

# ANNALES DE L'INSTITUT FOURIER

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*Annales de l'institut Fourier*, tome 20, n° 1 (1970), p. 433-456

[<http://www.numdam.org/item?id=AIF\\_1970\\_\\_20\\_1\\_433\\_0>](http://www.numdam.org/item?id=AIF_1970__20_1_433_0)

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## THE MARTIN BOUNDARIES OF EQUIVALENT SHEAVES

by J. C. TAYLOR

### Introduction.

Let  $X$  be a locally compact space on which a sheaf  $\underline{H}$  of vector spaces of continuous real-valued functions is defined satisfying the basic axioms of Brelot [3]<sup>(1)</sup>. In addition, assume that the following conditions hold :  $X$  has a countable base ; there is a positive potential defined by  $\underline{H}$  ; and  $\underline{H}$  satisfies the hypothesis of proportionality, that is for  $y \in X$  any two potentials with support  $\{y\}$  are proportional.

Then, following the original construction of R.S. Martin [13], it is possible to define a Martin compactification of  $X$ . A priori this compactification depends on  $\underline{H}$ . The purpose of this article is to initiate a study of the dependence.

The question is not an empty one as the following examples show. Let  $B$  be the closed unit ball in  $\mathbb{R}^3$  and set  $X = B \setminus (S \cup L)$ , where  $S$  is the unit sphere and  $L$  is the closed line segment joining  $(0, 0, 0)$  to  $(0, 0, 1)$ . Then, if  $\underline{H}$  is the sheaf defined by Laplace's equation  $\Delta h = 0$ , the Martin compactification of  $X$  is  $B$ . This follows from the fact that the Martin compactification of  $B \setminus S$  is  $B$  and that  $L$  is a closed set of capacity zero.

Now let  $Y = B \setminus (S \cup C)$ , where  $C$  is the closed convex cone defined by  $x^2 + y^2 \leq z^2$  and  $z \geq 0$ . Then from results of de la Vallée Poussin [17] it follows that the Martin compactification for  $Y$  associated with Laplace's equation is the closure  $\bar{Y}$  of  $Y$  in  $\mathbb{R}^3$ . Following a suggestion of Choquet, the differential equation  $\Delta h = 0$

<sup>(1)</sup> Throughout this article it will be assumed that  $X$  satisfies the customary connectivity conditions.

on  $Y$  can be transported to  $X$  by means of a diffeomorphism. This defines an elliptic operator  $L$  on  $X$  whose associated Martin compactification is clearly homeomorphic to  $\bar{Y}$ . The Martin boundary in this case is homeomorphic to  $S$ , which is not homeomorphic to  $S \cup L$ . Hence, these two Martin compactifications of  $X$  are distinct.

The principal result of this article is the result (theorem 2) that if two sheaves  $\underline{H}_1$  and  $\underline{H}_2$  are equivalent, that is they agree on the complement of a compact subset of  $X$  [11], then the Martin compactifications of  $X$  coincide. In this coincidence, the corresponding sets of minimal points coincide (theorem 2). This has as a consequence the result that the cones  $\underline{S}_2^+$  and  $\underline{S}_2^+$ , equipped with the  $T$ -topology, are isomorphic.

The last part of the article discusses the relation between the Martin compactification of  $X \setminus A$ ,  $A$  compact, and  $\bar{X} \setminus A$  where  $\bar{X}$  is the Martin compactification of  $X$  (corollary 3 to theorem 5). Further, it is shown that the Martin compactification is of type  $S$  (corollary 4 to theorem 5) and that the ends of  $X$  are related to direct decomposition of the cone of positive harmonic functions.

I wish to thank M. Sieveking for a very useful discussion in the course of which we obtained the proof of theorem 1.

I would also like to thank Professor M. Brelot for his continued interest in this work, and for his persistent belief that a proof for theorem 2 could be found without the use of adjoint harmonic functions [16].

## 2. Elementary properties of $Q$ -compactification.

Let  $X$  be a locally compact space and let  $(K_a)_{a \in A}$  be a family of continuous functions  $K_a : X \longrightarrow \bar{\mathbb{R}}$ . Then, as is well known (c.f. [6]), there is a unique compactification  $\bar{X}$  of  $X$  such that (1) each function  $K_a$  extends continuously to  $\bar{X}$  and (2) the extended functions separate the points of  $\Delta = \bar{X} \setminus X^{(2)}$ . The space  $\bar{X}$  can be realized as the closure of the image of  $X$  under the embedding of  $X$  in  $\Pi\{\mathbb{R}_f | f \in \underline{C}_K(X) + A\}$  by the mapping  $e$ , which is defined as

(<sup>2</sup>) "Unique" means that if  $\bar{X}_1$  and  $\bar{X}_2$  are any two compactifications satisfying (1) and (2), there is a homeomorphism  $\varphi : \bar{X}_1 \longrightarrow \bar{X}_2$  with  $\varphi(x) = x$  for each  $x \in X$ .

follows :  $(pr_f \circ e)(x)$  equals  $f(x)$  if  $f \in \underline{C}_K(X)$  and equals  $K_\alpha(x)$  if  $f = \alpha \in A^{(3)}$ .

The following formal properties of Q-compactifications are easily verified.

1) Let  $\varphi : X \longrightarrow Y$ ,  $Y$  locally compact, be proper and let  $(K_\alpha)_{\alpha \in A}$  be a family of continuous functions on  $X$ ,  $(L_\beta)_{\beta \in B}$  a family of continuous functions on  $Y$ . Denote by  $\bar{X}$  and  $\bar{Y}$  the corresponding compactifications. If for each  $\beta \in B$  there is an  $\alpha \in A$  with  $L_\beta \circ \varphi = K_\alpha$ , then there is a unique continuous map  $\bar{\varphi} : \bar{X} \longrightarrow \bar{Y}$  with  $\bar{\varphi}(x) = \varphi(x)$ , for all  $x \in X$ .

2) Let  $(K_\alpha)_{\alpha \in A}$  be a given family of continuous functions on  $X$  and denote by  $(K'_\beta)_{\beta \in B}$  a second family such that each  $K'_\beta$  extends continuously to  $\bar{X}$ . Define  $(K''_\gamma)_{\gamma \in A+B}$  by setting  $K''_\gamma = K_\alpha$  if  $\gamma = \alpha \in A$  and  $K''_\gamma = K'_\beta$  if  $\gamma = \beta \in B$ . Let  $\bar{X}''$  be the compactification determined by  $(K''_\gamma)$ . Then  $\bar{X}'' = \bar{X}'$ .

3) Let  $\varphi, X, Y, (K_\alpha)_{\alpha \in A}$  and  $(L_\beta)_{\beta \in B}$  be as in 1. Assume that for each  $\beta \in B$ ,  $L_\beta \circ \varphi = K'_\beta$  extends continuously to  $\bar{X}$ . Then there is a unique continuous map  $\bar{\varphi} : \bar{X} \longrightarrow \bar{Y}$  which extends  $\varphi$ .

4) Let  $(K'_\alpha)_{\alpha \in A}$  and  $(K''_\alpha)_{\alpha \in A}$  be two families of continuous functions such that, for each  $\alpha$ , there is a compact set  $D_\alpha$  with  $K'_\alpha(x) = K''_\alpha(x)$  if  $x \in X \setminus D_\alpha$ . Let  $\bar{X}'$  and  $\bar{X}''$  be the corresponding compactifications of  $X$ . Then  $\bar{X}' = \bar{X}''$ .

These elementary properties established, it is easy to prove the following propositions.

**PROPOSITION 1.** — *Let  $X$  be a locally compact space and  $(K_\alpha)_{\alpha \in A}$  a family of continuous functions  $K_\alpha : X \setminus D_\alpha \longrightarrow \bar{R}$ , where  $D_\alpha$  is a compact subset of  $X$ . Then there is a unique compactification  $\bar{X}$  of  $X$  such that :*

- 1) *each  $K_\alpha$  extends continuously to  $\bar{X} \setminus D_\alpha$  ; and*
- 2) *the extended functions separate the points of  $\bar{X} \setminus X$ .*

**Proof.** — It is an immediate consequence of 4), since for each  $K_\alpha$  there is a continuous function  $K'_\alpha$  which agrees with  $K_\alpha$  on the complement of a compact neighbourhood of  $D_\alpha$ .

(<sup>3</sup>) If  $A, B$  are sets, then  $A + B$  denotes their disjoint sum.

PROPOSITION 2. — Let  $X$ ,  $(K_\alpha)_{\alpha \in A}$  be as in proposition 1 and let  $Y$  be a locally compact space with a family  $(L_\beta)_{\beta \in B}$  of functions  $L_\beta : Y \setminus E_\beta \longrightarrow \bar{R}$ ,  $E_\beta$  compact in  $Y$ .

If  $\varphi$  is proper and such that each  $L_\beta \circ \varphi$  extends continuously to  $\bar{X} \setminus \varphi^{-1}(E_\beta)$ , then there is a unique continuous extension  $\bar{\varphi} : \bar{X} \longrightarrow \bar{Y}$  of  $\varphi$ .

*Proof.* — It is a consequence of 3) and 4).

If  $\bar{X}$  is the compactification of  $X$  determined by  $(K_\alpha)_{\alpha \in A}$  let  $\bar{K}_\alpha$  denote the extended function. These functions separate the points of  $\bar{X} \setminus X$  strictly if for  $x_1 \neq x_2$ , two points of  $\bar{X} \setminus X$ , there exist  $\alpha_1, \alpha_2$  with  $\bar{K}_{\alpha_1}(x_1) \bar{K}_{\alpha_2}(x_2) \neq \bar{K}_{\alpha_1}(x_2) \bar{K}_{\alpha_2}(x_1)$ .

### 3. A general theorem.

Let  $X_1, X_2$ , be locally compact spaces with countable bases and denote by  $\underline{H}_1$  and  $\underline{H}_2$  sheaves on the corresponding spaces which satisfy the axioms of Bauer [2] and are such that both harmonic spaces are strict.

Let  $K^i : X_i \times X_i \longrightarrow \bar{R}^+$  be a function with the following properties :

1)  $y \longrightarrow K^i(x, y) = *K_x^i(y)$  is continuous outside a compact set  $A_x$  ;

2)  $x \longrightarrow K^i(x, y) = K_y^i(x)$  is superharmonic.

Denote by  $\varphi : X_1 \longrightarrow X_2$  a surjective proper map and let  $A_2 \subset X_2$ , and  $A_1 = \varphi^{-1}(A_2)$ , be compact sets such that for  $y \in X_1 \setminus A_1$  :

3)  $K_y^1 - P_A K_y^1 = f(y) [K_{\varphi(y)}^2 - P_A K_{\varphi(y)}^2] \circ \varphi + d_y(x)$  where  $d_y(x)$  tends to zero in  $x$  as  $y$  tends to the point at infinity and, for any superharmonic function  $v$ ,  $P_E v = R_v^E$ .

THEOREM 1. — Let  $\bar{X}_i$  be the compactification of  $X_i$  determined by the family  $(*K_x^i)_{x \in X}$ . Assume that the extensions of the functions  $*K_x^2$  separate the points of  $\bar{X}_2 \setminus X_2$  strictly. There is a unique continuous extension  $\bar{\varphi} : \bar{X}_1 \longrightarrow \bar{X}_2$  of  $\varphi$  if the following holds for  $i = 1, 2$  :

4) if a net  $(y_\gamma)$  converges to a point  $y \in \bar{X}_1 \setminus X_1$ , then the functions  $K_{y_\gamma}^t$  converge pointwise to a harmonic function, the convergence being uniform on  $A_t$ , where  $K_y^t(x) = K^t(x, y)$ .

*Proof.* — In view of proposition 2, it suffices to show that if the net  $(y_\gamma)$  on  $X_1$  converges to a point  $y_1 \in \bar{X}_1 \setminus X_1$ , then  $\lim_\gamma *K_x^2(\varphi(y_\gamma))$  exists for each  $x \in X_2$ .

Since  $y_\gamma$  converges to  $y_1$ , the superharmonic functions  $K_{y_\gamma}^1$  converge pointwise to a harmonic function  $h_1$ . The convergence being uniform on  $A_1$ , the functions  $P_{A_1} K_{y_\gamma}^1$  converge to  $P_A h_1$ . It follows from 3) that  $f(y_\gamma) [K_{\varphi(y_\gamma)}^2 - P_{A_2} K_{\varphi(y_\gamma)}^2]$  converges on  $X_2 \setminus A_2$ . For convenience denote  $\varphi(y)$  by  $z$  and let  $(z_{\gamma'})$  and  $(z_{\gamma''})$  be two subnets of the net  $(z_\gamma)$  which converge to points  $z'$  and  $z''$  of  $\bar{X}_2$ , necessarily in  $\bar{X}_2 \setminus X_2$  as  $\varphi$  is proper. Denote by  $h'_2$  and  $h''_2$  the corresponding harmonic functions.

Condition 4) implies that  $P_{A_2} K_{z_{\gamma'}}^2$  converges to  $P_{A_2} h'_2$  and that  $P_{A_2} K_{z_{\gamma''}}^2$  converges to  $P_{A_2} h''_2$ . However, since

$$f(y_\gamma) [K_{z_\gamma}^2 - P_{A_2} K_{z_\gamma}^2]$$

converges, it follows that  $f$  converges along both subnets and that  $\alpha' [h'_2 - P_{A_2} h'_2] = \alpha'' [h''_2 - P_{A_2} h''_2]$  where

$$\alpha' = \lim_{\gamma'} f(y_{\gamma'}) \quad \text{and} \quad \alpha'' = \lim_{\gamma''} f(y_{\gamma''}).$$

Rewriting this  $\alpha' h'_2 + \alpha'' P_{A_2} h''_2 = \alpha'' h''_2 + \alpha' P_{A_2} h'_2$ , it follows from the Riesz Decomposition theorem that  $\alpha' h'_2 = \alpha'' h''_2$  (regularize both sides). Since  $h'_2(x) = \lim_{\gamma'} *K_x^2(y_{\gamma'})$  and similarly for  $h''_2(x)$ , it follows from the strict separation assumption that  $h'_2 = h''_2$ .

In other words,  $(z_\gamma)$  converges in  $\bar{X}_2$ , that is, for each

$$x \in X_2, \lim_\gamma *K_x^2(\varphi(y_\gamma)) \text{ exists.}$$

*Remark.* — The assumption of a countable base for  $X$  does not enter into the proof. It is made so as to fulfil the hypotheses of the theory of Bauer. The result holds for the theory without this assumption.

#### 4. Application to the Martin Boundary.

Let  $X$  be a non compact locally compact space with a countable base, and let  $\underline{H}$  be a sheaf on  $X$  that satisfies the basic axioms of Brelot [3]<sup>(4)</sup>. Assume that a positive potential is defined by  $\underline{H}$  and that  $\underline{H}$  satisfies the hypothesis of proportionality.

Madame Hervé [10] (Proposition 18.1) proved the existence of a lower semi-continuous function  $G : X \times X \longrightarrow \bar{\mathbb{R}}^+$ , continuous off the diagonal and such that for each

$$y \in X, \quad x \longrightarrow G(x, y) = p_y(x)$$

is a potential with support  $\{y\}$ . Such a function will be called a Green's function for  $\underline{H}$ . If  $f$  is continuous and strictly positive on  $X$  define  $Gf(x, y) = \bar{G}(x, y)f(y)$ . Then  $Gf$  is a Green's function and every Green's function has this form.

Let  $x_0 \in X$  and define  $K(x, y)$  to be 1 if  $x = x_0 = y$  and to be  $G(x, y)/G(x_0, y)$  otherwise.

The compactification of  $X$  defined by  $(*K_x)_{x \in X}$ , where

$$*K_x(y) = K(x, y),$$

will be called the *Martin compactification* of  $X$  and will be denoted by  $M(X, \underline{H})$  or  $\bar{X}$ . It is clearly independent of the choice of Green's function  $\bar{G}$ .

Let  $\Lambda$  be a compact base for the cone  $\underline{S}^+$  equipped with the  $T$ -topology [10]. For  $y \in X$  denote by  $p_y$  the unique potential in  $\Lambda$  with support  $\{y\}$ . Gowrisankaran [9] (theorem IV.I) proved that the mapping  $y \longrightarrow p_y$  embeds  $X$  in  $\Lambda$ . Identifying  $X$  with its image let  $\tilde{X}$  denote the closure of  $X$  in  $\Lambda$ . It is not hard to see from Scolie 21.1 of [10] that  $\tilde{X}$  is the compactification of  $X$  determined by  $(p_x^*)_{x \in X}$ , where  $p_x^*(y) = p_y(x)$ .

**PROPOSITION 3.** — *The compactifications  $\bar{X}$  and  $\tilde{X}$  coincide. Hence,  $\bar{X}$  is independent of the choice of  $x_0$ .*

<sup>(4)</sup> As was pointed out by C. Constantinescu, the assumption of a countable base is not necessary. However, in order to avoid it it is necessary to establish some lemmas corresponding to results of Madame Hervé [10]. These lemmas are established in the appendix.

*Proof.* — According to proposition 22.1 of [10], the function  $G(x, y) = p_y(x)$  is a Green's kernel for  $\underline{H}$ . Let  $x_0 \in X$  and define  $K(x, y)$  as above. Then as long as  $y \notin \{x, x_0\}$ ,  $p_y(x_0) K_y(x) = p_y(x)$  or  $p_{x_0}^*(y) K_x^*(y) = p_x^*(y)$ . Since  $y \longrightarrow p_{x_0}^*(y)$  never vanishes on  $\bar{X} \setminus X$  (note that  $x \longrightarrow p_x^*(y)$  is harmonic on  $X$  for each  $y \in \bar{X} \setminus X$ ), it follows that all the functions  $*K_x$  extend continuously to  $\tilde{X} \setminus \{x, x_0\}$  as functions  $*\tilde{K}_x$ .

Suppose that  $y_1, y_2$  are two points in  $\tilde{X} \setminus X$  and that for all  $x \in X$ ,  $*\tilde{K}_x(y_1) = *\tilde{K}_x(y_2)$ . Denote by  $p_{y_1}$  and  $p_{y_2}$  the corresponding harmonic functions in  $\Lambda$ . Then  $[p_{y_1}(x)/p_{y_1}(x_0)] = [p_{y_2}(x)/p_{y_2}(x_0)]$  for all  $x \in X$ . Since the functions  $p_{y_i}$  are in  $\Lambda$ , it follows that  $p_{y_1} = p_{y_2}$  and so  $y_1 = y_2$ . Consequently,  $\tilde{X}$  and  $\bar{X}$  are the same compactification of  $X$  in the sense defined in 2.

The general theorem is now applied to prove the following result.

**THEOREM 2.** — *Let  $\underline{H}_1$  and  $\underline{H}_2$  be two sheaves on  $X$  that satisfy the above hypotheses. Assume there is a compact set  $A \subset X$  such that the sheaves coincide on  $X \setminus A$ . Then  $M(X, \underline{H}_1) = M(X, \underline{H}_2)$ .*

*Proof.* — Let  $G^i$  be a Green's function for  $\underline{H}_i$  and let

$$p_y^i(x) = G^i(x, y).$$

Define  $q_y^i = p_y^i - P_A p_y^i$ ,  $y \in X \setminus A$ . Then by theorem 16.4 of [6]  $q_y^i$  is a potential on  $X \setminus A$  of support  $\{y\}$  which is positive only on the connected component of  $A$  that contains  $y$ . Furthermore, by the same theorem, the hypothesis of proportionality is satisfied on  $\bar{C}A$ .

Pick  $x_0 \in A$  and consider the two functions  $K^1$  and  $K^2$  defined by  $G^1$  and  $G^2$ . Since  $K_y^i - P_A K_y^i = [1/p_y^i(x_0)] q_y^i$  there is a continuous function  $f(y)$  with  $K_y^1 - P_A K_y^1 = f(y) [K_y^2 - P_A K_y^2]$ .

Since, in the case of Martin compactifications, the harmonic functions corresponding to the boundary points all take 1 at  $x_0$ , it follows that the extensions to  $M(X, \underline{H}_i)$  of the functions  $*K_x^i$  separate strictly the points of the ideal boundary.

It is well known that condition 4) is satisfied and so the conditions of theorem 1 are satisfied with  $\varphi(x) = x$  for all  $x \in X$ .



Hence, there is a unique continuous map  $\bar{\varphi} :$

$$M(X, \underline{H}_1) \longrightarrow M(X, \underline{H}_2)$$

which extends the identity map on  $X$ . The argument being symmetrical, it follows formally from the denseness of  $X$  in a compactification and the uniqueness condition that  $\varphi$  is a homeomorphism. In other words, the compactifications coincide.

*Remarks.* — Since theorem 1 holds in Bauer's theory, it is reasonable to ask for a similar theorem there. A slight modification of Sieveking's definition of a Martin space [15] leads to similar results.

However, the non compactness of  $X \cup \Delta$  requires a hypothesis [4'] that ensures if a net  $(y_\gamma)$  converges to a point in  $\Delta^1$ , then it converges to a point in  $\Delta^2$ ,  $\Delta^i$  being boundaries for  $\underline{H}_i$ ,  $i = 1, 2$ .

## 5. The extension of harmonic functions.

Let  $\underline{H}$  be a sheaf satisfying the axioms of Bauer [2] which is strict. It can be assumed that 1 is superharmonic.

Denote by  $A, B$  compact subsets of  $X$  and by  $O$  a relatively compact open set with  $A \subset \overset{\circ}{B} \subset B \subset O$ .

If  $U$  is open in  $X$  denote by  $H_U$  the kernel defined by the Dirichlet problem for  $U$ , that is, if  $\varphi$  is a continuous function with compact support on  $X$  then  $H_U(x, \varphi)$  equals  $\varphi(x)$  if  $x \notin U$  and equals  $H_U^U(x)$  where  $f = \varphi|_{\partial U}$  if  $x \in U$  (see [11]).

The open set  $O$  can be chosen so that  $H_O H_{\overset{\circ}{B}} 1 \leq \lambda < 1$  on  $B$  since  $H_{\overset{\circ}{B}} 1$  coincides with a potential except possibly on  $\partial B$ .

Define  $T : \underline{C}(\partial O) \longrightarrow \underline{C}(\partial O)$  by setting  $Tf = (H_{\overset{\circ}{B}} H_O f)|_{\partial O}$  and define  $S : \underline{C}(\partial B) \longrightarrow \underline{C}(\partial B)$  by setting

$$Sg = (H_O H_{\overset{\circ}{B}} g)|_{\partial B}.$$

Then  $S$  and  $T$  are positive linear operators such that  $\|S\| \leq \lambda$ ,  $\|T\| \leq 1$  and  $T^n = H_{\overset{\circ}{B}} S^{n-1} H_O$ . Hence  $\|T^n\| \leq \lambda^{n-1}$ . As a result the series

$\sum_{n \geq 0} T^n$  converges to an operator which is the inverse of  $(I - T)$ .

Therefore  $(I - T)^{-1}$  exists and is a positive operator.

Using these results, it is easy to prove the following proposition due to Nakai [14] in Brelot's theory.

**PROPOSITION 4.** — *Let  $h$  be a continuous function on  $X \setminus A$ , harmonic on  $X \setminus B$ . Then there is a unique harmonic function  $\bar{h}$  on  $X$  such that  $\bar{h} - H_{CB} \bar{h} = h - H_{CB} h$ . The function  $\bar{h}$  is positive if  $h$  is positive.*

*Proof.* — Let  $f$  be the unique continuous function on  $\partial O$  with  $(I - T)f = (h - H_{CB} h)|_{\partial O}$ . Define  $\bar{h}$  by setting  $\bar{h}(x)$  equal to  $H_O(x, f)$  if  $x \in O$  and equal to

$$(H_{CB} H_O)(x, f) + h(x) - H_{CB}(x, h) \quad \text{if } x \notin B.$$

Consider the harmonic function

$$h_1 = H_{CB} H_O f + h - H_{CB} h = h + H_{CB}(H_O f - h)$$

and let  $O_1 = O \setminus B$ . Now the continuity of  $h$  implies that  $H_O h_1 = h$  and from the comparison theorem it follows that

$$H_{O_1} H_{CB}(H_O f - h) = H_{CB}(H_O f - h).$$

Hence,  $h_1 = H_{O_1} h_1$ . On  $\partial O_1$ ,  $h_1 = H_O f$  and so on  $O_1$ ,

$$h_1 = H_{O_1} H_O f = H_O f.$$

As a result,  $\bar{h}$  is a well defined harmonic function on  $X$ . Furthermore,  $H_{CB} \bar{h} = H_{CB} H_O f$ , and so  $\bar{h} - H_{CB} \bar{h} = h - H_{CB} h$ .

Assume that  $\bar{h}$  is a harmonic function on  $X$  for which

$$\bar{h} - H_{CB} \bar{h} = h - H_{CB} h.$$

Then  $h - H_{CB} h = \bar{h} - H_{CB} H_O \bar{h}$ , which coincides on  $\partial O$  with  $(I - T)\bar{h}$ . Consequently,  $\bar{h}$  is uniquely defined.

If  $h \geq 0$  then  $h - H_{CB} h \geq 0$  and as a result  $(I - T)\bar{h}$ , and consequently  $\bar{h}$  are also positive.

It follows from the proposition that  $\bar{h}$  is independent of the set  $O$  containing  $B$ . Let  $B_1$  be compact  $B \subset B_1$ . The fact that

$$H_{C_{B_1}} H_{C_B} = H_{C_B}$$

implies that  $\bar{h}$  is independent of the compact set  $B \supset A$  provided  $h$  is harmonic on  $X \setminus B$ . Define the linear operator  $E$  :

$$\underline{\underline{H}}(X \setminus A) \longrightarrow \underline{\underline{H}}(X)$$

by setting  $E h = \bar{h}$ .

In what follows,  $B$  and  $O$  will be sets such that  $A \subset \mathring{B} \subset B \subset O$ ,  $B$  compact,  $O$  open, relatively compact with  $H_O H_{C_B} 1 \leq \lambda < 1$  on  $B$ .

PROPOSITION 5. — *The operator  $E$  has the following properties :*

- 1) *it is linear and positive ;*
- 2) *it is continuous and open in the topology of uniform convergence on compact sets ;*
- 3) *if  $h = h' | X \setminus A$ ,  $h' \in \underline{\underline{H}}(X)$ , then  $E(h) = h'$  ;*
- 4)  *$E(h) = 0$ , if and only if for some compact  $B \supset A$ ,  $h = H_{C_B} h$ .*

*Proof.* — Statements 1) and 3) have been proved. To prove 2), let  $(h_n)$  be a sequence in  $\underline{\underline{H}}(X \setminus A)$  converging to  $h$ . Then  $h_n - H_{C_B} h_n$  converges uniformly to  $\bar{h} - H_{C_B} h$  on the compact subsets of  $X \setminus B$ . The continuity of  $(I - T)^{-1}$  implies that  $\bar{h}_n$  converges to  $\bar{h}$ .

Let  $P$  be open in  $\underline{\underline{H}}(X \setminus A)$  and let  $\bar{h}_0 = E(h_0)$ ,  $h_0 \in P$ . There exist  $\varepsilon > 0$  and  $K$  compact in  $X \setminus A$  with  $h \in P$  if  $|h(x) - h_0(x)| < \varepsilon$  for all  $x \in K$ . Clearly  $K$  can be assumed to contain  $\partial O$ . Let  $D = K \cup \bar{O}$ .

If  $\bar{h} \in \underline{\underline{H}}(X)$  and  $|\bar{h}(x) - \bar{h}_0(x)| < \varepsilon$  for all  $x \in D$  then

$$(\bar{h} - \bar{h}_0) | (X \setminus A) + h_0 = h \in \underline{\underline{H}}(X \setminus A)$$

is in  $P$ . From 3) it follows that  $E(h) = \bar{h}$  and so  $E$  is open.

Assume that  $h = H_{C_B} h$  for some compact set  $B \supset A$ . Then  $O = h - H_{C_B} h$  which implies  $(I - T) \bar{h} = 0$ . As a result,  $\bar{h} = 0$ . Assume now that  $\bar{h} = 0$ . Then since  $h - H_{C_B} h = \bar{h} - H_{C_B} \bar{h}$  it follows that  $h = H_{C_B} h$ .

PROPOSITION 6. — *Let  $\underline{\underline{H}}_1$  and  $\underline{\underline{H}}_2$  be two strict harmonic sheaves on  $X$ . Assume that there is a compact set  $A \subset X$  such that the sheaves agree on  $X \setminus A$ .*

Then with respect to the topology of uniform convergence on compact sets, the topological vector spaces  $\underline{\underline{H}}_1(X)$  and  $\underline{\underline{H}}_2(X)$  are isomorphic.

*Proof.* — Let  $v_i$  be a continuous strictly positive superharmonic function on  $X$  (relative to  $\underline{\underline{H}}_i$ ),  $i = 1, 2$ . Define  $E_i$  :

$$\underline{\underline{H}}_i(X \setminus A) \longrightarrow \underline{\underline{H}}_i(X)$$

by setting  $E_i(h) = v_i E(h/v_i)$ . These operators are continuous and open.

Assume  $\varphi(x) > 0$  for all  $x \in X$  is a continuous real-valued function. If  $U$  is open in  $X$  and if  $H_U^\varphi$  is the kernel defined by the Dirichlet problem for  $U$  relative to the sheaf  $\varphi^{-1} \underline{\underline{H}}$ , then

$$\varphi[H_U^\varphi(f/\varphi)] = H_U f.$$

Consequently,  $E_1$  and  $E_2$  have the same kernel.

This shows that the mapping  $J : \underline{\underline{H}}_1(X) \longrightarrow \underline{\underline{H}}_2(X)$  defined by setting  $J(E_1(h)) = E_2(h)$ , for all  $h \in \underline{\underline{H}}_1(X \setminus A)$  is an isomorphism.

*Remark.* — Again the assumption of a countable base is not necessary. For example, the arguments hold in the theory of Brelot without this assumption.

## 6. Applications.

All sheaves considered here will be assumed to satisfy the initial hypotheses of section 3.

Let  $\underline{\underline{H}}_1$  and  $\underline{\underline{H}}_2$  be two sheaves on  $X$  that satisfy the hypothesis of theorem 2. Denote by  $\bar{X}$  the common Martin compactification of  $X$  and by  $\Delta$  the Martin boundary  $\bar{X} \setminus X$ .

**THEOREM 3.** — *A point  $y_0 \in \Delta$  is minimal with respect to  $\underline{\underline{H}}_1$  if and only if it is minimal with respect to  $\underline{\underline{H}}_2$ .*

*Proof.* — In the proof of theorem 2, it was shown that there is a continuous function  $f$  with  $K_y^1 - P_A K_y^1 = f(y) [K_y^2 - P_A K_y^2]$  for all  $y \notin A$ .

From the proof of theorem 1, it follows that if a net  $(y_\gamma)$  on  $X$  converges to  $\bar{y} \in \Delta$  then  $\lim_\gamma f(y_\gamma) = f(\bar{y})$  exists. Hence, if

$$K_{\bar{y}}^i(x) = \lim_\gamma {}^*K_x^i(y_\gamma),$$

it follows that  $K_{\bar{y}}^1 - P_A K_{\bar{y}}^1 = f(\bar{y}) [K_{\bar{y}}^2 - P_A K_{\bar{y}}^2]$ .

Since  $P_A = H_{\mathcal{C}_A}$ , this shows that  $J(K_{\bar{y}}^1) = f(\bar{y}) K_{\bar{y}}^2$  where  $J$  is the isomorphism of proposition 4. As  $J$  is a positive operator  $K_{\bar{y}}^1$  is minimal if and only if  $K_{\bar{y}}^2$  is minimal.

Denote by  $\Lambda_i = (l_i = 1)$  compact bases of  $\underline{S}_i^+$  for  $i = 1, 2$  where  $l_1$  and  $l_2$  are positive continuous linear functionals. Then theorem 2 and proposition 3 imply that there is a unique homeomorphism  $\varphi : \overline{\mathcal{G}(\Lambda_1)} \longrightarrow \overline{\mathcal{G}(\Lambda_2)}$  with  $\varphi(p_y^1) = p_y^2$  for all  $y \in X$  ( $p_y^i$  being the potential with support  $\{y\}$  in  $\Lambda_i$ ). The fact, proved in theorem 3, that  $J(K_{\bar{y}}^1) = f(\bar{y}) K_{\bar{y}}^2$  implies the following result.

LEMMA. — If  $h_1 \in \overline{\mathcal{G}(\Lambda_1)} \cap \underline{H}_1^+$ , then

$$\varphi(h_1) = [1/(l_2 \circ J)(h_1)] J(h_1).$$

*Proof.* — Let  $h_2 = \varphi(h_1)$  and for any positive harmonic function  $h$  set  $h^0 = [1/h(x_0)] h$ . Let  $K^i$  denote the kernel defined by  $(p_y^i)_{y \in X}$  and  $x_0$ . Then  $\varphi(h_1) = h_2$  if  $\lim_\gamma K_{y_\gamma}^1 = h_1^0$  implies  $\lim_\gamma K_{y_\gamma}^2 = h_2^0$ .

Hence, if  $\alpha = \lim_\gamma f(y_\gamma)$  then  $J(h_1^0) = \alpha h_2^0$ . Since  $h_i = [1/l_i(h_i^0)] h_i^0$  if  $h_i \in \Lambda_i$ , this implies that  $h_2 = [1/(l_2 \circ J)(h_1)] J(h_1)$ .

THEOREM 4. — Let  $\underline{H}_1$  and  $\underline{H}_2$  be equivalent sheaves on  $X$ . Then the topological cones  $\underline{S}_1^+$  and  $\underline{S}_2^+$  are isomorphic.

*Proof.* — According to a remark of Alfsen [1] (p. 120), there is a continuous affine map  $\Phi : \Lambda_1 \longrightarrow \Lambda_2$  extending  $\varphi$  providing  $\varphi$  has the following property : if  $\mu, \mu' \in \mathcal{M}_1^+(\mathcal{G}(\Lambda_1))$  have the same barycentre then the image measures  $\hat{\varphi}_\mu = \nu$  and  $\varphi_{\mu'} = \nu'$  have the same barycentre.

Let  $\nu_1 = p_1 + h_1$  be the barycentre of  $\mu$  and of  $\mu'$ . Then both measures coincide on  $\{p_y^1 | y \in X\}$ . Since  $\varphi$  maps this set on  $\{p_y^2 | y \in X\}$ , it follows that  $\nu$  and  $\nu'$  coincide on this set. By the lemma  $\varphi$  coincides on  $\overline{\mathcal{G}(\Lambda_1)} \cap \underline{H}_1^+$  with the composition of  $J$  with the affine map

$$h_2 \longrightarrow [1/l_2(h_2)] h_2 .$$

Hence, it follows that if  $\nu_2 = p_2 + h_2$  and  $\nu'_2 = p'_2 + h'_2$  are the barycentres of  $\nu$  and  $\nu'$ , then  $p_2 = p'_2$  and  $h_2 = h'_2$ .

Extending  $\Phi$  to  $\underline{S}_1^+$  by setting  $\Phi(\nu) = l_1(\nu) \cdot \Phi([1/l_1(\nu)] \nu)$ , gives the required isomorphism.

*Remark.* — This theorem implies theorems 2 and 3. It would therefore be desirable to have a direct proof of this result.

## 7. The Martin Compactification of $X \setminus A$ .

Let  $\underline{H}$  be a sheaf on  $X$  satisfying the hypotheses of section 4 for which 1 is superharmonic. Let  $A \subset X$  be a compact set and let  $O$  denote a connected component of  $X \setminus A$  which is not relatively compact.

Pick  $x_0 \in O$  and let  $K(x, y)$  be the kernel obtained by normalizing the potentials  $p_y$ . Then if  $q_y = p_y - P_A p_y$ , the kernel  $K^O(x, y)$  on  $O \times O$  defined by normalizing the potentials  $q_y$  equals

$f(y) [K_y(x) - P_A K_y(x)]$ , with  $f(y) = p_y(x_0)/q_y(x_0)$  if  $q_y(x_0)$  is finite and 1 if  $q_y(x_0)$  equals  $+\infty$ .

Let  $\bar{X} = X \cup \Delta$  denote the Martin compactification of  $X$  and let  $\bar{O}$  denote the closure of  $O$  in  $\bar{X}$ .

LEMMA 1. — *The functions  $*K_x^O$ ,  $x \in O$ , extend continuously to  $\bar{O} \setminus A$  and separate the points of  $\bar{O} \cap \Delta$ .*

*Proof.* — Since  $X$  is locally connected  $[\partial \bar{O}] \setminus A \subset \Delta$ . Let  $(y_\gamma)$  be a net on  $O$  which converges in  $\bar{X}$  to  $y \in \Delta$ . The functions

$$[1/f(y_\gamma)] K_{y_\gamma}^O$$

converge on  $O$  to the harmonic function  $h - P_A h$ , where  $h$  corresponds to  $y$ .

Let  $(y_{\gamma'})$  and  $(y_{\gamma''})$  be two subnets for which  $h' = \lim_{\gamma'} K_{y_{\gamma'}}^O$  and  $h'' = \lim_{\gamma''} K_{y_{\gamma''}}^O$  exist. Then  $\alpha' = \lim_{\gamma'} f(y_{\gamma'})$  and  $\alpha'' = \lim_{\gamma''} f(y_{\gamma''})$

both exist and are non zero. Now  $\alpha''h' = \alpha'h''$  and as

$$h'(x_0) = h''(x_0) = 1,$$

it follows that  $h' = h''$ . Hence,  $\lim_{\gamma} *K_x^O(y_\gamma)$  exists for each  $x \in O$ .

Let  $y, y'$  be two points of  $\bar{O} \cap \Delta$  and let  $h, h'$  be the corresponding harmonic functions on  $X$ . The function  $f$  extends continuously to  $\bar{O} \setminus A$  as do the functions  $*K_x^O$ . If the extensions of these functions do not distinguish  $y$  from  $y'$ , then

$$f(y) [h - P_A h] = f(y') [h' - P_A h'].$$

Consequently  $f(y) h = f(y') h'$  and so  $f(y) = f(y')$ . As a result,  $h = h'$  and so  $y = y'$ .

Denote by  $*\bar{K}_x^O$  the extension of  $*K_x^O$  to  $\bar{O} \setminus A$  and let  $Y$  be the compactification of  $O$  obtained by compactifying  $\bar{O} \setminus A$  with respect to  $(*\bar{K}_x^O)_{x \in O}$ .

LEMMA 2. —  $Y$  is the Martin compactification of  $O$ .

*Proof.* — The Martin compactification of  $O$  is the one defined by  $(*K_x^O)_{x \in O}$ . Since these functions clearly extend continuously to  $Y$ , it suffices to show that if  $y \in \bar{O} \cap \Delta$  and  $y' \in Y \setminus [\bar{O} \setminus A]$  their extensions distinguish these two points.

Assume this is false. Then there are nets  $(y_\gamma)$  and  $(y_{\gamma'})$  on  $O$  such that  $y = \lim_{\gamma} y_\gamma$  and  $y' = \lim_{\gamma'} y_{\gamma'}$ , with  $h^O = \lim_{\gamma} K_{y_\gamma}^O = \lim_{\gamma'} K_{y_{\gamma'}}^O$ , a harmonic function on  $O$ .

Because  $y \in \Delta$ ,  $y = \lim_{\gamma} y_\gamma$  in  $\bar{X}$  and so, if  $h$  is the corresponding harmonic function on  $X$ ,  $f(y) [h - P_A h] = h^O$  on  $O$ . Further since  $h = \lim_{\gamma} K_{y_\gamma}$ ,  $h - P_A h = 0$  on  $\bar{CO}$ . Adopting the convention of extending all functions on  $O$  to  $X \setminus A$  by defining them to be zero off  $O$ , it follows that  $f(y) [h - P_A h] = h^O$  on  $X \setminus A$ . Hence,

$$f(y) h = E(h^O),$$

$E$  the operator of proposition 5.

Since  $\Delta$  is a compact subset of  $Y$  it follows that the limit points in  $O$  of all the convergent subnets of  $(y_{\gamma'})$  lie on  $\partial A$ . Let  $(y_{\gamma''})$  be such a net. Let  $B$  be a compact neighbourhood of  $A$  and let  $E$  :

$\underline{H}(X \setminus B) \longrightarrow \underline{H}(X)$  be the operator of proposition 5. Then, viewing all functions as defined on  $X \setminus B$ ,

$$\begin{aligned} E(h^0) &= \lim_{\gamma''} E(K_{y_{\gamma''}}^0) = \lim_{\gamma''} E(f(y_{\gamma''}) [K_{y_{\gamma''}} - P_A K_{y_{\gamma''}}]) = \\ &= \lim_{\gamma''} E(f(y_{\gamma''}) K_{y_{\gamma''}}) . \end{aligned}$$

However, if  $y_{\gamma''}$  is close enough to  $A$ ,  $P_B K_{y_{\gamma''}} = K_{y_{\gamma''}}$  and so  $E(K_{y_{\gamma''}}) = 0$ . Therefore,  $h = 0$ , which is a contradiction.

This completes the proof of the first part of the following result.

**THEOREM 5.** —  $\bar{O} \setminus A$  is an open subspace of the Martin compactification of  $O$ . Further, a point  $y \in \bar{O} \cap \Delta$  is minimal if and only if it is minimal as a point of the Martin boundary of  $O$ .

*Proof.* — The proof of the second assertion uses the following lemma,  $E$  being the operator of proposition 5.

**LEMMA 3.** — Let  $h' \geq 0$  be harmonic on  $X \setminus A$  and such that  $P_B h'$  decreases to zero as  $B$ , a compact neighbourhood of  $A$ , decreases to  $A$ . Then if  $h = E(h')$ ,  $h - P_A h = h'$ .

Assuming the lemma 3 let  $y \in \bar{O} \cap \Delta$ . It corresponds to a positive harmonic function  $h$  on  $X$ . In the identification of  $\bar{O} \cap \Delta$  with a subset of the Martin boundary of  $O$ , the function  $h$  is replaced by

$$f(y) [h - P_A h] .$$

The second assertion states that  $h$  is minimal on  $X$  if and only if  $h - P_A h$  is minimal on  $O$ .

Assume  $h$  is minimal on  $X$  and let  $h - P_A h \geq h' \geq 0$  where  $h'$  is harmonic on  $O$ . If  $h_1 = E(h')$ , viewing  $h'$  as extended by zero to  $X \setminus A$ , there exists  $\lambda$ ,  $0 \leq \lambda \leq 1$ , with  $h_1 = \lambda h$ . Hence,

$$h_1 - P_A h_1 = \lambda (h - P_A h) .$$

If  $B \supset A$  is compact,  $P_B (h - P_A h) = P_B h - P_A h$  and so  $P_B h'$  decreases to zero as  $B \downarrow A$ . Lemma 3 implies that  $h_1 - P_A h_1 = h'$ .

Assume now that  $h - P_A h = h'$  is minimal on  $O$  and that  $h \geq h_1 \geq 0$ . Then if  $h_1 - P_A h_1 = h'_1$ ,  $0 \leq h'_1 \leq h'$  and so  $h'_1 = \lambda h'$ ,  $0 \leq \lambda \leq 1$ . As a result,  $h_1 = E(h'_1) = \lambda E(h') = \lambda h$ .



To prove the lemma, note that for any compact neighbourhood  $B$  of  $A$ ,  $h' - h = P_B h' - P_B h$  (this follows from the definition of  $E$ ). Since  $P_B h$  decreases to  $P_A h$ , it follows that  $h' = h - P_A h$ .

**COROLLARY 1.** — *If  $X \setminus A$  is connected and  $\bar{X}$  is the Martin compactification of  $X$ , then  $\bar{X} \setminus A$  is an open subspace of the Martin compactification of  $X \setminus A$ . Further, a point  $y \in \Delta = \bar{X} \setminus X$  is minimal if and only if it is minimal as a point of the Martin boundary of  $X \setminus A$ .*

**COROLLARY 2.** — *Let  $O, O'$  be two connected components of  $X \setminus A$  that are not relatively compact. Then  $\bar{O} \cap \bar{O}' \cap \Delta = \emptyset$ .*

*Proof.* — If  $h$  corresponds to  $y \in \bar{O} \cap \bar{O}' \cap \Delta$ , then  $h - P_A h = 0$ . To see this note that  $h - P_A h$  is a limit of functions of the form  $K_y - P_A K_y$ ,  $y \in O$  (respectively,  $y \in O'$ ) which vanish on  $\bar{CO}$  (respectively,  $\bar{CO}'$ ).

The following lemma together with the above corollary imply that the Martin compactification is of type  $S$  (See [6] p. 99).

**LEMMA 4.** — *Let  $A \subset X$  be compact. Then  $X \setminus A$  has only a finite number of connected components which are not relatively compact. Further, if  $y \in \Delta$  there is a unique component  $O$  of this type with  $y \in \bar{O}$ .*

*Proof.* — Let  $U$  be a relatively compact open set containing  $A$ . Cover  $\partial U$  with a finite number  $U_1, \dots, U_n$  of connected open sets with  $\bar{U}_i \cap A = \emptyset$ .

Since  $X$  is connected, for any connected component  $O$  of  $X \setminus A$ ,  $\bar{O} \cap \partial A \neq \emptyset$ . If in addition  $O \cap [X \setminus U] \neq \emptyset$ , then  $O$  meets some  $U_i$ . Consequently, at most  $n$  connected components of  $X \setminus A$  meet  $X \setminus \bar{U}$ . The uniqueness in the last statement follows from corollary 2 to theorem 5.

**COROLLARY 3.** — *Let  $\Sigma$  be the topological sum of the Martin compactification of the connected components of  $X \setminus A$ . Then  $\bar{X} \setminus A$  is an open subspace of  $\Sigma$ .*

*Proof.* — It follows from corollary 2 and lemma 4 that the connected components of  $\bar{X} \setminus A$  are the sets  $\bar{O} \setminus A$ , where  $O$  is a connected component of  $X \setminus A$ .

In view of theorem 5, each set  $\bar{O} \setminus A$  can be identified with an open subspace of the Martin compactification of  $O$  (trivially if  $\bar{O} \subset X$ ).

COROLLARY 4. — *The Martin compactification of  $\bar{X}$  is of type S.*

*Proof.* — If  $f$  is a continuous function on  $X$  such that for some compact set  $A \subset X$ ,  $f$  is constant on the connected components of  $X \setminus A$ , then clearly  $f$  extends continuously to  $\bar{X}$ . The result follows from Satz 9.1 of [6].

COROLLARY 5. — *For  $y \in \Delta$  let  $O(A, y)$  be the unique connected component  $O$  of  $X \setminus A$  with  $y \in \bar{O}$ . Denote by  $\Gamma(y)$  the intersection of the sets  $\overline{O(A, y)}$  as  $A$  runs over the compact subsets of  $X$ . Then  $\Gamma(y)$  is the connected component of  $y$  in  $\Delta$ .*

*Proof.* — The sets  $\Gamma(y)$ ,  $y \in \Delta$ , are connected and pairwise disjoint. Corollary 2 to theorem 5 implies that for each  $y \in \Delta$   $\Gamma(y)$  contains the connected component of  $y$  in  $\Delta$ .

COROLLARY 6. — *The cardinal number of the set of connected components of  $\Delta$  is at most  $2^{s/s_0}$*

*Remarks.* — 1) The requirement that  $1$  be superharmonic is no restriction since for an arbitrary sheaf  $\underline{H}$  satisfying the hypotheses of section 4, there exists a positive continuous superharmonic function.

2) In [6] corollary 4 was proved for hyperbolic Riemann surfaces. The proof there depends on a description of the Martin boundary which does not apply in general. For it to hold, the sheaf  $\underline{H}$  has to have an adjoint (see [16]).

3) Corollary 6 holds without the assumption of a countable base (as do the other results) in view of the result of Cornea [8] which ensures that  $X$  is  $\sigma$ -compact.

### 8. Direct Decomposition and the Ends of X.

The points of the compactification of X determined by the functions defined in corollary 4 of theorem 5 are often called the *ends* of the space X. As is pointed out in [6], since the Martin compactification is of type S, there is a one-one correspondence between the ends of X and the connected components of  $\Delta$ .

Denote by C the cone of positive harmonic functions on X. Let  $(h_i)_{i \in I}$  be a family of elements of C. Assume that

$$\left\{ \sum_{i \in F} h_i \mid F \subset I \text{ finite} \right\}$$

is bounded above in C. Then the supremum of this family of finite sums will be defined to be  $\sum_{i \in I} h_i$ . Using this concept of infinite

sum, the cone C is said to be the *direct sum of the family*  $(C_i)_{i \in I}$  of convex subcones  $C_i$  of C if for each  $h \in C$  there is a unique family  $(h_i)_{i \in I}$  with  $h_i \in C_i$ ,  $\forall i \in I$  and  $h = \sum_{i \in I} h_i$ .

A convex subcone  $C_1$  of C will be said to be a *direct summand* of C if there is a convex subcone  $C_2$  with C the direct sum  $C_1 \oplus C_2$  of  $C_1$  and  $C_2$ . If C is the direct sum of  $(C_i)_{i \in I}$  then each  $C_i$  is a direct summand.

If  $C = C_1 \oplus C_2$  and both  $C_1, C_2$  are closed in the topology of uniform convergence on compact sets then C will be called a *topological direct sum*. In this case  $C_1, C_2$  will be called *topological direct summands*.

The purpose of this section is to discuss the relationship between direct sums and the ends of X. The basic result is the following theorem.

**THEOREM 6.** — *Let  $A \subset X$  be compact and let  $O_1, \dots, O_n$  be the connected components of  $X \setminus A$  which are not relatively compact. Set  $D_i = \Delta \cap \overline{O_i}$  and let  $C_i$  denote the set of all positive harmonic functions on X represented by measures whose support lies in  $D_i$ . Then  $C_i$  is a topological direct summand of C and  $C = C_1 \oplus \dots \oplus C_n$ .*

*Proof.* — Since  $\Delta = \bigcup_{i=1}^n D_i$  it follows that if  $h \in C$  then  $h = \sum_{i=1}^n h_i$  with  $h_i \in C_i$ , which is clearly a closed convex subcone of  $C$ .

Let  $h \in C$  be represented by two measures  $\mu$  and  $\mu'$  and let  $\mu_i = \mu|D_i$ ,  $\mu'_i = \mu'|D_i$ . Then  $\mu_i$  and  $\mu'_i$  represent the same harmonic function. Consider  $h = \sum_{i=1}^n h_i = \sum_{i=1}^n h'_i$  where  $h_i$  corresponds to  $\mu_i$  and  $h'_i$  to  $\mu'_i$  (i.e.  $h_i(x) = \int_{D_i} K_y(x) d\mu(y)$  etc.). Then

$$h - P_A h = \sum_{i=1}^n (h_i - P_A h_i) = \sum_{i=1}^n (h'_i - P_A h'_i).$$

Since  $h_i - P_A h_i$  and  $h'_i - P_A h'_i$  both equal  $h - P_A h$  on  $O_i$  and zero on  $\bar{C}O_i$  (this follows because  $K_y - P_A K_y$  vanishes on  $\bar{C}O_i$  if  $y \in D_i$ ) as a result  $h_i - P_A h_i = h'_i - P_A h'_i$ . Hence,  $h_i = h'_i$ .

Consequently,  $C = C_1 \oplus \dots \oplus C_n$ . Since  $C_2 \oplus \dots \oplus C_n$  is closed, it follows that  $C_1$  (and similarly each  $C_i$ ) is a topological direct summand.

**COROLLARY 1.** — *Let  $D \subset \Delta$  be closed and a union of connected components. Let  $\mu, \mu'$  be two measures on  $\Delta$  that represent the same positive harmonic function. Then  $\mu|D$  and  $\mu'|D$  represent the same harmonic function.*

*Proof.* — From the theorem it follows that this is true if  $D$  is a finite union of sets of the form  $D_i$ .

Denote by  $D(A)$ ,  $A \subset X$  compact, the union of the sets  $D_i$  (defined in the theorem) that meet  $D$ . Then if  $(A_n)_n$  is an increasing sequence of compact sets with  $X = \bigcup A_n$ ,  $D = \bigcap D(A_n)$  and

$$D(A_n) \supset D(A_{n+1})$$

for all  $n$ . Let  $\mu_n = \mu|D(A_n)$ . Then  $\mu_n$  converges weakly to  $\mu|D$ . Since the same is true for  $\mu'$ , the result follows.

**COROLLARY 2.** — *Let  $\Gamma$  be a connected component of  $\Delta$  and let  $C_\Gamma$  be the set of all positive harmonic functions represented by measures supported by  $\Gamma$ . Then  $C$  is the direct sum of the cones  $C_\Gamma$ ,  $\Gamma$  running through the collection of connected components of  $\Delta$ .*

*Proof.* — Using the notation of the previous corollary,

$$\Gamma = \cap \Gamma(A_n).$$

Hence, by corollary 1 if  $h \in C$  is represented by  $\mu$  and  $\mu'$ , then  $\mu|_{\Gamma}$  and  $\mu'|_{\Gamma}$  represent the same function  $h_{\Gamma}$ .

Clearly,  $h = \sum_{\Gamma} h_{\Gamma}$  and as this representation is unique  $C$  is the direct sum of the cones  $C_{\Gamma}$ .

**COROLLARY 3.** — *Each connected component  $\Gamma$  of  $\Delta$  contains minimal points.*

*Proof.* — If  $h \in C_{\Gamma}$  and  $\mu$  represents  $h$  it follows from the uniqueness of its representation that  $\mu$  is supported by  $\Gamma$ . Let  $y \in \Gamma$  and let  $h$  be the corresponding harmonic function. Since it is represented by a canonical measure  $\mu$  carried by  $\Delta_1$  the result follows.

The cones  $C_{\Gamma}$ , corresponding to the connected components of  $\Delta$  are in some sense “canonical”. The following result leads to a characterization of the  $C_{\Gamma}$  in terms of the cone  $C$  when  $\bar{\Delta}_1 = \Delta$ .

**PROPOSITION 7.** — *Let  $C$  be the topological direct sum of  $C_1$  and  $C_2$ . Then there are disjoint closed sets  $D_i$  with  $D_1 \cup D_2 = \Delta$  and  $C_i$  equal to the cone  $C_{D_i}$  of positive harmonic functions represented by measures carried by  $D_i$  providing  $\Delta_1$  is dense in  $\Delta$ .*

*Conversely, if  $\Delta = D_1 \cup D_2$  with  $D_1$  and  $D_2$  disjoint closed sets then  $C$  is the topological direct sum of  $C_{D_1}$  and  $C_{D_2}$ .*

*Proof.* — The converse follows from corollary 1 to theorem 6.

Viewing  $\Delta$  as a subset of  $C$  let  $D_i = C_i \cap \bar{\Delta}_1$ . The sets  $D_1, D_2$  are disjoint, closed and since their union contains  $\Delta_1, D_1 \cup D_2 = \bar{\Delta}_1 = \Delta$ . Hence,  $C = C_{D_1} \oplus C_{D_2}$ .

Let  $C'_i$  be the cone of functions whose canonical measure is carried by  $D_i$ . Then  $C = C'_1 \oplus C'_2$  and as  $C'_i \subset C_{D_i}$ ,  $C'_i = C_{D_i}$  for  $i = 1, 2$ . Further, since  $C_i$  is closed it follows that  $C'_i \subset C_i$ . As a result,  $C_i = C_{D_i}$  for  $i = 1, 2$ .

In view of this proposition, it follows that the cones  $C_{\Gamma}$  have the following property : if  $C$  is the topological direct sum of  $C_1$

and  $C_2$  then  $C_\Gamma \subset C_1$  or  $C_\Gamma \subset C_2$ . Subcones of  $C$  with this property will be called *compatible*.

**THEOREM 7.** — Assume  $\bar{\Delta}_1 = \Delta$ . Then the cones  $C_\Gamma$  are the maximal compatible subcones of  $C$ . If  $C_\Gamma$  is a topological direct summand it is a minimal one.

Further, the following statements are equivalent :

- 1)  $X$  has a finite number of ends ;
- 2)  $\Delta$  has a finite number of connected components ;
- 3)  $C$  has only a finite number of topological direct summands ;
- 4) each  $C_\Gamma$  is a topological direct summand.

In this case,  $C$  is the topological direct sum of the cones  $C_\Gamma$ .

*Proof.* — It follows from proposition 7 that if a cone  $C_\Gamma$  is a topological direct summand, then it is minimal.

Let  $C_0$  be a compatible subcone of  $C$ . If  $A \subset X$  is compact then there is a unique non relatively compact connected component  $O$  of  $X \setminus A$  with  $C_0 \subset C_D$ , where  $D = \bar{O} \cap \Delta$  and  $C_D$  is defined as in proposition 7. Hence if  $\Gamma$  is the intersection of these sets  $D$ ,  $C_0 \subset C_\Gamma$ .

It is clear that 1), 2) and 3) are equivalent and that 2) implies 4). Assume  $\Delta$  has an infinite number of connected components  $(\Gamma_\alpha)_{\alpha \in I}$ . For each  $\alpha$  let  $y_\alpha \in \Gamma_\alpha$  and let  $y$  be a limit point of  $\{y_\alpha | \alpha \in I\}$ . Assume that  $\Gamma_{\alpha_0}$  is the connected component of  $y$ . Since  $C_{\Gamma_{\alpha_0}}$  is a topological direct summand  $\sum_{\alpha \neq \alpha_0} C_{\Gamma_\alpha} = C_1$  is a closed subcone.

This contradicts the fact that  $y$  is a limit point since the harmonic functions  $h_\alpha$  corresponding to  $y_\alpha$  lie in  $C_1$  if  $\alpha \neq \alpha_0$ .

When  $\Delta$  has a finite number of connected components then, for some compact  $A \subset X$ , the connected components of  $\Delta$  coincide with the sets  $\bar{O} \cap \Delta$ ,  $O$  a non relatively compact connected component of  $X \setminus A$ . Consequently, the last statement follows from theorem 6.

*Example.* — Let  $B$  be an open ball in  $\mathbb{R}^n (n \geq 2)$  and let  $D \subset B$  be a sequence of points all of whose limit points lie outside  $B$ .

Set  $X = B \setminus D$ . The Martin boundary of  $X$  equals  $D \cup S$ ,  $S$  the sphere bounding  $B$ . Each  $x \in D$  is a connected component of  $\Delta$  whose corresponding subcone is a topological direct summand. The cone corresponding to the component  $S$  does not have this property.

*Remarks.* — In [4] Constantinescu and Cornea show that for any integer  $n > 0$ , there exists a hyperbolic Riemann surface  $X$  with one end and  $\Delta_1$  a set of  $n$  points. Since  $\Delta$  is connected

$$\overline{\Delta}_1 = \Delta_1 \neq \Delta.$$

Furthermore, for these examples theorem 7 clearly breaks down.

It would be of interest to have sufficient conditions that ensure  $\overline{\Delta}_1 = \Delta$ .

## 9. Appendix.

Let  $X$  and  $\underline{H}$  be as in section 4 without the assumption of a countable base. Denote by  $\underline{S}^+$  the cone of positive superharmonic functions equipped with the  $\underline{T}$ -topology. Then  $\underline{S}^+$  has a compact base  $\Lambda$  by corollary 3.2 of [7].

Let  $E^+$  denote the set of superharmonic functions that are either harmonic or potentials with point support.

LEMMA 5. —  $E^+ \cap \Lambda$  is compact.

*Proof.* — Let  $\mathcal{U}$  be an ultrafilter on  $E^+ \cap \Lambda$  and denote by  $s_u$  the function defined in [7] (p. 1335). Define  $\varphi: E^+ \longrightarrow X \cup \{a\}$ ,  $a$  the Alexandroff point at infinity by setting  $\varphi(s) = a$  if  $s$  is harmonic and  $\varphi(s) = y$  if the support of  $s$  is  $\{y\}$ .

The image ultrafilter  $\varphi\mathcal{U}$  converges to  $a$  or to a point  $y \in X$ . In the first case  $s_u$  is harmonic, and in the second it has its support contained in  $\{y\}$  (see theorem 2.1 of [7]).

Since in theorem 3.1 of [7] it is shown that  $\mathcal{U}$  converges to  $s_u$  in the  $T$ -topology, the result follows.

It was proved in [6] (theorem 5) that for each  $y \in X$  there is

a potential with support  $\{y\}$ , and so there is a unique potential  $p_y$  in  $\Lambda$  with this property.

PROPOSITION 8. — *The mapping  $y \longrightarrow p_y$  embeds  $X$  in  $\Lambda$ . Further the function  $G(x, y) = p_y(x)$  is a lower semicontinuous function on  $X \times X$ , continuous off the diagonal.*

*Proof.* — The proofs of propositions 18.1 and 19.1 of [10] do not use the second axiom of countability.

Lemma 5 therefore allows the argument of proposition 22.1 of [10] to apply without change.

It remains to note that proposition 19.1 of [10] implies that  $p_y \longrightarrow y$  is a continuous function.

*Remark.* — From what has been said above, proposition 3 holds without the second axiom of countability.

## BIBLIOGRAPHY

- [1] E.M. ALFSEN, Boundary values for homeomorphisms of compact convex sets, *Math. Scand.* 19 (1966), 113-121.
- [2] H. BAUER, Harmonische Räume und ihre Potentialtheorie, Springer Lecture Notes 22, Berlin 1966.
- [3] M. BRELOT, Lectures on potential theory, Tata Inst. of Fund. Res., Bombay 1960.
- [4] C. CONSTANTINESCU and A. CORNEA, Über den idealen Rand und einige seiner Anwendung bei der Klassifikation der Riemannschen Flächen, *Nagoya Mathematical Journal* 13 (1958), 169-233.
- [5] C. CONSTANTINESCU and A. CORNEA, Ideale Ränder Riemannscher Flächen, Springer-Verlag, Berlin 1963.
- [6] C. CONSTANTINESCU and A. CORNEA, On the axiomatic of harmonic functions (I), *Ann. Inst. Fourier* 13 (1963), 373-388.



- [7] C. CONSTANTINESCU, A topology on the cone of non-negative superharmonic functions, *Rev. Roum. Math. Pures et Appl.* 10 (1965), 1331-1348.
- [8] A. CORNEA, Sur la dénombrabilité à l'infini d'un espace harmonique de Brelot, *C.R. Acad. Sci. Paris* 264 (1967), 190-191.
- [9] K. GOWRISANKARAN, Extreme harmonic functions and boundary value problems, *Ann. Inst. Fourier* 13 (2) (1963), 307-356.
- [10] R.M. HERVE, Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel, *Ann. Inst. Fourier* 12 (1962), 415-571.
- [11] P. LOEB, An axiomatic treatment of pairs of elliptic differential equations, *Ann. Inst. Fourier* 16 (2) (1966), 167-208.
- [12] J. KÖHN and M. SIEVEKING, Zum Cauchyschen und Dirichletschen Problem, *Math. Ann.* 177 (1968), 133-142.
- [13] R.S. MARTIN, Minimal positive harmonic functions, *Trans. Amer. Math. Soc.* 49 (1941), 137-172.
- [14] B. RODIN and L. SARIO, Principal functions, Van Nostrand, Princeton 1968.
- [15] M. SIEVEKING, Integraldarstellung superharmonischer Funktionen mit anwendung auf parabolische Differentialgleichungen, Seminar über Potential theorie, Springer Lecture Notes 69, Berlin 1968.
- [16] J.C. TAYLOR, The Martin boundary and adjoint harmonic functions, publ. in Contributions to extension theory of topological structures, V E B Deutscher Verlag, Berlin 1969.
- [17] Ch. de la VALLEE POUSSIN, Propriétés des fonctions harmoniques dans un domaine ouvert limité par des surfaces à courbure bornée, *Ann. Ec. Norm. de Pise* 2 (1933), 167-197.

Manuscrit reçu le 8 décembre 1969

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