{\alpha^\vee}^{\mathrm{num}}) \neq 0$.

Lemma 4.5.

(1) *Let* $\tau \in \mathcal{U}_{\mathcal{F}}$*. Then for all* $w \in W^v$*,* I_τ *and* $I_{w.\tau}$ *are isomorphic as* ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$*-modules.*
(2) *Let* $\tau \in T_{\mathcal{F}}$ *be such that* I_τ *is irreducible. Then* $\tau \in \mathcal{U}_{\mathcal{F}}$*.*

Proof. — Let $\tau \in \mathcal{U}_{\mathcal{F}}$. Let $w \in W^v$ and $\widetilde{\tau} = w.\tau$. Let $s \in \mathscr{S}$. Assume that $s.\widetilde{\tau} \neq \widetilde{\tau}$. Then by Remark 4.1, $\zeta_s^{\mathrm{den}}(\widetilde{\tau}) \neq 0$ and $\zeta_s^{\mathrm{den}}(s.\widetilde{\tau}) \neq 0$. Therefore $\zeta_s(\tau)$, $\zeta_s(s.\widetilde{\tau})$ are well-defined and hence $F_s(\widetilde{\tau})$, $F_s(\widetilde{\tau})$ are well-defined. Let $\phi(\widetilde{\tau}, s.\widetilde{\tau}) = \Upsilon_{F_s(\tilde{\tau})\otimes_{s.\tilde{\tau}}1} : I_{\tilde{\tau}} \to I_{s.\tilde{\tau}}$ and $\phi(s.\widetilde{\tau}, \widetilde{\tau}) = \Upsilon_{F_s(\tilde{\tau})\otimes_{\tilde{\tau}}1} : I_{s.\tilde{\tau}} \to I_{\tilde{\tau}}$. Then by Lemma 4.4,

$$\begin{aligned} &\phi(s.\widetilde{\tau}, \widetilde{\tau}) \circ \phi(\widetilde{\tau}, s.\widetilde{\tau}) = \widetilde{\tau}(\zeta_s \zeta_s^s)\,\mathrm{Id}_{I_{\tilde{\tau}}} \\ \text{and}\quad &\phi(\widetilde{\tau}, s.\widetilde{\tau}) \circ \phi(s.\widetilde{\tau}, \widetilde{\tau}) = \widetilde{\tau}(\zeta_s \zeta_s^s)\,\mathrm{Id}_{I_{s.\tilde{\tau}}}. \end{aligned}$$

By definition of $\mathcal{U}_{\mathcal{F}}$, $\widetilde{\tau}(\zeta_s\zeta_s^s) = \widetilde{\tau}(\zeta_s)\widetilde{\tau}(\zeta_s^s) \neq 0$ and thus $\phi(\widetilde{\tau}, s.\widetilde{\tau})$ and $\phi(s.\widetilde{\tau}, \widetilde{\tau})$ are isomorphisms. Consequently $I_{\tilde{\tau}}$ is isomorphic to $I_{s.\tilde{\tau}}$ and (1) follows by induction.

Let $\tau \in T_{\mathcal{F}}$ be such that I_τ is irreducible. Let $s \in \mathscr{S}$.

Suppose $\tau(\zeta_s^{\mathrm{den}}) = 0$. Then by assumption, $\tau(\zeta_s^{\mathrm{num}}) \neq 0$. Moreover by Remark 4.1, $I_{s.\tau} = I_\tau$.

Suppose now $\tau(\zeta_s^{\mathrm{den}}) \neq 0$. Then (with the same notations as in Lemma 4.4), $\phi(s.\tau, \tau) \neq 0$ and $\mathrm{Im}\big(\phi(s.\tau,\tau)\big)$ is a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-submodule of I_τ: $\mathrm{Im}\big(\phi(s.\tau,\tau)\big) = I_\tau$. Therefore $\phi(\tau, s.\tau) \circ \phi(s.\tau, \tau) \neq 0$. Thus by Lemma 4.4, $\phi(\tau, s.\tau)$ is an isomorphism and $\tau(\zeta_s\zeta_s^s) \neq 0$. In particular, $\tau(\zeta_s^{\mathrm{num}}) \neq 0$.

Therefore in any cases, I_τ is isomorphic to $I_{s.\tau}$ and $\tau(\zeta_s^{\mathrm{num}}) \neq 0$. By induction we deduce that $I_{w.\tau}$ is isomorphic to I_τ. Thus $I_{w.\tau}$ is irreducible for all $w \in W^v$. Thus $w.\tau(\zeta_s^{\mathrm{num}}) \neq 0$ for all $w \in W^v$ and $s \in \mathscr{S}$, which proves that $\tau \in \mathcal{U}_{\mathcal{F}}$. □

Lemma 4.6. — *Let* $\tau \in T_{\mathcal{F}}$ *be such that* $I_{w.\tau} \simeq I_\tau$ *(as a* ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$*-module) for all* $w \in W^v$*. Then for all* $w \in W^v$*, there exists a vector space isomorphism* $I_\tau(\tau) \simeq I_\tau(w.\tau)$*.*

Proof. — Let $w \in W^v$. Then by hypothesis, $\mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_\tau, I_\tau) \simeq \mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_{w.\tau}, I_{w.\tau})$. Let $\phi : I_\tau \to I_{w.\tau}$ be a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module isomorphism. Then ϕ induces an isomorphism of vector spaces $I_\tau(w.\tau) \simeq I_{w.\tau}(w.\tau)$. By Lemma 3.5,

$$\begin{aligned} I_\tau(\tau) \simeq \mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_\tau, I_\tau) &\simeq \mathrm{Hom}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}-\mathrm{mod}}(I_{w.\tau}, I_{w.\tau}) \\ &\simeq I_{w.\tau}(w.\tau) \simeq I_\tau(w.\tau). \end{aligned}$$ □

4.3. An irreducibility criterion for I_τ

In this subsection, we give a characterization of irreducibility for I_τ, for $\tau \in T_{\mathbb{C}}$.

If $\mathcal{B}$ is a $\mathbb{C}$-algebra with unity e and $a \in \mathcal{B}$, one sets

$$\mathrm{Spec}(a) = \{\lambda \in \mathbb{C} |\ a - \lambda e \text{ is not invertible}\}.$$

Recall the following theorem of Amitsur (see Théorème B.I of [31]):

Theorem 4.7. — *Let $\mathcal{B}$ be a $\mathbb{C}$-algebra with unity e. Assume that the dimension of $\mathcal{B}$ over $\mathbb{C}$ is countable. Then for all $a \in \mathcal{B}$,* $\mathrm{Spec}(a) \neq \emptyset$.

Recall that $\mathcal{U}_{\mathbb{C}}$ is the set of $\tau \in T_{\mathbb{C}}$ such that for all $\alpha^\vee \in \Phi^\vee$, $\tau(\zeta_{\alpha^\vee}^{\mathrm{num}}) \neq 0$.

Theorem 4.8. — *Let $\tau \in T_{\mathbb{C}}$. Then the following are equivalent:*

(1) *I_τ is irreducible,*
(2) *$I_\tau(\tau) = \mathbb{C}.1 \otimes_\tau 1$ and $\tau \in \mathcal{U}_{\mathbb{C}}$,*
(3) $\mathrm{End}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}-\mathrm{mod}}(I_\tau) = \mathbb{C}.\mathrm{Id}$ *and $\tau \in \mathcal{U}_{\mathbb{C}}$.*

Proof. — Assume that $\mathcal{B} = \mathrm{End}_{{}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}-\mathrm{mod}}(I_\tau) \neq \mathbb{C}\,\mathrm{Id}$. By Lemma 3.5 and the fact that I_τ has countable dimension, $\mathcal{B}$ has countable dimension. Let $\phi \in \mathcal{B} \setminus \mathbb{C}\,\mathrm{Id}$. Then by Amitsur Theorem, there exists $\gamma \in \mathrm{Spec}(\phi)$. Then $\phi - \gamma\,\mathrm{Id}$ is non-injective or non-surjective and therefore $\mathrm{Ker}(\phi - \gamma\,\mathrm{Id})$ or $\mathrm{Im}(\phi - \gamma\,\mathrm{Id})$ is a non-trivial ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$-module, which proves that I_τ is reducible. Using Lemma 4.5 we deduce that (1) implies (3).

By Lemma 3.5, (2) is equivalent to (3).

Let $\tau \in T_{\mathbb{C}}$ satisfying (2). Then by Lemma 4.5 and Lemma 4.6, $\dim I_\tau(w.\tau) = 1$ for all $w \in W^v$. By Lemma 4.5, for all $w \in W^v$, there exists an isomorphism of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$-modules $f_w : I_{w.\tau} \to I_\tau$. As $\mathbb{C}.f_w(1 \otimes_{w.\tau} 1) \subset I_\tau(w.\tau)$ we deduce that $I_\tau(w.\tau) = \mathbb{C}.f_w(1 \otimes_{w.\tau} 1)$ for all $w \in W^v$.

Let $M \neq \{0\}$ be a ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$-submodule of I_τ. Let $x \in M \setminus \{0\}$. Then $M' = \mathbb{C}[Y].x$ is a finite dimensional $\mathbb{C}[Y]$-module. Thus by Lemma 3.1), there exists $\xi \in M' \setminus \{0\}$ such that $Z^\lambda.\xi \in \mathbb{C}.\xi$ for all $\lambda \in Y$. Then $\xi \in I_\tau(\tau')$ for some $\tau' \in T_{\mathbb{C}}$. By Lemma 3.2, $\tau' = w.\tau$, for some $w \in W^v$. Thus $\xi \in \mathbb{C}^* f_w(1 \otimes_{w.\tau} 1)$. One has

$${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}.\xi = f_w({}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}.1 \otimes_{w.\tau} 1) = f_w(I_{w.\tau}) = I_\tau \subset M.$$

Hence I_τ is irreducible, which finishes the proof of the theorem. □

Remark 4.9. — Actually, our proof of the equivalence between (2) and (3), and of the fact that (2) implies (1) is valid when $\mathcal{F}$ is a field, without assuming $\mathcal{F} = \mathbb{C}$.

Recall that an element $\tau \in T_{\mathcal{F}}$ is called regular if $w.\tau \neq \tau$ for all $w \in W^v$.

Corollary 4.10 (see [25, Théorème 4.3.5]). — *Let $\tau \in T_{\mathcal{F}}$ be regular. Then I_τ is irreducible if and only if $\tau \in \mathcal{U}_{\mathcal{F}}$.*

Proof. — By Lemma 4.5, if I_τ is irreducible, then $\tau \in \mathcal{U}_{\mathcal{F}}$.

Assume that $\tau \in \mathcal{U}_{\mathcal{F}}$. Then by Proposition 3.4 (2), $\dim I_\tau(\tau) = 1$ and we conclude with Theorem 4.8 and Remark 4.9. $\square$

Remark 4.11. — Assume that $\mathcal{F} = \mathbb{C}$ and that $\sigma_s = \sigma'_s = \sqrt{q}$ for all $s \in \mathscr{S}$, for some $q \in \mathbb{Z}_{\geqslant 2}$. Let $(y_j)_{j\in J}$ be a $\mathbb{Z}$-basis of Y. Then the map $T_{\mathbb{C}} \to (\mathbb{C}^*)^J$ defined by $\tau \in T_{\mathbb{C}} \mapsto (\tau(y_j))_{j\in J}$ is a group isomorphism. We equip $T_{\mathbb{C}}$ with a Lebesgue measure through this isomorphism. Then the set of measurable subsets of $T_{\mathbb{C}}$ having full measure does not depend on the choice of the $\mathbb{Z}$-basis of Y. Then $\mathcal{U}_{\mathbb{C}} = \bigcap_{\alpha^\vee\in\Phi^\vee}\{\tau \in T_{\mathbb{C}} \mid \tau(\alpha^\vee) \neq q\}$ has full measure in $T_{\mathbb{C}}$. Moreover $T_{\mathbb{C}}^{\mathrm{reg}} \supset \bigcap_{\lambda\in Y\setminus\{0\}}\{\tau \in T_{\mathbb{C}} \mid \tau(\lambda) \neq 1\}$ has full measure in $T_{\mathbb{C}}$ and thus $\{\tau \in T_{\mathbb{C}} \mid I_\tau \text{ is irreducible}\}$ has full measure in $T_{\mathbb{C}}$.

Recall that $\mathscr{R} = \{wsw^{-1} \mid w \in W^v, s \in \mathscr{S}\}$ is the set of reflections of W^v. For $\tau \in T_{\mathbb{C}}$, set $W_\tau = \{w \in W^v \mid w.\tau = \tau\}$, $\Phi^\vee_{(\tau)} = \{\alpha^\vee \in \Phi^\vee_+ \mid \zeta^{\mathrm{den}}_{\alpha^\vee}(\tau) = 0\}$, $\mathscr{R}_{(\tau)} = \{r = r_{\alpha^\vee} \in \mathscr{R} \mid \alpha^\vee \in \Phi^\vee_{(\tau)}\}$ and

$$W_{(\tau)} = \langle \mathscr{R}_{(\tau)}\rangle = \langle\{r = r_{\alpha^\vee} \in \mathscr{R} \mid \zeta^{\mathrm{den}}_{\alpha^\vee}(\tau) = 0\}\rangle \subset W^v.$$

By Remark 4.1, $W_{(\tau)} \subset W_\tau$. It is moreover normal in W_τ. When $\alpha_s(Y) = \mathbb{Z}$ for all $s \in \mathscr{S}$, then $W_{(\tau)} = \langle W_\tau \cap \mathscr{R}\rangle$.

Corollary 4.12. — *Let $\tau \in T_{\mathcal{F}}$ be such that $W_\tau = W_{(\tau)} = \{1, t\}$ for some reflection t. Then I_τ is irreducible if and only if $\tau \in \mathcal{U}_{\mathcal{F}}$.*

Proof. — By Lemma 4.5, if I_τ is irreducible, then $\tau \in \mathcal{U}_{\mathcal{F}}$. Conversely, let $\tau \in \mathcal{U}_{\mathcal{F}}$ be such that $W_\tau = W_{(\tau)} = \{1, t\}$, for some $t \in \mathscr{R}$. Write $t = v^{-1}sv$ for $s \in \mathscr{S}$ and $v \in W^v$. Let $\widetilde{\tau} = v.\tau$. One has $s.\widetilde{\tau} = \widetilde{\tau}$ and $W_{\widetilde{\tau}} = \{1, s\}$. By Lemma 3.2, $I_{\widetilde{\tau}}(\widetilde{\tau}) \subset I_{\widetilde{\tau}}^{\leqslant s}$.

Let $\lambda \in Y$. Then $Z^\lambda.H_s \otimes_{\widetilde{\tau}} 1 = \widetilde{\tau}(\lambda)H_s \otimes_{\widetilde{\tau}} 1 + \widetilde{\tau}(Q_s(Z)(Z^\lambda - Z^{s.\lambda}))1 \otimes_{\widetilde{\tau}} 1$.

Suppose $\sigma_s = \sigma'_s$. Then as $W_{(\widetilde{\tau})} = v.W_{(\tau)}.v^{-1} = \{1, s\}$, one has $\widetilde{\tau}(\alpha_s^\vee) = 1$. By Remark 2.7, $\widetilde{\tau}((Q_s(Z)(Z^\lambda - Z^{s.\lambda})) = (\sigma_s - \sigma_s^{-1})\alpha_s(\lambda)$. As there exists $\lambda \in Y$ such that $\alpha_s(\lambda) \neq 0$, we deduce that $H_s \otimes_{\widetilde{\tau}} 1 \notin I_{\widetilde{\tau}}(\widetilde{\tau})$ and thus $I_{\widetilde{\tau}}(\widetilde{\tau}) = \mathcal{F}.1 \otimes_{\widetilde{\tau}} 1$. Similarly, if $\sigma_s \neq \sigma'_s$ then $I_{\widetilde{\tau}}(\widetilde{\tau}) = \mathcal{F}.1 \otimes_{\widetilde{\tau}} 1$. By Theorem 4.8 and Remark 4.9, we deduce that $I_{\widetilde{\tau}}$ is irreducible. By Lemma 4.5 we deduce that I_τ is isomorphic to $I_{\widetilde{\tau}}$ and thus I_τ is irreducible. $\square$

4.4. Weight vectors regarded as rational functions

In this subsection, we introduce and study elements $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$, $w \in W^v$, such that for all $\chi \in T_{\mathcal{F}}$ such that $F_w(\chi)$ is well-defined, $F_w(\chi) \otimes_\chi 1 \in I_\chi(w.\chi)$.

For $w \in W^v$, let $\pi_w^T : {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}}) \to \mathcal{F}(Y)$ be the right $\mathcal{F}(Y)$-module morphism defined by $\pi_w^T(T_v) = \delta_{v,w}$ for all $v \in W^v$.

Lemma 4.13. — *Let $\mathcal{F}'$ be an uncountable field containing $\mathcal{F}$. Let $P \in \mathcal{F}[Y]$ be such that $P(\tau) = 0$ for all $\tau \in T_{\mathcal{F}'}^{\mathrm{reg}}$. Then $P = 0$.*

Proof. — Let $\mathcal{F}_0 \subset \mathcal{F}$ be a countable field (one can take $\mathcal{F}_0 = \mathbb{Q}$ or $\mathcal{F}_0 = \mathbb{F}_\ell$ for some prime power ℓ). Write $P = \sum_{\lambda \in Y} a_\lambda Z^\lambda$, with $a_\lambda \in \mathcal{F}$ for all $\lambda \in Y$. Let $(y_j)_{j \in J}$ be a $\mathbb{Z}$-basis of Y and $X_j = Z^{y_j}$ for all $j \in J$. Let $\mathcal{F}_1 = \mathcal{F}_0(a_\lambda | \lambda \in Y)$. Let $(x_j)_{j \in J} \in (\mathcal{F}')^J$ be algebraically independent over $\mathcal{F}_1$. Let $\tau \in T_{\mathcal{F}'}$ be defined by $\tau(y_j) = x_j$ for all $j \in J$.

Let us prove that $\tau \in T_{\mathcal{F}}^{\mathrm{reg}}$. Let $w \in W^v \setminus \{1\}$. Let $\lambda \in Y$ be such that $w^{-1}.\lambda - \lambda \neq 0$. Write $w^{-1}.\lambda - \lambda = \sum_{j \in J} n_j y_j$ with $n_j \in \mathbb{Z}$ for all $j \in J$. Let $Q = \prod_{j \in J} Z_j^{n_j} \in \mathcal{F}_1[Z_j, j \in J]$. Then $Q \neq 1$ and thus $\tau(w^{-1}.\lambda - \lambda) = Q((x_j)_{j \in J}) \neq 1$. Thus $w.\tau \neq \tau$ and $\tau \in T_{\mathcal{F}'}^{\mathrm{reg}}$. Thus $P(\tau) = 0$ and by choice of $(x_j)_{j \in J}$ this implies $P = 0$. □

Let $w \in W^v$. Let $w = s_1 \dots s_r$ be a reduced expression of w. Set $F_w = F_{s_r} \dots F_{s_1} = (B_{s_r} + \zeta_{s_r}) \dots (B_{s_1} + \zeta_{s_1}) \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$. By the lemma below, this does not depend on the choice of the reduced expression of w.

Lemma 4.14 (see [29, Lemma 4.3]). — *Let $w \in W^v$.*

1. *The element $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$ is well-defined, i.e. it does not depend on the choice of a reduced expression for w.*
2. *One has $F_w - T_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{<w}$.*
3. *If $\theta \in \mathcal{F}(Y)$, then $\theta * F_w = F_w * \theta^{w^{-1}}$.*
4. *If $\tau \in T_{\mathcal{F}}$ is such that $\zeta_{\beta^\vee} \in \mathcal{F}(Y)_\tau$ for all $\beta^\vee \in N_{\Phi^\vee}(w)$, then $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$ and $F_w(\tau).1 \otimes_\tau 1 \in I_\tau(w.\tau)$.*
5. *Let $\tau \in T_{\mathcal{F}}^{\mathrm{reg}}$. Then $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$.*

Proof. — Let us prove (4) by induction on $\ell(w)$. By Lemma 4.2, $\theta * F_w = F_w * \theta^{w^{-1}}$ for all $\theta \in \mathcal{F}(Y)$. Let $n \in \mathbb{Z}_{\geqslant 0}$ and assume that (4) is true for all $w \in W^v$ such that $\ell(w) \leqslant n$. Let $w \in W^v$ be such that $\ell(w) \leqslant n+1$. Write $w = sv$, with $s \in \mathscr{S}$ and $\ell(v) \leqslant n$. By Lemma 2.4, $N_{\Phi^\vee}(sv) = N_{\Phi^\vee}(v) \cup \{v^{-1}.\alpha_s^\vee\}$. Let $\tau \in T_{\mathcal{F}}$ be such that $\zeta_{\alpha^\vee} \in \mathcal{F}(Y)_\tau$ for all $\alpha^\vee \in N_{\Phi^\vee}(w)$. One has $F_w = (B_s + \zeta_s) * F_v$. As $F_v \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$ and ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$ is a left $\mathcal{H}_{W^v,\mathcal{F}}$-submodule of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})$, $B_s * F_v \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$. One has $\zeta_s * F_v = F_v * \zeta_s^{v^{-1}} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$ and hence $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$.

Let $\tau \in T_{\mathcal{F}}$ be such that $\zeta_{\alpha^\vee} \in \mathcal{F}(Y)_\tau$ for all $\alpha^\vee \in N_{\Phi^\vee}(w)$. Let $\theta \in \mathcal{F}[Y]$. Then

$$(\theta * F_w)(\tau) = (F_w * \theta^{w^{-1}})(\tau) = \tau(\theta^{w^{-1}})\tau(F_w(\tau)),$$

which finishes the proof of (4).

Let $\tau \in T_{\mathcal{F}}^{\mathrm{reg}}$ and $\alpha^\vee \in \Phi^\vee$. Write $\alpha^\vee = w.\alpha_s^\vee$ for $w \in W^v$ and $s \in \mathscr{S}$. Then $s.w^{-1}.\tau \neq w^{-1}.\tau$ and by Remark 4.1, $w^{-1}.\tau(\zeta_s^{\mathrm{den}}) \neq 0$ or equivalently $\tau(\zeta_{\alpha^\vee}^{\mathrm{den}}) \neq 0$. By (4) we deduce that $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$ for all $\tau \in T_{\mathcal{F}}^{\mathrm{reg}}$, which proves (5).

Let us prove (2). Let $v \in W^v$ be such that $h := F_v - T_v \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{<v}$ and $s \in \mathscr{S}$ be such that $sv > v$. Then

$$F_{sv} = (T_s - \sigma_s^2 + \zeta_s) * (T_v + h) = T_{sv} + (-\sigma_s^2 + \zeta_s) * T_v + (-\sigma_s^2 + \zeta_s) * h + T_s * h.$$

By Lemma 2.8,

$$(-\sigma_s^2 + \zeta_s) * T_v, (-\sigma_s^2 + \zeta_s) * h \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{\leqslant v}.$$

By [22, Corollary 1.3.19], $s.[1, v) \subset [1, sv)$ and thus $T_s * h \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{<sw}$ thus $F_{sv} - T_{sv} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})^{<sv}$. By induction we deduce (2).

Let $w = s_1 \dots s_r = s_1' \dots s_r'$ be reduced expressions of w. Let F_w be associated to $s_1 \dots s_r$ and F_w' be associated to $s_1' \dots s_r'$. Let $\mathcal{F}'$ be a uncountable field containing $\mathcal{F}$. Then by Proposition 3.4 (2), for all $\tau \in T_{\mathcal{F}'}^{\mathrm{reg}}$ there exists $\theta(\tau) \in \mathcal{F}'^*$ such that $F_w(\tau) = \theta(\tau) F_w'(\tau)$. Let $v \in W^v$ be such that $\pi^v(F_w') \neq 0$ and $\theta_v = \frac{\pi_v^H(F_w)}{\pi_v^H(F_w')} \in \mathcal{F}(Y)$. Then $\theta_v(\tau) = \theta(\tau)$ for all $\tau \in T_{\mathcal{F}'}^{\mathrm{reg}}$. But by (2), $\theta(\tau) = 1$ for all $\tau \in T_{\mathcal{F}'}^{\mathrm{reg}}$. Thus by Lemma 4.13, $\theta = 1 = \theta_v$ and $F_w' = F_w$. $\square$

Remark 4.15. — When $\sigma_s = \sigma_s'$ for all $s \in \mathscr{S}$, the condition (4) is equivalent to $\tau(\beta^\vee) \neq 1$ for all $\beta^\vee \in N_{\Phi^\vee}(w)$.

4.5. One implication of Kato's criterion

Recall the definition of $W_{(\tau)}$ from Section 4.3.

In this subsection, we prove that if I_τ is irreducible, then $W_\tau = W_{(\tau)}$.

Lemma 4.16. — *Let $\tau \in T_{\mathbb{C}}$ be such that $W_\tau \neq W_{(\tau)}$. Let $w \in W_\tau \setminus W_{(\tau)}$ be of minimal length. Then $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$.*

Proof. — Write $w = s_k \dots s_1$, where $k = \ell(w)$ and $s_1, \dots, s_k \in \mathscr{S}$. Let $j \in [\![0, k-1]\!]$. Set $w_j = s_j \dots s_1$. Suppose that $w_j.\zeta_{s_{j+1}}^{\mathrm{den}}(\tau) = 0$. Then $r_{w_j.\alpha_{s_{j+1}}^\vee} = s_1 \dots s_j s_{j+1} s_j \dots s_1 \in W_{(\tau)}$. Moreover as $W_{(\tau)} \subset W_\tau$, we have $s_{j+1} \dots s_1.\tau = s_j \dots s_1.\tau$. Therefore

$$\tau = w.\tau = s_k \dots s_j \dots s_1.\tau = s_k \dots \widehat{s}_{j+1} \dots s_1.\tau,$$

and $w' = s_k \dots \widehat{s}_{j+1} \dots s_1 \in W_\tau$. By definition of w, $w' \in W_{(\tau)}$. Consequently

$$w = s_k \dots \widehat{s}_{j+1} \dots s_1.s_1 \dots s_j.s_{j+1}.s_j \dots s_1 = w' r_{w_j.\alpha^\vee_{s_{j+1}}} \in W_{(\tau)}:$$

a contradiction. Therefore $w_j.\zeta^{\mathrm{den}}_{s_{j+1}}(\tau) \neq 0$ and by Lemma 2.4 and Lemma 4.14, $F_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathcal{F}})_\tau$. □

Proposition 4.17. — *Let $\tau \in T_{\mathbb{C}}$ be such that $W_\tau \neq W_{(\tau)}$. Then I_τ is reducible.*

Proof. — Let $w \in W_\tau \setminus W_{(\tau)}$ be of minimal length. Then by Lemma 4.16 and Lemma 4.14, $F_w(\tau) \otimes_\tau 1 \in I_\tau(\tau)$. Moreover, $\pi^T_w\big(F_w(\tau) \otimes_\tau 1\big) = 1$ and thus $F_w(\tau) \otimes_\tau 1 \notin \mathbb{C}1 \otimes_\tau 1$. We conclude with Theorem 4.8. □

4.6. Link with the works of Matsumoto and Kato

Assume that W^v is finite. Then $\mathcal{H}_{\mathbb{C}} = {}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$. Let $\tau \in T_{\mathbb{C}}$. Then by Section 2.4, $\dim_{\mathbb{C}} I_\tau = |W^v|$. One has $Z^\lambda.1 \otimes_\tau 1 = \tau(\lambda) 1 \otimes_\tau 1$ for all $\lambda \in Y$ and $\mathcal{H}_{\mathbb{C}}.1 \otimes_\tau 1 = I_\tau$. Thus by [25, Théorème 4.1.10] the definition we used is equivalent to Matsumoto's one.

Assume that $\mathcal{H}_{\mathbb{C}}$ is associated with a split reductive group over a field with residue cardinal q. Then by (BL2), one has:

$$\forall\, s \in \mathscr{S}, \forall\, w \in W^v,\ T_s * T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ (q-1)T_w + qT_{sw} & \text{if } \ell(sw) = \ell(w)-1. \end{cases}$$

Set $1'_\tau = \sum_{w \in W^v} T_w \otimes_\tau 1$. Then if $s \in \mathscr{S}$, $T_s.1'_\tau = q1'_\tau$. Then by [20, (1.19)], $1'_\tau$ is proportional to the vector 1_τ defined in [20]. Kato proves Theorem 1.1 by studying whether the following property is satisfied: "for all $w \in W^v$, $\mathcal{H}_{\mathbb{C}}.1'_{w.\tau} = I_{w.\tau}$" (see [20, Lemma 2.3]). When W^v is infinite, we do not know how to define an analogue of $1'_\tau$ and thus we do not know how to adapt Kato's proof.

5. Description of generalized weight spaces

In this section, we describe $I_\tau(\tau, \mathrm{gen})$, when $\tau \in T_{\mathbb{C}}$ is such that $W_{(\tau)} = W_\tau$. We then deduce Kato's criterion for size 2 matrices.

Let us sketch our proof of this criterion. By Theorem 4.8 and Proposition 4.17, it suffices to study $I_\tau(\tau)$ when $\tau \in \mathcal{U}_{\mathbb{C}}$ is such that $W_\tau = W_{(\tau)}$. For this, we begin by describing $I_\tau(\tau, \mathrm{gen})$. Let $\tau \in T_{\mathbb{C}}$ satisfying the above condition. By Dyer's theorem, $(W_{(\tau)}, \mathscr{S}_\tau)$ is a Coxeter system, for some $\mathscr{S}_\tau \subset W_{(\tau)}$. Let $r \in \mathscr{S}_\tau$. We study the singularity of F_r at τ, that is, we determine an (explicit) element $\theta \in \mathbb{C}(Y)$ such that $F_r - \theta$ is defined at τ (see Lemma 5.19). Using this, we then describe $I_\tau(\tau, \mathrm{gen})$. We then deduce that when $W_\tau = W_{(\tau)}$ is the infinite dihedral group then $I_\tau(\tau)$ is irreducible. After classifying the subgroups of the infinite dihedral group (see Lemma 5.34), we deduce Kato's criterion for size 2 matrices.

In Section 5.1, we study the torus $T_\mathbb{C}$.

In Section 5.2, we introduce a new basis of $\mathcal{H}_{W^v,\mathbb{C}}$ which enables us to have information on the poles of the coefficients of the F_w.

In Section 5.3, we give a recursive formula which enables us to have information on the poles of the coefficients of the F_w.

In Section 5.4, we study the singularity of F_r at τ, for $r \in \mathscr{S}_\tau$.

In Section 5.5, we give a description of $I_\tau(\tau, \mathrm{gen})$, when $W_\tau = W_{(\tau)}$.

In Section 5.6, we prove that when $W_\tau = W_{(\tau)}$ is the infinite dihedral group and $\tau \in \mathcal{U}_\mathbb{C}$, then I_τ is irreducible.

In Section 5.7, we prove Kato's criterion for size 2 Kac–Moody matrices.

This section is strongly inspired by [29].

In certain proofs, when $\mathcal{F} = \mathbb{C}$, we will make additional assumptions on the σ_s and σ'_s, $s \in \mathscr{S}$. To avoid these assumptions, we can assume that $\sigma_s, \sigma'_s \in \mathbb{C}$ and $|\sigma_s| > 1, |\sigma'_s| > 1$ for all $s \in \mathscr{S}$.

5.1. The complex torus $T_\mathbb{C}$

We assume that $|\sigma_s| \in \mathbb{R}_{>1}$ for all $s \in \mathscr{S}$. Let $(y_j)_{j\in J}$ be a $\mathbb{Z}$-basis of Y. The map $T_\mathbb{C} \to (\mathbb{C}^*)^J$ mapping each $\tau \in T_\mathbb{C}$ on $(\tau(y_j))_{j\in J}$ is a bijection. We identify $T_\mathbb{C}$ and $(\mathbb{C}^*)^J$. We equip $T_\mathbb{C}$ with the usual topology on $(\mathbb{C}^*)^J$. This does not depend on the choice of a basis $(y_j)_{j\in J}$.

Lemma 5.1. — *The set $\{\tau \in T_\mathbb{C} \mid \forall\ (w, \lambda) \in W^v \setminus \{1\} \times (C^v_f \cap Y),$ $w.\tau(\lambda) \neq \tau(\lambda)\}$ is dense in $T_\mathbb{C}$. In particular, $T_\mathbb{C}^{\mathrm{reg}}$ is dense in $T_\mathbb{C}$.*

Proof. — Let $\lambda \in C^v_f \cap Y$. By [4, V.Chap 4 Section 6 Proposition 5], for all $w \in W^v \setminus \{1\}$, $w.\lambda \neq \lambda$. Let $(\gamma_j)_{j\in J} \in (\mathbb{C}^*)^J$ be algebraically independent over $\mathbb{Q}$ and $\tau_\gamma \in T_\mathbb{C}$ be defined by $\tau_\gamma(y_j) = \gamma_j$ for all $j \in J$. Then $w.\tau_\gamma(\lambda) \neq \tau_\gamma(\lambda)$ for all $w \in W^v \setminus \{1\}$. Let $\tau \in T_\mathbb{C}$. Let $(\gamma^{(n)}) \in \left((\mathbb{C}^*)^J\right)^{\mathbb{Z}_{\geqslant 0}}$ be such that $\gamma^{(n)}$ is algebraically independent over $\mathbb{Q}$ for all $n \in \mathbb{Z}_{\geqslant 0}$ and such that $\gamma^{(n)} \to (\tau(y_j))_{j\in J}$. Then $\tau_{\gamma^{(n)}} \to \tau$ and we get the lemma. ☐

Let $A \subset \mathbb{R}$ be a ring. We set $Q^\vee_A = \bigoplus_{s\in\mathscr{S}} A\alpha^\vee_s \subset \mathbb{A}$.

Lemma 5.2. — *Let $(\gamma_s) \in (\mathbb{C}^*)^{\mathscr{S}}$. Then there exists $\tau \in T_{\mathbb{C}}$ such that $\tau(\alpha_s^\vee) = \gamma_s$ for all $s \in \mathscr{S}$.*

Proof. — Let us prove that there exists $n \in \mathbb{Z}_{\geqslant 1}$ such that $\frac{1}{n}Q_{\mathbb{Z}}^\vee \supset Y \cap Q_{\mathbb{Q}}^\vee$. The module $Y \cap Q_{\mathbb{Q}}^\vee$ is a $\mathbb{Z}$-submodule of the free module Y. Thus it is a free module and its rank is lower or equal to the rank of Y. Let $(y_j)_{j\in J}$ be a $\mathbb{Z}$-basis of $Y \cap Q_{\mathbb{Q}}^\vee$. As $\alpha_s^\vee \in Y \cap Q_{\mathbb{Q}}^\vee$ for all $s \in \mathscr{S}$, we have we have $\mathrm{vect}_{\mathbb{Q}}(Y \cap Q_{\mathbb{Q}}^\vee) = Q_{\mathbb{Q}}^\vee$. Therefore for all $j \in J$, there exists $(m_{j,s}) \in \mathbb{Q}^{\mathscr{S}}$ such that $y_j = \sum_{j\in J} m_{j,s}\alpha_s^\vee$ and thus there exists $n \in \mathbb{Z}_{\geqslant 1}$ such that $\frac{1}{n}Q_{\mathbb{Z}}^\vee \supset Y \cap Q_{\mathbb{Q}}^\vee$.

Let S be a complement of $Y \cap Q_{\mathbb{Q}}^\vee$ in $Y \otimes \mathbb{Q}$. For $s \in \mathscr{S}$, choose $\gamma_s^{\frac{1}{n}} \in \mathbb{C}^*$ such that $(\gamma_s^{\frac{1}{n}})^n = \gamma_s$. Let $\widetilde{\tau} : \frac{1}{n}Q_{\mathbb{Z}}^\vee \oplus S \to \mathbb{C}^*$ be defined by $\widetilde{\tau}(\sum_{s\in\mathscr{S}} \frac{a_s}{n}\alpha_s^\vee + x) = \prod_{s\in\mathscr{S}}(\gamma_s^{\frac{1}{n}})^{a_s}$ for all $(a_s) \in \mathbb{Z}^{\mathscr{S}}$ and $x \in S$. Let $\tau = \widetilde{\tau}_{|Y}$. Then $\tau \in T_{\mathbb{C}}$ and $\tau(\alpha_s^\vee) = \gamma_s$ for all $s \in \mathscr{S}$. □

5.2. A new basis of $\mathcal{H}_{W^v,\mathbb{C}}$

In [21], Kazhdan and Lusztig defined the Kazhdan–Lusztig basis $(C_w)_{w\in W^v}$ of $\mathcal{H}_{W^v,\mathbb{C}}$ in the case where $\sigma_s = \sigma$ for all $s \in \mathscr{S}$. This basis is defined by its properties with respect to some involution of $\mathcal{H}_{W^v,\mathbb{C}}$ and by the fact that $C_w - T_w \in \bigoplus_{v<w} \mathbb{C}T_v$, for $w \in W^v$ (see [21, Theorem 1.1] for a precise statement). This basis was then defined in the general case (where the σ_s, $s \in \mathscr{S}$ need not be all equal) see [24, 6] for example. We now define a basis $(B_w)_{w\in W^v}$ of $\mathcal{H}_{W^v,\mathbb{C}}$ from the Kazhdan–Lusztig basis $(C_w)_{w\in W^v}$ and then compute the coefficient in front of B_1 of the expansion of F_w in the basis $(B_v)_{v\in W^v}$, for $w \in W^v$ (see Lemma 5.4). This will enable us to have information on the coefficient $\pi_1^H(F_w) \in \mathbb{C}(Y)$, for $w \in W^v$ (see Lemma 5.4 and Lemma 5.19). Our computation relies on certain multiplicative properties of (B_w) (see Lemma 5.3) and we will not need the precise definition of the Kazhdan–Lusztig basis.

Let $(C_w)_{w\in W^v}$ be the basis introduced in [24, 6]. For $w \in W^v$, we set $B_w = (-1)^{\ell(w)}\sigma_w C_w$, where σ_w is defined in Remark 2.7(6). Then for $s \in \mathscr{S}$, one has $B_s = T_s - \sigma_s^2$ and thus this notation is coherent with the notation B_s introduced in Section 4.1.

Lemma 5.3. — *The basis $(B_w)_{w\in W^v}$ satisfies:*

(1) *$B_s = T_s - \sigma_s^2$ for all $s \in \mathscr{S}$,*
(2) *$B_w - T_w \in \mathcal{H}_{W^v,\mathbb{C}}^{<w}$ for all $w \in W^v$,*

(3) *For all $w \in W^v$ and $s \in \mathscr{S}$ we have:*

$$B_w B_s = \begin{cases} -(1+\sigma_s^2)B_w & \text{if } ws < w \\ B_{ws} + \sum_{vs<v<w} b(v,w)B_v & \text{if } ws > w, \end{cases}$$

for some $b(v,w) \in \mathbb{C}$.

Proof.

(2). — It is a consequence of [24, 2. Proposition].

(3). — Let $w \in W^v$ and $s \in \mathscr{S}$ be such that $ws < w$. By [24, 6.4], $C_w(H_s + \sigma_s^{-1}) = 0$, thus $(-1)^{\ell(w)}\sigma_w C_w(T_s+1) = 0$ and hence $B_w(T_s+1-\sigma_s^2-1) = B_w B_s = -(\sigma_s^2+1)B_w$.

Let $w \in W^v$ and $s \in \mathscr{S}$ be such that $ws > w$. Then by [24, 6.3], one has $C_w(-C_s) \in C_{ws} + \bigoplus_{vs<v<w} \mathbb{C}C_v$ and thus

$$\begin{aligned}(-1)^{\ell(w)}\sigma_w C_w(-\sigma_s C_s) = B_w B_s \in (-1)^{\ell(w)+1}\sigma_{ws}C_{ws} + \bigoplus_{vs<v<w} \mathbb{C}B_v \\ = B_{ws} + \bigoplus_{vs<v<w} \mathbb{C}B_v,\end{aligned}$$

which proves the lemma. □

As $(B_w)_{w\in W^v}$ is a $\mathbb{C}$-basis of $\mathcal{H}_{W^v,\mathbb{C}}$, $(B_w)_{w\in W^v}$ is a $\mathbb{C}(Y)$-basis of the right module ${}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})$.

Let $w \in W^v$. Write $F_w = \sum_{v\in W^v} B_v p_{v,w}$, where $(p_{v,w}) \in \mathbb{C}(Y)^{(W^v)}$. By an induction on $\ell(w)$ using Lemma 5.3 (2) we have $\bigoplus_{v\leqslant w} H_v\mathbb{C}(Y) = \bigoplus_{v\leqslant w} B_v\mathbb{C}(Y)$ for all $w \in W^v$. Thus for all $v \in W^v$ such that $v \not\leqslant w$, one has $p_{v,w} = 0$. In [29, 5.3], Reeder gives recursive formulae for the $p_{v,w}$. The following lemma is a particular case of them.

For $v \in W^v$, define $\pi_v^B : {}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C}) \to \mathbb{C}(Y)$ by $\pi_v^B(\sum_{u\in W^v} B_u f_u) = f_v$ for all $(f_u) \in \mathbb{C}(Y)^{(W^v)}$.

Lemma 5.4. — *Let $w \in W^v$. Then $p_{1,w} = \zeta_w := \prod_{\beta^\vee \in N_{\Phi^\vee}(w)} \zeta_{\beta^\vee}$.*

Proof. — We prove it by induction on $\ell(w)$.

Let $v \in W^v$ and assume that $p_{1,v} = \zeta_v$. Let $s \in \mathscr{S}$ be such that $vs > v$. By Lemma 4.2 one has

$$\begin{aligned} F_{vs} &= F_v * F_s \\ &= \left(\sum_{u\in W^v} B_u p_{u,v}\right) * F_s \\ &= \sum_{u\in W^v} B_u * F_s p_{u,v}^s = \sum_{u\in W^v} B_u * B_s p_{u,v}^s + \sum_{u\in W^v} B_u p_{u,v}^s \zeta_s. \end{aligned}$$

By Lemma 5.3, we have

$$\pi_1^B\left(\sum_{u\in W^v} B_u * B_s p_{u,v}^s\right)=0$$
$$\text{and}\quad \pi_1^B\left(\sum_{u\in W^v} B_u p_{u,v}^s\zeta_s\right)=p_{1,v}^s\zeta_s.$$

By Lemma 2.4, $N_{\Phi^\vee}(vs)=s.N_{\Phi^\vee}(v)\sqcup\{\alpha_s^\vee\}$ and thus $\pi_1^B(F_{vs})=p_{1,vs}=p_{1,v}^s\zeta_s=\zeta_{vs}$ which proves the lemma. □

Remark 5.5. — In the proof of Lemma 5.4, we only used the properties of $(B_w)_{w\in W^v}$ described in Lemma 5.3 and not its precise definition. In [29, Lemma 5.2], Reeder gives an elementary proof of the existence of a basis $(B_w)_{w\in W^v}$ satisfying Lemma 5.3. Its proof can be adapted to our framework to construct a basis (B_w) without using Kazhdan–Lusztig basis.

5.3. An expression for the coefficients of the F_w in the basis (T_v)

In this subsection, we give a recursive formula for the coefficients of the F_w in the basis $(T_v)_{v\in W^v}$ (see formula (5.1) below and Lemma 5.7). We will deduce information concerning the elements $v\in W^v$ such that $\pi_v^T(F_w)$ is well-defined at τ, for a given $\tau\in T_{\mathbb{C}}$ (see Lemma 5.8).

Let $\lambda\in Y$ and $w\in W^v$. By (BL4), Remark 2.7 (2) and an induction on $\ell(w)$, there exists $(P_{v,w,\lambda}(Z))_{v\in W^v}\in\mathbb{C}[Y]^{(W^v)}$ such that $Z^\lambda * T_w=\sum_{v\in W^v} T_v * P_{v,w,\lambda}(Z)$. Moreover $P_{w,w,\lambda}=Z^{w^{-1}.\lambda}$ and for all $v\in W^v\backslash[1,w]$, $P_{v,w,\lambda}=0$.

Let $\lambda\in C_f^v\cap Y$. Then by [4, V.Chap 4 Section 6 Proposition 5], for all $v,w\in W^v$ such that $v\neq w$, one has $v.\lambda\neq w.\lambda$. Let $w\in W^v$. Let $w=s_1\dots s_k$ be a reduced expression. Set $Q_{w,w,\lambda}(Z)=1\in\mathbb{C}(Y)$. For $v\in W^v\setminus[1,w]$, set $Q_{v,w,\lambda}(Z)=0$. Define $(Q_{v,w,\lambda}(Z))_{v\in[1,w]}$ by decreasing induction by setting:

$$Q_{v,w,\lambda}(Z)=\frac{1}{Z^{w^{-1}.\lambda}-Z^{v^{-1}.\lambda}}\sum_{w\geqslant u>v} Q_{u,w,\lambda}P_{v,u,\lambda}\in\mathbb{C}(Y). \tag{5.1}$$

Lemma 5.6. — *Let $\lambda\in C_f^v\cap Y$, $w\in W^v$ and $\tau\in T_{\mathbb{C}}^{\mathrm{reg}}$ be such that $v.\tau(\lambda)\neq\tau(\lambda)$ for all $v\in W^v\backslash\{1\}$. Let $x\in I_\tau$ be such that $Z^\lambda.x=w.\tau(\lambda).x$. Then $x\in I_\tau(w.\tau)$.*

Proof. — By Proposition 3.4 (2), we can write $x=\sum_{v\in W^v} x_v$ where $x_v\in I_\tau(v.\tau)$ for all $v\in W^v$. One has $Z^\lambda.x-w.\tau(\lambda).x=0=\sum_{v\in W^v}(v.\tau(\lambda)-w.\tau(\lambda))x_v$. As $v.\tau(\lambda)\neq w.\tau(\lambda)$ for all $v\neq w$, we deduce that $x=x_w$. □

Lemma 5.7. — *Let $v, w \in W^v$. Then $\pi_v^T(F_w) = Q_{v,w,\lambda}$, for any $\lambda \in C_f^v \cap Y$. In particular, $Q_{v,w,\lambda}$ does not depend on the choice of λ.*

Proof. — Let $\lambda \in C_f^v$ and $h = \sum_{v\in W^v} T_v Q_{v,w,\lambda} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$. One has:

$$\begin{aligned} Z^\lambda * h &= Z^\lambda * \sum_{v\in W^v} T_v Q_{v,w,\lambda} \\ &= \sum_{u,v\in W^v} T_u P_{u,v,\lambda} Q_{v,w,\lambda} \\ &= \sum_{u\in W^v} T_u \sum_{v\in W^v} P_{u,v,\lambda} Q_{v,w,\lambda}. \end{aligned}$$

Let $u \in W^v$. Then:

$$\begin{aligned} \sum_{v\in W^v} P_{u,v,\lambda} Q_{v,w,\lambda} &= P_{u,u,\lambda} Q_{u,w,\lambda} + \sum_{v>u} P_{u,v,\lambda} Q_{v,w,\lambda} \\ &= Z^{u^{-1}.\lambda} + \left(Z^{w^{-1}.\lambda} - Z^{u^{-1}.\lambda}\right) Q_{u,w,\lambda} \\ &= Z^{w^{-1}.\lambda} Q_{u,w,\lambda}, \end{aligned}$$

and therefore $Z^\lambda.h = h.Z^{w^{-1}.\lambda}$.

Let $\lambda \in C_f^v \cap Y$ and $\tau \in T_{\mathbb{C}}^{\mathrm{reg}}$ be such that $u.\tau(\lambda) \neq \tau(\lambda)$ for all $u \in W^v \setminus \{1\}$. Then $\mathrm{ev}_\tau(Z^\lambda * h) = \mathrm{ev}_\tau(h * Z^{v^{-1}.\lambda}) = w.\tau(\lambda).h(\tau)$. By Lemma 5.6 we deduce that $h(\tau) \in I_\tau(w.\tau)$. By Proposition 3.4 (2) and Lemma 4.14 we deduce that $h(\tau) = F_w(\tau)$. By Lemma 5.1, we deduce that $h = F_w$, which proves the lemma. □

Lemma 5.8. — *Let $w \in W^v$, $\tau \in T_{\mathbb{C}}$ and $v \in [1, w]$. Assume that for all $u \in [v, w)$, $u.\tau \neq w.\tau$. Then for all $u \in [v, w]$, $\pi_u^T(F_w) \in \mathbb{C}(Y)_\tau$.*

Proof. — We do it by decreasing induction on v. Suppose that for all $u \in (v, w)$, $\pi_u^T(F_w) \in \mathbb{C}(Y)_\tau$. Let $\lambda \in C_f^v \cap Y$ be such that $v.\tau(\lambda) \neq w.\tau(\lambda)$, which exists because $C_f^v \cap Y$ generates Y. By Lemma 5.7 we have

$$\pi_v^T(F_w) = Q_{v,w,\lambda} = \frac{1}{Z^{w^{-1}.\lambda} - Z^{v^{-1}.\lambda}} \sum_{w\geqslant u>v} Q_{u,w,\lambda} P_{v,u,\lambda}.$$

We deduce that $\pi_v^T(F_w) \in \mathbb{C}(Y)_\tau$ because by assumption $Q_{u,w,\lambda} \in \mathbb{C}(Y)_\tau$ for all $u \in (v, w]$. Lemma follows. □

5.4. τ-simple reflections and intertwining operators

Let $\tau \in T_{\mathbb{C}}$. Following [29, 14], we introduce τ-simple reflections (see Definition 5.9). If $\mathscr{S}_\tau$ is the set of τ-simple reflections, then $(W_{(\tau)}, \mathscr{S}_\tau)$ is a

Coxeter system. We study, for such a reflection r, the singularity of F_r at τ: we prove that $F_r - \zeta_r$ is in ${}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau$ (see Lemma 5.19). This enables us to define $K_r(\tau) = (F_r - \zeta_r)(\tau) \in \mathcal{H}_{W^v,\mathbb{C}}$. This will be useful to describe $I_\tau(\tau, \mathrm{gen})$.

We now define τ-simple reflections. Our definition slightly differs from [29, Definition 14.2]. These definitions are equivalent (see Lemma 5.13).

Definition 5.9. — *Let $\tau \in T_\mathbb{C}$. A coroot $\beta^\vee \in \Phi^\vee_\tau$ and its corresponding reflection $r_{\beta^\vee}$ are said to be* τ-simple *if $N_\mathscr{R}(r_{\beta^\vee}) \cap W_{(\tau)} = \{r_{\beta^\vee}\}$. We denote by $\mathscr{S}_\tau$ the set of τ-simple reflections.*

Recall that $\Phi^\vee_{(\tau)} = \{\alpha^\vee \in \Phi^\vee_+ \mid \zeta^{\mathrm{den}}_{\alpha^\vee}(\tau) = 0\}$ and $\mathscr{R}_{(\tau)} = \{r_{\alpha^\vee} \mid \alpha^\vee \in \Phi^\vee_{(\tau)}\}$.

5.4.1. Coxeter structure of $W_{(\tau)}$ and comparison of the definitions of τ-simplicity

We use the same notation as in Section 2.2.3. Then $\mathscr{S}_\tau = \mathscr{S}(W_{(\tau)})$ and thus $(W_{(\tau)}, \mathscr{S}_\tau)$ is a Coxeter system.

Let $\leqslant_\tau$ and ℓ_τ be the Bruhat order and the length on $(W_{(\tau)}, \mathscr{S}_\tau)$.

Lemma 5.10. — *Let $x, y \in W_{(\tau)}$ be such that $x \leqslant_\tau y$. Then $x \leqslant y$.*

Proof. — By definition, if $x, y \in W_{(\tau)}$, then $x \leqslant_\tau y$ (resp. $x \leqslant y$) if there exist $n \in \mathbb{Z}_{\geqslant 0}$ and $x_0 = x, x_1, \ldots, x_n = y \in W_{(\tau)}$ (resp. W^v) such that (x_i, x_{i+1}) is an arrow of the graph of [10, Definition 1.1] for all $i \in [\![0, n-1]\!]$. We conclude with [10, Theorem 1.4] □

Remark 5.11. — The orders $\leqslant$ and $\leqslant_\tau$ can be different on $W_{(\tau)}$: there can exist $v, w \in W_{(\tau)}$ such that v and w are not comparable for $\leqslant_\tau$ and $v < w$. For example if $W^v = \{s_1, s_2\}$ is the infinite dihedral group, $r_1 = s_1$ and $r_2 = s_2 s_1 s_2$ (see Lemma B.2), then $r_1 < r_2$ but r_1 and r_2 are not comparable for $<_\tau$.

Set $\Phi^\vee_{(\tau),+} = \Phi^\vee_{(\tau)} \cap \Phi^\vee_+$ and $\Phi^\vee_{(\tau),-} = \Phi^\vee_{(\tau)} \cap \Phi^\vee_-$. For $w \in W_{(\tau)}$, set $N_{\Phi^\vee_{(\tau)}}(w) = N_{\Phi^\vee}(w) \cap \Phi^\vee_{(\tau)}$.

Lemma 5.12. — *Let $w \in W_{(\tau)}$. Then $w.\Phi^\vee_{(\tau)} = \Phi^\vee_{(\tau)}$ and $w.\mathscr{R}_{(\tau)}.w^{-1} = \mathscr{R}_{(\tau)}$.*

Proof. — Let $\alpha^\vee \in \Phi^\vee_{(\tau)}$. One has $\zeta^{\mathrm{den}}_{w.\alpha^\vee} = (\zeta^{\mathrm{den}}_{\alpha^\vee})^w$ and hence

$$\zeta^{\mathrm{den}}_{\alpha^\vee}(\tau) = (\zeta^{\mathrm{den}}_{\alpha^\vee})^w(\tau) = (\zeta^{\mathrm{den}}_{\alpha^\vee})(w^{-1}.\tau) = 0$$

because $w \in W_{(\tau)} \subset W_\tau$. Thus $w.\alpha^\vee \in \Phi^\vee_{(\tau)}$ and $r_{v.\alpha^\vee} = v r_{\alpha^\vee} v^{-1} \in \mathscr{R}_{(\tau)}$, which proves the lemma. □

We now prove that our definition of τ-simplicity is equivalent to the definition of [29, 14.2]. This equivalence will be useful in our study of the weight spaces of I_τ and thus in the study of the irreducibility of I_τ. Indeed, our definition of τ-simplicity is well adapted to the study of the Coxeter structure of $W_{(\tau)}$ whereas Reeder's one is well adapted to the study of the singularity F_r at τ.

Lemma 5.13.

(1) *One has* $\mathscr{S}_\tau \subset \mathscr{R} \cap W_{(\tau)} = \mathscr{R}_{(\tau)}$.
(2) *Let* $r = r_\beta^\vee \in \mathscr{R}$. *Then* $r \in \mathscr{S}_\tau$ *if and only if* $N_{\Phi^\vee}(r_{\beta^\vee}) \cap \Phi^\vee_{(\tau)} = \{\beta^\vee\}$.
(3) *Let* $w \in W_{(\tau)}$. *Let* $w = r_1 \dots r_k$ *be a reduced writing of* $W_{(\tau)}$, *with* $k = \ell_\tau(w)$ *and* $r_1, \dots, r_k \in \mathscr{S}_\tau$. *Then*
$$|N_{\Phi^\vee_{(\tau)}}(w)| = \{\alpha^\vee_{r_k}, r_k.\alpha^\vee_{r_{k-1}}, \dots, r_k \dots r_2.\alpha^\vee_{r_1}\}$$
and $|N_{\Phi^\vee}(w) \cap \Phi^\vee_{(\tau)}| = k = \ell_\tau(w)$.

Proof. — We begin by proving a part of (3). By Lemma 5.10 and [22, Lemma 1.3.13], for $v \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$, one has $\ell_\tau(vr) > \ell_\tau(v)$ if and only if $vr >_\tau v$ if and only if $vr > v$ if and only if $v.\alpha_r^\vee \in \Phi_+^\vee$ if and only if $v.\alpha_r^\vee \in \Phi^\vee_{(\tau),+}$.

One has $N_{\Phi^\vee_{(\tau)}}(w) = \{\alpha^\vee \in \Phi^\vee_{(\tau),+} | w.\alpha^\vee \in \Phi^\vee_{(\tau),-}\}$. Then using the same proof as in [22, Lemma 1.3.14], one has $N_{\Phi^\vee_{(\tau)}}(w) \supset \{\alpha^\vee_{r_k}, r_k.\alpha^\vee_{r_{k-1}}, \dots, r_k \dots r_2.\alpha^\vee_{r_1}\}$ and $|\{\alpha^\vee_{r_k}, r_k.\alpha^\vee_{r_{k-1}}, \dots, r_k \dots r_2.\alpha^\vee_{r_1}\}| = k = \ell_\tau(w)$.

We now prove (1) and (2). Let $f : \Phi_+^\vee \to \mathscr{R}$ be the map defined by $f(\alpha^\vee) = r_{\alpha^\vee}$ for $\alpha^\vee \in \Phi_+^\vee$. Then by Section 2.2, f is a bijection. Let $r = r_{\beta^\vee} \in \mathscr{S}_\tau$. One has $f\big(N_{\Phi^\vee}(r) \cap \Phi^\vee_{(\tau)}\big) = N_{\mathscr{R}}(r) \cap \mathscr{R}_{(\tau)}$. Moreover, $\mathscr{R}_{(\tau)} \subset W_{(\tau)} \cap \mathscr{R}$. Thus
$$f^{-1}\big(N_{\mathscr{R}}(r) \cap W_{(\tau)}\big) = \{\beta^\vee\} \supset f^{-1}\big(N_{\mathscr{R}}(r) \cap \mathscr{R}_{(\tau)}\big) = N_{\Phi^\vee}(r) \cap \Phi^\vee_{(\tau)}.$$
Moreover, $|N_{\Phi^\vee}(r) \cap \Phi^\vee_{(\tau)}| \geqslant 1$ and thus $|N_{\Phi^\vee}(r) \cap \Phi^\vee_{(\tau)}| = \{\beta^\vee\}$. In particular, $\beta^\vee \in \Phi^\vee_{(\tau)}$ and $r \in \mathscr{R}_{(\tau)}$. Thus $\mathscr{S}_\tau \subset \mathscr{R}_{(\tau)}$.

By [9, Theorem 3.3 (i)], $\mathscr{R} \cap W_{(\tau)} = \bigcup_{w \in W_{(\tau)}} w \mathscr{S}_\tau w^{-1}$ and thus by Lemma 5.12, $\mathscr{R} \cap W_{(\tau)} \subset \bigcup_{w \in W_{(\tau)}} w.\mathscr{R}_{(\tau)}.w^{-1} = \mathscr{R}_{(\tau)}$. As by definition, $\mathscr{R}_{(\tau)} \subset W_{(\tau)} \cap \mathscr{R}$, we deduce that $\mathscr{R}_{(\tau)} = W_{(\tau)} \cap \mathscr{R}$, which proves (1).

Let $r = r_\beta^\vee \in \mathscr{R}$. Suppose that $N_{\Phi^\vee}(r_{\beta^\vee}) \cap \Phi^\vee_{(\tau)} = \{\beta^\vee\}$. Then
$$f\big(N_{\Phi^\vee}(r_{\beta^\vee}) \cap \Phi^\vee_{(\tau)}\big) = \{r_{\beta^\vee}\} = N_{\mathscr{R}}(r_{\beta^\vee}) \cap \mathscr{R}_{(\tau)} = N_{\mathscr{R}}(r_{\beta^\vee}) \cap W_{(\tau)},$$
which proves (2).

Let $\alpha^\vee \in N_{\Phi^\vee_{(\tau)}}(w)$. Then there exists $j \in [\![2, k]\!]$ such that $r_j \dots r_k.\alpha^\vee \in \Phi^\vee_{(\tau),+}$ and $r_{j-1} \dots r_k.\alpha^\vee \in \Phi^\vee_{(\tau),-}$. Thus $r_{j-1} \dots r_k.\alpha^\vee \in N_{\Phi^\vee_{(\tau)}}(r_j) =$

$\{\alpha_{r_j}^\vee\}$ and hence $\alpha^\vee = r_k \dots r_{j-1}.\alpha_{r_j}^\vee$, which concludes the proof of the lemma. $\square$

5.4.2. Singularity of F_r at τ for a τ-simple reflection

Lemma 5.14. — *Let $\tau \in T_{\mathbb{C}}$ and $r_{\beta^\vee} \in \mathscr{S}_\tau$. Then there exists $h' \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ such that $F_{r_{\beta^\vee}} = h'.(\zeta_{\beta^\vee}^{\mathrm{den}})^{-1}$.*

Proof. — Using [3, 1. Exercise 10], we write $r_{\beta^\vee} = wsw^{-1}$ with $w \in W^v$, $s \in \mathscr{S}$ and $\ell(wsw^{-1}) = 2\ell(w)+1$. One has $\beta^\vee = w.\alpha_s^\vee$. Let $r_{\beta^\vee} = s_m \dots s_1$ be a reduced expression of $r_{\beta^\vee}$, with $m \in \mathbb{Z}_{\geqslant 0}$ and $s_1, \dots, s_m \in \mathscr{S}$. Let $k \in [\![0, m-1]\!]$ and $v = s_k \dots s_1$. Suppose that $F_v = h_k'.(\zeta_{\beta^\vee}^{\mathrm{den}})^{-\eta(k)}$ where $h_k' \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ and $\eta(k) \in \mathbb{Z}_{\geqslant 0}$. Then $F_{s_{k+1}v} = F_{s_{k+1}} * F_v = (B_{s_{k+1}} + \zeta_{s_{k+1}}) * F_v$. One has $\zeta_{s_{k+1}} * F_v = F_v.\zeta_{s_{k+1}}^{v^{-1}}$ by Lemma 4.14.

By Lemma 5.13 if $\zeta_{s_{k+1}}^{v^{-1}}$ is not defined in τ then $k = \ell(w)$. As $B_{s_{k+1}} \in \mathcal{H}_{W^v,\mathbb{C}}$ and ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ is a left $\mathcal{H}_{W^v,\mathbb{C}}$-module, we can write $F_{s_{k+1}v} = h_{k+1}'.(\zeta_{\beta^\vee}^{\mathrm{den}})^{-\eta(k+1)}$ where $h_{k+1}' \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ and $\eta(k+1) \leqslant \eta(k)$ if $k \neq \ell(w)$ and $\eta(k+1) \leqslant \eta(k)+1$ if $k = \ell(w)$, which proves the lemma. $\square$

Lemma 5.15. — *Let $h \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$ and $\tau \in T_{\mathbb{C}}$. Then*

$$\max\{u \in W^v \mid \pi_u^H(h) \notin \mathbb{C}(Y)_\tau\} = \max\{u \in W^v \mid \pi_u^B(h) \notin \mathbb{C}(Y)_\tau\}.$$

Proof. — Let $v \in \max\{u \in W^v \mid \pi_u^H(h) \notin \mathbb{C}(Y)_\tau\}$. By 5.3 (2),

$$\pi_v^B(h) = \sum_{u \geqslant v} \pi_v^B(H_u)\pi_u^H(h) = \pi_v^B(H_v)\pi_v^H(h) + \sum_{u > v} \pi_v^B(H_u)\pi_u^H(h).$$

Moreover, by Lemma 5.3 (1), $\pi_v^B(H_v) \in \mathbb{C}^*$. Thus $\pi_v^B(h) \notin \mathbb{C}(Y)_\tau$. Similarly if $v' \in \max\{u \in W^v, u \geqslant v \mid \pi_u^B(h) \notin \mathbb{C}(Y)_\tau\}$, then $\pi_{v'}^H(h) \notin \mathbb{C}(Y)_\tau$. Hence $v \in \max\{u \in W^v \mid \pi_u^B(h) \notin \mathbb{C}(Y)_\tau\}$ and consequently

$$\max\{u \in W^v \mid \pi_u^H(h) \notin \mathbb{C}(Y)_\tau\} \subset \max\{u \in W^v \mid \pi_u^B(h) \notin \mathbb{C}(Y)_\tau\}.$$

By a similar reasoning we get the other inclusion. $\square$

Lemma 5.16. — *Let $w \in W^v$. Suppose that for some $s \in \mathscr{S}$, we have $w.\lambda - \lambda \in \mathbb{R}\alpha_s^\vee$ for all $\lambda \in Y$. Then $w \in \{\mathrm{Id}, s\}$.*

Proof. — Let $\beta^\vee \in N_{\Phi^\vee}(w)$. Write $\beta^\vee = \sum_{t \in \mathscr{S}} n_t \alpha_t^\vee$, with $n_t \in \mathbb{Z}_{\geqslant 0}$ for all $t \in \mathscr{S}$. Then $w.\beta^\vee \in \Phi_-^\vee$ and by assumption, $n_t = 0$ for all $t \in \mathscr{S} \setminus \{s\}$. Therefore $\beta^\vee \in \mathbb{Z}_{\geqslant 0}\alpha_s^\vee \cap \Phi^\vee = \{\alpha_s^\vee\}$. We conclude with Lemma 2.4. $\square$

Lemma 5.17. — *Let $\chi \in T_{\mathbb{C}}$. Assume that there exists $\beta^\vee \in \Phi_+^\vee$ such that $r_{\beta^\vee} \in W_\chi$. Then there exists $(\chi_n) \in (T_{\mathbb{C}})^{\mathbb{Z}_{\geqslant 0}}$ such that:*

- $\chi_n \to \chi$,
- $W_{\chi_n} = \langle r_{\beta^\vee} \rangle$ *for all* $n \in \mathbb{Z}_{\geqslant 0}$,
- $\chi_n(\beta^\vee) = \chi(\beta^\vee)$ *for all* $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. — We first assume that $\beta^\vee = \alpha_s^\vee$, for some $s \in \mathscr{S}$. Let $(y_j)_{j\in J}$ be a $\mathbb{Z}$-basis of Y. For all $j \in J$, choose $z_j \in \mathbb{C}$ such that $\chi(y_j) = \exp(z_j)$. Let $g : \mathbb{A} \to \mathbb{C}$ be the linear map such that $g(y_j) = z_j$ for all $j \in J$. Let V be a complement of $Q_{\mathbb{R}}^\vee$ in $\mathbb{A}$. Let $n \in \mathbb{Z}_{\geqslant 1}$. Let $b_s^{(n)} = g(\alpha_s^\vee)$ and $(b_t^{(n)}) \in \mathbb{C}^{\mathscr{S}\setminus\{s\}}$ be such that $|b_t^{(n)} - g(\alpha_t^\vee)| < \frac{1}{n}$ and such that the $\exp(b_t^{(n)})$, $t \in \mathscr{S} \setminus \{s\}$ are algebraically independent over $\mathbb{Q}$. Let $g_n : \mathbb{A} \to \mathbb{C}$ be the linear map such that $g_n(\alpha_t^\vee) = b_t^{(n)}$ for all $t \in \mathscr{S}$ and $g_n(v) = g(v)$ for all $v \in V$. For $n \in \mathbb{Z}_{\geqslant 0}$ set $\chi_n = (\exp \circ g_n)_{|Y} \in T_{\mathbb{C}}$. For all $x \in \mathbb{A}$, $g_n(x) \to g(x)$ and thus $\chi_n \to \chi$.

Let $n \in \mathbb{Z}_{\geqslant 1}$. Then $\chi(\alpha_s^\vee) = \chi_n(\alpha_s^\vee)$ and thus $s \in W_{\chi_n}$. Let $w \in W_{\chi_n}$. Then $w^{-1}.\lambda - \lambda \in \mathbb{Z}\alpha_s^\vee$ for all $\lambda \in Y$. By Lemma 5.16 we deduce that $w \in \{\mathrm{Id}, s\}$. Therefore $W_{\chi_n} = \{\mathrm{Id}, s\}$.

We no more assume that $\beta^\vee = \alpha_s^\vee$ for some $s \in \mathscr{S}$. Write $\beta^\vee = w.\alpha_s^\vee$ for some $w \in W^v$ and $s \in \mathscr{S}$. Let $\widetilde{\chi} = w^{-1}.\chi$. Then $s \in W_{\widetilde{\chi}}$. Thus there exists $(\widetilde{\chi}_n) \in (T_{\mathbb{C}})^{\mathbb{Z}_{\geqslant 0}}$ such that $\widetilde{\chi}_n \to \widetilde{\chi}$ and $W_{\widetilde{\chi}_n} = \{\mathrm{Id}, s\}$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Let $(\chi_n) = (w.\widetilde{\chi}_n)$. Then $\chi_n \to \chi$ and $W_{\chi_n} = \{1, r_{\beta^\vee}\}$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

Moreover, $\chi(\beta^\vee) \in \{-1, 1\}$ and $\chi_n(\beta^\vee) \in \{-1, 1\}$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Maybe considering a subsequence of (χ_n), we may assume that there exists $\epsilon \in \{-1, 1\}$ such that $\chi_n(\beta^\vee) = \epsilon$ for all $n \in \mathbb{Z}_{\geqslant 0}$. As $\chi_n \to \chi$, $\chi_n(\beta^\vee) = \epsilon \to \chi(\beta^\vee)$, which proves the lemma. $\square$

Let $\mathbb{C}[Q_{\mathbb{Z}}^\vee] = \bigoplus_{\lambda \in Q_{\mathbb{Z}}^\vee} \mathbb{C}Z^\lambda \subset \mathbb{C}[Y]$. This is the group algebra of $Q_{\mathbb{Z}}^\vee$. Let $\mathbb{C}(Q_{\mathbb{Z}}^\vee) \subset \mathbb{C}(Y)$ be the field of fractions of $\mathbb{C}[Q_{\mathbb{Z}}^\vee]$ and $\mathcal{H}(Q_{\mathbb{Z}}^\vee) = \bigoplus_{w \in W^v} H_w \mathbb{C}(Q_{\mathbb{Z}}^\vee) \subset {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$. This is a $(\mathcal{H}_{W^v,\mathbb{C}} - \mathbb{C}(Q_{\mathbb{Z}}^\vee))$-bimodule of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$ and a left $\mathbb{C}(Q_{\mathbb{Z}}^\vee)$-submodule of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$. Consequently $F_w \in \mathcal{H}(Q_{\mathbb{Z}}^\vee)$ for all $w \in W^v$.

Let $A = \mathbb{C}[Z^{\alpha_s^\vee} \mid s \in \mathscr{S}] \subset \mathbb{C}[Q_{\mathbb{Z}}^\vee]$. This is a unique factorization domain and $\mathbb{C}(Q_{\mathbb{Z}}^\vee)$ is the field of fractions of A.

Lemma 5.18. — *Let $\beta^\vee \in \Phi^\vee$. Then $Z^{\beta^\vee} - 1$ and $Z^{\beta^\vee} + 1$ are irreducible in A.*

Proof. — Write $\beta^\vee = w.\alpha_s^\vee$, where $w \in W^v$ and $s \in \mathscr{S}$. Then $Z^{\beta^\vee} = (Z^{\alpha_s^\vee})^w$. $\square$

Lemma 5.19 (see [29, Proposition 14.3]). — *Let $\tau \in T_{\mathbb{C}}$ and $r = r_{\beta^\vee} \in \mathscr{S}_\tau$. Then $F_{r_\beta^\vee} - \zeta_{\beta^\vee} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$.*

Proof. — One has $F_{r_\beta^\vee} - \zeta_{\beta^\vee} \in \mathcal{H}(Q_{\mathbb{Z}}^\vee)$. Write $F_{r_\beta^\vee} - \zeta_{\beta^\vee} = \sum_{u\in W^v} H_u \frac{f_u}{g_u}$, with $f_u, g_u \in A$ and $f_u \wedge g_u = 1$ for all $u \in W^v$. Let $u \in (1, r_{\beta^\vee})$. Let us prove that $\zeta_{\beta^\vee}^{\mathrm{den}} \wedge g_u = 1$. Suppose that $\zeta_{\beta^\vee}^{\mathrm{den}} \wedge g_u \neq 1$. Then there exists $\eta \in \{-1, 1\}$ such that $Z^{\beta^\vee} + \eta$ divides g_u.

Let $\chi \in T_{\mathbb{C}}$ be such that $\chi(\beta^\vee) = -\eta$. By Remark 4.1, $r_{\beta^\vee} \in W_\chi$. Let $(\chi_n) \in (T_{\mathbb{C}})^{\mathbb{Z}_{\geqslant 0}}$ be such that $\chi_n \to \chi$ and $W_{\chi_n} = \{1, r_{\beta^\vee}\}$ for all $n \in \mathbb{Z}_{\geqslant 0}$, and $\chi_n(\beta^\vee) = -\eta$ for all $n \in \mathbb{Z}_{\geqslant 0}$ (the existence of such a χ is provided by Lemma 5.17). One has $g_u(\chi_n) = 0$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Moreover by Lemma 5.8, $\pi_u^H(F_{r_\beta^\vee}) = \frac{f_u}{g_u} \in \mathbb{C}(Y)_{\chi_n}$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Therefore, $f_u(\chi_n) = 0$ for all $n \in \mathbb{Z}_{\geqslant 0}$ and thus $f_u(\chi) = 0$.

By the Nullstellensatz (see [23, IX, Theorem 1.5] for example), there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that $Z^{\beta^\vee} + \eta$ divides f_u^n in A. By Lemma 5.18, $Z^{\beta^\vee} + \eta$ is irreducible in A and thus $Z^{\beta^\vee} + \eta$ divides f_u: a contradiction. Therefore $\zeta_{\beta^\vee}^{\mathrm{den}} \wedge g_u = 1$. By Lemma 5.14, $g_u(\tau) \neq 0$.

Therefore $\{u \in W^v \mid \pi_u^H(F_{r_{\beta^\vee}} - \zeta_{r_{\beta^\vee}}) \notin \mathbb{C}(Y)_\tau\} \subset \{1\}$. By Lemma 5.15 we deduce that $\{u \in W^v \mid \pi_u^B(F_{r_{\beta^\vee}} - \zeta_{r_{\beta^\vee}}) \notin \mathbb{C}(Y)_\tau\} \subset \{1\}$. Using Lemma 5.4 we deduce that $\{u \in W^v \mid \pi_u^B(F_{r_{\beta^\vee}} - \zeta_{r_{\beta^\vee}}) \notin \mathbb{C}(Y)_\tau\} = \emptyset$. By Lemma 5.15, $\{u \in W^v \mid \pi_u^H(F_{r_{\beta^\vee}} - \zeta_{r_{\beta^\vee}}) \notin \mathbb{C}(Y)_\tau\} = \emptyset$, which proves the lemma. □

5.5. Description of generalized weight spaces

In this subsection, we describe $I_\tau(\tau, \mathrm{gen})$ for $\tau \in \mathcal{U}_{\mathbb{C}}$ when $W_{(\tau)} = W_\tau$, using the $K_{r_1} \dots K_{r_k}(\tau)$, for $r_1, \dots, r_k \in \mathscr{S}_\tau$ (see Theorem 5.27).

For $r \in \mathscr{R}$, one sets $K_r = F_r - \zeta_{\alpha_r^\vee} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$. By Lemma 4.14 we have:

$$\theta * K_r = K_r * \theta^r + (\theta^r - \theta)\zeta_r \text{ for all } \theta \in \mathbb{C}(Y). \tag{5.2}$$

Lemma 5.20. — *Let $w_1, w_2 \in W^v$. Then there exists $P \in \mathbb{C}(Y)^\times$ such that $F_{w_1} * F_{w_2} = F_{w_1 w_2} * P$. If moreover $\tau \in \mathcal{U}_{\mathbb{C}}$, then one can write $P = \frac{f}{g}$ with $f, g \in \mathbb{C}[Y]^\times$ and $f(w.\tau) \neq 0$ for all $w \in W^v$.*

Proof. — Let $u, v \in W^v$. Let us prove that if $\chi \in T_{\mathbb{C}}^{\mathrm{reg}}$, then $F_u * F_v \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi$. Write $F_u = \sum_{u' \leqslant u} H_{u'}\theta_{u'}$, where $\theta_{u'} \in \mathbb{C}(Y)$ for all $u' \leqslant u$. Then by Lemma 4.14,

$$F_u * F_v = \sum_{u' \leqslant u} H_{u'}\theta_{u'} * F_v = \sum_{u' \leqslant u} H_{u'} * F_v * (\theta_{u'})^{v^{-1}}.$$

By Lemma 4.14, $\theta_{u'} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi$ for all $\chi \in T_{\mathbb{C}}^{\mathrm{reg}}$ and thus $(\theta_{u'})^{v^{-1}} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi$ for all $\chi \in T_{\mathbb{C}}^{\mathrm{reg}}$. Let $\chi \in T_{\mathbb{C}}^{\mathrm{reg}}$. As ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi$ is an $\mathcal{H}_{W^v,\mathbb{C}}-\mathbb{C}(Y)_\chi$ bimodule, we deduce that $F_u * F_v \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi$.

Let $u, v \in W^v$. Let us prove that there exists $Q \in \mathbb{C}(Y)$ such that $F_u * F_v = F_{uv} * Q$. Let $\lambda \in Y$. Then by Lemma 4.14, one has $Z^\lambda F_u * F_v = F_u * F_v * Z^{(uv)^{-1}.\lambda}$. Therefore for all $\chi \in T_{\mathbb{C}}^{\mathrm{reg}}$, there exists $a(\chi) \in \mathbb{C}$ such that $F_u * F_v(\chi) = a(\chi) F_{uv}(\chi)$. Write $F_u * F_v = \sum_{w\in W^v} H_w * \theta_w$ and $F_{uv} = \sum_{w\in W^v} H_w * \widetilde{\theta}_w$, where $(\theta_w), (\widetilde{\theta}_w) \in \mathbb{C}(Y)^{(W^v)}$. Let $Q = \frac{\theta_{uv}}{\widetilde{\theta}_{uv}} = \theta_{uv}$. Let $w \in W^v$ be such that $\widetilde{\theta}_w = 0$. Then for all $\chi \in T_{\mathbb{C}}^{\mathrm{reg}}$, $\theta_w(\chi) = 0$ and by Lemma 5.1, $\theta_w = 0 = Q\widetilde{\theta}_w$. Let $w \in W^v$ be such that $\theta_w \neq 0$. Then $U := \{\chi \in T_{\mathbb{C}} \mid \theta_w \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\chi \text{ and } \theta_w(\chi) \neq 0\}$ is open and dense in $T_{\mathbb{C}}$. By Remark 4.11, $T_{\mathbb{C}}^{\mathrm{reg}}$ has full measure in $T_{\mathbb{C}}$ and thus $U \cap T_{\mathbb{C}}^{\mathrm{reg}}$ is dense in $T_{\mathbb{C}}$. Moreover $\theta_w(\chi) = Q(\chi)\widetilde{\theta}(\chi)$ for all $\chi \in U \cap T_{\mathbb{C}}^{\mathrm{reg}}$ and thus $\widetilde{\theta}_w = Q\theta_w$. Consequently, there exists $Q \in \mathbb{C}(Y)$ such that $F_u * F_v = F_{uv} * Q$.

Let $\tau \in \mathcal{U}_{\mathbb{C}}$. Let $w_1 \in W^v$. Let $u \in W^v$ be such that there exists $\theta = \frac{f}{g} \in \mathbb{C}(Y)^\times$ such that $F_{w_1} * F_u = F_{w_1 u} * \theta$, with $f(w.\tau) \neq 0$ for all $w \in W^v$. Let $s \in \mathscr{S}$ be such that $us > u$. Then by Lemma 4.3,

$$F_{w_1} * F_{us} = F_{w_1 u} * \theta * F_s = F_{w_1 u} * F_s * \theta^s.$$

Suppose $w_1 us > w_1 u$. Then $F_{w_1 u} * F_s = F_{w_1 us}$ and thus $F_{w_1} * F_{us} = F_{w_1 us} * \theta^s$ and $f^s(w.\tau) \neq 0$ for all $w \in W^v$. Suppose $w_1 us < w_1 u$. Then $F_{w_1 u} * F_s = F_{w_1 us} * (F_s)^2$ and thus by Lemma 4.3, $F_{w_1} * F_{us} = F_{w_1 us} * (\theta^s \zeta_s \zeta_s^s)$. By definition of $\mathcal{U}_{\mathbb{C}}$, one can write $F_{w_1} * F_{us} = F_{w_1 us} * \frac{\tilde{f}}{\tilde{g}}$ with $\widetilde{f}, \widetilde{g} \in \mathbb{C}[Y]^\times$ such that $\widetilde{f}(w.\tau) \neq 0$ for all $w \in W^v$ and the lemma follows. □

Remark 5.21. — In [29, Lemma 4.3 (2)], Reeder gives an explicit expression of $F_u * F_v$, for $u, v \in W^v$.

Let $r \in \mathscr{R}$. Let $\Omega_r : \mathbb{C}(Y) \to \mathbb{C}(Y)$ be defined by $\Omega_r(\theta) = \zeta_r(\theta^r - \theta)$ for all $\theta \in \mathbb{C}(Y)$.

Lemma 5.22. — *Let $r \in \mathscr{S}_\tau$. Then $\Omega_r(\mathbb{C}(Y)_\tau) \subset \mathbb{C}(Y)_\tau$.*

Proof. — Write $r = r_{\beta^\vee}$, where $\beta^\vee \in \Phi^\vee$. Then one has $r(\lambda) = \lambda - \beta(\lambda)\beta^\vee$ for all $\lambda \in Y$. Let $\lambda \in Y$. Then with the same computation as in Remark 2.7 (2), we have that $\Omega_r(Z^\lambda) \in \mathbb{C}(Y)_\tau$. Thus $\Omega_r(\theta) \in \mathbb{C}(Y)_\tau$ for all $\theta \in \mathbb{C}[Y]$.

Let $\theta \in \mathbb{C}(Y)_\tau$. Write $\theta = \frac{f}{g}$, where $f, g \in \mathbb{C}[Y]$ and $g(\tau) \neq 0$. Then $\zeta_r(\theta^r - \theta) = \zeta_r(\frac{f^r g - (f^r g)^r}{g g^r})$. Moreover, $g^r(\tau) = g(r.\tau) = g(\tau) \neq 0$ and as $f^r g \in \mathbb{C}[Y]$, we have that $\zeta_r(\theta^r - \theta) \in \mathbb{C}(Y)_\tau$. □

We now assume that $\tau \in \mathcal{U}_{\mathbb{C}}$.

For each $w \in W_{(\tau)}$ we fix a reduced writing $w = r_1 \dots r_k$, with $k = \ell(w)$ and $r_1, \dots, r_k \in \mathscr{S}_\tau$ and we set $\underline{w} = (r_1, \dots, r_k)$. Let $K_{\underline{w}} = K_{r_1} \dots K_{r_k} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$.

Lemma 5.23. — *Let $r \in \mathscr{S}_\tau$. Then ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau * K_r \subset {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$. In particular, $K_{\underline{w}} \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ for all $w \in W_{(\tau)}$.*

Proof. — Let $w \in W^v$ and $\theta \in \mathbb{C}(Y)_\tau$. Then $H_w\theta * K_r = H_w K_r \theta^r + H_w * \Omega_r(\theta)$. Using Lemma 5.19, Lemma 5.22 and the fact that ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ is a $\mathcal{H}_{W^v,\mathbb{C}} - \mathbb{C}(Y)_\tau$-bimodule, we deduce that $H_w\theta * K_r \in {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$. Hence ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau * K_r \subset {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$. □

Lemma 5.24. — *Let $w \in W_{(\tau)}$. Then* $\max\operatorname{supp}\big(K_{\underline{w}}(\tau)\big) = \{w\}$, *where* $\max$ *is defined with respect to the order $\leqslant$ on W^v.*

Proof. — Write $\underline{w} = (r_1, \dots, r_k)$ with $r_1, \dots, r_k \in \mathscr{S}_\tau$. Then

$$K_{\underline{w}} = (F_{r_{i_1}} - \zeta_{r_{i_1}}) \dots (F_{r_{i_k}} - \zeta_{r_{i_k}}) = F_{r_{i_1}} * F_{r_{i_2}} * \dots * F_{r_{i_k}} + \sum_{v <_\tau w} F_v P_v,$$

for some $P_v \in \mathbb{C}(Y)$. By Lemma 5.20, there exist $f, g \in \mathbb{C}[Y]^\times$ such that $F_{r_{i_1}} * F_{r_{i_2}} * \dots * F_{r_{i_k}} = F_w * \frac{f}{g}$ and $f(\tau) \neq 0$. One has $\pi_w^T(F_w) = 1$ and by Lemma 5.10, $\pi_v^T(F_v) = 0$ for all $v \in [1, w)_{\leqslant_\tau}$. Thus using Lemma 5.23, one can moreover assume $g(\tau) \neq 0$. Therefore $\pi_w^T(K_{\underline{w}}) = \frac{f}{g} \in \mathbb{C}(Y)_\tau$ and $f(\tau) \neq 0$, which proves the lemma. □

Let $\mathcal{K}(W_{(\tau)}) = \bigoplus_{w \in W_{(\tau)}} F_w \mathbb{C}(Y)$. By Lemma 5.20 and Lemma 4.14, $\mathcal{K}(W_{(\tau)})$ is a subalgebra of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$. Let $\mathcal{K}_\tau = \mathcal{K}(W_{(\tau)}) \cap {}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$. For $w \in W_{(\tau)}$, set $\mathcal{K}(W_{(\tau)})^{<_\tau w} = \bigoplus_{v \in W_{(\tau)}, v <_\tau w} F_w \mathbb{C}(Y)$ and $\mathcal{K}_\tau^{<_\tau w} = \bigoplus_{v <_\tau w} K_{\underline{v}} \mathbb{C}(Y)_\tau$.

Lemma 5.25. — *Let $\theta \in \mathbb{C}(Y)_\tau$ and $w \in W_{(\tau)}$. Then there exists $k_{\underline{w}}(\theta) \in \mathcal{K}_\tau^{<_\tau w}$ such that $\theta * K_{\underline{w}} = K_{\underline{w}} * \theta^{w^{-1}} + k_{\underline{w}}(\theta)$.*

Proof. — If $w = 1$, this is clear. Suppose $w >_\tau 1$. Write $w = vr$ with $v \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$ such that $v <_\tau w$. Suppose that $\theta * K_{\underline{v}} = K_{\underline{v}} * \theta^{v^{-1}} + k_{\underline{v}}(\theta)$ with $k_{\underline{v}}(\theta) \in \mathcal{K}_\tau^{<_\tau v}$. One has

$$\begin{aligned} \theta * K_{\underline{w}} = \theta * K_{\underline{v}} * K_r &= \big(K_{\underline{v}} \theta^{v^{-1}} + k_{\underline{v}}(\theta)\big) * K_r \\ &= K_{\underline{w}} * \theta^{w^{-1}} + K_{\underline{v}} * \Omega_r(\theta^{v^{-1}}) + k_{\underline{v}}(\theta) * K_r. \end{aligned}$$

The sets $\mathcal{K}(W_{(\tau)})^{\leqslant_\tau v} = \bigoplus_{v' \leqslant_\tau v} F_{v'} \mathbb{C}(Y)$ and ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})_\tau$ are right $\mathbb{C}(Y)_\tau$-submodules of ${}^{\mathrm{BL}}\mathcal{H}(T_{\mathbb{C}})$ and thus by Lemma 5.23 and Lemma 5.22, $K_{\underline{v}} * \Omega_r(\theta^{v^{-1}}) \in \mathcal{K}_\tau^{\leqslant_\tau v} \subset \mathcal{K}_\tau^{<_\tau w}$.

By Lemma 5.23, $k_{\underline{v}}(\theta) * K_r \in {}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau$. By Lemma 4.14 and [22, Corollary 1.3.19], $k_{\underline{v}}F_r \in \mathcal{K}(W_{(\tau)})^{<_\tau \max(vr,v)} = \mathcal{K}(W_{(\tau)})^{<_\tau w}$. Consequently $k_{\underline{v}} * K_r \in \mathcal{K}_\tau^{<_\tau w}$ and $K_{\underline{v}}\Omega_r(\theta^{v^{-1}}) + k_{\underline{v}}(\theta)K_r \in \mathcal{K}_\tau^{<_\tau w}$, which proves the lemma. □

Lemma 5.26. — *One has* $\mathcal{K}_\tau = \bigoplus_{w\in W_{(\tau)}} K_{\underline{w}}\mathbb{C}(Y)_\tau$.

Proof. — By Lemma 5.23, $\mathcal{K}_\tau \supset \bigoplus_{w\in W_{(\tau)}} K_{\underline{w}}\mathbb{C}(Y)_\tau$. For $w \in W_{(\tau)}$, set $\mathcal{K}(W_{(\tau)})^{\leqslant_\tau w} = \bigoplus_{v \leqslant_\tau w} F_v\mathbb{C}(Y) \subset \mathcal{K}(W_{(\tau)})$. Let $w \in W_{(\tau)}$. Suppose that for all $v \in [1,w)_{\leqslant_\tau}$, one has $\mathcal{K}_\tau^{\leqslant_\tau v} = \bigoplus_{v'\in[1,v]_{\leqslant_\tau}} K_{\underline{v'}}\mathbb{C}(Y)_\tau$. By Lemma 5.24, one can write $\pi_w^T(K_{\underline{w}}) = \frac{f}{g}$, with $f,g \in \mathbb{C}[Y]$ such that $f(\tau)g(\tau) \neq 0$. Let $x \in \mathcal{K}_\tau^{\leqslant_\tau w}$ and $\theta = \pi_w^T(x) \in \mathbb{C}(Y)_\tau$. By Lemma 5.23, $\theta\frac{g}{f}K_{\underline{w}} \in {}^{\mathrm{BL}}\mathcal{H}(T_\mathbb{C})_\tau$. Moreover, $x - \theta\frac{g}{f}K_{\underline{w}} \in \sum_{v\in[1,w)_{\leqslant_\tau}} \mathcal{K}_\tau^{\leqslant_\tau v}$. Therefore, $x \in \bigoplus_{v\in[1,w]_{\leqslant_\tau}} K_{\underline{v}}\mathbb{C}(Y)_\tau$ and the lemma follows. □

Theorem 5.27. — *Let* $\tau \in \mathcal{U}_\mathbb{C}$ *be such that* $W_{(\tau)} = W_\tau$. *Then* $I_\tau(\tau,\mathrm{gen}) = \mathrm{ev}_\tau(\mathcal{K}_\tau) \otimes_\tau 1$.

Proof. — Let $w \in W_{(\tau)}$ and $\theta \in \mathbb{C}(Y)_\tau$. As $w \in W_\tau$, $\theta^{w^{-1}} \in \mathbb{C}(Y)_{w.\tau} = \mathbb{C}(Y)_\tau$ and $\tau(\theta^{w^{-1}}) = \tau(\theta)$. Then by Lemma 5.25, $(\theta - \tau(\theta))K_{\underline{w}}(\tau) \otimes_\tau 1 \in \mathcal{K}^{<_\tau w}(\tau) \otimes_\tau 1$. By an induction using Lemma 5.26 we deduce that $\mathcal{K}_\tau(\tau) \otimes_\tau 1 \subset I_\tau(\tau,\mathrm{gen})$.

Let $w \in W^v$ and $E_w = \big(\mathrm{ev}_\tau(\mathcal{K}_\tau) \otimes_\tau 1\big) \cap I_\tau^{\leqslant w}$. By Lemma 5.24, $\dim E_w = |W_{(\tau)} \cap \{v \in W^v \mid v \leqslant w\}|$. By Proposition 3.4, $\dim I_\tau(\tau,\mathrm{gen})^{\leqslant w} = |\{v \in W_\tau \mid v \leqslant w\}| = \dim E_w$. As $(W^v, \leqslant)$ is a directed poset, $I_\tau = \bigcup_{w\in W^v} I_\tau^{\leqslant w}$, which proves the theorem. □

5.6. Irreducibility of I_τ when $W_\tau = W_{(\tau)}$ is the infinite dihedral group

In this subsection, we prove that if $\tau \in \mathcal{U}_\mathbb{C}$ is such that $W_\tau = W_{(\tau)}$ and $W_{(\tau)}$ is isomorphic to the infinite dihedral group, then I_τ is irreducible (see Lemma 5.33). Let us sketch the proof of this lemma. We prove that $I_\tau(\tau) = \mathbb{C}1 \otimes_\tau 1$. For $w \in W_{(\tau)}$, let $\pi_w^K : I_\tau(\tau,\mathrm{gen}) \to \mathbb{C}$ be defined as $\pi_w^K\big(\sum_{v\in W^v} K_{\underline{v}}(\tau)x_v\big) = x_w$, for all $(x_v) \in \mathbb{C}^{(W_{(\tau)})}$, which is well-defined by Lemma 5.24 and Theorem 5.27. We suppose that $I_\tau(\tau) \setminus \mathbb{C}1 \otimes_\tau 1$ is nonempty and we consider one of its elements x. We reach a contradiction by computing $\pi_w^K(x)$, where $w \in W_{(\tau)}$ is such that $\ell_\tau(w) = \max\{\ell_\tau(v) \mid v \in \mathrm{supp}(x) \cap W_{(\tau)}\} - 1$.

Let $\tau \in \mathcal{U}_{\mathbb{C}}$. Assume that $(W_{(\tau)}, \mathscr{S}_\tau)$ is isomorphic to the infinite dihedral group (in particular, $|\mathscr{S}_\tau| = 2$ and every element of $W_{(\tau)}$ admits a unique reduced writing).

The following lemma is easy to prove.

Lemma 5.28. — *Let $w \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$ be such that $\ell_\tau(wr) = \ell_\tau(w) + 1$. Let $u \in [1, w)_{\leqslant_\tau}$. Then $ur \neq w$.*

Lemma 5.29. — *Let $\tau \in \mathcal{U}_{\mathbb{C}}$. Let $r = r_{\beta^\vee} \in \mathscr{S}_\tau$, where $\beta^\vee \in \Phi^\vee$. Then there exists $a \in \mathbb{C}^*$ such that for all $\lambda \in Y$,*

$$\tau\big((Z^{r.\lambda} - Z^\lambda)\zeta_r\big) = a\tau(\lambda)\beta(\lambda).$$

Proof. — One has

$$\zeta_r = \frac{1}{\zeta^{\mathrm{den}}_{\beta^\vee}}. \prod_{\alpha^\vee \in N_{\Phi^\vee}(r)} \zeta^{\mathrm{num}}_{\alpha^\vee}. \prod_{\alpha^\vee \in N_{\Phi^\vee}(r)\setminus\{\beta^\vee\}} \frac{1}{\zeta^{\mathrm{den}}_{\alpha^\vee}}.$$

By Lemma 5.13 and by definition of $\mathcal{U}_{\mathbb{C}}$,

$$\tau\left(\prod_{\alpha^\vee \in N_{\Phi^\vee}(r)\setminus\{\beta^\vee\}} \zeta^{\mathrm{den}}_{\alpha^\vee}\right) \neq 0 \quad \text{and} \quad \tau\left(\prod_{\alpha^\vee \in N_{\Phi^\vee}(r)} \zeta^{\mathrm{num}}_{\alpha^\vee}\right) \neq 0.$$

If $\sigma_{\beta^\vee} = \sigma'_{\beta^\vee}$, one has $\frac{Z^{r.\lambda}-Z^\lambda}{\zeta^{\mathrm{den}}_{\beta^\vee}} = \frac{Z^{r.\lambda}-Z^\lambda}{1-Z^{\beta^\vee}} = Z^\lambda \frac{Z^{-\beta(\lambda)\beta^\vee}-1}{1-Z^{\beta^\vee}}$. By Lemma 5.13, $r \in \mathscr{R}_{(\tau)}$ and thus $\tau(\beta^\vee) = 1$. Thus by the same computation as in Remark 2.7, $\tau(\frac{Z^{r.\lambda}-Z^\lambda}{1-Z^{\beta^\vee}}) = \beta(\lambda)\tau(\lambda)$. Using a similar computation when $\sigma_{\beta^\vee} \neq \sigma'_{\beta^\vee}$, we deduce the lemma. □

Lemma 5.30. — *Let $w \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$ be such that $\ell_\tau(wr) = \ell_\tau(w) + 1$. Then there exists $a \in \mathbb{C}^*$ such that for all $\lambda \in Y$, one has:*

$$\pi^K_w\big(Z^\lambda * K_{\underline{wr}}(\tau) \otimes_\tau 1\big) = a\tau(\lambda)\alpha_r(w^{-1}.\lambda).$$

Proof. — Let $\lambda \in Y$. Write $Z^\lambda * K_{\underline{w}} = K_{\underline{w}} * Z^{w^{-1}.\lambda} + k$, where $k \in \mathcal{K}^{<_\tau w}_\tau$, which is possible by Lemma 5.25. One has

$$\begin{aligned} Z^\lambda * K_{\underline{wr}} &= (K_{\underline{w}} * Z^{w^{-1}.\lambda} + k) * K_r \\ &= K_{\underline{wr}} * Z^{rw^{-1}.\lambda} + K_{\underline{w}}\big((Z^{rw^{-1}.\lambda} - Z^{w^{-1}.\lambda})\zeta_r\big) + k * K_r. \end{aligned}$$

Therefore, using Lemma 5.28 and Lemma 5.29 we deduce

$$\pi^K_w\big(Z^\lambda K_{\underline{wr}}(\tau) \otimes_\tau 1\big) = \tau\big((Z^{rw^{-1}.\lambda} - Z^{w^{-1}.\lambda})\zeta_r\big) = a\tau(\lambda)\beta(w^{-1}.\lambda),$$

for some $a \in \mathbb{C}^*$. □

Lemma 5.31. — *Let $w \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$ be such that $\ell_\tau(rw) = \ell_\tau(w) + 1$. One has $\pi^K_w(K_r * \mathcal{K}(W_{(\tau)})^{\leqslant_\tau w}) = \{0\}$.*

Proof. — Let $u \in W_{(\tau)}$ and $r \in \mathscr{S}_\tau$ be such that $ru >_\tau u$. Then by Lemma 5.20 and [22, Corollary 1.3.19], $F_r * \mathcal{K}(W_{(\tau)})^{\leqslant_\tau u} \subset \mathcal{K}(W_{(\tau)})^{\leqslant_\tau \max(u,ru)}$ and thus $K_r * \mathcal{K}(W_{(\tau)})^{\leqslant_\tau u} \subset \mathcal{K}(W_{(\tau)})^{\leqslant_\tau \max(u,ru)}$.

Let $v \in [1,w)_{\leqslant_\tau}$. If $rv >_\tau v$, then by Lemma 5.20, there exists $Q \in \mathbb{C}(Y)$ such that $F_r * F_v = F_{rv} * Q$ and thus $K_r * F_v \in F_{rv} * Q + F_v\mathbb{C}(Y)$. By Lemma 5.28, $rv \neq w$. Using Lemma 5.24 and the fact w and rv have the same length, we deduce that $\pi_w^K(K_r * F_v) = 0$.

If $rv <_\tau v$, then $K_r * F_v \in \mathcal{K}(W_{(\tau)})^{\leqslant_\tau v}$ and thus $\pi_w^K(K_r * F_v) = 0$ which finishes the proof of the lemma. □

Lemma 5.32. — *Let $w \in W_\tau$, $r \in \mathscr{S}_\tau$ be such that $\ell_\tau(rw) = \ell_\tau(w) + 1$. Then there exists $b \in \mathbb{C}^*$ such that for all $\lambda \in Y$:*

$$\pi_w^K(Z^\lambda.K_{\underline{rw}}(\tau) \otimes_\tau 1) = b\tau(\lambda)\alpha_r(\lambda).$$

Proof. — One has

$$Z^\lambda K_{\underline{rw}} = (Z^\lambda * K_r) * K_{\underline{w}} = \big(K_r.Z^{r.\lambda} + (Z^{r.\lambda} - Z^\lambda)\zeta_r\big) * K_{\underline{w}}(\tau).$$

One has $Z^{r.\lambda} * K_{\underline{w}} \in \mathcal{K}(W_{(\tau)})^{\leqslant_\tau w}$. Thus by Lemma 5.31, $\pi_w^K(K_r.Z^{r.\lambda} * K_{\underline{w}}) = 0$. Moreover, by Lemma 5.29, there exists $b \in \mathbb{C}^*$ such that

$$\pi_w^K\big((Z^{r.\lambda} - Z^\lambda)\zeta_r K_{\underline{w}}(\tau) \otimes_\tau 1\big) = w.\tau\big((Z^{r.\lambda} - Z^\lambda)\zeta_r\big) = b\tau(\lambda)\alpha_r(\lambda),$$

which proves the lemma. □

Lemma 5.33. — *Let $\tau \in \mathcal{U}_\mathbb{C}$ be such that $W_\tau = W_{(\tau)}$ and such that there exists $r_1, r_2 \in \mathscr{S}_\tau$ such that $(W_{(\tau)}, \{r_1, r_2\})$ is isomorphic to the infinite dihedral group. Then I_τ is irreducible.*

Proof. — Let us prove that $I_\tau(\tau) = \mathbb{C}.1 \otimes_\tau 1$. Let $x \in I_\tau \setminus \mathbb{C}.1 \otimes_\tau 1$ and assume that $x \in I_\tau(\tau)$. Let $n = \max\{\ell_\tau(w) \,|\, w \in \mathrm{supp}(x)\}$. Let $w \in W_{(\tau)}$ be such that $\ell_\tau(w) = n-1$. Then there exist $r, r' \in \mathscr{S}_\tau$ such that $\{v \in W_{(\tau)} \,|\, \ell_\tau(v) = n\} = \{rw, wr'\}$. By Theorem 5.27, $x \in \sum_{v \in W_{(\tau)}} \mathbb{C}K_{\underline{v}}(\tau) \otimes_\tau 1$. Let $\gamma = \pi_{rw}^K(x)$ and $\gamma' = \pi_{wr'}^K(x)$.

Set $\gamma_w = \pi_w^K(x)$. Then by Lemma 5.30 and Lemma 5.32, there exist $a, a' \in \mathbb{C}^*$ such that for all $\lambda \in Y$,

$$\pi_w^K(Z^\lambda.x) = \tau(\lambda)\big(a\gamma\alpha_r(\lambda) + a'\gamma' w.\alpha_{r'}(\lambda) + \gamma_w\big) = \tau(\lambda)\gamma_w.$$

Therefore $\{\alpha_r, w.\alpha_{r'}\}$ is linearly dependent and hence $w.\alpha_{r'} \in \{\pm\alpha_r\} = \{\alpha_r, r.\alpha_r\}$. By Lemma 2.3 we deduce $rw = wr'$: a contradiction because $|\{rw, wr'\}| = |\{v \in W_{(\tau)} \,|\, \ell_\tau(v) = n\}| = 2$.

Therefore $I_\tau = \mathbb{C}1 \otimes_\tau 1$ and by Theorem 4.8, I_τ is irreducible. □

5.7. Kato's criterion when the Kac–Moody matrix has size 2

In this subsection, we prove Kato's irreducibility criterion when $|\mathscr{S}| = 2$ (see Theorem 5.35). As the case where W^v is finite is a particular case of Kato's theorem [20, Theorem 2.2] we assume that W^v is infinite.

This is equivalent to assuming that the Kac–Moody matrix of the root generating system $\mathcal{S}$ is of the form $\left(\begin{smallmatrix} 2 & a \\ b & 2 \end{smallmatrix}\right)$, with $a, b \in \mathbb{Z}_{<0}$ and $ab \geqslant 4$ ([22, Proposition 1.3.21]). The system $(W^v, \mathscr{S})$ is then the infinite dihedral group. Write $\mathscr{S} = \{s_1, s_2\}$. Then every element of W^v admits a unique reduced writing involving s_1 and s_2.

Let G be a group and $a, b \in G$. For $k \in \mathbb{Z}_{\geqslant 0}$, we define $P_k(a,b) = aba\ldots$ where the products has k terms.

Lemma 5.34. — *The subgroups of W^v are exactly the ones of the following list:*

(1) $\{1\}$
(2) $\langle r \rangle = \{1, r\}$, *for some* $r \in \mathscr{R}$
(3) $Z_k = \langle P_{2k}(s_1, s_2) \rangle = \langle P_{2k}(s_2, s_1) \rangle \simeq \mathbb{Z}$ *for* $k \in \mathbb{Z}_{\geqslant 1}$
(4) $R_{k,m} = \langle P_{2k+1}(s_1, s_2), P_{2m+1}(s_2, s_1) \rangle \simeq W^v$ *for* $k, m \in \mathbb{Z}_{\geqslant 0}$.

Proof. — Let $\{1\} \neq H \subset W^v$ be a subgroup. Let $n = \min\{\ell(w) \,|\, w \in H \setminus \{1\}\}$.

First assume that n is even and set $k = \frac{n}{2}$. Then $P(s_1, s_2, n) = P(s_2, s_1, n)^{-1}$ and as these are the only elements having length n in W^v, $H \supset Z_k$. Let $w = P_n(s_1, s_2)$. Let $h \in H \setminus \{1\}$. Write $\ell(h) = an + r$ with $a \in \mathbb{Z}_{\geqslant 1}$ and $r \in [\![0, r-1]\!]$. Then there exists $\epsilon \in \{-1, 1\}$ such that $h = w^{\epsilon a}.h'$, with $\ell(h') = r$. Moreover, $h' \in H$ and thus $h' = 1$. Therefore $H = Z_k$.

We now assume that n is odd. Maybe considering vHv^{-1} for some $v \in W^v$ and exchanging the roles of s_1 and s_2, we may assume that $s_1 \in H$. Assume $H \neq \langle s_1 \rangle$. Let $n' = \min\{\ell(w) | w \in H \setminus \langle s_1 \rangle\}$. Let $w \in H \setminus \langle s_1 \rangle$ be such that $\ell(w) = n'$. Then the reduced writing of w begins and ends with s_2. Thus $n' = 2n'' + 1$ for some $n'' \in \mathbb{Z}_{\geqslant 0}$. Then it is easy to see that $H = R_{1,n''}$, which finishes the proof. □

We prove in Appendix B that there exist size 2 Kac–Moody matrices such that for each subgroup of W^v, there exists $\tau \in T_{\mathbb{C}}$ such that $W_{(\tau)}$ is isomorphic to this subgroup.

Theorem 5.35. — *Assume that the matrix of the root generating system $\mathcal{S}$ is of size 2. Let $\tau \in T_{\mathbb{C}}$. Then I_τ is irreducible if and only if $\tau \in \mathcal{U}_{\mathbb{C}}$ and $W_\tau = W_{(\tau)}$.*

Proof. — If W^v is finite, this is a particular case of Kato's theorem ([20, Theorem 2.2]). Suppose that W^v is infinite. By Lemma 4.5 and Proposition 4.17, if I_τ is irreducible, then $\tau \in \mathcal{U}_\mathbb{C}$ and $W_\tau = W_{(\tau)}$. Reciprocally, suppose $\tau \in \mathcal{U}_\mathbb{C}$ and $W_\tau = W_{(\tau)}$. Then by Lemma 5.34, either $W_{(\tau)} = \{1\}$, or $W_{(\tau)} = \langle r \rangle$ for some $r \in \mathscr{R}$ or $W_{(\tau)} = \langle r_1, r_2 \rangle$ for some $r_1, r_2 \in \mathscr{R}$ and $(W_{(\tau)}, \{r_1, r_2\})$ is isomorphic to the infinite dihedral group. In the first two cases, I_τ is irreducible by Corollary 4.10 or Corollary 4.12. Suppose $W_{(\tau)} = \langle r_1, r_2 \rangle$. Then by Remark 2.5 (1), $(W_{(\tau)}, \mathscr{S}_\tau)$ is isomorphic to the infinite dihedral group and I_τ is irreducible by Lemma 5.33. □

Comments on the proofs of Kato's criterion. There are several proofs of Kato's criterion in the literature. In [28], Reeder proves this criterion (see Corollary 8.7 therein). In his proof, he uses the R-group $R_\tau = \{w \in W_\tau \mid w(\Phi^\vee_{(\tau)} \cap \Phi^\vee_+) = \Phi^\vee_{(\tau)} \cap \Phi^\vee_+\}$. This group is reduced to $\{1\}$ when $W_\tau = W_{(\tau)}$. His proof uses Harish–Chandra completeness theorem, which (under certain hypothesis on τ) majorizes the dimension of the space of intertwining operators of I_τ. Unfortunately, it seems that there exists up to now no equivalent of Harish–Chandra completeness theorem available in the Kac–Moody framework.

In [32], Rogawski gives a proof of a particular case of Kato's criterion (see Corollary 3.2 therein). However, it seems that its proof uses the fact that every element x of $I_\tau(\tau)$ can be written as a sum $x = \sum_{j \in J} x_j$ where J is a finite set and for all $j \in J$, $|\max \operatorname{supp}(x_j)| = 1$ and $x_j \in I_\tau(\tau)$. I do not know how to prove such a property.

In [29], Reeder gives two proofs of Kato's criterion or of weak versions of it (see Corollary 4.6 and Theorem 14.7 therein). Our proof of Theorem 5.35 is strongly inspired by the proof of [29, Theorem 14.7].

6. Towards principal series representations of G

Suppose that $\mathcal{H}_\mathbb{C}$ is associated with a reductive group G. Then for every open compact subgroup K' of G and every smooth representation V, $V^{K'}$ is naturally equipped with the structure of an $\mathcal{H}_{K',\mathbb{C}}$ module, where $\mathcal{H}_{K',\mathbb{C}}$ is the Hecke algebra associated with K' with coefficients in $\mathbb{C}$. Moreover, the assignment $V \mapsto V^{K'}$ induces a bijection between the following sets:

- equivalence classes of irreducible smooth representations V of G such that $V^{K'} \neq \{0\}$,
- isomorphism classes of simple $\mathcal{H}_{K',\mathbb{C}}$-modules (see [7, 4.3] for example).

In the Kac–Moody case, we do not know how to define "smooth" for a representation of G. We know that for any topological group structure on G, K_I is not compact open (see [1, Theorem 3.1]). The hope is that there should be a link between representations of G satisfying some regularity conditions and representations of $\mathcal{H}_{\mathbb{C}}$ or ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$.

Let $\epsilon \in \{+, \emptyset\}$. In this section, we associate to every $\tau \in T_{\mathcal{F}}^{\epsilon}$ a representation $\widehat{I(\tau^{\epsilon})^{\epsilon}}$ of G^{ϵ}. The principal series representation associated with τ should correspond to the space of elements of $\widehat{I(\tau^{\epsilon})^{\epsilon}}$ which satisfy some regularity condition. We define an action of $\mathcal{H}_{\mathcal{F}}$ on some subspace $I_{\tau^{\epsilon},G^{\epsilon}}$ of $\left(\widehat{I(\tau^{\epsilon})^{\epsilon}}\right)^{K_I}$. We then prove that $I_{\tau^{\epsilon},G^{\epsilon}}$ is isomorphic (as an $\mathcal{H}_{\mathcal{F}}$-module) to the representation $I^{+}_{\tau^{\epsilon}|G^{+}}$ introduced in Section 2. We then study the extendability of $\widehat{I(\tau^{\epsilon})^{\epsilon}}$ and $I_{\tau^{\epsilon},G^{\epsilon}}$ to representations of G and ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$.

For simplicity, we only introduce split Kac–Moody groups, although our results also apply to almost-split Kac–Moody groups over local fields, see [36].

In Section 6.1, we introduce split Kac–Moody groups over local fields, masures, their Iwahori–Hecke algebras and principal series representations.

In Section 6.2 we prove that the actions of $\mathcal{H}_{\mathcal{F}}$ on $I_{\tau,G^{+}}$ and $I_{\tau,G}$ are well-defined and prove that $I_{\tau,G^{+}}$ is isomorphic to I_{τ}.

In Section 6.3 we study under which condition $I_{\tau,G^{+}}$ and I_{τ}^{+} extend to representations of G and of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$, for $\tau \in T_{\mathcal{F}}^{+}$. We give examples of $\tau \in T_{\mathcal{F}}$ (for particular choices of G) such that $I_{\tau,G^{+}}$ and I_{τ}^{+} do not extend to representations of G and of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$.

6.1. Kac–Moody groups over local fields and masures

6.1.1. Split Kac–Moody groups over local fields and masure

Let $\mathbf{G}_{\mathcal{S}}$ be the group functor associated in [38] with the generating root datum $\mathcal{S}$, see also [30, 8]. Let $(\mathcal{K}, \omega)$ be a non-Archimedean local field where $\omega : \mathcal{K} \twoheadrightarrow \mathbb{Z} \cup \{+\infty\}$ is a valuation. Let $G = \mathbf{G}_{\mathcal{S}}(\mathcal{K})$ be the *split Kac–Moody group over $\mathcal{K}$ associated with $\mathcal{S}$*. The group G is generated by the following subgroups:

- the fundamental torus $T = \mathbf{T}(\mathcal{K})$, where $\mathbf{T} = \mathrm{Spec}(\mathbb{Z}[X])$,
- the root subgroups $U_\alpha = \mathbf{U}_\alpha(\mathcal{K})$, each isomorphic to $(\mathcal{K}, +)$ by an isomorphism x_α.

In [11] and [35] (see also [36]) the authors associate a masure $\mathcal{I}$ on which the group G acts. We recall briefly the construction of this masure. Let N be the normalizer of T in G. Then they define an action of N on $\mathbb{A}$, see [11, 3.1]. For $n \in N$ denote by $\nu(n) : \mathbb{A} \to \mathbb{A}$ the affine automorphism of $\mathbb{A}$ induced by the action of N on $\mathbb{A}$. Then $\nu(t)$ is a translation, for every $t \in T$ and $\nu(N) = W^v \ltimes Y$. For every $\mathbf{w} \in W^v \ltimes Y$, we choose $n_{\mathbf{w}} \in N$ such that $\nu(n_{\mathbf{w}}) = \mathbf{w}$.

The masure $\mathcal{I}$ is defined to be the set $G \times \mathbb{A}/ \sim$, for some equivalence relation $\sim$ (see [11, Definition 3.15]). Then G acts on $\mathcal{I}$ by $g.[h, x] = [gh, x]$ for $g, h \in G$ and $x \in \mathbb{A}$, where $[h, x]$ denotes the class of (h, x) for $\sim$. The map $x \mapsto [1, x]$ is an embedding of $\mathbb{A}$ in $\mathcal{I}$ and we identify $\mathbb{A}$ with its image. Then N is the stabilizer of $\mathbb{A}$ in G and it acts on $\mathbb{A}$ by ν. If $\alpha \in \Phi$ and $a \in \mathcal{K}$, then $x_\alpha(a) \in U_\alpha$ fixes the half-apartment $D_{\alpha,\omega(a)} = \{y \in \mathbb{A} | \ \alpha(y) + \omega(a) \geqslant 0\}$ and for all $y \in \mathbb{A} \setminus D_{\alpha,\omega(a)}$, $x_\alpha(a).y \notin \mathbb{A}$.

An *apartment* is a set of the form $g.\mathbb{A}$, for $g \in G$. We have $\mathcal{I} = \bigcup_{g \in G} g.\mathbb{A}$. Then $\mathcal{I}$ satisfies axioms (MA i), (MA ii) and (MA iii) of [15, Appendix A] or [16].These axioms describe the following properties.

(MA i) Let A be an apartment of $\mathcal{I}$. Then $A = g.\mathbb{A}$, for some $g \in G$. We can then transport every notion which is preserved by $\nu(N) = W^v \ltimes Y$ to A (in particular, we can define a segment, a hyperplane, ... in A).

(MA ii) This axiom asserts that if A and A' are two apartments such that $A \cap A'$ is "large enough", then $A \cap A'$ is a finite intersection of half-apartments (i.e. of sets of the form $h.D_{\alpha,k}$, for $\alpha \in \Phi$, $k \in \mathbb{Z}$, if $A = h.\mathbb{A}$) and there exists $g \in G$ such that $A' = g.A$ and g fixes $A \cap A'$. When G is an affine Kac–Moody group, this is true for every pair of apartments A, A', without any assumption on $A \cap A'$.

(MA iii) This axiom asserts that for some pairs of filters on $\mathcal{I}$, there exists an apartment containing them. This axiom is the building theoretic translation of some decompositions of G (e.g. Iwasawa decomposition).

A *filter* on a set E is a nonempty set $\mathcal{V}$ of nonempty subsets of E such that, for all subsets S, S' of E, if S, $S' \in \mathcal{V}$ then $S \cap S' \in \mathcal{V}$ and, if $S' \subset S$, with $S' \in \mathcal{V}$ then $S \in \mathcal{V}$.

Let E, E' be sets, $E' \subset E$ and $\mathcal{V}$ be a filter on E'. One says that a set $\Omega \subset E$ contains $\mathcal{V}$ if there exists $\Omega' \in \mathcal{V}$ such that $\Omega' \subset \Omega$ (or equivalently if $\Omega \in \mathcal{V}$ if $E = E'$). Let $f : E \to E$. One says that f fixes $\mathcal{V}$ if there exists $\Omega' \in \mathcal{V}$ such that f fixes Ω'.

6.1.2. Cartan decomposition, Tits preorder on $\mathcal{I}$ and sub-semi-group G^+

Let $K = \mathbf{G}_{\mathcal{S}}(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of $\mathcal{K}$. Then K is the fixator of $0 \in \mathbb{A} \subset \mathcal{I}$ in G. For $\lambda \in Y$, choose $n_\lambda \in T$ such that n_λ induces the translation on $\mathbb{A}$ by the vector λ. Unless G is reductive, the Cartan decomposition of G does not hold: $\bigsqcup_{\lambda \in Y^{++}} K n_\lambda K \subsetneq G$, where $Y^{++} = \overline{C_f^v} \cap Y$. For $x, y \in \mathbb{A}$, one writes $x \leqslant y$ if $y - x \in \mathcal{T}$ (where $\mathcal{T}$ is the Tits cone). If $x, y \in \mathcal{I}$, one writes $x \leqslant y$ if there exists $g \in G$ such that $g.x, g.y \in \mathbb{A}$ and $g.x \leqslant g.y$. This defines a G-invariant preorder on $\mathcal{I}$ by [34, Théorème 5.9]. We call it the *Tits preorder on* $\mathcal{I}$. Let $G^+ = \{g \in G | g.0 \geqslant 0\}$ (see [6, 1.2.2] for a more explicit description of G^+, when G is affine). Then G^+ is a sub-semi-group of G (as $\leqslant$ is transitive) and we have $G^+ = \bigsqcup_{\lambda \in Y^{++}} K n_\lambda K$: the Cartan decomposition holds on G^+. Note that when G is reductive, $G = G^+$ since $\mathcal{T} = \mathbb{A}$. A *type* 0 *vertex* is a point of the form $g.0$ for some $g \in G$. We set $\mathcal{I}_0 = G.0$. Then the map $g \mapsto g.0$ induces a bijection between G/K and $\mathcal{I}_0$.

Let $x, y \in \mathcal{I}$ be such that $x \leqslant y$. Let A_1, A_2 be apartments containing x and y. Let $[x, y]_{A_1}$ (resp. $[x, y]_{A_2}$) be the segment in A_1 (resp. A_2) joining x to y. Then by [34, Proposition 5.4], $[x, y]_{A_1} = [x, y]_{A_2}$ and there exists $g \in G$ such that $g.A_1 = A_2$ and g fixes $[x, y]_{A_1}$. We thus simply write $[x, y]$. Let $h \in G$ be such that $h.A_1 = \mathbb{A}$. Then as $\leqslant$ is G-invariant, $h.x \leqslant h.y$ and thus $h.y - h.x \in \mathcal{T}$. Replacing h by nh for some $n \in N$, we may assume that $h.y - h.x \in \overline{C_f^v}$. One sets $d^{Y^{++}}(x, y) = h.y - h.x \in \overline{C_f^v}$. We thus get a G-invariant *vectorial distance* $d^{Y^{++}} : \mathcal{I} \times_{\leqslant} \mathcal{I} \to \overline{C_f^v}$, where $\mathcal{I} \times_{\leqslant} \mathcal{I}$ is the set of pairs $x, y \in \mathcal{I}$ such that $x \leqslant y$. It is denoted d^v in [12]. When moreover $x, y \in \mathcal{I}_0$, then $d^{Y^{++}}(x, y) \in Y^{++}$. This distance parametrizes the K double cosets: if $g \in G^+$ and $\lambda \in Y^+$, then $g \in K n_\lambda K$ if and only if $d^{Y^{++}}(0, g.0) = \lambda$.

6.1.3. Local faces and chambers

Recall the definition of vectorial faces from Section 2.1. A *local face of* $\mathbb{A}$ (we omit the adjective "local" in the sequel) is a filter on $\mathbb{A}$ associated with a point x and with a vectorial face F^v. The point x is the *vertex* of F and F^v is its direction. More precisely the chamber $F = F_{x, F^v}$ associated to x and F^v is the filter on $\mathbb{A}$ consisting of the sets $\Omega \cap (x + F^v)$, where Ω is a neighborhood of x in $\mathbb{A}$. We call F *positive* (resp. *negative*) if F^v is. When F^v is a vectorial chamber (resp. a vectorial panel, that is when F^v is a codimension one face of a vectorial chamber), we call F a *chamber* (resp.

panel). As the sets of local faces, of positive faces, of local chambers,… are stable under the action of $W^v \ltimes Y$, we extend these notions to $\mathcal{I}$: a local face F (resp. positive, negative) is a filter on $\mathcal{I}$ generated by $g.F$ for some local face (resp. positive, negative) F_0 and some $g \in G$. Its *vertex* is $\mathrm{vert}(F) = g.\lambda$, where λ is the vertex of F_0. This does not depend on the choices of g and F_0 such that $F = g.F_0$.

We denote by C_0^+ the local positive chamber associated with 0 and C_f^v. A type 0 positive local chamber is a filter of the form $g.C_0^+$ for some $g \in G$. Equivalently, this is a positive chamber based at a type 0 vertex. We denote by $\mathscr{C}_0^+$ the set of positive type 0 chambers of $\mathcal{I}$.

We say that a chamber C of $\mathbb{A}$ *dominates* a panel P of $\mathbb{A}$ if C and P are based at the same vertex and if $P^v \subset \overline{C^v}$, where C^v and P^v are the vectorial faces defining C and P.

We say that a chamber C of $\mathcal{I}$ *dominates* a panel P of $\mathcal{I}$ if there exists $g \in G$ such that $g.C, g.P \subset \mathbb{A}$ and such that $g.C$ dominates $g.P$. Then every type 0 local panel is dominated by exactly $q+1$ chambers, where q is the cardinal of the residue cardinal of $\mathcal{K}$. In particular, $\mathcal{I}$ has *finite thickness*: every panel is dominated by finitely many chambers. This property is crucial in order to apply the finiteness results of [12] and [2].

Let $W^+ = W^v \ltimes Y^+$. Then W^+ is a sub-semi-group of $W^v \ltimes Y$.If $C, C' \in \mathscr{C}_0^+$, we write $C \leqslant C'$ if $\mathrm{vert}(C) \leqslant \mathrm{vert}(C')$. Let $\mathscr{C}_0^+ \times_{\leqslant} \mathscr{C}_0^+ = \{(C,C') \in \mathscr{C}_0^+ | C \leqslant C'\}$. Let $(C,C') \in \mathscr{C}_0^+ \times_{\leqslant} \mathscr{C}_0^+$. Then by [34, Proposition 5.5] or [16, Proposition 5.17], there exists an apartment $A = g.\mathbb{A}$ containing C and C'. Then $g.C \subset \mathbb{A}$ and thus there exists $\mathbf{w} \in W^v \ltimes Y$ such that $g.C = \mathbf{w}.C_0^+$. Maybe replacing g by $n_{\mathbf{w}}^{-1} g$, we may assume that $g.C = C_0^+$. Then $g.C' \geqslant C$ and thus there exists $\mathbf{v} \in W^+$ such that $g.C' = \mathbf{v}.C_0^+$. One sets $d^{W^+}(C,C') = \mathbf{v}$. By [34, Proposition 5.5] or [15, Theorem 4.4.17], $\mathbf{v}$ does not depend on the choice of A. This defines a G-invariant "W-distance" $d^{W^+} : \mathscr{C}_0^+ \times_{\leqslant} \mathscr{C}_0^+ \to W^+$.

Let C, C' be two chambers of the same sign and based at the same vertex. We say that C and C' are *adjacent* if they dominate a common panel. A gallery Γ between C and C' is a finite sequence $\Gamma = (C_1, \dots, C_n)$ such that $n \in \mathbb{Z}_{\geqslant 0}$, $C_1 = C$, $C_n = C'$ and C_i, C_{i+1} are adjacent for every $i \in [\![1, n-1]\!]$. The gallery Γ is called *minimal* if n is the minimum length among all the lengths of the galleries joining C to C'. If the vertex of C and C' is in $\mathcal{I}_0$, then the length of a minimal gallery between C and C' is $\ell(w)$, where $w = d^{W^+}(C,C') \in W^v$.

6.1.4. Iwahori subgroup and Iwahori–Hecke algebras associated with G

Let K_I be the fixator of C_0^+ in G. This is the *Iwahori subgroup* of G (see also [6, (3.8)] for a more explicit description in the affine case). The map $g \mapsto g.C_0^+$ induces a bijection between G/K_I and $\mathscr{C}_0^+$. For $\mathbf{w} \in W^v \ltimes Y$, we choose $n_{\mathbf{w}} \in N$ such that $n_{\mathbf{w}}$ induces $\mathbf{w}$ on $\mathbb{A}$. Then we have the Bruhat decomposition (see [2, 1.11]):

$$G^+ = \bigsqcup_{\mathbf{w}\in W^+} K_I n_{\mathbf{w}} K_I.$$

In terms of masures, this decomposition has the following interpretation: for every $C, C' \in \mathscr{C}_0^+$ such that $\mathrm{vert}(C) \leqslant \mathrm{vert}(C')$, there exists an apartment containing C and C'. Note that d^{W^+} parametrizes the K_I double cosets: if $g \in G^+$, then $g \in K_I n_{\mathbf{w}} K_I$ if and only if $\mathbf{w} = d^{W^+}(C_0^+, g.C_0^+)$.

Let $\mathscr{R}$ be a ring. For $\mathbf{w} \in W^+$, we denote by $T_{\mathbf{w}}$ the indicator function of $K_I n_{\mathbf{w}} K_I$. Then the *Iwahori–Hecke algebra of G* with coefficients in $\mathscr{R}$ is the free $\mathscr{R}$-module $\mathcal{H}_{G,\mathscr{R}}$ with basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$ equipped with the product $*$ such that $T_{\mathbf{v}} * T_{\mathbf{w}} = \sum_{\mathbf{u}\in W^+} a^{\mathbf{u}}_{\mathbf{v},\mathbf{w}}$, with $a^{\mathbf{u}}_{\mathbf{v},\mathbf{w}} = |(K_I n_{\mathbf{v}} K_I \cap n_{\mathbf{u}} K_I n_{\mathbf{w}}^{-1} K_I)/K_I|$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W^+$. The fact that such an algebra is well-defined is [2, Theorem 2.4] (the definition of the $T_{\mathbf{w}}$ in [2, 2] is slightly different but we obtain the same algebra).

Let $\mathcal{F}$ be a field as in Definition 2.6. Let q be the residue cardinal of $\mathcal{K}$. As in [2, 5.7], we assume that there exists $\delta^{1/2} \in T_{\mathcal{F}}$ such that $\delta^{1/2}(\alpha_s^\vee) = \sqrt{q}$ for every $s \in \mathscr{S}$. If $\mathcal{F} = \mathbb{C}$, such a map exists by Lemma 5.2. For $w \in W^v \subset W^+$, set $H_w = q^{-\frac{1}{2}\ell(w)} T_w \in \mathcal{H}_{G,\mathcal{F}}$. For $\lambda \in Y^{++}$, set $Z^\lambda = \delta^{-\frac{1}{2}}(\lambda) T_\lambda \in \mathcal{H}_{G,\mathcal{F}}$. By [2, 5], we have the following proposition.

Proposition 6.1. — *Let $\iota : \{Z^\lambda \mid \lambda \in Y^{++}\} \cup \{T_w \mid w \in W^v\} \subset \mathcal{H}_{G,\mathcal{F}} \to {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ be defined by $\iota(Z^\lambda) = Z^\lambda$ and $\iota(T_w) = T_w$ for $\lambda \in Y^{++}$ and $w \in W^v$. Then ι extends uniquely to an algebra morphism $\iota : \mathcal{H}_{G,\mathcal{F}} \to {}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$. Moreover, $\iota(\mathcal{H}_{G,\mathcal{F}}) = \mathcal{H}_{\mathcal{F}}$ and ι is injective.*

6.1.5. Iwasawa decomposition and retractions centered at $\epsilon\infty$

Let $\epsilon \in \{-, +\}$ and $U_\epsilon = \langle U_\alpha \mid \alpha \in \Phi_\epsilon \rangle$. We denote by $\epsilon\infty$ the germ of ϵC_f^v at infinity: this is the filter on $\mathcal{I}$ composed with the sets containing a translate of ϵC_f^v. Then U_ϵ fixes $\epsilon\infty$, which means that for every $u \in U_\epsilon$, there exists $x \in \mathbb{A}$ such that u fixes $x + \epsilon C_f^v$.

Let C be a chamber of $\mathcal{I}$. Then there exists an apartment containing C and $\epsilon\infty$. This means that there exists $\Omega \in C$, $y \in \mathbb{A}$ and an apartment containing $\Omega \cup y + \epsilon C_f^v$. In particular for every $x \in \mathcal{I}$, there exists an apartment containing x and $\epsilon\infty$. When $C \in \mathscr{C}_0^+$ and $x \in \mathcal{I}_0$, these results correspond to the following decompositions:

$$G = \bigsqcup_{\mathbf{w} \in W^v \ltimes Y} U_\epsilon n_{\mathbf{w}} K_I \text{ and } G = \bigsqcup_{\lambda \in Y} U_\epsilon n_\lambda K.$$

Let $x \in \mathcal{I}$. Let A be an apartment containing x and $\epsilon\infty$. Then by (MA ii), there exists $h \in G$ such that $h.A = \mathbb{A}$ and h fixes $A \cap \mathbb{A}$. We set $\rho_{\epsilon\infty}(x) = h.x$. This is well-defined, independently of the choices of A and h. Then $\rho_{\epsilon\infty}(x)$ is the unique element of $U_\epsilon.x \cap \mathbb{A}$. Then $\rho_{\epsilon\infty} : \mathcal{I} \to \mathbb{A}$ is a retraction called the *retraction onto* $\mathbb{A}$ *centered at* $\epsilon\infty$.

6.1.6. Towards principal series representations of G^+ and G

Let $B = TU_+$ be the *positive standard Borel subgroup of* G. In term of masures, B is stabilizer of $+\infty$ in G (by [15, Lemma 3.4.1]), which means that B is the set of $g \in G$ such that there exists $a, a' \in \mathbb{A}$ such that $g.(a + C_f^v) = (a' + C_f^v)$ and such that there exists a translation f of $\mathbb{A}$ such that $g.x = f(x)$ for every $x \in a + C_f^v$. Let $B^+ = G^+ \cap B$ and $T^+ = T \cap G^+$.

Lemma 6.2. — *We have $T^+ \subset B^+ \subset T^+ U_+$.*

Proof. — Let $g \in B^+$. Write $g = tu$ with $t \in T$ and $u \in U_+$. Then as t normalizes U_+ (by [30, 8.3.3]), there exists $u' \in U_+$ such that $g = u't$. Then $\rho_{+\infty}(g.0) = t.0$. Moreover by [34, Corollaire 2.8], $\rho_{+\infty}(g.0) \geqslant 0$ and thus $t.0 \geqslant 0$, which proves the lemma. □

Remark 6.3. — Unless G is reductive, $T^+U_+ \supsetneq B^+$. Indeed, let us prove that U_+ is not contained in G^+. Let $s \in \mathscr{S}$. Take $a \in \mathcal{K}$ such that $\omega(a) = -2$. Set $u = x_{\alpha_s}(a) \in U_+$. Let $A' = u.\mathbb{A}$. Then $A' \cap \mathbb{A}$ is the half-apartment $D_{\alpha_s,-2} = \{x \in \mathbb{A} \,|\, \alpha_s(x) - 2 \geqslant 0\}$. Let $D_{A'}$ be the half-apartment of A' opposite to $D_{\alpha_s,-2}$. By [34, Proposition 2.92)], $\widetilde{A} := D_{-\alpha_s,2} \cup D_{A'}$ is an apartment of $\mathcal{I}$. As $0 \notin D_{\alpha_s,-2}$, $u.0 \in D_{A'}$. Then $\widetilde{A} \ni 0, u.0$. Let $g \in G$ be such that $g.\widetilde{A} = \mathbb{A}$ and such that g fixes $D_{-\alpha_s,2}$. Let $r : \mathbb{A} \to \mathbb{A}$ be defined by $r(x) = s.x + 2\alpha_s^\vee$ for $x \in \mathbb{A}$. Then by [17, Lemma 3.4], $g.u.0 = r.0 = 2\alpha_s^\vee$. By the lemma below, $g.u.0$ and $0 = g.0$ are not comparable for $\leqslant$. We deduce that $u.0$ and 0 are not comparable for $\leqslant$, which proves that $u \notin G^+$.

Recall the definition of indecomposable Kac–Moody matrices from [18, Section 1.1].

Lemma 6.4. — *Assume that G is associated with an indecomposable Kac–Moody matrix A which is not a Cartan matrix. Then for all $s \in \mathscr{S}$, $\alpha_s^\vee \in \mathbb{A} \setminus (\mathcal{T} \cup -\mathcal{T})$.*

Proof. — We first assume that A is of affine type (see [18, Theorem 4.3] for the definition). Then there exists $\delta \in \bigoplus_{s\in\mathscr{S}} \mathbb{R}_+\alpha_s$ such that $\mathcal{T} = \delta^{-1}(\mathbb{R}_+^*) \sqcup \bigcap_{s\in\mathscr{S}} \alpha_s^{-1}(\{0\})$ (see [15, Corollary 2.3.8]). By [18, Proposition 5.2 a) and Theorem 5.6 b)], $w.\delta = \delta$ for every $w \in W^v$. Let $x \in \mathbb{A}$ be such that $\delta(x) = 0$ and $x \geqslant 0$. Then there exists $w \in W^v$ such that $w.x \in \overline{C_f^v}$. Then $\delta(x) = \delta(w.x) = 0$. Thus $w.x \in \bigcap_{s'\in\mathscr{S}} \alpha_{s'}^{-1}(\{0\})$. As $\alpha_s(\alpha_s^\vee) = 2$, $\alpha_s^\vee \notin \mathcal{T}$. As $s.\alpha_s^\vee = -\alpha_s^\vee$ we have $\alpha_s^\vee \in \mathbb{A} \setminus (\mathcal{T} \cup -\mathcal{T})$.

We now assume that A is of indefinite type. Then by [18, Proposition 5.8 c)] and [12, 2.9 Lemma], $\alpha_s^\vee \in \mathbb{A} \setminus \overline{\mathcal{T}}$. As $s.\alpha_s^\vee = -\alpha_s^\vee$ we deduce that $\alpha_s^\vee \in \mathbb{A} \setminus (\overline{\mathcal{T}} \cup -\overline{\mathcal{T}})$. □

Let $T_{\mathcal{F}}^+ = \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}^*)$. Let $\tau \in T_{\mathcal{F}}$ (resp. $\tau \in T_{\mathcal{F}}^+$). We regard τ as a homomomorphism $T \to \mathcal{F}^*$ (resp. as a monoid morphism $T^+ \to \mathcal{F}$) by setting $\tau(t) = \tau(t.0)$ for every $t \in T$ (resp. $t \in T^+$). We extend τ to a homomorphism $B \to \mathcal{F}^*$ (resp. to a monoid morphism $B^+ \to \mathcal{F}$) by setting $\tau(tu) = \tau(t)$, for every $t \in T$ and $u \in U_+$ (resp $\tau(tu) = \tau(t)$ for every $t \in T^+$ and $u \in U_+$ such that $tu \in B^+$). By [33, Proposition 1.5 (DR5)] (note that there is a misprint in this proposition, Z is in fact T), $T \cap U_+ = \{1\}$. This implies that $\tau : B \to \mathcal{F}^*$ is well-defined. The fact that τ is a homomorphism follows from the fact that t normalizes U for every $t \in T$ (by [30, 8.3.3]).

Lemma 6.5.

(1) *Let $g \in G$ and $v \in W^v$. Then $g \in Bn_vK_I$ if and only if $\rho_{+\infty}(g.C_0^+) \in v.C_0^+ + Y$. In particular $G = \bigsqcup_{v\in W^v} Bn_vK_I$.*
(2) *We have $G^+ = \bigsqcup_{v\in W^v} B^+n_vK_I$.*

Proof. — There exists $v \in W^v$ and $\lambda \in Y$ such that $\rho_{+\infty}(g.C_0^+) = v.C_0^+ + \lambda$. Thus there exists $t \in T$ and $v \in W^v$ such that $\rho_{+\infty}(g.C_0^+) = tn_v.C_0^+$. Hence $g.C_0^+ = utn_v.C_0^+$ and $g \in utn_vK_I \subset Bn_vK_I$, for some $u \in U_+$. Conversely if $g \in Bn_vK_I$, then $\rho_{+\infty}(g.C_0^+) \in v.C_0^+ + Y$, which proves (1).

As G^+ is a sub-semi-group of G, $\bigsqcup_{v\in W^v} B^+n_vK_I \subset G^+$. Let $g \in G^+$. By (1), we can write $g = bn_vk$, with $b \in B$, $v \in W^v$ and $k \in K_I$. Then $b.0 = g.0 \geqslant 0$ and hence $b \in B^+$, which proves (2). □

6.2. Action of $\mathcal{H}_{\mathcal{F}}$ on I_{τ,G^+} and $I_{\tau,G}$

6.2.1. Well-definedness of the action

Let $\epsilon \in \{+, \emptyset\}$. For $\tau \in T_{\mathcal{F}}^{\epsilon}$, we define $\widehat{I(\tau)^{\epsilon}}$ to be the set of functions f from G^{ϵ} to $\mathcal{F}$ such that for all $b \in B^{\epsilon}$ and $g \in G^{\epsilon}$, one has $f(bg) = (\delta^{1/2}\tau)(b)f(g)$. The group G (resp. semi-group G^+) acts on $\widehat{I(\tau)}$ (resp. $\widehat{I(\tau)^+}$) by right translation. When G is reductive, the principal series representation associated with τ is the subset $I(\tau)$ of functions of $\widehat{I(\tau)}$ which are locally constant. Then $I_\tau = I(\tau)^{K_I}$. When G is not reductive, we do not know which condition could replace "locally constant". The hope is that the principal series representation of G associated with τ should be the set of functions of $\widehat{I(\tau)}$ satisfying some "regularity condition".

Let $\tau \in T_{\mathcal{F}}^{\epsilon}$. Let $\widehat{I(\tau)_{\text{fin}}^{\epsilon}}$ be the set of $f \in \widehat{I(\tau)^{\epsilon}}$ such that there exists a finite set $F \subset W^v$ such that $\operatorname{supp}(f) \subset \bigcup_{v \in F} B n_v K_I$. Let $I_{\tau, G^{\epsilon}} = (\widehat{I(\tau)_{\text{fin}}^{\epsilon}})^{K_I}$ be the set of elements of $\widehat{I(\tau)_{\text{fin}}^{\epsilon}}$ which are invariant under the action of K_I. For $v, w \in W^v$, define $f_w \in I_{\tau, G^{\epsilon}}$ by $f_w(n_v) = 1$ if and only $v = w$. Then by Lemma 6.5, $(f_w)_{w \in W^v}$ is a basis of $I_{\tau, G^{\epsilon}}$.

Fix $\tau \in T_{\mathcal{F}}^{\epsilon}$. Following [7, 4.2.2], we would like to define an action of $\mathcal{H}_{\mathcal{F}}$ on $I_{\tau, G^{\epsilon}}$ by

$$\phi.f = \sum_{g \in G^+/K_I} \phi(g) g.f, \quad \forall\, (\phi, f) \in \mathcal{H}_{\mathcal{F}} \times I_{\tau, G^{\epsilon}}.$$

However, we need to prove that such an action is well-defined. The main difficulties are to prove that if $\phi \in \mathcal{H}_{\mathcal{F}}$, $f \in I_{\tau, G^{\epsilon}}$ and $h \in G$, then:

$$\sum_{g \in G^+/K_I} \phi(g) f(hg)$$

only involves finitely many terms and that $\phi.f$ also has finite support. The aim of this section is to prove these results. For this, we use the masure $\mathcal{I}$, finiteness results of [11] and [12] and the theory of Hecke paths introduced by Kapovich and Millson in [19]. In [11] and [12], the authors mainly use $\rho_{-\infty}$. As we use $\rho_{+\infty}$, we adapt their results to our framework.

Let $\lambda \in Y^{++}$. A *λ-path* of $\mathbb{A}$ is a continuous piecewise linear map $\pi : [0, 1] \to \mathbb{A}$ such that for every $t \in]0, 1[$, $\pi'_-(t), \pi'_+(t) \in W^v.\lambda$ (where $\pi'_-(t)$ and $\pi'_+(t)$ denote the left-hand and right-hand derivatives of π at t) and $\pi'_+(0), \pi'_-(1) \in W^v.\lambda$. A Hecke path of $\mathbb{A}$ of shape λ with respect to C_f^v is a λ-path satisfying [12, 1.8 Definition], with β_i satisfying $\beta_i(C_f^v) < 0$. Hecke paths are the images by retractions of preordered segments in $\mathcal{I}$. More precisely:

Theorem 6.6 (see [11, Theorem 6.2]). — *Let $x, y \in \mathcal{I}$ be such that $x \leqslant y$ and $\lambda = d^{Y^{++}}(x,y) \in \overline{C_f^v}$. Let $\gamma : [0,1] \to A$ be an affine parametrization of the segment x, y. Then $\rho_{+\infty} \circ \gamma$ is a Hecke path of shape λ with respect to C_f^v from $\rho_{+\infty}(x)$ to $\rho_{+\infty}(y)$.*

By definition of Hecke paths and by [22, Lemma 1.3.13], we have the following lemma.

Lemma 6.7. — *Let $\lambda \in \overline{C_f^v}$ and $\pi : [0,1] \to \mathbb{A}$ be a Hecke path of shape λ with respect to C_f^v. For $t \in [0,1]$ where it makes sense, we write $\pi'_+(t) = w'_+(t).\lambda$ and $\pi'_-(t) = w'_-(t).\lambda$, where $w'_-(t), w'_+(t) \in W^v$ have minimal lengths for these properties. Then for all $t, t' \in [0,1]$ such that $0 \leqslant t < t' \leqslant 1$, we have $w'_-(t) \leqslant w'_+(t) \leqslant w'_-(t') \leqslant w'_+(t')$, where we delete the derivatives that do not make sense (for $t = 0$ or $t' = 1$).*

Theorem 6.8 (see [12, 5.2]). — *Let $x \in \mathcal{I}_0$, $\lambda \in Y^{++}$ and $\mu \in Y$. Then*

$$\{y \in \mathcal{I}_0 \mid y \geqslant x,\ d^{Y^{++}}(x,y) = \lambda \text{ and } \rho_{+\infty}(y) = \mu\}$$

is finite.

Lemma 6.9. — *Let $y \in \mathcal{I}_0$ and C be a type 0 positive local chamber of $\mathbb{A}$. Then*

$$\{C' \in \mathscr{C}_0^+ \mid \mathrm{vert}(C') = y \text{ and } \rho_{+\infty}(C') = C\}$$

is finite.

Proof. — Let A be an apartment containing y and $+\infty$. Then by (MA ii), there exists $g \in G$ such that $g.A = \mathbb{A}$ and g fixes $A \cap \mathbb{A}$. Maybe working with $\rho_{+\infty,A} = g^{-1}.\rho_{+\infty}$ instead of $\rho_{+\infty}$, we can thus assume that y is in $\mathbb{A}$. Let $C' \in \mathscr{C}_0^+$ be such that $\mathrm{vert}(C') = y$ and $\rho_{+\infty}(C') = C$. Let A' be an apartment containing C' and $+\infty$. Then A' contains y and by (MA ii), A' contains $y + C_0^+$. Let $h \in G$ be such that h fixes $A' \cap \mathbb{A}$ and $h.A' = \mathbb{A}$. Then $\rho_{+\infty}(C') = h.C'$. Therefore

$$d^{W^+}(C', y+C_0^+) = d^{W^+}\big(h.C', h.(y+C_0^+)\big) = d^{W^+}(C, y+C_0^+) \in W^v. \tag{6.1}$$

Using [1, Lemma 5.5] we deduce that $\{C' \in \mathscr{C}_0^+ \mid \mathrm{vert}(C') = y$ and $\rho_{+\infty}(C') = C\}$ is finite. $\square$

Let $x \in \mathcal{I}_0$ and $C \in \mathscr{C}_0^+$ be such that $C \geqslant x$ (i.e. $\mathrm{vert}(C) \geqslant x$). By [16, Proposition 5.17], there exists an apartment A containing x and C. Then there exists $g \in G$ such that $g.A = \mathbb{A}$, $g.x = 0$ and $g.C_0^+ \in Y + C_0^+$. Then $g.\mathrm{vert}(C) \geqslant g.0$ and thus $g.\mathrm{vert}(C) \in Y^+$. One sets $d^{Y^+}(0, C) = g.\mathrm{vert}(C)$. This does not depend on the choices we made by [15, Theorem 4.4.17]. This defines a G-invariant "distance" $d^{Y^+} : \mathcal{I}_0 \times_{\leqslant} \mathscr{C}_0^+ \to Y^+$.

Lemma 6.10. — *Let $v \in W^v$, $\lambda \in Y^+$. Then*

$$E := \{C \in \mathscr{C}_0^+ \mid C \geqslant 0, \rho_{+\infty}(C) \in v.C_0^+ + Y \text{ and } d^{Y^+}(0, C) = \lambda\}$$

is finite.

Suppose moreover that $\lambda \in Y^{++}$ and that $v = 1$. Then $E = \{\lambda + C_0^+\}$.

Proof. — In order to prove that E is finite, we begin by proving that $\mathrm{vert}(E) := \{\mathrm{vert}(C) \mid C \in E\}$ is finite. To that end, our idea is to study, for each $C \in E$, the path $\widetilde{\pi} = \rho_{+\infty} \circ \widetilde{\gamma} : [0, 1] \to \mathbb{A}$, where $\widetilde{\gamma}$ is the segment joining 0 to $\mathrm{vert}(C)$. We want to prove that $\widetilde{\pi}'_-(1)$ lies in a finite set depending only on v and λ. In order to use the assumption that $\rho_{+\infty}(C) \in Y + v.C_0^+$, it is convenient to extend slightly the segment $\widetilde{\gamma}$ and this is why we consider a segment $\gamma : [0, 1] \to \mathcal{I}$ such that $\gamma(0) = 0$ and $\gamma(\frac{1}{2}) = \mathrm{vert}(C)$.

Let $C \in E$. Let A be an apartment containing 0 and C. Let $g \in G$ be such that $g.\mathbb{A} = A$, $g.0 = 0$ and $g.(\lambda + C_0^+) = C$. Let $\gamma : [0, 1] \to A$ be defined by $\gamma(t) = g.2t\lambda$. Then $\pi = \rho_{+\infty} \circ \gamma$ is a Hecke path with respect to $+\infty$ of shape 2λ. Let $w_\lambda \in W^v$ be such that $(w_\lambda)^{-1}.\lambda \in Y^{++}$ and such that w_λ has minimum length for this property. Set $C_\lambda = g.(\lambda + w_\lambda.C_0^+)$. Then:

$$\begin{aligned} d^{W^+}(C, C_\lambda) &= d^{W^+}\big(g.(\lambda + C_0^+), g.(\lambda + w_\lambda.C_0^+)\big) \\ &= d^{W^+}(\lambda + C_0^+, \lambda + w_\lambda.C_0^+) = w_\lambda. \end{aligned}$$

Take a minimal gallery Γ from C to C_λ. Then Γ has length $\ell(w_\lambda)$ and $\rho_{+\infty}(\Gamma)$ is a gallery from $\rho_{+\infty}(C)$ to $\rho_{+\infty}(C_\lambda)$. Therefore

$$w := d^{W^+}\big(\rho_{+\infty}(C), \rho_{+\infty}(C_\lambda)\big) \in W^v \quad \text{and} \quad \ell(w) \leqslant \ell(w_\lambda).$$

Moreover, by definition of E, $\rho_{+\infty}(C) = \nu + v.C_0^+$, for some $\nu \in Y$. Consequently, $\rho_{+\infty}(C_\lambda) = \nu + vw.C_0^+$. Therefore for $\epsilon \in \left]0, \frac{1}{2}\right]$ small enough, $\pi\big([\frac{1}{2}, \frac{1}{2} + \epsilon]\big) \subset \nu + vw.\overline{C_f^v}$ and thus $\pi'_+(\frac{1}{2}) = 2vw.\lambda$. By Lemma 6.7, $\pi'_-(\frac{1}{2}) = u.\lambda$ for some $u \in W^v$ such that $\ell(u) \leqslant \ell(v) + \ell(w_\lambda)$.

Let now $\widetilde{\gamma} : [0, 1] \to A$ be defined by $\widetilde{\gamma}(t) = g.t\lambda$ for $t \in [0, 1]$ and $\widetilde{\pi} = \rho_{+\infty} \circ \widetilde{\gamma}$. Then by what we proved above, $\widetilde{\pi}'_-(0) = u.\lambda$. By [2, Lemma 1.8] we have

$$\begin{gathered} u.\lambda = \widetilde{\pi}_-(1) \leqslant_{Q^\vee} \widetilde{\pi}(1) - \widetilde{\pi}(0) = \rho_{+\infty}\big(\mathrm{vert}(C)\big) \leqslant_{Q^\vee} \lambda^{++}, \\ \ell(u) \leqslant \ell(v) + \ell(w_\lambda), \end{gathered}$$

where λ^{++} is the unique element of $Y^{++} \cap W^v.\lambda$. We deduce that

$$F := \rho_{+\infty}\big(\mathrm{vert}(E)\big) = \{\rho_{+\infty}\big(\mathrm{vert}(C)\big) \mid C \in E\}$$

is finite.

Let $\nu \in F$. Let $E_\nu = \{C \in E \,|\, \rho_{+\infty}(C) = \nu + v.C_0^+\}$. If $C \in E_\nu$, then $d^{Y^{++}}\big(0, \mathrm{vert}(C)\big) = \lambda^{++}$ and $\rho_{+\infty}(\mathrm{vert}(C)) = \nu$. Using Theorem 6.8 we deduce that $\{\mathrm{vert}(C) \,|\, C \in E_\nu\}$ is finite. By Lemma 6.9, E_ν is finite and thus $E = \bigcup_{\nu \in F} E_\nu$ is finite.

Suppose now that $v = 1$ and that $\lambda \in Y^{++}$. Take $C \in E$. We use the same notation as in the beginning of the proof. Then we have $\pi'_-(\frac{1}{2}) = \lambda = 1.\lambda$ and by Lemma 6.7 we deduce that there exists $\epsilon > 0$ such that $\pi(t) = 2t\lambda$ for every $t \in [0, \frac{1}{2} + \epsilon]$. Moreover $\gamma(0) \in \mathbb{A}$ and thus by [14, Lemma 3.4] we deduce that $\gamma([0, \frac{1}{2} + \epsilon]) \subset \mathbb{A}$. Therefore $C \subset \mathbb{A}$. Thus $\rho_{+\infty}(C) = C = \nu + C_0^+$ for some $\nu \in Y$. Moreover $d^{Y^+}(0, C) = \lambda + C_0^+$ and thus $\nu = \lambda$, which proves that $E = \{\lambda + C_0^+\}$ and completes the proof of the lemma. □

In the next lemma, we use the projection of a chamber on a vertex introduced in [2, 1.9]. Let $x \in \mathbb{A}$ and C be a positive chamber of $\mathbb{A}$ such that $y := \mathrm{vert}(C) \geqslant x$. Let C^v be the positive vectorial chamber of $\mathbb{A}$ such that $C = F_{y, C^v}$. Take $\xi \in C^v$. Then there exists a positive vectorial chamber $\widetilde{C}^v \subset \mathbb{A}$ such that $x + \widetilde{C}^v \supset \mathrm{conv}(x,]y, y + \epsilon\xi])$, for $\epsilon > 0$ small enough, where conv denotes the convex hull. Then the chamber $\mathrm{pr}_x(C) = F_{x, \widetilde{C}^v}$ is the projection of C on x. Let now $x \in \mathcal{I}$ and C be a positive chamber of $\mathcal{I}$ such that $\mathrm{vert}(C) \geqslant x$. Then there exists $g \in G$ such that $g.x, g.C \subset \mathbb{A}$. We set $\mathrm{pr}_x(C) = g^{-1}.\big(\mathrm{pr}_{g.x}(g.C)\big)$. This is the *projection of C on x*. Then by [15, Theorem 4.4.17], $\mathrm{pr}_x(C)$ does not depend on the choice of g, every apartment containing x and C contains $\mathrm{pr}_x(C)$ and every $h \in G$ fixing x and C fixes $\mathrm{pr}_x(C)$.

Lemma 6.11. — *Let $\mathbf{w} \in W^+$ and $v \in W^v$. Then:*

(1) $\bigcup_{u \in W^v} (n_u K_I n_{\mathbf{w}} K_I \cap B n_v K_I)/K_I$ *is finite,*
(2) $\{u \in W^v \,|\, n_u K_I n_{\mathbf{w}} K_I \cap B n_v K_I \neq \emptyset\}$ *is finite.*

Proof. — Set $F = \bigcup_{u \in W^v} (n_u K_I n_{\mathbf{w}} K_I \cap B n_v K_I)/K_I$. Let $u \in W^v$ and $g \in n_u K_I n_{\mathbf{w}} K_I$. Set $C = g.C_0^+$. Then $d^{W^+}(u.C_0^+, C) = \mathbf{w}$. Thus there exists $h \in G$ such that $h^{-1}.\mathbb{A}$ contains $u.C_0^+, C$ and such that $h.u.C_0^+ = C_0^+, h.C = \mathbf{w}.C_0^+$. Write $\mathbf{w} = \lambda w$ (i.e. $\mathbf{w}.x = \lambda + w.x$ for every $x \in \mathbb{A}$). Set $h' = n_{w^{-1}} h$. Then $h'^{-1}.\mathbb{A} = h^{-1}.\mathbb{A}$ contains $0, C$, $h'.0 = 0$ and $h'.C = w^{-1}.\lambda + C_0^+$. Thus $d^{Y^+}(0, C) = w^{-1}.\lambda$. Therefore

$$F.C_0^+ \subset \{C \in \mathscr{C}_0^+ \,|\, C \geqslant 0, \rho_{+\infty}(C) \in v.C_0^+ + Y \text{ and } d^{Y^+}(0, C) = w^{-1}.\lambda\}.$$

By Lemma 6.10, $F.C_0^+$ is finite, which proves that F is finite.

Let $u \in W^v$ be such that there exists $g \in n_u K_I n_{\mathbf{w}} K_I \cap B n_v K_I$. Let $P = \{\mathrm{pr}_0(C') \,|\, C' \in F.C_0^+\}$. Let $C = g.C_0^+$. Then as $d^{W^+}(u.C_0^+, C) = \mathbf{w}$, there exists $h \in G$ such that $h^{-1}.\mathbb{A}$ contains $u.C_0^+, C$, $h.u.C_0^+ = C_0^+$ and $h.C = \mathbf{w}.C$. Then $h.\operatorname{pr}_0(C) = \operatorname{pr}_0(\mathbf{w}.C_0^+)$. Therefore

$$\begin{aligned} w' &:= d^{W^+}\big(h.u.C_0^+, h.\operatorname{pr}_0(C)\big) \\ &= d^{W^+}\big(u.C_0^+, \operatorname{pr}_0(C)\big) \\ &= d^{W^+}\big(C_0^+, \operatorname{pr}_0(\mathbf{w}.C_0^+)\big) \in W^v. \end{aligned}$$

Consequently there exists $C' \in P$ such that $d^{W^+}(u.C_0^+, C') = w'$. Consequently,

$$\ell(u) = \ell\big(d^{W^+}(u.C_0^+, C_0^+)\big) \leqslant \ell(w') + \max_{C' \in P} \ell\big(d^{W^+}(C', C_0^+)\big).$$

This proves (2). □

Definition 6.12. — *Let $\epsilon \in \{+, \emptyset\}$ and $\tau \in T_{\mathcal{F}}^{\epsilon}$. Let $\phi \in \mathcal{H}_{\mathcal{F}}$ and $f \in I_{\tau, G^\epsilon}$. Define $\phi.f \in I_{\tau,G}$ by*

$$\phi.f = \sum_{g \in G^+/K_I} \phi(g) g.f.$$

Then . is well-defined and induces an action of $\mathcal{H}_{\mathcal{F}}$ on I_{τ,G^ϵ}.

Proof. — To prove that $\phi.f$ is a well-defined element of I_{τ,G^ϵ}, it suffices to prove it for $\phi = T_{\mathbf{w}}$ and $f = f_v$, for $v \in W^v$ and $\mathbf{w} \in W^+$. Let $g \in G^+$ and $h \in G^\epsilon$. Suppose that $T_{\mathbf{w}}(g) f_v(hg) \neq 0$. Then $g \in K_I n_{\mathbf{w}} K_I \cap h^{-1} B n_v K_I$. Write $h = b n_u k$, with $b \in B^\epsilon$ and $k \in K_I$. Then $K_I n_{\mathbf{w}} K_I \cap k^{-1} n_u^{-1} B n_v K_I \neq \emptyset$. Therefore

$$g \in K_I n_{\mathbf{w}} K_I \cap k^{-1} n_u^{-1} B n_v K_I = k^{-1}(K_I n_{\mathbf{w}} K_I \cap k^{-1} n_u^{-1} B n_v K_I). \tag{6.2}$$

By Lemma 6.11,

$$\sum_{g \in G^+/K_I} T_{\mathbf{w}}(g) f_v(hg) = \sum_{g \in K_I n_{\mathbf{w}} K_I \cap k^{-1} n_u^{-1} B n_v K_I / K_I} T_{\mathbf{w}}(g) f_v(hg)$$

is well-defined. Thus $T_{\mathbf{w}}.f_v$ is a well-defined map $G^\epsilon \to \mathcal{F}$. The fact that it is right K_I-invariant and that $T_{\mathbf{w}}.f(bh) = \delta^{1/2}\tau(b) T_{\mathbf{w}}.f(h)$, for $B \in B^\epsilon$ are clear.

Let $u \in W^v$. Suppose that $T_{\mathbf{w}}.f_v(n_u) \neq 0$. Then by (6.2), $K_I n_{\mathbf{w}} K_I \cap n_u^{-1} B n_v K_I \neq \emptyset$. By Lemma 6.11 we deduce that $\{u \in W^v \mid T_{\mathbf{w}}.f_v(n_u) \neq 0\}$ is finite, which proves that $T_{\mathbf{w}}.f_v$ is an element of I_{τ,G^ϵ}.

The fact that $(\phi * \phi').f = \phi.(\phi'.f)$ for every $f \in I_{\tau,G^\epsilon}$, $\phi, \phi' \in \mathcal{H}_{\mathcal{F}}$ is an easy consequence of the fact that $\phi * \phi'(h) = \sum_{g \in G^+/K_I} \phi(g)\phi'(g^{-1}h)$ for every $h \in G^+/K_I$. □

6.2.2. Isomorphism between I_τ^ϵ and I_{τ,G^ϵ}

Let $\tau : Y^+ \to \mathcal{F}$ be a monoid morphism. Then τ induces an algebra morphism $\tau : \mathcal{F}[Y^+] \to \mathcal{F}$ and thus this defines a representation $I_\tau^+ = \mathrm{Ind}_{\mathcal{F}[Y^+]}^{\mathcal{H}_\mathcal{F}}(\tau) = \mathcal{H}_\mathcal{F} \otimes_{\mathcal{F}[Y^+]} \mathcal{F}$. Let $\epsilon \in \{+, \emptyset\}$. The aim of this section is to prove that if $\tau \in (T_\mathcal{F})^\epsilon$ then the map $I_\tau^\epsilon \to I_{\tau,G^\epsilon}$ defined by $h.1 \otimes_\tau 1 \mapsto h.f_1$, for $h \in \mathcal{H}_\mathcal{F}$ is well-defined and is an isomorphism of $\mathcal{H}_\mathcal{F}$-modules (see Proposition 6.17). To that end, we prove that $Z^\lambda.f_1 = \tau(\lambda) f_1$ for $\lambda \in Y^+$. For this we begin by proving that if $\lambda \in Y^{++}$, then $Z^\lambda.f_1 = \tau(\lambda) f_1$. In the reductive case, this is sufficient to deduce the result for any $\lambda \in Y = Y^+$, since Z^λ is invertible for $\lambda \in Y^{++}$. In the Kac–Moody case however, Z^λ is not necessarily invertible for $\lambda \in Y^{++}$. We thus prove that if $f \in I_{\tau,G^\epsilon}$ is such that $Z^\lambda.f = 0$ for $\lambda \in Y^{++}$ sufficiently dominant, then $f = 0$.

Lemma 6.13. — *Let $w \in W^v$. Then $T_w.f_1 = f_{w^{-1}}$.*

Proof. — Let $v \in W^v$.Then $T_w.f_1(n_v) = \sum_{g \in G^+/K_I} T_w(g) f_1(n_v g)$. Suppose that $T_w.f_1(n_v) \neq 0$. Then there exists $g \in K_I n_w K_I \cap n_v^{-1} B K_I$ and thus $n_v K_I n_w K_I \cap B K_I \neq \emptyset$.

Let $h \in n_v K_I n_w K_I \cap B K_I$ and $C = h.C_0^+$. Then $d^{W^+}(v.C_0^+, C) = w$ and $\rho_{+\infty}(C) \in Y + C_0^+$. Therefore $\mathrm{vert}(C) = 0$ and hence $\rho_{+\infty}(C) = C_0^+$. By formula (6.1) of the proof of Lemma 6.9, we have $C = C_0^+$. Consequently $C = C_0^+$, $v = w^{-1}$, $\mathrm{supp}(T_w.f_1) \subset B n_{w^{-1}} K_I$ and $T_w.f(n_{w^{-1}}) = 1$. Therefore $T_w.f_1 = f_{w^{-1}}$. $\square$

Lemma 6.14. — *Let $w \in W^v$ and $\lambda \in Y \cap C_f^v$. Then:*

(1) $\mathrm{supp}(T_\lambda.f_w) \subset \bigcup_{v \leqslant w} B n_v K_I$.
(2) $T_\lambda.f_w(n_w) \neq 0$.

Proof. — Let $v \in W^v$. Suppose that $T_\lambda.f_w(n_v) \neq 0$. Then

$$X := n_v K_I n_\lambda K_I \cap B n_w K_I$$

is non-empty. Let $g \in X$. Let $\gamma : [0,1] \to \mathcal{I}$ be defined by $\gamma(t) = g.t.\lambda$ for $t \in [0,1]$. Let $\pi = \rho_{+\infty} \circ \gamma$. Then π is a Hecke path of shape λ from 0 to $\rho_{+\infty}\big(\mathrm{vert}(C)\big)$. For $t \in [0,1]$ where it makes sense, write $\pi'_-(t) = w_-(t).\lambda$, $\pi'_+(t) = w'_+(t).\lambda$, where $w'_-(t)$ and $w'_+(t)$ have minimum lengths for these properties. By the proof of Lemma 6.10, $w'_-(1) \leqslant w$ (we have $w_\lambda = 1$ in this case). Using Lemma 6.7 we deduce that $w'_+(0) \leqslant w$. Let $C_{\pi(0^+)}$ (resp. $C_{\gamma(0^+)}$) be the local chamber based at 0 and containing $\pi(t)$ (resp. $\gamma(t)$)

for $t \in [0,1]$ near 0. Then

$$\begin{aligned} d^{W^+}(C_0^+, C_{\gamma(0^+)}) &= d^{W^+}\big(\rho_{+\infty}(C_0^+), \rho_{+\infty}(C_{\gamma(0^+)})\big) \\ &= d^{W^+}(C_0^+, C_{\pi(0^+)}) \\ &= w'_+(0). \end{aligned}$$

Let us prove that $C_{\gamma(0^+)} = v.C_0^+$. Let A be an apartment containing $v.C_0^+$ and C. Let $h \in G$ be such that $h.A = \mathbb{A}$ and such that h fixes $v.C_0^+$. Then

$$\begin{aligned} d^{W^+}(C_0^+, \lambda + C_0^+) &= d^{W^+}\big(h^{-1}.C_0^+, h^{-1}.(\lambda + C_0^+)\big) \\ &= \lambda \\ &= d^{W^+}\big(v.C_0^+, h^{-1}.(\lambda + C_0^+)\big) \\ &= d^{W^+}(v.C_0^+, C). \end{aligned}$$

As A contains $v.C_0^+, C$ and $h^{-1}.(\lambda+C_0^+)$, we deduce that $h^{-1}.(\lambda+C_0^+) = C$. In particular, $h^{-1}.\lambda = g.\lambda$ and thus by [34, Proposition 5.4], $\gamma(t) = h^{-1}.t.\lambda$ for all $t \in [0,1]$. Let Ω' be a neighborhood of 0 in $\mathbb{A}$ such that h pointwise fixes $\Omega = \Omega' \cap v.C_f^v$. Then for $t \in [0,1]$ small enough, $\gamma(t) \in \Omega$ and thus $C_{\gamma(0^+)} = v.C_0^+$. Consequently, $\gamma(t) \in \mathbb{A}$ for $t \in [0,1]$ small enough, thus $C_{\gamma(0^+)} \subset \mathbb{A}$, thus $C_{\gamma(0^+)} = C_{\pi(0^+)} = v.C_0^+$ and hence $v = w'_+(0) \leqslant w$. Therefore:

$$\operatorname{supp}(T_\lambda.f_w) \subset \bigcup_{v \leqslant w} B n_v K_I.$$

Suppose now that $v = w$. Then with the same notation as above, one has $w'_+(0) = w$. Therefore $w \leqslant w'_-(t) \leqslant w$ and $w \leqslant w'_+(t) \leqslant w$ for every $t \in [0,1]$ and hence π is the line segment from 0 to $w.\lambda$. Therefore if $g \in n_w K_I n_\lambda K_I \cap B n_w K_I$, then $\rho_{+\infty}(g.C_0^+) = w.(\lambda + C_0^+)$. Consequently

$$n_w K_I n_\lambda K_I \cap B n_w K_I \subset U_+ n_{w.\lambda} n_w K_I,$$

and $n_{w.\lambda} \in T$. Thus

$$\begin{aligned} T_\lambda.f_w(n_w) &= \sum_{g \in K_I n_\lambda K_I \cap n_w^{-1} B n_w K_I / K_I} f_w(n_w g) \\ &= |n_w K_I n_\lambda K_I \cap B n_w K_I / K_I| \tau\delta^{1/2}(w.\lambda). \end{aligned}$$

Moreover $n_w n_\lambda \in n_w K_I n_\lambda K_I \cap B n_w K_I$, which proves that $T_\lambda.f_w(n_w) \neq 0$. □

Lemma 6.15. — *Let $f \in I_{\tau, G^\epsilon}$. Suppose that for some $\mu \in Y \cap C_f^v$, $T_\mu.f = 0$. Then $f = 0$.*

Proof. — Write $f = \sum_{w\in W^v} a_w f_w$, where $(a_w) \in \mathcal{F}^{W^v}$ has finite support. Suppose that $f \neq 0$. Let $w \in \operatorname{supp}\big((a_v)\big)$ be maximal for the Bruhat order. Then by Lemma 6.14, $T_\mu.f(n_w) = a_w T_\mu.f_w(n_w) \neq 0$. We reach a contradiction and thus $f = 0$. □

Lemma 6.16. — *Let $\lambda \in Y^+$. Then $Z^\lambda.f_1 = \tau(\lambda).f_1$.*

Proof. — First assume that $\lambda \in Y^{++}$. Then $Z^\lambda = \delta^{-1/2}(\lambda)T_\lambda$, by [2, 5.7 and Theorem 5.5]. By Lemma 6.14, $\operatorname{supp}(T_\lambda.f_1) = BK_I$ and thus $T_\lambda.f_1 \in \mathcal{F}f_1$.

We have $n_\lambda K_I \in K_I n_\lambda K_I \cap BK_I$. Let $g \in K_I n_\lambda K_I \cap BK_I$. Let $C = g.C_0^+$. Then $\rho_{+\infty}(C) \in Y + C_0^+$ and $d^{Y^+}(0,C) = \lambda$. Thus by Lemma 6.10, $C = \lambda + C_0^+$. Hence $g \in n_\lambda K_I$ and $K_I n_\lambda K_I \cap BK_I = n_\lambda K_I$. Therefore $T_\lambda.f_1(1) = f_1(\lambda) = \delta^{1/2}\tau(\lambda)$. Hence $T_\lambda.f_1 = \delta^{1/2}\tau(\lambda)f_1$ and $Z^\lambda.f_1 = \tau(\lambda)f_1$.

Let now $\lambda \in Y^+$. Then by [2, Theorem 5.5] and the fact that $Z^\lambda = \delta^{-1/2}(\lambda)X^\lambda$, one has $T_\mu.Z^\lambda.f_1 = \delta^{-1/2}(\lambda)T_{\lambda+\mu}.f_1 = \tau(\lambda+\mu)\delta^{1/2}(\mu)f_1 = T_\mu.(\tau(\lambda).f_1)$ for $\mu \in Y^{++}$ sufficiently dominant. Thus by Lemma 6.15, $Z^\lambda.f_1 = \tau(\lambda).f_1$, which proves the lemma. □

Proposition 6.17. — *Let $\epsilon \in \{+,\emptyset\}$. Let $\tau \in T_{\mathcal{F}}^\epsilon$. Then the map $\phi : I_\tau^\epsilon \to I_{\tau,G^\epsilon}$ defined by $\phi(h.1\otimes_\tau 1) \mapsto h.f_1$ for $h \in \mathcal{H}_{\mathcal{F}}$ is well-defined and is an isomorphism of $\mathcal{H}_{\mathcal{F}}$-modules.*

Proof. — By Lemma 3.5 and Lemma 6.16, ϕ is well-defined. Let $x \in I_\tau^\epsilon$ be such that $\phi(x) = 0$. Write $x = \sum_{v\in W^v} a_v T_v \otimes_\tau 1$, with $(a_v) \in \mathcal{F}^{W^v}$. Then $\phi(x) = \sum_{v\in W^v} a_v T_v.f_1$. Suppose that $x \neq 0$. Let $w \in W^v$ be such that $a_w \neq 0$ and such that w is maximal for this property (for the Bruhat order). Then by Lemma 6.14 and Lemma 6.13, $\phi(x)(n_{w^{-1}}) = a_w T_w.f_1(n_{w^{-1}}) \neq 0$: a contradiction. Therefore $x = 0$ and ϕ is injective. By Lemma 6.13 and Lemma 6.5, $(T_w.f_1)_{w\in W^v}$ is a basis of I_{τ,G^ϵ}. Consequently ϕ is surjective, which proves the proposition. □

6.3. Extendability of representations of G^+ and $\mathcal{H}_{\mathcal{F}}$

In this subsection, we study the extendability of I_{τ^+,G^+} (resp. $I^+_{\tau^+}$) to a representation of G (resp. ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$), for $\tau \in T_{\mathcal{F}}^+$. We obtain a criterion depending on the extendability of τ^+ to an element of $T_{\mathcal{F}}$ (see Proposition 6.28).

6.3.1. Extendability of elements of $T_{\mathcal{F}}^+$

Recall that if $\tau : Y^+ \to \mathcal{F}$ is a monoid morphism $I_\tau^+ = \mathrm{Ind}_{\mathcal{F}[Y^+]}^{\mathcal{H}_\mathcal{F}}(\tau) = \mathcal{H}_\mathcal{F} \otimes_{\mathcal{F}[Y^+]} \mathcal{F}$ is a representation of $\mathcal{H}_\mathcal{F}$. If I_τ^+ is not the restriction of a representation of ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$ we call I_τ^+ a *non-extendable principal series representation of* $\mathcal{H}_\mathcal{F}$. In this section we study the existence of non-extendable principal series representations of $\mathcal{H}_\mathcal{F}$. We prove that in some cases (for example when $\mathcal{H}_\mathcal{F}$ is associated with an affine root generating system or to a size 2 Kac–Moody matrix) every principal series representations of $\mathcal{H}_\mathcal{F}$ can be extended to a representation of ${}^{\mathrm{BL}}\mathcal{H}_\mathcal{F}$ (see Lemma 6.20). We prove that there exist Kac–Moody matrices such that $\mathcal{H}_\mathcal{F}$ admits non-extendable principal series representations (see Lemma 6.24).

Let $\mathrm{res}_{Y^+} : \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}) \to \mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F})$ be defined by $\mathrm{res}_{Y^+}(\tau) = \tau_{|Y^+}$ for all $\tau \in \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F})$.

Lemma 6.18. — *The map* $\mathrm{res}_{Y^+} : \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathcal{F}^*) = \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}^*) \to \mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F}^*)$ *is a bijection.*

Proof. — Let $\tau \in \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}^*)$. Let $\nu \in C_f^v$. Let $\lambda \in Y$ and $n \in \mathbb{Z}_{\geqslant 0}$ be such that $\lambda + n\nu \in \mathcal{T}$. Then $\tau(\lambda) = \frac{\tau(\lambda+n\nu)}{\tau(n\nu)}$ and thus $\mathrm{res}_{|Y^+}$ is injective.

Let $\tau^+ \in \mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F}^*)$. Let $\lambda \in Y$. Write $\lambda = \lambda_+ - \lambda_-$, with $\lambda_+, \lambda_- \in Y^+$. Set $\tau(\lambda) = \frac{\tau^+(\lambda_+)}{\tau^+(\lambda_-)}$, which does not depend on the choices of λ_- and λ_+. Then $\tau \in \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}^*)$ is well-defined and $\mathrm{res}_{|Y^+}(\tau) = \tau^+$, which finishes the proof. $\square$

Lemma 6.19. — *Let* $\tau \in \mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F})$ *and* $\chi \in T_\mathcal{F}$.

(1) *Suppose* $\mathrm{Hom}_{\mathcal{H}_\mathcal{F}-\mathrm{mod}}(I_\tau^+, I_\chi) \neq \{0\}$. *Then there exists* $w \in W^v$ *such that* $\tau = w.\chi_{|Y^+}$.

(2) *Suppose* $\mathrm{Hom}_{\mathcal{H}_\mathcal{F}-\mathrm{mod}}(I_\chi, I_\tau^+) \neq \{0\}$. *Then there exists* $w \in W^v$ *such that* $\tau = w.\chi_{|Y^+}$.

Proof.

(1). — Let $\phi \in \mathrm{Hom}_{\mathcal{H}_\mathcal{F}-\mathrm{mod}}(I_\tau^+, I_\chi) \setminus \{0\}$. Let $x = \phi(1 \otimes_{\tau^+} 1)$. Then $Z^\lambda.x = \tau(\lambda).x$ for all $\lambda \in Y^+$. By Lemma 2.8, $Z^\lambda.x \neq 0$ for all $\lambda \in Y^+$. Thus $\tau(\lambda) \neq 0$ for all $\lambda \in Y^+$.

Let $\mu \in Y$. Let $\nu \in C_f^v \cap Y$ be such that $\mu + \nu \in Y^+$. Then $Z^\mu.x = \frac{\tau(\mu+\nu)}{\tau(\nu)}.x$. Therefore there exists $\chi' \in T_\mathcal{F}$ such that $x \in I_\chi(\chi')$. By Lemma 3.2, $\chi' \in W^v.\chi$. Moreover, $\chi'_{|Y^+} = \tau$, which proves (1).

(2). — Let $\phi \in \mathrm{Hom}_{\mathcal{H}_\mathcal{F}-\mathrm{mod}}(I_\chi, I_\tau^+) \setminus \{0\}$. Let $x = \phi(1 \otimes_\chi 1)$. Then $Z^\lambda.x = \chi(\lambda).x$ for all $\lambda \in Y^+$. By a lemma similar to Lemma 3.2 we deduce that $\chi_{|Y^+} \in W^v.\tau$, which proves the lemma. $\square$

One has $\mathrm{Hom}_{\mathrm{Mon}}\big(Y, (\mathcal{F}, .)\big) = \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathcal{F}^*) \cup \{0\}$. Set $\mathbb{A}_{in} = \bigcap_{s\in\mathscr{S}} \ker(\alpha_s)$. Let $\mathring{\mathcal{T}}$ be the interior of the Tits cone.

Lemma 6.20. — *Let $\tau^+ \in \mathrm{Hom}_{\mathrm{Mon}}\big(Y, (\mathcal{F}, .)\big)$. Assume that there exists $\lambda \in Y^+$ such that $\tau^+(\lambda) = 0$. Then $\tau^+(\mathring{\mathcal{T}} \cap Y) = \{0\}$. In particular, if $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$, then $\mathrm{Hom}_{\mathrm{Mon}}\big(Y^+, (\mathcal{F}, .)\big) = \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}^*) \cup \{0\}$.*

Proof. — Let $\mu \in \mathring{\mathcal{T}} \cap Y$. Then for $n \gg 0$, $n\mu \in \lambda + \mathcal{T}$. Indeed, $n\mu - \lambda = n(\mu - \frac{\lambda}{n}) \in \mathcal{T}$ for $n \gg 0$. Hence $\tau^+(n\mu) = (\tau^+(\mu))^n = 0$. ☐

A face $F^v \subset \mathcal{T}$ is called *spherical* if its fixator in W^v is finite.

Remark 6.21.

(1) If $\mathbb{A}$ is associated to an affine Kac–Moody matrix, then $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$ (see [15, Corollary 2.3.8] for example).
(2) If $\mathbb{A}$ is associated to a size 2 indefinite Kac–Moody matrix, then $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$. Indeed, by [30, Théorème 5.2.3], $\mathring{\mathcal{T}}$ is the union of the spherical vectorial faces. By [34, 1.3], if $J \subset \mathscr{S}$ and $w \in W^v$, the fixator of $w.F^v$ is $w.W^v(J).w^{-1}$. Therefore the only non-spherical face of $\mathcal{T}$ is $\mathbb{A}_{in}$ and hence $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$.
(3) Let $A = (a_{i,j})_{i,j\in[\![1,3]\!]}$ be a Kac–Moody matrix such that for all $i \neq j$, $a_{i,j}a_{j,i} \geqslant 4$. Then by [22, Proposition 1.3.21], W^v is the free group with 3 generators s_1, s_2, s_3 of order 2. Thus for all $J \subset \mathscr{S}$ such that $|J| = 2$, $F^v(J)$ is non-spherical. Hence $\mathcal{T} \supsetneq \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$.

6.3.2. Construction of an element of $\mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F}) \setminus \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F})$

We now prove that there exist Kac–Moody matrices for which

$$\mathrm{Hom}_{\mathrm{Mon}}(Y^+, \mathcal{F}) \neq \mathrm{Hom}_{\mathrm{Mon}}(Y, \mathcal{F}).$$

Assume that $\mathbb{A}$ is associated to an invertible indefinite size 3 Kac–Moody matrix (see [18, Theorem 4.3] for the definition of indefinite). Then one has $\mathbb{A} = \mathbb{A}' \oplus \mathbb{A}_{in}$, where $\mathbb{A}' = \bigoplus_{i\in I} \mathbb{R}\alpha_i^\vee$. Maybe considering $\mathbb{A}/\mathbb{A}_{in}$, we may assume that $\mathbb{A}_{in} = \{0\}$.

Recall that $\mathcal{T}$ is the disjoint union of the positive vectorial faces of $\mathbb{A}$.

Lemma 6.22. — *Assume that there exists a non-spherical vectorial face $F^v \neq \{0\}$. Let $x \in \mathcal{T}$ and $y \in \mathcal{T} \setminus \overline{F^v}$. Then $[x, y] \cap F^v \subset \{x\}$.*

Proof. — Assume that $y \in \mathring{\mathcal{T}}$. Then $(x, y] \subset \mathring{\mathcal{T}}$ and thus $[x, y] \cap F^v \subset \{x\}$.

Assume that $y \notin \mathring{\mathcal{T}}$. For $a \in \mathcal{T}$, we denote by F^v_a the vectorial face of $\mathcal{T}$ containing a. If $F^v_x = F^v_y$, then $[x, y] \subset F^v_x$. As $F^v_y \neq F^v$, we deduce that $[x,y]\cap F^v=\emptyset$. We now assume that $F^v_x\neq F^v_y$. As W^v is countable, the number of positive vectorial faces is countable and thus there exist $u\neq u'\in[x,y]$ such that $F^v_u=F^v_{u'}$. Then the dimension of the vector space spanned by F^v_u is at least 2. Thus there exists $w\in W^v$ such that $F^v_u=w.F^v(J)$, for some $J\subset\mathscr{S}$ such that $|J|\leqslant 1$. Then the fixator of F^v_u is $w.W_J.w^{-1}$, where $W_J=\langle J\rangle$. Then W_J is finite and thus F^v_u is spherical. Consequently, $(x,y)=(x,u]\cup[u,y)\subset\mathring{\mathcal{T}}$ and the lemma follows. □

Lemma 6.23. — *Assume that there exists a non-spherical vectorial face $F^v\neq\{0\}$. Then $\mathcal{T}\setminus\overline{F^v}$ and $\mathcal{T}\setminus\{0\}$ are convex.*

Proof. — Let $x,y\in\mathcal{T}\setminus F^v$. Suppose that $[x,y]\cap F^v\neq\emptyset$. By Lemma 6.22, $y\in\overline{F^v}=F^v\cup\{0\}$ and hence $y=0$. Let F^v_x be the vectorial face containing x. Then $[x,y)\subset F^v_x$ and hence $[x,y)\cap F^v=\emptyset$: a contradiction. Thus $\mathcal{T}\setminus F^v$ is convex.

By [12, 2.9 Lemma], there exists a basis $(\delta_s)_{s\in\mathscr{S}}$ of $\bigoplus_{s\in\mathscr{S}}\mathbb{R}\alpha_s^\vee$ such that $\delta_s(\mathcal{T})\geqslant 0$ for all $s\in\mathscr{S}$. Thus $\mathcal{T}\setminus\{0\}$ is convex and hence $\mathcal{T}\setminus\overline{F^v}=\mathcal{T}\setminus F^v\cap\mathcal{T}\setminus\{0\}$ is convex. □

Lemma 6.24. — *Assume that $\mathbb{A}$ is associated with an indefinite Kac–Moody matrix of size 3 such that there exists a non-spherical face different from $\mathbb{A}_{in}$. Assume moreover that $(\alpha_s^\vee)_{s\in\mathscr{S}}$ is a basis of $\mathbb{A}$. Then $\mathrm{Hom}_{\mathrm{Mon}}\left(Y^+,(\mathcal{F},.)\right)\supsetneq\mathrm{Hom}_{\mathrm{Mon}}\left(Y^+,\mathcal{F}^*\right)\cup\{0\}$.*

Proof. — Let $\tau^+=\mathbb{1}_{\overline{F^v}}:\mathcal{T}\to\mathcal{F}$. Let us prove that $\tau^+\in\mathrm{Hom}_{\mathrm{Mon}}\big(\mathcal{T},(\mathcal{F},.)\big)$. Let $x,y\in\mathcal{T}$. If $x,y\in\mathcal{T}\setminus\overline{F^v}$, then $x+y=2.\frac{1}{2}(x+y)\in\mathcal{T}\setminus\overline{F^v}$ by Lemma 6.23 and thus $\tau^+(x+y)=0=\tau^+(x)\tau^+(y)$.

Suppose $x\in F^v$ and $y\in\mathcal{T}\setminus\overline{F^v}$, then $x+y=2.\frac{1}{2}(x+y)\in\mathcal{T}\setminus\overline{F^v}$ by Lemma 6.22. Thus $\tau^+(x+y)=0=\tau^+(x)\tau^+(y)$.

Suppose $x=\{0\}$ and $y\in\mathcal{T}\setminus\overline{F^v}$. Let F^v_y be the vectorial face containing y. Then $(x,y]\subset F^v_y$ and hence $x+y\in F^v_y$: $\tau^+(x+y)=0=\tau^+(x)\tau^+(y)$. Consequently, $\tau^+\in\mathrm{Hom}_{\mathrm{Mon}}\big(\mathcal{T},(\mathcal{F},.)\big)$.

Maybe considering $w.F^v$, for some $w\in W^v$, we can assume $F^v\subset\overline{C^v_f}$. Then there exist $s_1,s_2,s_3\in\mathscr{S}$ such that $\mathscr{S}=\{s_1,s_2,s_3\}$ and $F^v=\alpha_{s_1}^{-1}(\{0\})\cap\alpha_{s_2}^{-1}(\{0\})\cap\alpha_{s_3}^{-1}(\mathbb{R}^*_+)$. Let $\lambda\in\mathbb{A}$ be such that $\alpha_{s_1}(\lambda)=\alpha_{s_2}(\lambda)=0$ and $\alpha_{s_3}(\lambda)=1$. There exists $n\in\mathbb{Z}_{\geqslant 1}$ such that $\lambda\in\frac{1}{n}Y$. Thus $\tau^+_{|Y^+}\in\mathrm{Hom}_{\mathrm{Mon}}\left(Y^+,(\mathcal{F},.)\right)\setminus(\mathrm{Hom}_{\mathrm{Mon}}\left(Y^+,\mathcal{F}^*\right)\cup\{0\})$. □

6.3.3. Extension of the representations from G^+ to G

We now study under which condition the representation I_{τ,G^+} of G^+ extends to a representation of G, for $\tau\in T^+_{\mathcal{F}}$.

Lemma 6.25. — *Let $g \in G$. Then for $t \in T$ such that $t.0$ is sufficiently dominant, $tg \in G^+$.*

Proof. — Let $g \in G$ and $x = g.0$. There exists an apartment containing $-\infty$ and x, i.e. there exists $g \in G$ such that $g.\mathbb{A} \cap \mathbb{A}$ contains $a - C^v_f$, for some $a \in \mathbb{A}$. For $q \in C^v_f$ sufficiently dominant, $a - q \leqslant x$. In particular, there exists $y \in \mathbb{A}$ such that $y \leqslant x$. For $\lambda \in Y^{++}$ sufficiently dominant, $y + \lambda \geqslant 0$. Then $n_\lambda.y = y + \lambda \geqslant 0$. As $\leqslant$ is G-invariant, $n_\lambda.y \leqslant n_\lambda.x$ and thus $0 \leqslant n_\lambda.x = n_\lambda g.0$. Therefore $n_\lambda g \in G^+$. □

Let $x, y \in \mathcal{I}$. We write $x \mathring{<} y$ (resp. $x \mathring{\leqslant} y$) if there exists $g \in G$ such that $gx, g.y \in \mathbb{A}$ and $y - x \in \mathring{\mathcal{T}}$ (resp. $y - x \in \mathring{\mathcal{T}} \cup \{0\}$). This does not depend on the choice of g.

If G is reductive, then $x \leqslant y$ for every $x, y \in \mathcal{I}$. We now assume that G is not reductive. Then for every $x \in \mathbb{A}$, for every $y \in x + C^v_f$, one has $x \mathring{<} y$ and $y \not\leqslant x$.

Lemma 6.26. — *Let $x, y, z \in \mathcal{I}$. Suppose that $x \leqslant y$, $y \mathring{<} z$ and $z \not\leqslant y$. Then $x \mathring{<} z$.*

Proof. — Let A be an apartment containing y and z. Let F_y be a positive face of A based at y and containing $[y, y']$ for $y' \in [y, z]$ near y. Then by [15, Theorem 4.4.17], there exists an apartment A' containing F_y and x. Then A' contains $[y, y']$ for some $y' \in [y, z]$ near y. In the apartment A', one has $y \mathring{<} y'$ and $x \leqslant y$. Consequently $x \mathring{<} y'$ (because $\mathring{\mathcal{T}} + \mathcal{T} \subset \mathring{\mathcal{T}}$). We thus have $x \mathring{\leqslant} y'$ and $y' \mathring{\leqslant} z$. Using [34, Théorème 5.9] we deduce that $x \mathring{\leqslant} z$. As $x \leqslant y$ and $z \not\leqslant y$, we have $x \neq z$, which proves the result. □

Lemma 6.27.

(1) *Let $\tau \in T^+_{\mathcal{F}}$ be such that τ is the restriction of some element of $T_{\mathcal{F}}$ (still denoted τ). Then every element of $\widehat{I(\tau)^+}$ uniquely extends to an element of $\widehat{I(\tau)}$.*

(2) *Let $\tau \in T^+_{\mathcal{F}}$ be such that τ is not the restriction of some element of $T_{\mathcal{F}}$. Then for every $f : G \to \mathcal{F}$ such that for all $g \in G^+$ and $b \in B^+$, $f(bg) = (\delta^{1/2}\tau)(b)f(g)$, one has $f = 0$.*

(3) *Let $\tau \in T^+_{\mathcal{F}}$ be such that τ is not the restriction of some element of $T_{\mathcal{F}}$. Then there exists $t \in T$ such that for every $f \in I_{\tau,G^+}$, $t.f = 0$.*

Proof.

(1). — Let $f \in \widehat{I(\tau)^+}$. Suppose that there exists $\widetilde{f} \in \widehat{I(\tau)}$ extending f. Let $g \in G$. Let $t \in T$ be such that $tg \in G^+$. Then $\widetilde{f}(tg) = (\delta^{1/2}\tau)(t)\widetilde{f}(g) = f(tg)$ and thus $\widetilde{f}(g) = (\delta^{1/2}\tau(t))^{-1} f(tg)$. Thus $\widetilde{f}$ is unique if it exists.

We now set $f'(g) = (\delta^{1/2}\tau(t))^{-1}f(tg)$, for $t \in T$ such that $t.0$ is dominant and such that $tg \in G^+$, which exists by Lemma 6.25. Let us prove that f' is well-defined. Let $t, t' \in T$ be such that $tg, t'g \in G^+$ and such that $t.0, t'.0 \in Y^{++}$. Then

$$f(tt'g) = (\tau\delta^{1/2})(t')f(tg) = (\tau\delta^{1/2})(t)f(t'g)$$

so that $f(t'g)\big(\tau\delta^{1/2}(t')\big)^{-1} = f(tg)\big(\tau\delta^{1/2}(t)\big)^{-1}$. This prove that f' is well-defined. In particular, f' extends f.

Let now $t \in T$ and $g \in G$. Let us prove that $f'(tg) = \tau\delta^{1/2}(t)f'(g)$. Let $t' \in T$ be such that $t'g, t'tg \in G^+$. Then

$$\begin{aligned} f'(g) = f(tt'g)\big(\delta^{1/2}\tau(tt')\big)^{-1} = \tau\delta^{1/2}(t')f'(tg)\big(\tau\delta^{1/2}(tt')\big)^{-1} \\ = f'(tg)\big(\tau\delta^{1/2}(t)\big)^{-1}, \end{aligned}$$

which proves that $f'(tg) = \tau\delta^{1/2}(t)f'(g)$.

Let now $g \in G^+$ and $u \in U_+$. Let $t \in T$ be such that $tg, tu \in G^+$. Then $f'(tug) = \tau\delta^{1/2}(t)f'(ug)$ and $f'(tug) = \tau\delta^{1/2}(tu)f'(g) = \tau\delta^{1/2}(t)f(g)$. Thus

$$\tau\delta^{1/2}(t)f'(g) = \tau\delta^{1/2}(t)f'(ug)$$

and hence $f'(ug) = f'(g)$ for every $u \in U_+$ and $g \in G^+$.

Let now $g \in G$ and $u \in U_+$. Let $t \in T$ be such that $tug, tg \in G^+$. As t normalizes U_+, we can write $tu = u't$ for some $u' \in U_+$. Then

$$\begin{aligned} f'(ug) = f'(tug)\big(\tau\delta^{1/2}(t)\big)^{-1} = f'(u'tg)\big(\tau\delta^{1/2}(t)\big)^{-1} \\ = f'(tg)\big(\tau\delta^{1/2}(t)\big)^{-1} = f'(g). \end{aligned}$$

Let $b \in B$ and $g \in G$. Write $b = tu$, with $t \in T$ and $u \in U_+$. Then we have

$$f'(bg) = f'(tug) = \tau\delta^{1/2}(t)f'(ug) = \tau\delta^{1/2}(t)f'(g) = \tau\delta^{1/2}(b)f'(g)$$

and thus $f' \in \widehat{I(\tau)}$ and f' extends f. This proves (1).

(2). — Let $\tau \in T_{\mathcal{F}}^+$ be such that τ is not the restriction of some element of $T_{\mathcal{F}}$. Then by Lemma 6.18, there exists $t \in T$ such that $\tau(t) = 0$. Let $f : G \to \mathcal{F}$ be such that for all $g \in G^+$ and $b \in B^+$, $f(bg) = (\delta^{1/2}\tau)(b)f(g)$. Let $g \in G$. Then $f(g) = f(tt^{-1}g) = \tau\delta^{1/2}(t)f(t^{-1}g) = 0$, which proves (2).

(3). — By Lemma 6.20, one has $\tau(t') = 0$ for every $t' \in T$ such that $t'.0 \in \mathring{\mathcal{T}}$. Let $t \in T$ be such that $t.0 \in C^v_{\mathring{f}}$. Let $g \in G^+$ and $f \in I_{\tau,G^+}$. Then $t.0 \mathring{>} 0$ and $t.0 \not\leqslant 0$. Therefore $gt.0 \mathring{>} g.0$ and $gt.0 \not\leqslant g.0$. Moreover $g.0 \geqslant 0$ and thus by Lemma 6.26 we have $gt.0 \mathring{>} 0$. Using Lemma 6.5 we write $gt = bn_vk$, with $b \in B^+$, $v \in W^v$ and $k \in K_I$. Then $gt.0 = b.0$, which proves that $b.0 \mathring{>} 0$. Write $b = u't'$, with $u' \in U_+$ and $t' \in T$. Then by Theorem 6.6, $\rho_{+\infty}(b.0) = t'.0 \mathring{>} 0$ and thus $\tau(t') = 0$. Therefore $f(gt) = t.f(g) = \tau\delta^{1/2}(t')f(n_v k) = 0$, which proves (3). $\square$

Proposition 6.28. — *Let $\tau^+ \in T_{\mathcal{F}}^+$.*

(1) *Suppose that τ^+ is not the restriction to Y^+ of an element of $T_{\mathcal{F}}$. For every $f \in \widehat{I(\tau^+)} \setminus \{0\}$, for every G-module M, the restriction of M to G^+ is not isomorphic to $G^+.f$. For every $x \in I^+_{\tau^+} \setminus \{0\}$, for every ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module M, the restriction of M to $\mathcal{H}_{\mathcal{F}}$ is not isomorphic to $\mathcal{H}_{\mathcal{F}}.x$.*

(2) *Suppose that τ^+ is the restriction to Y^+ of a (necessarily unique) element τ of $T_{\mathcal{F}}$. Every element f^+ of $\widehat{I(\tau^+)^+}$ can be extended uniquely to an element f of $\widehat{I(\tau)}$. Then $f^+ \mapsto f$ is an isomorphism of G^+-modules. The action of $\mathcal{H}_{\mathcal{F}}$ on $I^+_{\tau^+}$ extends uniquely to an action of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ on $I^+_{\tau^+}$. Then $I^+_{\tau^+}$ is naturally isomorphic to I_τ as a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module.*

Proof.

(1). — By Lemma 6.18, there exists $\lambda \in Y^+$ such that $\tau^+(\lambda) = 0$. Then if $x \in I^+_{\tau^+} \setminus \{0\}$, $Z^\lambda.x = 0$. If M is a ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$-module, one has $Z^{-\lambda}.Z^\lambda.y = y \neq 0$ for every $y \in M \setminus \{0\}$. The similar statement for G^+ is a consequence of Lemma 6.27(3).

(2). — The statement for $\widehat{I(\tau^+)^+}$ follows from Lemma 6.27(1). The statement for I_τ follows from Proposition 2.12. By Proposition 6.17, the actions of $\mathcal{H}_{\mathcal{F}}$ on I_{τ,G^+} and $I_{\tau,G}$ extend to actions of ${}^{\mathrm{BL}}\mathcal{H}_{\mathcal{F}}$ on I_{τ,G^+} and $I_{\tau,G}$. $\square$

Appendix A. Existence of one dimensional representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$

In this section, we prove the existence of one dimensional representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$, when $\sigma_s = \sigma'_s = \sigma$, for all $s \in \mathscr{S}$.

Lemma A.1. — *Assume that $\mathcal{F} = \mathbb{C}$ and that there exists $\sigma \in \mathbb{C}$ such that $\sigma_s = \sigma'_s = \sigma$ for all $s \in \mathscr{S}$ and such that $|\sigma| \neq 1$. Let $\epsilon \in \{-1, 1\}$ and $\tau \in T_{\mathbb{C}}$ be such that $\tau(\alpha_s^\vee) = \sigma^{2\epsilon}$ for all $s \in \mathscr{S}$. Then I_τ admits a unique maximal proper submodule M. Moreover, $I_\tau = M \oplus \mathbb{C}1 \otimes_\tau 1$ and if $x \in I_\tau/M$, then $Z^\lambda.x = \tau(\lambda).x$ and $H_w.x = (\epsilon\sigma^\epsilon)^{\ell(w)}.x$ for all $(w, \lambda) \in W^v \times Y$.*

Proof. — By Lemma 5.2, such a τ exists. Let $q=\sigma^2$. Let $\mathrm{ht}:Y\to\mathbb{Q}$ be a $\mathbb{Z}$-linear map such that $\mathrm{ht}(\alpha_s^\vee)=1$ for all $s\in\mathscr{S}$. Then one has $\tau(\alpha^\vee)=q^{\epsilon\,\mathrm{ht}(\alpha^\vee)}$ for all $\alpha^\vee\in\Phi^\vee$.

Let $s\in\mathscr{S}$. With the same notation as in Lemma 4.4, let $\phi_s=\phi(s.\tau,\tau):I_{s.\tau}\to I_\tau$. Then by Lemma 4.4 $M_s:=\mathrm{Im}(\phi_s)$ is a proper submodule of I_τ. Moreover, $H_s-\epsilon\sigma^\epsilon\otimes_\tau 1\in M_s$. Let $M=\sum_{s\in\mathscr{S}}M_s$. Let $w\in W^v\setminus\{1\}$ and $w=s_1\dots s_k$ be a reduced expression. Let $v=ws_k$. Then $H_v.(H_{s_k}-\epsilon\sigma^\epsilon)=H_w-\epsilon\sigma^\epsilon H_v\in M_{s_k}$. Therefore, for all $w\in W^v\setminus\{1\}$, there exists $x_w\in M$ such that $\pi_w^H(x_w)=1$ and $x_w\in M\cap I_\tau^{\leqslant w}$. By induction on $\ell(w)$ we deduce that $M+\mathbb{C}1\otimes_\tau 1=I_\tau$.

By [12, Lemma 2.4 a)], $\tau\in T_{\mathbb{C}}^{\mathrm{reg}}$. Moreover, by Proposition 3.4 (2),

$$I_\tau=\bigoplus_{w\in W^v}I_\tau(w.\tau)$$

and if we choose $\xi_v\in I_\tau(v.\tau)\setminus\{0\}$ for all $v\in W^v$, then $(\xi_v)_{v\in W^v}$ is a basis of I_τ. For $w\in W^v$, let $\pi_w^\xi:I_\tau\to\mathbb{C}$ be the linear map defined by $\pi_w^\xi(\xi_v)=\delta_{v,w}$ for all $v\in W^v$. As $\xi_1\in\mathbb{C}1\otimes_\tau 1$, one has $\pi_1^\xi(M_s)=\{0\}$ for all $s\in\mathscr{S}$. Thus $I_\tau=M\oplus\mathbb{C}1\otimes_\tau 1$. Moreover, $M\subset(\pi_1^\xi)^{-1}(\{0\})$ and by dimension $M=\pi_1^\xi(\{0\})$. We deduce that M is the unique maximal proper submodule of I_τ and the lemma follows. $\square$

Remark A.2. — Actually, the representations constructed in Lemma A.1 generalize the well known trivial representation (when $\epsilon=1$) and Steinberg representation (when $\epsilon=-1$). For simplicity, we assumed all the σ_s,σ'_s to be equal, but this is not necessary. We can also construct these representations directly by setting $\mathrm{triv}(H_s)=\sigma_s$, $\mathrm{triv}(Z^{\alpha_s^\vee})=\sigma_s\sigma'_s$, $\mathrm{St}(H_s)=-\sigma_s^{-1}$, $\mathrm{St}(Z^{\alpha_s^\vee})=\sigma_s^{-1}\sigma_s'^{-1}$. Using the fact that the relations (BL1) to (BL4) are preserved by triv and St, we can extend them to representations of ${}^{\mathrm{BL}}\mathcal{H}_{\mathbb{C}}$ over $\mathbb{C}$.

Appendix B. Examples of possibilities for W_τ for size 2 Kac–Moody matrices

In this section, we prove that there exist size 2 Kac–Moody matrices such that for each subgroup H of W^v, there exist $\tau\in T_{\mathbb{C}}$ such that W_τ is isomorphic to H. We assume that $\alpha_s(Y)=\mathbb{Z}$ for all $s\in\mathscr{S}$ and thus $W_{(\tau)}=W_\tau$. We already proved the existence of regular elements in Lemma 5.1. If $\tau\in T_{\mathbb{C}}$ is such that $\tau(\alpha_{s_1}^\vee)=1$ and $\tau(\alpha_{s_2}^\vee)$ is not a root of 1, then $W_\tau=\{1,s_1\}$.

Lemma B.1. — *Let $A=(a_{i,j})_{(i,j)\in[\![1,2]\!]^2}$ be a Kac–Moody matrix. Assume that $a_{1,2}$ and $a_{2,1}$ are even and such that $a_{1,2}a_{2,1}$ is greater than 6. Let γ_2 be a primitive $\frac{1}{2}(a_{1,2}a_{2,1}-4)$-th root of 1. Let $\gamma_1=\gamma_2^{\frac{1}{2}a_{1,2}}$. Let $\tau: Y=\mathbb{Z}\alpha_1^\vee\oplus\mathbb{Z}\alpha_2^\vee\to\mathbb{C}^*$ be the group morphism defined by $\tau(\alpha_i^\vee)=\gamma_i$ for both $i\in\{1,2\}$. Then $W_\tau=\langle s_1s_2\rangle\simeq\mathbb{Z}$.*

Proof. — Let $\tau'\in T_\mathbb{C}$ and $\gamma_i'=\tau'(\alpha_i^\vee)$ for both $i\in\{1,2\}$. For $\lambda\in Y$, one has $(s_2-s_1).\lambda=\alpha_1(\lambda)\alpha_1^\vee-\alpha_2(\lambda)\alpha_2^\vee$. Thus

$$\begin{aligned}s_1.\tau'=s_2.\tau' &\iff \forall\,\lambda\in Y, \tau'(\alpha_1(\lambda)\alpha_1^\vee-\alpha_2(\lambda)\alpha_2^\vee)=1\\ &\iff \forall\,\lambda\in Y, \gamma_1'^{\alpha_1(\lambda)}=\gamma_2'^{\alpha_2(\lambda)}\\ &\iff (\gamma_1')^2=(\gamma_2')^{a_{1,2}} \text{ and } (\gamma_2')^2=(\gamma_1')^{a_{2,1}}.\end{aligned}$$

Thus $s_1.s_2.\tau=\tau$. Moreover $s_2.\tau\neq\tau$ and hence $W_\tau=\langle s_1s_2\rangle$. □

If $\tau=\mathbb{1}:Y\to\{1\}$, then $W_\tau=1$. The following lemma proves that W_τ can be a proper subgroup of W^v isomorphic to the infinite dihedral group.

Lemma B.2. — *Let $A=(a_{i,j})_{(i,j)\in[\![1,2]\!]^2}$ be an irreducible Kac–Moody matrix which is not a Cartan matrix. One has $a_{1,2}a_{2,1}\geqslant 4$ and maybe considering ${}^tA$, one may assume $a_{1,2}\leqslant -2$. Write $W^v=\langle s_1,s_2\rangle$. Let γ_2 be an $a_{1,2}$-th primitive root of 1 and $\tau\in T_\mathbb{C}$ be defined by $\tau(\alpha_{s_1}^\vee)=1$ and $\tau(\alpha_{s_2}^\vee)=\gamma_2$. Then $W_\tau=\langle s_1,s_2s_1s_2\rangle$.*

Proof. — Let $\widetilde{\tau}=s_2.\tau$. Let us prove that $s_1.\widetilde{\tau}=\widetilde{\tau}$, i.e. that $\widetilde{\tau}(\alpha_{s_1}^\vee)=1$. One has $\widetilde{\tau}(\alpha_{s_1}^\vee)=\tau(s_2.\alpha_{s_1}^\vee)=\tau(\alpha_{s_1}^\vee-\alpha_{s_2}(\alpha_{s_1}^\vee)\alpha_{s_2}^\vee)=\tau(\alpha_{s_2}^\vee)^{-a_{1,2}}=1$. Thus $W_\tau\ni\{s_1,s_2s_1s_2\}$. Therefore $W^v/W_\tau=\{W_\tau,t.W_\tau\}$. Moreover $t\notin W_\tau$, thus $[W^v:W_\tau]=2$ and hence $W_\tau=\langle s_1,s_2s_1s_2\rangle$. □

Index

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