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Probabilistic local Cauchy theory of the cubic nonlinear wave equation in negative Sobolev spaces

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PROBABILISTIC LOCAL CAUCHY THEORY OF THE CUBIC NONLINEAR WAVE EQUATION IN NEGATIVE SOBOLEV SPACES

**by Tadahiro OH,
Oana POCOVNICU & Nikolay TZVETKOV (*)**

Dedicated to the memory of Professor Ioan I. Vrabie (1951–2017)

Abstract. — We study the three-dimensional cubic nonlinear wave equation (NLW) with random initial data below $L^2(\mathbb{T}^3)$. By considering the second order expansion in terms of the random linear solution, we prove almost sure local well-posedness of the renormalized NLW in negative Sobolev spaces. We also prove a new instability result for the defocusing cubic NLW without renormalization in negative Sobolev spaces, which is in the spirit of the so-called triviality in the study of stochastic partial differential equations. More precisely, by studying (un-renormalized) NLW with given smooth deterministic initial data plus a certain truncated random initial data, we show that, as the truncation is removed, the solutions converge to 0 in the distributional sense for *any* deterministic initial data.

Résumé. — On étudie l'équation des ondes non linéaire cubique (NLW) en dimension 3 avec une donnée initiale aléatoire en-dessous de $L^2(\mathbb{T}^3)$. En considérant le développement d'ordre 2 en termes de la solution aléatoire linéaire, on prouve le caractère presque sûrement localement bien posé de NLW renormalisée dans les espaces de Sobolev d'indices négatifs. On montre aussi un nouveau résultat d'instabilité pour l'équation NLW cubique défocalisante sans renormalisation dans les espaces de Sobolev d'indices négatifs, dans l'esprit du caractère non trivial dans l'étude des équations aux dérivées partielles stochastiques. Plus précisément, en étudiant NLW non renormalisée avec des données initiales régulières déterministes plus une donnée initiale aléatoire tronquée, on montre que, dès que la troncature est supprimée, les solutions tendent vers 0 au sens des distributions pour toute donnée initiale déterministe.

Keywords: nonlinear wave equation, Gaussian measure, local well-posedness, renormalization, triviality.
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1. Introduction

1.1. Main result

We consider the Cauchy problem for the defocusing cubic nonlinear wave equation (NLW) on the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$:

$$\begin{cases} \partial_t^2 u - \Delta u + u^3 = 0 \\ (u, \partial_t u)\,|_{t=0} = (u_0, u_1) \in \mathcal{H}^s\left(\mathbb{T}^3\right), \end{cases} \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ and $\mathcal{H}^s(\mathbb{T}^3) = H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$. Here, $H^s(\mathbb{T}^3)$ denotes the standard Sobolev space on $\mathbb{T}^3$ endowed with the norm:

$$\|f\|_{H^s(\mathbb{T}^3)} = \left\| \langle n \rangle^s \widehat{f}(n) \right\|_{\ell^2(\mathbb{Z}^3)},$$

where $\widehat{u}(n)$ is the Fourier coefficient of u and $\langle \cdot \rangle = (1+|\cdot|^2)^{\frac{1}{2}}$. The classical well-posedness result (see for example [53]) for (1.1) reads as follows.

Theorem 1. — *Let $s \geqslant 1$. Then, for every $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$, there exists a unique global-in-time solution u to* (1.1) *in $C(\mathbb{R}; H^s(\mathbb{T}^3))$. Moreover, the dependence of the solution map: $(u_0, u_1) \mapsto u(t)$ on initial data and time $t \in \mathbb{R}$ is continuous.*

The proof of Theorem 1 follows from Sobolev's inequality: $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$ and the conservation of the energy for (1.1). Recall that the scaling symmetry: $u(t,x) \mapsto \lambda u(\lambda t, \lambda x)$ for (1.1) posed on $\mathbb{R}^3$ induces the scaling-critical Sobolev regularity $s_{\mathrm{crit}} = \frac{1}{2}$. By using the Strichartz estimates (see Lemma 2.4 below), one may indeed show that the Cauchy problem (1.1) remains locally well-posed in $\mathcal{H}^s(\mathbb{T}^3)$ for $s \geqslant \frac{1}{2}$ [36]. On the other hand, it is known that the Cauchy problem (1.1) is ill-posed for $s < \frac{1}{2}$ [13, 18, 23, 45, 54]. We refer to [23, 45, 53] for the proofs of these facts.

One may then ask whether a sort of well-posedness of (1.1) survives below the scaling-critical regularity, i.e. for $s < \frac{1}{2}$. As it was shown in the work [13, 14] by Burq and the third author, the answer to this question is positive if one considers *random* initial data. In this paper, we will primarily consider the following random initial data:

$$u_0^\omega = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{in\cdot x} \quad \text{and} \quad u_1^\omega = \sum_{n \in \mathbb{Z}^3} \frac{h_n(\omega)}{\langle n \rangle^{\alpha-1}} e^{in\cdot x}, \tag{1.2}$$

where the series $\{g_n\}_{n \in \mathbb{Z}^3}$ and $\{h_n\}_{n \in \mathbb{Z}^3}$ are two families of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ conditioned that[(1)] $g_n = \overline{g_{-n}}$, $h_n = \overline{h_{-n}}$, $n \in \mathbb{Z}^3$. More precisely, with the notation $\mathbb{N} = \{1, 2, 3, \cdots\}$, we first define the index set Λ by

$$\Lambda = \big(\mathbb{Z}^2 \times \mathbb{N}\big) \cup (\mathbb{Z} \times \mathbb{N} \times \{0\}) \cup (\mathbb{N} \times \{(0,0)\}) \cup \{(0,0,0)\}. \tag{1.3}$$

We then define $\{g_n, h_n\}_{n \in \Lambda}$ to be a family of independent standard Gaussian random variables which are complex-valued for $n \neq 0$ and are real-valued for $n = 0$. We finally set $g_n = \overline{g_{-n}}$, $h_n = \overline{h_{-n}}$ for $n \in \mathbb{Z}^3 \setminus \Lambda$.

The partial sums for the series (u_0^ω, u_1^ω) in (1.2) form a Cauchy sequence in $L^2(\Omega; \mathcal{H}^s(\mathbb{T}^3))$ for every $s < \alpha - \frac{3}{2}$ and therefore the random initial data (u_0^ω, u_1^ω) in (1.2) belongs almost surely to $\mathcal{H}^s(\mathbb{T}^3)$ for the same range of s. On the other hand, one may show that the probability of the event $(u_0^\omega, u_1^\omega) \in \mathcal{H}^{\alpha - \frac{3}{2}}(\mathbb{T}^3)$ is zero. See [13, Lemma B.1]. As a result, when $\alpha > \frac{5}{2}$, one may apply the classical global well-posedness result in Theorem 1 for the random initial data (u_0^ω, u_1^ω) given by (1.2) since $(u_0^\omega, u_1^\omega) \in \mathcal{H}^1(\mathbb{T}^3)$ almost surely. For $\alpha > 2$, one may still apply the more refined (deterministic) local well-posedness result in $\mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$ mentioned above. For $\alpha \leqslant 2$, however, the

(1) In particular, g_0 and h_0 are real-valued.

Cauchy problem (1.1) becomes ill-posed. Despite this ill-posedness result, the analysis in [14, 53] implies the following statement.

Theorem 2. — *Let $\alpha > \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$. Let $\{u_N\}_{N\in\mathbb{N}}$ be a sequence of the smooth global solutions*(2) *to* (1.1) *with the following random C^∞-initial data:*

(1.4)
$$u^\omega_{0,N}(x) = \sum_{|n|\leqslant N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x} \quad \text{and} \quad u^\omega_{1,N}(x) = \sum_{|n|\leqslant N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x},$$

where $\{g_n\}_{n\in\mathbb{Z}^3}$ and $\{h_n\}_{n\in\mathbb{Z}^3}$ are as in (1.2)*. Then, as $N\to\infty$, u_N converges almost surely to a* (*unique*) *limit u in $C(\mathbb{R}; H^s(\mathbb{T}^3))$, satisfying NLW* (1.1) *in the distributional sense.*

Here, by uniqueness, we firstly mean that the entire sequence $\{u_N\}_{N\in\mathbb{N}}$ converges to u, not up to some subsequence. Compare this with the case of weak solution techniques (see for example [11, 12]), which usually only give convergence up to subsequences. Furthermore, when we regularize the random initial data (u_0^ω, u_1^ω) in (1.2) by mollification, it can be shown that the limit u is independent of the choice of mollification kernels. See Remark 1.1. Lastly, as we see in Subsection 1.3, the limit u admits a decomposition $u = z_1 + v$, where z_1 is the random linear solution, emanating from the random initial data (u_0^ω, u_1^ω), and v is the *unique* solution to the perturbed NLW:

$$\begin{cases} \mathcal{L}v + (v+z_1)^3 = 0 \\ (v, \partial_t v)\,|_{t=0} = (0,0), \end{cases}$$

Similar comments apply to the limiting distribution u in Theorem 3 below.

For $\alpha \leqslant \frac{3}{2}$, u_0^ω in (1.2) is almost surely no longer a classical function and it should be interpreted as a random Schwartz distribution lying in a Sobolev space of negative index. Therefore for $\alpha \leqslant \frac{3}{2}$, the study of (1.1) with the random initial data (1.2) is no longer within the scope of applicability of [14, 53]. The goal of this paper is to extend the results in [14, 53] to the random initial data when they are no longer classical functions. More precisely, we prove the following statement.

Theorem 3. — *Let $\frac{5}{4} < \alpha \leqslant \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$. There exists a divergent sequence $\{\alpha_N\}_{N\in\mathbb{N}}$ of positive numbers such that the following holds true; there exist small $T_0 > 0$ and positive constants C, c, κ such that for every $T \in (0, T_0]$, there exists a set Ω_T of complemental probability smaller*

(2) Theorem 1 guarantees existence of smooth global solutions $\{u_N\}_{N\in\mathbb{N}}$ to (1.1).

than $C\exp(-c/T^\kappa)$ such that if we denote by $\{u_N\}_{N\in\mathbb{N}}$ the smooth global solutions to

$$\begin{cases}\partial_t^2 u_N - \Delta u_N + u_N^3 - \alpha_N u_N = 0\\ (u_N, \partial_t u_N)\,|_{t=0} = \left(u^\omega_{0,\,N}, u^\omega_{1,\,N}\right),\end{cases} \tag{1.5}$$

where the random initial data $(u^\omega_{0,\,N}, u^\omega_{1,\,N})$ is given by the truncated Fourier series in (1.4)*, then for every $\omega \in \Omega_T$, the sequence $\{u_N\}_{N\in\mathbb{N}}$ converges to some (unique) limiting distribution u in $C([-T,T];H^s(\mathbb{T}^3))$ as $N\to\infty$.*

We prove Theorem 3 by writing u_N in the second order expansion: $u_N = z_{1,\,N} + z_{2,\,N} + w_N$, where $z_{1,\,N}$ is the linear solution,(3) emanating from the random initial data (u_0^ω, u_1^ω), and $z_{2,\,N}$ denotes the additional term appearing in the Picard second iterate; see (1.17) below. We first use stochastic analysis to show convergence of $z_{j,\,N}$, $j=1,2$, and then show convergence of the residual term w_N by deterministic analysis.

In view of the asymptotic behavior $\alpha_N \to \infty$, one may be tempted to say that the limiting distribution u obtained in Theorem 3 is a solution to the following limit "equation":

$$\begin{cases}\partial_t^2 u - \Delta u + u^3 - \infty\cdot u = 0\\ (u, \partial_t u)\,|_{t=0} = (u_0^\omega, u_1^\omega),\end{cases}$$

where the random initial data (u_0^ω, u_1^ω) is as in (1.2). The expression $\infty\cdot u$ is merely formal and thus a natural question is to understand in which sense u satisfies the cubic NLW on $\mathbb{T}^3$. As we see in the next subsection, the limit u has the decomposition $u = z_1 + z_2 + w$ (see (1.22)), where z_j, $j=1,2$, denotes the limit of $z_{j,N}$ in a suitable sense and the residual term w satisfies the perturbed NLW equation; see (1.23) below. The uniqueness statement in Theorem 3 refers to the uniqueness of w as a solution to this perturbed NLW equation (along with the uniqueness of various stochastic terms appearing in (1.23) as the limits of their regularized versions); see Remark 1.6. See also Remark 1.1 below. We will describe our strategy in the next two subsections. We also refer readers to [47] for a related discussion in the two-dimensional case.

Given fixed $N\in\mathbb{N}$, by adapting the classical argument, it is easy to see that the truncated equation (1.5) is globally well-posed in $\mathcal{H}^s(\mathbb{T}^3)$ for $s\geqslant 1$. In particular, one needs to apply a Gronwall-type argument to exclude a possible finite-time blowup of the $\mathcal{H}^1$-norm of a solution. The main issue

(3) For a technical reason, we take $z_{1,\,N}$ to satisfy the linear Klein–Gordon equation. See (1.8) below.

here is that there is no good *uniform* (in N) bound for the solutions to (1.5). One may try to extend the local-in-time solutions constructed in Theorem 3 globally in time by using truncated energies in the spirit of the I-method, introduced in [19]. See [29] for such a globalization argument in the context of the two-dimensional stochastic NLW.

Our ultimate goal is to push the analysis in the proof of Theorem 3 to cover the case $\alpha = 1$, corresponding to the regularity of the natural Gibbs measure associated with the cubic NLW. In the field of singular stochastic parabolic PDEs, there has been a significant progress in recent years. In particular, a substantial effort [3, 15, 26, 31, 32, 39] was made to give a proper meaning to the stochastic quantization equation (SQE) on $\mathbb{T}^3$, formally written as

$$\partial_t u - \Delta u = -u^3 + \infty \cdot u + \xi. \tag{1.6}$$

Here, ξ denotes the so-called space-time white noise. On the one hand, the randomization effects in the present paper are close in spirit to the works cited above. On the other hand, the deterministic part of the analysis in the context of the heat and the wave equations represent significant differences because, as it is well known, the deterministic regularity theories for these two types of equations are quite different. In fact, in order to extend Theorem 3 to lower values of α, it is crucial to exploit dispersion at a multilinear level, a consideration specific to dispersive equations, and combine it with randomization effects. See, for example, a recent work [28] by Gubinelli, Koch, and the first author on the three-dimensional stochastic NLW with a quadratic nonlinearity. Furthermore, in order to treat lower values of α, it will be crucial to impose a structure on the residual part w. See Remark 1.4 for a further discussion.

Remark 1.1. — We say that $\eta \in C(\mathbb{R}^3; [0,1])$ is a mollification kernel if $\int \eta dx = 1$ and $\operatorname{supp} \eta \subset (-\pi, \pi]^3 \simeq \mathbb{T}^3$. Given a mollification kernel η, define η_ε by setting $\eta_\varepsilon(x) = \varepsilon^{-3}\eta(\varepsilon^{-1}x)$. Then, $\{\eta_\varepsilon\}_{0 < \varepsilon \leqslant 1}$ forms an approximate identity on $\mathbb{T}^3$. By slightly modifying the proof of Theorem 2, we can show that if we denote by u_ε, the solution to (1.1) with the initial data $(\eta_\varepsilon * u_0^\omega, \eta_\varepsilon * u_1^\omega)$, where (u_0^ω, u_1^ω) is as in (1.2), then, for $\alpha > \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$, u_ε converges in probability to some (unique) limit u in $C(\mathbb{R}; H^s(\mathbb{T}^3))$ as $\varepsilon \to 0$. Here, the limit u is independent of the choice of mollification kernels η. Similarly, when $\frac{5}{4} < \alpha \leqslant \frac{3}{2}$, a slight modification of the proof of Theorem 3 shows that there exists a divergent sequence α_ε (as $\varepsilon \to 0$) such that the solution u_ε to

$$\begin{cases} \partial_t^2 u_\varepsilon - \Delta u_\varepsilon + u_\varepsilon^3 - \alpha_\varepsilon u_\varepsilon = 0 \\ (u_\varepsilon, \partial_t u_\varepsilon)\,|_{t=0} = (\eta_\varepsilon * u_0^\omega, \eta_\varepsilon * u_1^\omega) \end{cases}$$

converges in probability to some (unique) limit u in $C([-T_\omega, T_\omega]; H^s(\mathbb{T}^3))$, where $T_\omega > 0$ almost surely. Once again, the limit u_ε is independent of the choice of mollification kernels η.

Remark 1.2. — As in [13, 14], it is possible to consider a more general class of random initial data. Let a deterministic pair $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$ be given by the following Fourier series:

$$u_0 = \sum_{n \in \mathbb{Z}^3} a_n e^{in \cdot x} \quad \text{and} \quad u_1 = \sum_{n \in \mathbb{Z}^3} b_n e^{in \cdot x}$$

with the constraint $a_{-n} = \overline{a_n}$ and $b_{-n} = \overline{b_n}$, $n \in \mathbb{Z}^3$. We consider the randomized initial data (u_0^ω, u_1^ω) given by

$$u_0^\omega = \sum_{n \in \mathbb{Z}^3} g_n(\omega) a_n e^{in \cdot x} \quad \text{and} \quad u_1^\omega = \sum_{n \in \mathbb{Z}^3} h_n(\omega) b_n e^{in \cdot x},$$

Then, by slightly modifying the proof of Theorem 3, it is easy to see that, for $s > -\frac{1}{6}$ (corresponding to $\alpha > \frac{4}{3}$ in (1.2)), we can introduce a time dependent divergent sequence $\{\alpha_N\}_{N \in \mathbb{N}}$ with $\alpha_N = \alpha_N(t)$ such that the solution u_N to (1.5) converges to some (unique) limit u in $C([-T_\omega, T_\omega]; H^s(\mathbb{T}^3))$, where $T_\omega > 0$ almost surely. For this range of s, we need only the first order expansion. See the next subsection. For lower values of s, one may need to impose some additional summability assumptions on $\{a_n\}_{n \in \mathbb{Z}^3}$ and $\{b_n\}_{n \in \mathbb{Z}^3}$ (in particular to replicate the proof of Proposition 4.2 to obtain an analogue of Theorem 3).

1.2. Outline of the proof of Theorem 3.

In the following, we present the main idea of the proof of Theorem 3. Fix $\alpha \leqslant \frac{3}{2}$. With the short-hand notation:(4)

$$\mathcal{L} := \partial_t^2 - \Delta + 1, \tag{1.7}$$

we denote by $z_{1,N} = z_{1,N}(t, x, \omega)$ the solution to the following linear Klein–Gordon equation:

$$\mathcal{L} z_{1,N}(t, x, \omega) = 0 \tag{1.8}$$

(4) For our subsequent analysis, it will be more convenient to study the linear Klein–Gordon equation rather than the linear wave equation.

with the random initial data $(u^\omega_{0,N}, u^\omega_{1,N})$ given by the truncated Fourier series in (1.4). In the following, we discuss spatial regularities of various stochastic terms for fixed $t \in \mathbb{R}$. For simplicity of notation, we suppress the t-dependence and discuss spatial regularities. It is easy to see from (1.4) that $z_{1,N}$ converges almost surely to some limit z_1 in $H^{s_1}(\mathbb{T}^3)$ as $N \to \infty$, provided that

$$s_1 < \alpha - \frac{3}{2}. \tag{1.9}$$

In particular, when $\alpha \leqslant \frac{3}{2}$, $z_{1,N}$ has negative Sobolev regularity (in the limiting sense) and thus $(z_{1,N})^2$ and $(z_{1,N})^3$ do not have well-defined limits (in any topology) as $N \to \infty$ since it involves products of two distributions of negative regularities.

Let u_N be the solution to the renormalized NLW (1.5) with the same truncated random initial data $(u^\omega_{0,N}, u^\omega_{1,N})$ in (1.4). By writing u as

$$u_N = z_{1,N} + v_N, \tag{1.10}$$

we see that the residual term $v_N = u_N - z_{1,N}$ satisfies the following equation:

$$\begin{cases} \mathcal{L} v_N + v_N^3 + 3 z_{1,N} v_N^2 \\ \qquad +3\left\{(z_{1,N})^2 - \sigma_N\right\} v_N + \left\{(z_{1,N})^3 - 3\sigma_N z_{1,N}\right\} = 0, \\ (v_N, \partial_t v_N)\,|_{t=0} = (0,0), \end{cases} \tag{1.11}$$

where the parameter σ_N is defined by

$$\sigma_N := \frac{\alpha_N + 1}{3}.$$

As it is well known, the key point in the equation (1.11) is that the terms

$$Z_{2,N} := (z_{1,N})^2 - \sigma_N \quad \text{and} \quad Z_{3,N} := (z_{1,N})^3 - 3\sigma_N z_{1,N} \tag{1.12}$$

are "renormalizations" of $(z_{1,N})^2$ and $(z_{1,N})^3$. Here, by "renormalizations", we mean that by choosing a suitable renormalization constant σ_N, the terms $Z_{2,N}$ and $Z_{3,N}$ converge almost surely in suitable negative Sobolev spaces as $N \to \infty$.

The regularity $s_1 < \alpha - \frac{3}{2}$ of $z_{1,N}$ (in the limit) and a simple paraproduct computation show that if the expressions $Z_{2,N} = (z_{1,N})^2 - \sigma_N$ and $Z_{3,N} = (z_{1,N})^3 - 3\sigma_N z_{1,N}$ have any well-defined limits as $N \to \infty$, then their regularities in the limit are expected to be

$$s_2 < 2\left(\alpha - \frac{3}{2}\right) \quad \text{and} \quad s_3 < 3\left(\alpha - \frac{3}{2}\right), \tag{1.13}$$

respectively. In fact, by choosing the renormalization constant σ_N as

$$\sigma_N := \mathbb{E}\left[(z_{1,N}(t,x,\omega))^2\right], \tag{1.14}$$

we show that $Z_{j,N}$ converges in $H^{s_j}(\mathbb{T}^3)$ almost surely. See Proposition 3.2. Note that the renormalization constant σ_N a priori depends on t, x but it turns out to be independent of t and x.(5) We will also see that, for $N \gg 1$, σ_N behaves like (i) $\sim N^{3-2\alpha}$ when $\alpha < \frac{3}{2}$ and (ii) $\sim \log N$ when $\alpha = \frac{3}{2}$. See (3.2) below.

Thanks to the Strichartz estimates (see Lemma 2.4 below), the deterministic Cauchy problem for

$$\mathcal{L}v + v^3 = 0$$

is locally well-posed in $\mathcal{H}^s(\mathbb{T}^3)$ for $s \geqslant \frac{1}{2}$. We may therefore hope to solve the equation (1.11) uniformly in $N \in \mathbb{N}$ by the method of [9, 13, 14], if we can ensure that the solution v_N to the following linear problem:

$$\mathcal{L}v_N + \left\{(z_{1,N})^3 - 3\sigma_N z_{1,N}\right\} = 0 \tag{1.15}$$

with the zero initial data $(v_N, \partial_t v_N)|_{t=0} = (0,0)$ remains bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$ as $N \to \infty$. Using one degree of smoothing under the wave Duhamel operator (see (2.6) below), we see that the solution to (1.15) is almost surely bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$ uniformly in $N \in \mathbb{N}$, provided

$$3\left(\alpha - \frac{3}{2}\right) + 1 > \frac{1}{2} \quad \Longrightarrow \quad \alpha > \frac{4}{3}.$$

Therefore, $\alpha = \frac{4}{3}$ seems to be the limit of the approach of [9, 13, 14].(6)

In order to go below the $\alpha = \frac{4}{3}$ threshold, a new argument is needed. The introduction of such an argument is the main idea of this paper. More precisely, we further decompose v_N in (1.10) as

$$v_N = z_{2,N} + w_N \tag{1.16}$$

for some residual term w_N, where $z_{2,N}$ is the solution to the following equation:

$$\begin{cases} \mathcal{L}z_{2,N} + \left\{(z_{1,N})^3 - 3\sigma_N z_{1,N}\right\} = 0 \\ (z_{2,N}, \partial_t z_{2,N})\,|_{t=0} = (0,0). \end{cases} \tag{1.17}$$

(5) While we show this fact by a direct computation in (3.2), it can be seen from the stationarity (in both t and x) of the stochastic process $\{z_{1,N}(t,x)\}_{(t,x)\in\mathbb{R}\times\mathbb{T}^3}$. See Remark 3.1.

(6) Here, we are not taking into account a possible multilinear smoothing for the solution v to (1.15).

Thanks to the one degree of smoothing, we see that $z_{2,N}$ converges to some limit in $H^s(\mathbb{T}^3)$, provided that

$$s = s_3 + 1 < 3\left(\alpha - \frac{3}{2}\right) + 1$$

In terms of the original solution u_N to (1.5), we have from (1.10) and (1.16) that

$$u_N = z_{1,N} + z_{2,N} + w_N. \tag{1.18}$$

Note that $z_{1,N} + z_{2,N}$ corresponds to the Picard second iterate for the truncated renormalized equation (1.5).

The equation for w_N can now be written as

(1.19)
$$\begin{cases} \mathcal{L}w_N + (w_N + z_{2,N})^3 \\ \qquad +3z_{1,N}\,(w_N + z_{2,N})^2 + 3\left\{(z_{1,N})^2 - \sigma_N\right\}(w_N + z_{2,N}) = 0, \\ (w_N, \partial_t w_N)\,|_{t=0} = (0,0). \end{cases}$$

By using the second order expansion (1.18), we have eliminated the most singular term $Z_{3,N} = (z_{1,N})^3 - 3\sigma_N z_{1,N}$ in (1.11). In the equation (1.19), there are several source terms(7) and they are precisely the quintic, septic, and nonic (i.e. degree nine) terms added in considering the Picard third iterate for (1.5). As we see below, the most singular term in (1.19) is the following quintic term:

$$Z_{5,N} := 3\left\{(z_{1,N})^2 - \sigma_N\right\} z_{2,N}, \tag{1.20}$$

where $z_{2,N}$ is the solution to (1.17). As we already mentioned, the term $Z_{2,N} = (z_{1,N})^2 - \sigma_N$ and the second order term $z_{2,N}$ pass to the limits in $H^s(\mathbb{T}^3)$ for $s < 2(\alpha - \frac{3}{2})$ and $s < 3(\alpha - \frac{3}{2}) + 1$, respectively. In order to make sense of the product of $Z_{2,N}$ and $z_{2,N}$ in (1.20) by deterministic paradifferential calculus (see Lemma 2.1 below), we need the sum of the two regularities to be positive, namely

$$2\left(\alpha - \frac{3}{2}\right) + 3\left(\alpha - \frac{3}{2}\right) + 1 > 0 \quad \Longrightarrow \quad \alpha > \frac{13}{10}.$$

Otherwise, i.e. for $\alpha \leqslant \frac{13}{10}$, we will need to make sense of the product (1.20), using stochastic analysis. See Proposition 4.2. In either case, when the second factor in (1.20) has positive regularity $3(\alpha - \frac{3}{2}) + 1 > 0$, i.e. $\alpha > \frac{7}{6}$,

(7) Namely, purely stochastic terms independent of the unknown w_N.

we show that the product (1.20) (in the limit) inherits the regularity from $Z_{2,N} = (z_{1,N})^2 - \sigma_N$, allowing us to pass to a limit in $H^s(\mathbb{T}^3)$ for

$$s < 2\left(\alpha - \frac{3}{2}\right).$$

Once we are able to pass the term $Z_{5,N}$ in (1.20) in the limit $N \to \infty$, the main issue in solving the equation (1.19) for w_N by the deterministic Strichartz theory is to ensure that the solution of

$$\begin{cases} \mathcal{L}w + 3\left\{(z_{1,N})^2 - \sigma_N\right\} z_{2,N} = 0 \\ (w, \partial_t w)\,|_{t=0} = (0,0) \end{cases} \tag{1.21}$$

remains bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$ as $N \to \infty$ (recall that $s = \frac{1}{2}$ is the threshold regularity for the deterministic local well-posedness theory for the cubic wave equation on $\mathbb{T}^3$). Using again one degree of smoothing under the wave Duhamel operator, we see that the solution to (1.21) is almost surely bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$, provided

$$2\left(\alpha - \frac{3}{2}\right) + 1 > \frac{1}{2} \quad \Longrightarrow \quad \alpha > \frac{5}{4}.$$

This explains the restriction $\alpha > \frac{5}{4}$ in Theorem 3. See also Remark 1.4. We point out that under the restriction $\alpha > \frac{5}{4}$, we can use deterministic paradifferential calculus to make sense of the product of $z_{1,N}$ and $z_{2,N}^2$ appearing in (1.19), uniformly in $N \in \mathbb{N}$.

In proving Theorem 3, we apply the deterministic Strichartz theory and show that w_N converges almost surely to some limit w. Along with the almost sure convergence of $z_{1,N}$ and $z_{2,N}$ to some limits z_1 and z_2, respectively, we conclude from the decomposition (1.18) that u_N converges almost surely to

$$u := z_1 + z_2 + w. \tag{1.22}$$

By taking a limit of (1.19) as $N \to \infty$, we see that w is almost surely the solution to

$$\begin{cases} \mathcal{L}w + (w + z_2)^3 + 3z_1\,(w + z_2)^2 + 3Z_2 w + 3Z_5 = 0 \\ (w, \partial_t w)\,|_{t=0} = (0,0), \end{cases} \tag{1.23}$$

where Z_2 and Z_5 are the limits of $Z_{2,N}$ in (1.12) and $Z_{5,N}$ in (1.20), respectively. This essentially explains the proof of Theorem 3.

Remark 1.3. — The expansion (1.22) provides finer descriptions of u at different scales; the roughest term z_1 is essentially responsible for the small scale behavior of u, while z_2 describes its mesoscopic behavior and the smoother remainder part w describes its large-scale behavior.

Remark 1.4. — The argument based on the first order expansion (1.10) goes back to the work of McKean [38] and Bourgain [9] in the study of invariant Gibbs measures for the nonlinear Schrödinger equations on $\mathbb{T}^d$, $d = 1, 2$. See also [13]. In the field of stochastic parabolic PDEs, this argument is usually referred to as the Da Prato–Debussche trick [20].

As we explained above, the novelty in this paper with respect to the previous work [9, 13, 14] is that the proof of Theorem 3 crucially relies on the second order expansion (1.18). We also mention two other recent works [7, 48], where such higher order expansions were used in the context of dispersive PDEs with random initial data. The higher order expansions used in [48] are at negative Sobolev regularity but they are related to a gauge transform, which is very different from the situation in the present paper. The difference between the present paper and [7] is that, in this paper, we work in Sobolev spaces of negative indices, while solutions in [7] have positive Sobolev regularities.(8) We point out that while the higher order expansions helped lowering the regularity of random initial data in [7], the third order expansion would *not* help us improve Theorem 3 for our problem.(9) This can be seen from the product $Z_2 w$ in (1.23). From the regularity s_2 of Z_2 in (1.13) and the regularity $\frac{1}{2}$ of w, we see that the sum of their regularity is positive (which is needed to make sense of the product $Z_2 w$) only for $\alpha > \frac{5}{4}$. This is exactly the range covered in Theorem 3. Note that a higher order expansion is used to eliminate certain explicit stochastic terms. Namely, even if we go into a higher order expansion, we can not eliminate this problematic term $Z_2 w$ since this term depends on the unknown w. In order to lower values of α, we need to impose a structure of the residual term w.

For conciseness of the presentation, we decided to present only the simplest argument based on the second order expansion. There are, however, several ways for a possible improvement on the regularity restriction in Theorem 3. (i) In studying the regularity and convergence properties of the second order stochastic term $z_{2,N}$ in (1.17), we simply use a "parabolic thinking", namely, we only count the regularity $s_1 < \alpha - \frac{3}{2}$ of each of three factors $z_{1,N}$ for $Z_{3,N}$ (modulo the renormalization) and put them together

(8) In particular, all the products make sense as functions in [7]. In negative Sobolev spaces, the main problem is to make sense of a product as a distribution.

(9) That is, unless we combine it with multilinear smoothing and imposing a further structure (such as a paracontrolled structure) on w.

with one degree of smoothing coming from the wave Duhamel integral operator *without* taking into account the explicit product structure and the oscillatory nature of the linear wave propagator. See Proposition 4.1 below. In the field of dispersive PDEs, however, it is crucial to exploit an explicit product structure and study interaction of waves at a multilinear level to show a further smoothing property. See, for example, [10, 28, 43]. In this sense, the argument presented in this paper leaves a room for an obvious improvement. (ii) In recent study of singular stochastic parabolic PDEs such as SQE (1.6) on $\mathbb{T}^3$, higher order expansions (in terms of the stochastic forcing in the mild formulation) were combined with the theory of regularity structures [31] or the paracontrolled calculus [3, 15, 39]. In fact, it is possible to employ the ideas from the paracontrolled calculus in studying nonlinear wave equations. See a recent work [28] on the three-dimensional stochastic NLW with a quadratic nonlinearity.

In a very recent preprint [10] (which appeared more than one year after the appearance of the current paper), Bringmann studied the defocusing cubic NLW with a Hartree-type nonlinearity on $\mathbb{T}^3$:

$$\partial_t^2 u - \Delta u + \left(V * u^2\right) u = 0, \tag{1.24}$$

where $V = \langle \nabla \rangle^{-\beta}$ is the Bessel potential of order $\beta > 0$. By adapting the paracontrolled approach of [28] to the Hartree cubic nonlinearity and exploiting multilinear smoothing,(10) Bringmann proved almost sure local (and global) well-posedness of (1.24) with the Gibbs measure initial data (essentially corresponding to the random initial data (u_0^ω, u_1^ω) in (1.2) with $\alpha = 1$), provided that $\beta > 0$. In the context of the renormalized cubic NLW on $\mathbb{T}^3$, it seems possible to adapt the methodology developed in [10] and extend Theorem 3 to $\alpha > 1$. When $\alpha = 1$ (i.e. (1.24) with $\beta = 0$), the argument in [10] breaks down in various places and thus further novels ideas are needed to treat the case $\alpha = 1$. We also mention a recent work [21] by Deng, Nahmod, and Yue, where they introduced the theory of random tensors in studying the random data Cauchy theory for the nonlinear Schrödinger equations. While this theory is fairly general, as it is pointed out in [10, Remark 1.6 and Subsection 4.4], there are some technical challenges in extending the theory in [21] to the wave case.

(10) Also, combining other tools such as the random matrix estimates from [21].

1.3. Factorization of the ill-posed solution map

In the following, let us consider initial data of the form:

$$(u, \partial_t u)\,|_{t=0} = (w_0, w_1) + (u_0^\omega, u_1^\omega)\,, \tag{1.25}$$

where (w_0, w_1) is a given pair of deterministic functions in $\mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$ and (u_0^ω, u_1^ω) is the random initial data given in (1.2). Recall that the random initial data in (1.25) belongs almost surely to $\mathcal{H}^{\min(\frac{1}{2}, s)}(\mathbb{T}^3)$ for $s < \alpha - \frac{3}{2}$. When $\alpha > 2$, the deterministic local well-posedness in $\mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$ yields a continuous solution map[(11)]

$$\Phi : (w_0, w_1) + (u_0^\omega, u_1^\omega) \in \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right) \longmapsto (u, \partial_t u) \in C\left([-T, T]; \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right)\right).$$

On the other hand, when $\alpha \leqslant 2$, the random initial data in (1.25) does not belong to $\mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$. In particular, the ill-posedness results in [23, 45, 54] show that, given any $(w_0, w_1) \in \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$, the solution map Φ is almost surely discontinuous.

For $\frac{3}{2} < \alpha \leqslant 2$, the proof of Theorem 2, presented in [14], based on the first order expansion (1.10) yields the following factorization of the ill-posed solution map Φ:

$$\begin{aligned}(w_0, w_1) + (u_0^\omega, u_1^\omega) &\longmapsto (w_0, w_1, z_1)\\ &\overset{\Psi_1}{\longmapsto} (v, \partial_t v) \in C\left([-T, T]; \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right)\right)\\ &\longmapsto u = z_1 + v \in C\left([-T, T]; H^{s_1}\left(\mathbb{T}^3\right)\right),\end{aligned} \tag{1.26}$$

where z_1 is the solution to the linear equation (1.8) with the random initial data (u_0^ω, u_1^ω) in (1.2) and $s_1 < \alpha - \frac{3}{2}$. Here, we view the first map in (1.26) as a lift map, where we use stochastic analysis to construct an enhanced data set (w_0, w_1, z_1), and the second map Ψ_1 is the deterministic solution map to the following perturbed NLW:

$$\begin{cases}\mathcal{L}v + (v + z_1)^3 = 0\\ (v, \partial_t v)\,|_{t=0} = (w_0, w_1),\end{cases}$$

where we view (w_0, w_1, z_1) as an *enhanced data set.*[(12)] Furthermore, the deterministic map $\Psi_1 : (w_0, w_1, z_1) \mapsto (v, \partial_t v)$ is continuous from

$$\mathcal{X}_1^{s_1}(T) := \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right) \times C\left([-T, T]; W^{s_1, \infty}\left(\mathbb{T}^3\right)\right)$$

to $C([-T, T]; \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right))$.

(11) Here, the local well-posedness time is indeed random but we simply write it as T. The same comment applies in the following.

(12) In particular, we view z_1 as a given deterministic space-time distribution of some specified regularity.

Remark 1.5. — In [14], using a conditional probability, Burq and the third author introduced the notion of probabilistic continuity and showed that the map: $(w_0, w_1) + (u_0^\omega, u_1^\omega) \mapsto u$ in (1.26) is indeed probabilistically continuous when $\frac{3}{2} < \alpha \leqslant 2$. It would be of interest to investigate if such probabilistic continuity also holds for lower values of α.

For $\frac{4}{3} < \alpha \leqslant \frac{3}{2}$, the first order expansion (1.10) along with renormalization yields the following factorization of the ill-posed solution map Φ:

$$\begin{aligned}(w_0, w_1) + (u_0^\omega, u_1^\omega) &\longmapsto (w_0, w_1, z_1, Z_2, Z_3) \\ &\overset{\Psi_2}{\longmapsto} (v, \partial_t v) \in C\left([-T, T]; \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right)\right) \\ &\longmapsto u = z_1 + v \in C\left([-T, T]; H^{s_1}\left(\mathbb{T}^3\right)\right),\end{aligned} \tag{1.27}$$

where Z_2 and Z_3 are the limits of $Z_{2,N}$ and $Z_{3,N}$ in (1.12). With s_j, $j = 1, 2, 3$, as in (1.9) and (1.13), the second map Ψ_2 is the deterministic continuous map, sending an enhanced data set $(w_0, w_1, z_1, Z_2, Z_3)$ in

$$\mathcal{X}_2^{s_1, s_2, s_3}(T) := \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right) \times \prod_{j=1}^{3} C\left([-T, T]; W^{s_j, \infty}\left(\mathbb{T}^3\right)\right)$$

to a solution $(v, \partial_t v) \in C([-T, T]; \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3))$ to the following perturbed NLW:

$$\begin{cases}\mathcal{L}v + v^3 + 3z_1 v^2 + 3Z_2 v + Z_3 = 0 \\ (v, \partial_t v)\,|_{t=0} = (w_0, w_1).\end{cases}$$

For $\frac{13}{10} < \alpha \leqslant \frac{4}{3}$, the proof of Theorem 3 based on the second order expansion (1.18) yields the following factorization of the ill-posed solution map Φ:

$$\begin{aligned}(w_0, w_1) + (u_0^\omega, u_1^\omega) &\longmapsto (w_0, w_1, z_1, Z_2, z_2) \\ &\overset{\Psi_3}{\longmapsto} (w, \partial_t w) \in C\left([-T, T]; \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right)\right) \\ &\longmapsto u = z_1 + z_2 + w \in C\left([-T, T]; H^{s_1}\left(\mathbb{T}^3\right)\right),\end{aligned} \tag{1.28}$$

where z_2 is the limit of $z_{2,N}$ defined in (1.17). Here, with $s_4 = s_3 + 1 < 3(\alpha - \frac{3}{2}) + 1$, the second map Ψ_3 is the deterministic continuous map, sending an enhanced data set $(w_0, w_1, z_1, Z_2, z_2)$ in

$$\mathcal{X}_3^{s_1, s_2, s_4}(T) := \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right) \times \prod_{j \in \{1, 2, 4\}} C\left([-T, T]; W^{s_j, \infty}\left(\mathbb{T}^3\right)\right)$$

to a solution $(w, \partial_t w) \in C([-T,T]; \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3))$ to the following perturbed NLW:

$$\begin{cases} \mathcal{L}w + (w+z_2)^3 + 3z_1(w+z_2)^2 + 3Z_2 w + 3Z_2 z_2 = 0 \\ (w, \partial_t w)\,|_{t=0} = (w_0, w_1). \end{cases} \tag{1.29}$$

Lastly, let us discuss the case $\frac{5}{4} < \alpha \leqslant \frac{13}{10}$. In this case, the product $Z_2 z_2$ in (1.29) can not be defined by deterministic paradifferential calculus and thus we need to define Z_5 as a limit of $Z_{5,\,N}$ in (1.20). Then, the proof of Theorem 3 based on the second order expansion (1.18) yields the following factorization of the ill-posed solution map Φ:

$$\begin{aligned} (w_0, w_1) + (u_0^\omega, u_1^\omega) &\longmapsto (w_0, w_1, z_1, Z_2, z_2, Z_5) \\ &\overset{\Psi_4}{\longmapsto} (w, \partial_t w) \in C\left([-T,T]; \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right)\right) \\ &\longmapsto u = z_1 + z_2 + w \in C\left([-T,T]; H^{s_1}\left(\mathbb{T}^3\right)\right). \end{aligned} \tag{1.30}$$

With $s_5 < 2(\alpha - \frac{3}{2})$, the second map Ψ_4 is the deterministic continuous map, sending an enhanced data set $(w_0, w_1, z_1, Z_2, z_2, Z_5)$ in

$$\mathcal{X}_4^{s_1,\,s_2,\,s_4,\,s_5}(T) := \mathcal{H}^{\frac{1}{2}}\left(\mathbb{T}^3\right) \times \prod_{j\,\in\,\{1,\,2,\,4,\,5\}} C\left([-T,T]; W^{s_j,\,\infty}\left(\mathbb{T}^3\right)\right)$$

to a solution $(w, \partial_t w) \in C([-T,T]; \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3))$ to the following perturbed NLW:

$$\begin{cases} \mathcal{L}w + (w+z_2)^3 + 3z_1(w+z_2)^2 + 3Z_2 w + Z_5 = 0 \\ (w, \partial_t w)\,|_{t=0} = (w_0, w_1). \end{cases} \tag{1.31}$$

We point out that the last decomposition (1.30) with (1.31) can also be used to study the cases $\frac{4}{3} < \alpha \leqslant \frac{3}{2}$ and $\frac{13}{10} < \alpha \leqslant \frac{4}{3}$. For simplicity of the presentation, we only discuss the last decomposition (1.30) with (1.31) in this paper, while the previous decompositions (1.27) and (1.28) provide simpler arguments when $\frac{13}{10} < \alpha \leqslant \frac{3}{2}$.

In all the cases mentioned above, we decompose the ill-posed solution map Φ into

(i) the first step, constructing enhanced data sets by stochastic analysis and
(ii) the second step, where purely deterministic analysis is performed in constructing a continuous map Ψ_j on enhanced data sets, solving perturbed NLW equations.

Such decompositions of ill-posed solution maps also appear in studying rough differential equations via the rough path theory [24, 37] and singular stochastic parabolic PDEs [26, 31].

Remark 1.6. — By the use of stochastic analysis, the terms z_1, z_2, Z_2, and Z_5 are defined as the unique limits of their truncated versions. Furthermore, by deterministic analysis, we prove that a solution w to (1.31) is pathwise unique in an appropriate class (see the space X_T defined in (2.8)). Therefore, under the decomposition $u = z_1 + z_2 + w$, the uniqueness of u claimed in Theorem 3 refers to (i) the uniqueness of z_1 and z_2 as the limits of $z_{1,N}$ and $z_{2,N}$ and (ii) the uniqueness of w as a solution to (1.31).

Remark 1.7. — Given $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let $\mathbf{P}_j$ be the (non-homogeneous) Littlewood–Paley projector onto the (spatial) frequencies $\{n \in \mathbb{Z}^3 : |n| \sim 2^j\}$ such that

$$f = \sum_{j=0}^{\infty} \mathbf{P}_j f.$$

Given two functions f and g on $\mathbb{T}^3$ of regularities s_1 and s_2, we have the following paraproduct decomposition of the product fg due to Bony [8]:

$$\begin{aligned} fg &= f \olessthan g + f \mathbin{\textcircled{\scriptsize =}} g + f \ogreaterthan g \\ &:= \sum_{j < k-2} \mathbf{P}_j f \, \mathbf{P}_k g + \sum_{|j-k| \leqslant 2} \mathbf{P}_j f \, \mathbf{P}_k g + \sum_{k < j-2} \mathbf{P}_j f \, \mathbf{P}_k g. \end{aligned} \tag{1.32}$$

The first term $f \olessthan g$ (and the third term $f \ogreaterthan g$) is called the paraproduct of g by f (the paraproduct of f by g, respectively) and it is always well defined as a distribution of regularity $\min(s_2, s_1 + s_2)$. On the other hand, the resonant product $f \mathbin{\textcircled{\scriptsize =}} g$ is well defined in general only if $s_1 + s_2 > 0$. See Lemma 2.1 below.

Let $\frac{5}{4} < \alpha \leqslant \frac{13}{10}$. In this case, the sum of the regularities $s_2 < 2(\alpha - \frac{3}{2})$ and $s_4 < 3(\alpha - \frac{3}{2}) + 1$ of Z_2 and z_2 is non-positive and thus we can not make sense of the product $Z_2 z_2$ by deterministic paradifferential calculus. As we pointed out above, however, the paraproducts $Z_2 \olessthan z_2$ and $Z_2 \ogreaterthan z_2$ are well-defined distributions. Hence, it suffices to define $Z_5^{\textcircled{\scriptsize =}}$ as a suitable limit of the resonant products $Z_{2,N} \mathbin{\textcircled{\scriptsize =}} z_{2,N}$ in order to pass $3Z_{2,N} z_{2,N}$ to the limit

$$Z_5 = 3Z_2 \olessthan z_2 + 3Z_5^{\textcircled{\scriptsize =}} + 3Z_2 \ogreaterthan z_2.$$

This shows that we can in fact replace the enhanced data set $(w_0, w_1, z_1, Z_2, z_2, Z_5)$ in (1.30) and Z_5 in (1.31) by $(w_0, w_1, z_1, Z_2, z_2, Z_5^{\textcircled{\scriptsize =}})$ and $3Z_2 \olessthan z_2 + 3Z_5^{\textcircled{\scriptsize =}} + 3Z_2 \ogreaterthan z_2$, respectively. See also the proof of Proposition 4.2 and Remark 4.4 below.

1.4. NLW without renormalization in negative Sobolev spaces

We conclude this introduction by discussing a new instability phenomenon for NLW (1.1) (that is, without renormalization) in negative Sobolev spaces. This phenomenon is closely related to the so-called *triviality* in the study of stochastic PDEs [1, 33]. See Remark 1.8.

Fix a deterministic pair $(w_0, w_1) \in \mathcal{H}^{\frac{3}{4}}(\mathbb{T}^3)$. In the following, we study the (un-renormalized) NLW (1.1) with initial data of the form:

$$(u, \partial_t u)\,|_{t=0} = (w_0, w_1) + (0, u_1^\omega)\,,$$

where u_1^ω is the random distribution given by (1.2). We consider this problem by studying the following truncated problem. Given $N \in \mathbb{N}$, let u_N be the solution to the (un-renormalized) NLW (1.1) with the following initial data:

$$(u_N, \partial_t u_N)\,|_{t=0} = (w_0, w_1) + \left(\widetilde{u}^\omega_{0,\,N}, \widetilde{u}^\omega_{1,\,N}\right).$$

Here, $(\widetilde{u}^\omega_{0,\,N}, \widetilde{u}^\omega_{1,\,N})$ denotes the truncated random initial data given by

$$\begin{aligned} \widetilde{u}^\omega_{0,\,N}(x) &= \sum_{|n| \leqslant N} \frac{g_n(\omega)}{\langle n \rangle_N \langle n \rangle^{\alpha-1}} e^{in\cdot x} \\ \text{and} \quad \widetilde{u}^\omega_{1,\,N}(x) &= \sum_{|n| \leqslant N} \frac{h_n(\omega)}{\langle n \rangle^{\alpha-1}} e^{in\cdot x}, \end{aligned} \tag{1.33}$$

where $\{g_n\}_{n\in\mathbb{Z}^3}$ and $\{h_n\}_{n\in\mathbb{Z}^3}$ are as in (1.2) and

$$\langle n \rangle_N = \sqrt{C_N + |n|^2} \tag{1.34}$$

for some suitable choice of a divergent constant $C_N > 0$. Our goal is to study the asymptotic behavior of u_N as $N \to \infty$.

Given $N \in \mathbb{N}$, define the linear Klein–Gordon operator $\mathcal{L}_N$ by setting

$$\mathcal{L}_N := \partial_t^2 - \Delta + C_N. \tag{1.35}$$

Then, u_N satisfies the following equation:

$$\begin{cases} \mathcal{L}_N u_N + u_N^3 - C_N u_N = 0 \\ (u_N, \partial_t u_N)\,|_{t=0} = (w_0, w_1) + \left(\widetilde{u}^\omega_{0,\,N}, \widetilde{u}^\omega_{1,\,N}\right). \end{cases} \tag{1.36}$$

We denote by $\widetilde{z}_{1,\,N}$ the solution to the following linear Klein–Gordon equation:

$$\mathcal{L}_N \widetilde{z}_{1,\,N} = 0 \tag{1.37}$$

with the truncated random initial data $(\widetilde{u}^{\omega}_{0,\,N}, \widetilde{v}^{\omega}_{0,\,N})$ in (1.33). Then, we have

$$\begin{aligned}\widetilde{z}_{1,N}(t,x,\omega) = &\sum_{|n|\leqslant N} \frac{\cos\left(t\langle n\rangle_{N}\right)}{\langle n\rangle_{N}\langle n\rangle^{\alpha-1}} g_n(\omega)e^{in\cdot x}\\ &+\sum_{|n|\leqslant N} \frac{\sin\left(t\langle n\rangle_{N}\right)}{\langle n\rangle_{N}\langle n\rangle^{\alpha-1}} h_n(\omega)e^{in\cdot x}.\end{aligned} \tag{1.38}$$

In particular, for each fixed $(t,x)\in\mathbb{R}\times\mathbb{T}^3$, $\widetilde{z}_{1,\,N}(t,x)$ is a mean-zero Gaussian random variable with variance:

$$\begin{aligned}\widetilde{\sigma}_N :=&\mathbb{E}\left[\left(\widetilde{z}_{1,\,N}\,(t,x)\right)^2\right]\\ =&\sum_{|n|\leqslant N}\frac{\left(\cos\left(t\langle n\rangle_{N}\right)\right)^2}{\langle n\rangle_{N}^2\langle n\rangle^{2(\alpha-1)}}+\sum_{|n|\leqslant N}\frac{\left(\sin\left(t\langle n\rangle_{N}\right)\right)^2}{\langle n\rangle_{N}^2\langle n\rangle^{2(\alpha-1)}}\\ =&\sum_{|n|\leqslant N}\frac{1}{\langle n\rangle_{N}^2\langle n\rangle^{2(\alpha-1)}}.\end{aligned} \tag{1.39}$$

In view of (1.36), we implicitly define $C_N>0$ by

$$\begin{aligned}C_N = 3\widetilde{\sigma}_N &= 3\sum_{|n|\leqslant N}\frac{1}{\langle n\rangle_{N}^2\langle n\rangle^{2(\alpha-1)}}\\ &=3\sum_{|n|\leqslant N}\frac{1}{(C_N+|n|^2)\,\langle n\rangle^{2(\alpha-1)}}\end{aligned} \tag{1.40}$$

such that the subtraction of $C_N u_N$ in (1.36) corresponds to (artificial) renormalization of the cubic nonlinearity u_N^3. In Lemma 6.1 below, we show that for each $N\in\mathbb{N}$, there exists unique $C_N\geqslant 1$ whose asymptotic behavior of C_N as $N\to\infty$ is given by

$$C_N\sim\begin{cases}\log N, & \text{for } \alpha=\frac{3}{2},\\ N^{3-2\alpha}, & \text{for } 1\leqslant\alpha<\frac{3}{2},\end{cases}$$

for all sufficiently large $N\gg 1$. In particular, $C_N\to\infty$ as $N\to\infty$ and thus we see that $(\widetilde{u}^{\omega}_{0,\,N}, \widetilde{u}^{\omega}_{1,\,N})$ in (1.33) almost surely converges to $(0,u_1^{\omega})$ in a suitable topology.

We are now ready to state an instability result for the (un-renormalized) NLW (1.1) in negative Sobolev spaces.

Theorem 4. — *Let $\frac{5}{4}<\alpha<\frac{3}{2}$ and $(w_0,w_1)\in\mathcal{H}^{\frac{3}{4}}(\mathbb{T}^3)$. By setting C_N by (1.40), there exist small $T_1>0$ and positive constants C, c, κ such that for every $T\in(0,T_1]$, there exists a set Ω_T of complemental probability smaller than $C\exp(-c/T^{\kappa})$ such that if we denote by $\{u_N\}_{N\in\mathbb{N}}$ the smooth* *global solutions to the defocusing cubic NLW* (1.1) *with the random initial data*

$$(u_N, \partial_t u_N)\,|_{t=0} = (w_0, w_1) + \big(\widetilde{u}^\omega_{0,\,N}, \widetilde{u}^\omega_{1,\,N}\big)\,, \tag{1.41}$$

where $(\widetilde{u}^\omega_{0,\,N}, \widetilde{u}^\omega_{1,\,N})$ *is given by* (1.33)*, then for every* $\omega \in \Omega_T$*, the sequence* $\{u_N\}_{N\in\mathbb{N}}$ *converges to* 0 *as space-time distributions on* $[-T,T]\times\mathbb{T}^3$ *as* $N\to\infty$.

When $\alpha = \frac{3}{2}$*, the same result holds but only along a subsequence* $\{N_k\}_{k\in\mathbb{N}}$*. Namely, there exists an almost surely positive random time* $T_\omega > 0$ *such that the sequence* $\{u_{N_k}\}_{k\in\mathbb{N}}$ *converges to* 0 *as space-time distributions on* $[-T_\omega, T_\omega]\times\mathbb{T}^3$ *as* $k\to\infty$ *(in the sense described above).*

The proof of Theorem 4 is based on the reformulation (1.36) and an adaptation of the argument employed in proving Theorem 3.

We point out that the instability stated in Theorem 4 is due to the lack of renormalization (in negative regularity). Indeed, let us briefly discuss the situation when a proper renormalization is applied. Consider the following renormalized NLW:

$$\partial_t^2 u_N - \Delta u_N + u_N^3 - 3\widetilde{\sigma}_N u_N = 0 \tag{1.42}$$

with the random initial data in (1.41). First, note that the initial data in (1.41) gives rise to an enhanced data set

$$\Xi_N = \Big(w_0, w_1, \widetilde{z}_{1,\,N}, \widetilde{Z}_{2,\,N}, \widetilde{z}_{2,\,N}, \widetilde{Z}_{5,\,N}\Big)\,,$$

where $\widetilde{Z}_{2,\,N}$ and $\widetilde{Z}_{5,\,N}$ are defined by

$$\widetilde{Z}_{2,\,N} := (\widetilde{z}_{1,\,N})^2 - \widetilde{\sigma}_N \qquad \text{and} \qquad \widetilde{Z}_{5,\,N} := \Big\{(\widetilde{z}_{1,\,N})^2 - \widetilde{\sigma}_N\Big\}\,\widetilde{z}_{2,\,N}$$

and $\widetilde{z}_{2,\,N}$ is the solution to

$$\begin{cases} \mathcal{L}_N \widetilde{z}_{2,\,N} + \Big\{(\widetilde{z}_{1,\,N})^3 - 3\widetilde{\sigma}_N\widetilde{z}_{1,\,N}\Big\} = 0 \\ (z_{2,\,N}, \partial_t z_{2,\,N})\,|_{t=0} = (0,0). \end{cases}$$

In Section 6, we show that Ξ_N converges almost surely[(13)] to the limiting enhanced data set

$$\Xi = \Big(w_0, w_1, \widetilde{z}_1, \widetilde{Z}_2, \widetilde{z}_2, \widetilde{Z}_5\Big)\,,$$

emanating from the initial data $(w_0, w_1) + (0, u_1^\omega)$. Then, by slightly modifying the proof of Theorem 3, we can show that the solutions u_N to (1.42)

[(13)] Only along a subsequence $\{N_k\}_{k\in\mathbb{N}}$ when $\alpha = \frac{3}{2}$.

converges to some non-trivial limiting distribution $u = \widetilde{z}_1 + \widetilde{z}_2 + w$, where w is the solution to

$$\begin{cases} \mathcal{L}w + (w + \widetilde{z}_2)^3 + 3\widetilde{z}_1 (w + \widetilde{z}_2)^2 + 3\widetilde{Z}_2 w + \widetilde{Z}_5 = 0 \\ (w, \partial_t w)|_{t=0} = (w_0, w_1). \end{cases}$$

Here, we see that $u \not\equiv 0$ since the non-zero linear solution $\widetilde{z}_1$ with initial data $(0, u_1^\omega)$ does not belong to $H^{\alpha-\frac{3}{2}}(\mathbb{T}^3)$ (for a fixed time) while $z_2 + w \in H^{\alpha-\frac{3}{2}}(\mathbb{T}^3)$ almost surely. This shows the instability result stated in Theorem 4 is peculiar to the case without renormalization when we work in negative regularities.

Remark 1.8. — The instability result in Theorem 4 essentially corresponds to triviality results in the study of stochastic PDEs, where the dynamics without renormalization trivializes (either to the linear dynamics or the trivial dynamics, i.e. $u \equiv 0$) as regularization on a singular random forcing is removed. See, for example, [1, 2, 33, 44, 46]. In particular, our proof of Theorem 4 is inspired by the argument in [33] due to Hairer, Ryser, and Weber for the two-dimensional stochastic nonlinear heat equation. In the context of the random data Cauchy theory, Theorem 4 is the first result on triviality without renormalization.

In the context of stochastic nonlinear wave equations, Albeverio, Haba, and Russo [1] studied a triviality issue for the two-dimensional stochastic NLW:

$$\partial_t^2 u - \Delta u + f(u) = \xi, \tag{1.43}$$

where ξ is the space-time white noise and f is a bounded smooth function. Roughly speaking, they showed that solutions to (1.43) with regularized noises tend to that to the stochastic linear wave equation:

$$\partial_t^2 u - \Delta u = \xi.$$

Note that a power-type nonlinearity (such as the cubic nonlinearity u^3) does not belong to the class of nonlinearities considered in [1]. Furthermore, the analysis in [1] was carried out in the framework of Colombeau generalized functions, and as such, their solution does not a priori belong to $C([0,T]; H^{-\varepsilon}(\mathbb{T}^2))$. In fact, it is not clear if their generalized function represents an actual distribution. We refer the interested readers to [34, Remark 1.5] by Hairer and Shen, commenting on the work [2] by Albeverio, Haba, and Russo on stochastic nonlinear heat equations. We emphasize that our proof of Theorem 4 is based on an adaptation of the solution theory for Theorem 3. In particular, for each $N \in \mathbb{N}$, we construct the solution u_N to the defocusing cubic renormalized NLW (1.42) with the random initial data in (1.41) in the natural space: $C([0,T];H^s(\mathbb{T}^3))$, $s < \alpha - \frac{3}{2}$, with a uniform bound in $N \in \mathbb{N}$. We also point out that a small adaptation of the proof of Theorem 4 (as in [44]) yield a triviality result for the following stochastic damped NLW with the defocusing cubic nonlinearity on $\mathbb{T}^3$:

$$\partial_t^2 u + \partial_t u - \Delta u + u^3 = \langle \nabla \rangle^{-\alpha} \xi \qquad \text{for} \quad \frac{5}{4} < \alpha \leqslant \frac{3}{2}\,.$$

After the appearance of the current paper, following the idea of our triviality result (Theorem 4), Okamoto, Robert, and the first author [44] proved triviality for the two-dimensional stochastic damped NLW with the defocusing cubic nonlinearity. The argument in [44] is based on an adaptation of the recent solution theory of the two-dimensional stochastic (damped) nonlinear wave equations in [27, 29]. We also mention a recent work [46] by Robert, Sosoe, Y. Wang, and the first author on a triviality result for the two-dimensional stochastic wave equation with the sine nonlinearity: $f(u) = \sin(\beta u)$, $\beta \in \mathbb{R}\setminus\{0\}$. While the sine nonlinearity belongs to the class of nonlinearities considered in [1], the triviality result in [46] is established in the natural class $C([0,T];H^{-\varepsilon}(\mathbb{T}^2))$.

In the context of nonlinear Schrödinger type equations, such instability results without renormalization in negative Sobolev spaces are known even *deterministically*; see [30, 49]. See also [16, 17] for a similar instability result on the complex-valued mKdV equation in the deterministic setting.

Remark 1.9. — When $\alpha = \frac{3}{2}$, Theorem 4 yields the almost sure convergence of a subsequence $\{u_{N_k}\}_{k\in\mathbb{N}}$ to 0 (on a random time interval). By changing the mode of convergence and the related topology, it is possible to obtain convergence of the full sequence $\{u_N\}_{N\in\mathbb{N}}$, even when $\alpha = \frac{3}{2}$. More precisely, by slightly modifying the proof of Theorem 4, we can show that, when $\alpha = \frac{3}{2}$, $\{u_N\}_{N\in\mathbb{N}}$ converges in probability to the trivial solution 0 in $H^{-\varepsilon}([-T_\omega, T_\omega]; H^{-\varepsilon}(\mathbb{T}^3))$ as $N \to \infty$. See [44] for details of the proof in the two-dimensional stochastic setting.

Remark 1.10. — In the discussion above, we needed to consider the random data $(\widetilde{u}^\omega_{0,N}, \widetilde{u}^\omega_{1,N})$ in (1.33) in place of $(u^\omega_{0,N}, u^\omega_{1,N})$ in (1.4) such that C_N can be chosen to be time independent. Note that the distribution of $(\widetilde{u}^\omega_{0,N}, \widetilde{u}^\omega_{1,N})$ in (1.33) is precisely an invariant measure for the linear dynamics: $\mathcal{L}_N u = 0$.

Remark 1.11. — While the local-in-time results in Theorems 2 and 3 also holds in the focusing case, the proof of Theorem 4 only holds for the defocusing case. In the focusing case, we expect some undesirable behavior for solutions to the (un-renormalized) cubic NLW in negative Sobolev spaces but with a different mechanism.

1.5. Organization of the paper

The remaining part of this manuscript is organized as follows. In the next section, we state deterministic and stochastic tools needed for our analysis. In Sections 3 and 4, we study regularity and convergence properties of the stochastic terms from Subsection 1.2. In Section 5, we then use the deterministic Strichartz theory to study the equation (1.19) for w_N and present the proof of Theorem 3. In Section 6, by modifying the analysis from the previous sections, we prove Theorem 4.

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2. Tools from deterministic and stochastic analysis

2.1. Basic function spaces and paraproducts

We define the L^p-based Sobolev space $W^{s,p}(\mathbb{T}^3)$ by the norm:

$$\|f\|_{W^{s,\,p}} = \left\|\mathcal{F}^{-1}\left(\langle n\rangle^s \widehat{f}(n)\right)\right\|_{L^p}.$$

When $p=2$, we have $H^s(\mathbb{T}^3) = W^{s,\,2}(\mathbb{T}^3)$.

Next, we recall the regularity properties of paraproducts and resonant products, viewed as bilinear maps. For this purpose, it is convenient to use the Besov spaces $B^s_{p,q}(\mathbb{T}^3)$ defined by the norm:

$$\|u\|_{B^s_{p,\,q}} = \left\|2^{sj}\,\|\mathbf{P}_j u\|_{L^p_x}\right\|_{\ell^q_j(\mathbb{N}_0)}.$$

Note that $H^s(\mathbb{T}^3) = B^s_{2,\,2}(\mathbb{T}^3)$.

Lemma 2.1.

(i) *(paraproduct and resonant product estimates) Let* $s_1, s_2 \in \mathbb{R}$ *and* $1 \leqslant p, p_1, p_2, q \leqslant \infty$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$*. Then, we have*

$$\|f \otimes g\|_{B^{s_2}_{p,q}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B^{s_2}_{p_2,q}}. \tag{2.1}$$

When $s_1 < 0$*, we have*

$$\|f \otimes g\|_{B^{s_1+s_2}_{p,q}} \lesssim \|f\|_{B^{s_1}_{p_1,q}} \|g\|_{B^{s_2}_{p_2,q}}. \tag{2.2}$$

When $s_1 + s_2 > 0$*, we have*

$$\|f \circleddash g\|_{B^{s_1+s_2}_{p,q}} \lesssim \|f\|_{B^{s_1}_{p_1,q}} \|g\|_{B^{s_2}_{p_2,q}}. \tag{2.3}$$

(ii) *Let* $s_1 < s_2 < s_3$ *and* $1 \leqslant p, q \leqslant \infty$*. Then, we have*

$$\|u\|_{W^{s_1,p}} \lesssim \|u\|_{B^{s_2}_{p,q}} \lesssim \|u\|_{W^{s_3,p}}. \tag{2.4}$$

The product estimates (2.1), (2.2), and (2.3) follow easily from the definition (1.32) of the paraproduct and the resonant product. See [4, 40] for details of the proofs in the non-periodic case (which can be easily extended to the current periodic setting). The embeddings (2.4) follow from the ℓ^q-summability of $\{2^{(s_k - s_{k+1})j}\}_{j \in \mathbb{N}_0}$ for $s_k < s_{k+1}$, $k = 1, 2$, and the uniform boundedness of the Littlewood-Paley projector $\mathbf{P}_j$. Thanks to (2.4), we can apply the product estimates (2.1), (2.2), and (2.3) in the Sobolev space setting (with a slight loss of regularity).

2.2. Product estimates, an interpolation inequality, and Strichartz estimates

For $s \in \mathbb{R}$, we set $\langle \nabla \rangle^s := (1 - \Delta)^{\frac{s}{2}}$. Then, we have the following standard product estimates. See [27] for their proofs.

Lemma 2.2. — *Let* $0 \leqslant s \leqslant 1$.

(i) *Let* $1 < p_j, q_j, r < \infty$*,* $j = 1, 2$ *such that* $\frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j}$*. Then, we have*

$$\|\langle \nabla \rangle^s (fg)\|_{L^r(\mathbb{T}^3)} \lesssim \|\langle \nabla \rangle^s f\|_{L^{p_1}(\mathbb{T}^3)} \|g\|_{L^{q_1}(\mathbb{T}^3)} + \|f\|_{L^{p_2}(\mathbb{T}^3)} \|\langle \nabla \rangle^s g\|_{L^{q_2}(\mathbb{T}^3)}.$$

(ii) *Let* $1 < p, q, r < \infty$ *such that* $s \geqslant 3(\frac{1}{p} + \frac{1}{q} - \frac{1}{r})$*. Then, we have*

$$\big\|\langle \nabla \rangle^{-s} (fg)\big\|_{L^r(\mathbb{T}^3)} \lesssim \big\|\langle \nabla \rangle^{-s} f\big\|_{L^p(\mathbb{T}^3)} \|\langle \nabla \rangle^s g\|_{L^q(\mathbb{T}^3)}.$$

Note that while Lemma 2.2 (ii) was shown only for $s = 3(\frac{1}{p} + \frac{1}{q} - \frac{1}{r})$ in [27], the general case $s \geqslant 3(\frac{1}{p} + \frac{1}{q} - \frac{1}{r})$ follows from a straightforward modification.

Next, we state an interpolation inequality. This lemma allows us to reduce an estimate on the L^∞-norm in time to that with the L^q-norm in time for some finite q.

LEMMA 2.3. — *Let $T > 0$ and $1 \leqslant q, r \leqslant \infty$. Suppose that $s_1, s_2, s_3 \in \mathbb{R}$ satisfy $s_2 \leqslant s_1$ and*

$$q(s_1 - s_3) > s_1 - s_2.$$

Then, we have

$$\|u\|_{L^\infty([-T,T];\,W^{s_3,\,r}(\mathbb{T}^3))} \lesssim \|u\|^{1-\frac{1}{q}}_{L^q([-T,T];\,W^{s_1,\,r}(\mathbb{T}^3))} \|u\|^{\frac{1}{q}}_{W^{1,\,q}([-T,T];\,W^{s_2,\,r}(\mathbb{T}^3))}.$$

Here, the $W^{1,\,q}([-T,T]; W^{s,\,r}(\mathbb{T}^3))$-norm is defined by

$$\|f\|_{W^{1,\,q}([-T,T];\,W^{s,\,r}(\mathbb{T}^3))} = \|f\|_{L^q([-T,T];\,W^{s,\,r}(\mathbb{T}^3))} + \|\partial_t f\|_{L^q([-T,T];\,W^{s,\,r}(\mathbb{T}^3))}.$$

The proof of Lemma 2.3 follows from duality in x and Gagliardo–Nirenberg's inequality in t along with standard analysis based on (spatial) Littlewood–Paley decompositions. See [12, the proofs of Lemmas 3.2 and 3.3] for the $r = 2$ case. The proof for the general case follows from a straightforward modification.

We now recall the Strichartz estimates. Let $\mathcal{L}$ be the Klein–Gordon operator in (1.7). We use $\mathcal{L}^{-1} = (\partial_t^2 - \Delta + 1)^{-1}$ to denote the Duhamel integral operator, corresponding to the forward fundamental solution to the Klein-Gordon equation:

$$\mathcal{L}^{-1} F(t) := \int_0^t \frac{\sin\left((t - t')\langle\nabla\rangle\right)}{\langle\nabla\rangle} F(t')dt'. \tag{2.5}$$

Namely, $u := \mathcal{L}^{-1}(F)$ is the solution to the following nonhomogeneous linear equation:

$$\begin{cases} \mathcal{L}u = F \\ (u, \partial_t u)|_{t=0} = (0, 0). \end{cases}$$

The most basic regularity property of $\mathcal{L}^{-1}$ is the energy estimate:

$$\left\|\mathcal{L}^{-1}(F)\right\|_{L^\infty([-T,T];\,H^s(\mathbb{T}^3))} \lesssim \|F\|_{L^1([-T,T];\,H^{s-1}(\mathbb{T}^3))}. \tag{2.6}$$

The Strichartz estimates are important extensions of (2.6) and have been studied extensively by many mathematicians. See [25, 35, 36] in the context of the wave equation on $\mathbb{R}^d$. Thanks to the finite speed of propagation, the Strichartz estimates on $\mathbb{T}^3$ follow from the corresponding estimates on $\mathbb{R}^3$, locally in time. We now state the Strichartz estimates which are relevant for the analysis in this paper. We refer to [53] for a detailed proof.

Lemma 2.4. — *Let $0 < T \leqslant 1$. Then, the following estimate holds:*

$$\|\mathcal{L}^{-1}(F)\|_{L^4([-T,T]\times\mathbb{T}^3)} + \|\mathcal{L}^{-1}(F)\|_{L^\infty\left([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3)\right)} \lesssim \min\left(\|F\|_{L^1\left([-T,T];H^{-\frac{1}{2}}(\mathbb{T}^3)\right)}, \|F\|_{L^{\frac{4}{3}}([-T,T]\times\mathbb{T}^3)}\right). \tag{2.7}$$

For $T > 0$, we denote by X_T the closed subspace of $C([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3))$ endowed with the norm:

$$\|u\|_{X_T} = \|u\|_{L^\infty\left([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3)\right)} + \|u\|_{L^4([-T,T]\times\mathbb{T}^3)}. \tag{2.8}$$

In the following, we use shorthand notations such as $L^q_T L^r_x := L^q([-T,T]; L^r(\mathbb{T}^3))$.

2.3. On discrete convolutions

Next, we recall the following basic lemma on a discrete convolution.

Lemma 2.5. —

(i) *Let $d \geqslant 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy*

$$\alpha + \beta > d \quad \text{and} \quad \alpha, \beta < d.$$

Then, we have

$$\sum_{n = n_1 + n_2} \frac{1}{\langle n_1\rangle^\alpha \langle n_2\rangle^\beta} \lesssim \langle n\rangle^{d-\alpha-\beta}$$

for any $n \in \mathbb{Z}^d$.

(ii) *Let $d \geqslant 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha + \beta > d$.*

Then, we have

$$\sum_{\substack{n = n_1 + n_2 \\ |n_1| \sim |n_2|}} \frac{1}{\langle n_1\rangle^\alpha \langle n_2\rangle^\beta} \lesssim \langle n\rangle^{d-\alpha-\beta}$$

for any $n \in \mathbb{Z}^d$.

Namely, in the resonant case (ii), we do not have the restriction $\alpha, \beta < d$. Lemma 2.5 follows from elementary computations. See, for example, [41, Lemmas 4.1 and 4.2] for the proof.

2.4. Wiener chaos estimate

Lastly, we recall the following Wiener chaos estimate [50, Theorem I.22]. See also [51, Proposition 2.4].

Lemma 2.6. — *Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of independent standard real-valued Gaussian random variables. Given $k \in \mathbb{N}$, let $\{P_j\}_{j \in \mathbb{N}}$ be a sequence of monomials in $\bar{g} = \{g_n\}_{n \in \mathbb{N}}$ of degree at most k, namely, $P_j = P_j(\bar{g})$ is of the form $P_j = c_j \prod_{i=1}^{k_j} g_{n_i}$ with $k_j \leqslant k$ and $n_1, \dots, n_{k_j} \in \mathbb{N}$. Then, for $p \geqslant 2$, we have*

$$\Big\| \sum_{j \in \mathbb{N}} P_j(\bar{g}) \Big\|_{L^p(\Omega)} \leqslant (p-1)^{\frac{k}{2}} \Big\| \sum_{j \in \mathbb{N}} P_j(\bar{g}) \Big\|_{L^2(\Omega)}.$$

This lemma is a direct corollary to the hypercontractivity of the Ornstein-Uhlenbeck semigroup due to Nelson [42]. Note that in the definition of P_j above, we may have $n_i = n_\ell$ for $i \neq \ell$. Namely, we do not impose independence of the factors g_{n_i} of P_j in Lemma 2.6. In the following, we apply Lemma 2.6 to multilinear terms involving $\{g_n\}_{n \in \mathbb{Z}^3}$ and $\{h_n\}_{n \in \mathbb{Z}^3}$ in (1.2) by first expanding g_n and h_n into their real and imaginary parts.

3. On the random free evolution and its renormalized powers

Recall from (1.4) and (1.8) that $z_{1,N}(t,x,\omega)$ denotes the solution to the linear Klein–Gordon equation:

$$\left(\partial_t^2 - \Delta + 1\right) z_{1,N}(t,x,\omega) = 0$$

with the truncated random initial data:

$$z_{1,N}(0,x,\omega) = \sum_{|n| \leqslant N} \frac{g_n(\omega)}{\langle n \rangle^{\alpha}} e^{in\cdot x}$$

and

$$\partial_t z_{1,N}(0,x,\omega) = \sum_{|n| \leqslant N} \frac{h_n(\omega)}{\langle n \rangle^{\alpha-1}} e^{in\cdot x},$$

where $\{g_n\}_{n\in\mathbb{Z}^3}$ and $\{h_n\}_{n\in\mathbb{Z}^3}$ are as in (1.2). Given $t\in\mathbb{R}$, define $g_n^t(\omega)$ by

$$g_n^t(\omega) := \cos(t\langle n\rangle)\, g_n(\omega) + \sin(t\langle n\rangle)\, h_n(\omega). \tag{3.1}$$

Then, we have

$$\begin{aligned} z_{1,N}(t,x,\omega) &= \cos(t\langle\nabla\rangle)\,(z_{1,N}(0,x,\omega)) + \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}\,(\partial_t z_{1,N}(0,x,\omega)) \\ &= \sum_{|n|\leqslant N} \frac{g_n^t(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}. \end{aligned}$$

Using the definitions of the Gaussian random variables $\{g_n\}_{n\in\mathbb{Z}^3}$ and $\{h_n\}_{n\in\mathbb{Z}^3}$, we see that $\{g_n^t\}_{n\in\mathbb{Z}^3}$ defined in (3.1) forms a family of independent standard complex-valued Gaussian random variables conditioned that[14] $g_n^t = \overline{g_{-n}^t}$. Then, the renormalization constant σ_N defined in (1.14) is computed as

$$\begin{aligned} \sigma_N &= \mathbb{E}\Big[(z_{1,N}(t,x,\omega))^2\Big] = \sum_{|n|\leqslant N} \frac{\mathbb{E}\Big[|g_n^t(\omega)|^2\Big]}{\langle n\rangle^{2\alpha}} \\ &= \sum_{|n|\leqslant N} \frac{1}{\langle n\rangle^{2\alpha}} \sim \begin{cases} \log N, & \text{for } \alpha = \frac{3}{2}, \\ N^{3-2\alpha}, & \text{for } \alpha < \frac{3}{2}, \end{cases} \end{aligned} \tag{3.2}$$

which tends to ∞ as $N\to\infty$.

Remark 3.1. — From the definitions of the Gaussian random variables g_n and h_n and their rotational invariance, we see that

$$\mathrm{Law}(z_{1,N}(t,x)) = \mathrm{Law}(z_{1,N}(0,0))$$

for any $(t,x)\in\mathbb{R}\times\mathbb{T}^3$. This also explains the independence of σ_N from t and x.

We now define the sequences $\{Z_{j,N}\}_{N\in\mathbb{N}}$, $j=1,2,3$, by

$$\begin{aligned} Z_{1,N} &:= z_{1,N}, \quad Z_{2,N} := (z_{1,N})^2 - \sigma_N, \\ \text{and} \quad Z_{3,N} &:= (z_{1,N})^3 - 3\sigma_N z_{1,N}. \end{aligned} \tag{3.3}$$

The main goal of this section is to prove the following proposition on the regularity and convergence properties of the stochastic terms $Z_{1,N}$, $Z_{2,N}$, and $Z_{3,N}$.

[14] In particular, g_0^t is real-valued.

Proposition 3.2. — *Let $1 < \alpha \leqslant \frac{3}{2}$ and set*

$$s_1 < \alpha - \frac{3}{2}, \quad s_2 < 2\left(\alpha - \frac{3}{2}\right), \quad \text{and} \quad s_3 < 3\left(\alpha - \frac{3}{2}\right). \tag{3.4}$$

Fix $j = 1, 2$, or 3. Then, given any $T > 0$, $Z_{j,N}$ converges almost surely to some limit Z_j in $C([-T,T]; W^{s_j,\infty}(\mathbb{T}^3))$ as $N \to \infty$. Moreover, given $2 \leqslant q < \infty$, there exist positive constants C, c, κ, θ such that for every $T > 0$, there exists a set Ω_T of complemental probability smaller than $C\exp(-c/T^\kappa)$ with the following properties; given $\varepsilon > 0$, there exists $N_0 = N_0(T,\varepsilon) \in \mathbb{N}$ such that

$$\left\|Z_{j,N}\right\|_{L^q([-T,T];W^{s_j,\infty}(\mathbb{T}^3))} \leqslant T^\theta \tag{3.5}$$

and

$$\left\|Z_{j,M} - Z_{j,N}\right\|_{C([-T,T];W^{s_j,\infty}(\mathbb{T}^3))} < \varepsilon \tag{3.6}$$

for any $\omega \in \Omega_T$ and any $M \geqslant N \geqslant N_0$, where we allow $N = \infty$ with the understanding that $Z_{j,\infty} = Z_j$.

We split the proof of this proposition into several parts. We first present preliminary lemmas and then prove Proposition 3.2 at the end of this section.

Lemma 3.3. — *Let $1 < \alpha \leqslant \frac{3}{2}$ and s_j, $j = 1, 2, 3$, satisfy (3.4). Then, given $2 \leqslant q < \infty$ and $2 \leqslant r \leqslant \infty$, there exists $\delta > 0$ such that the following estimates hold for $j = 1, 2, 3$:*

$$\left\|\langle\nabla\rangle^{s_j} Z_{j,N}\right\|_{L^p\left(\Omega;L^q([-T,T];L^r(\mathbb{T}^3))\right)} \leqslant CT^{\frac{1}{q}}p^{\frac{j}{2}}, \tag{3.7}$$

$$\left\|\langle\nabla\rangle^{s_j}\left(Z_{j,M} - Z_{j,N}\right)\right\|_{L^p\left(\Omega;L^q([-T,T];L^r(\mathbb{T}^3))\right)} \leqslant CN^{-\delta}T^{\frac{1}{q}}p^{\frac{j}{2}}, \tag{3.8}$$

for any $M \geqslant N \geqslant 1$, $T > 0$, and any finite $p \geqslant 1$, where the constant C is independent of M, N, T, p.

Proof. — In the following, we only prove the difference estimate (3.8) since the first estimate (3.7) follows in a similar manner.

When $r = \infty$, we can apply the Sobolev embedding theorem and reduce the $r = \infty$ case to the case of large but finite r at the expense of a slight loss of spatial derivative. This, however, does not cause an issue since the conditions on s_j are open. Hence, we assume $r < \infty$ in the following.

Let $p \geqslant \max(q, r)$. Since

$$\langle\nabla\rangle^{s_1} Z_{1,N} = \sum_{|n|\leqslant N} \frac{g_n^t(\omega)}{\langle n\rangle^{\alpha - s_1}} e^{in\cdot x},$$

we see that $\langle\nabla\rangle^{s_1}(Z_{1,N}-Z_{1,M})(t,x)$ is a mean-zero Gaussian random variable for fixed t and x. In particular, there exists a universal constant $C>0$ such that

$$\Big\|\langle\nabla\rangle^{s_1}\big(Z_{1,M}-Z_{1,N}\big)(t,x)\Big\|_{L^p(\Omega)} \leqslant Cp^{\frac{1}{2}}\Big\|\langle\nabla\rangle^{s_1}\big(Z_{1,M}-Z_{1,N}\big)(t,x)\Big\|_{L^2(\Omega)}. \tag{3.9}$$

Then, it follows from Minkowski's integral inequality and (3.9) that

$$\begin{aligned}\Big\|\Big\|\langle\nabla\rangle^{s_1}\big(Z_{1,M}-Z_{1,N}\big)\Big\|_{L^q_TL^r_x}\Big\|_{L^p(\Omega)} &\leqslant \Big\|\Big\|\langle\nabla\rangle^{s_1}\big(Z_{1,M}-Z_{1,N}\big)(t,x)\Big\|_{L^p(\Omega)}\Big\|_{L^q_TL^r_x}\\ &\leqslant CT^{\frac{1}{q}}p^{\frac{1}{2}}\left(\sum_{N<|n|\leqslant M}\frac{1}{\langle n\rangle^{2(\alpha-s_1)}}\right)^{\frac{1}{2}} \leqslant CN^{-\delta}T^{\frac{1}{q}}p^{\frac{1}{2}}\end{aligned} \tag{3.10}$$

for some $\delta>0$ under the regularity assumption (3.4). This proves (3.8) for $j=1$.

Next, we turn to the $j=2$ case. Let us write

$$\langle\nabla\rangle^{s_2}Z_{2,N}=\mathrm{I}_N+\mathrm{II}_N, \tag{3.11}$$

where

$$\mathrm{I}_N(t,x):=\sum_{\substack{|n_1|\leqslant N,\,|n_2|\leqslant N\\ n_1\neq -n_2}}\frac{g^t_{n_1}(\omega)g^t_{n_2}(\omega)}{\langle n_1+n_2\rangle^{-s_2}\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\alpha}}e^{i(n_1+n_2)\cdot x}$$

and

$$\begin{aligned}\mathrm{II}_N(t,x)&:=\sum_{|n|\leqslant N}\langle n\rangle^{-2\alpha}\Big(\big|g^t_n(\omega)\big|^2-\mathbb{E}\Big[\big|g^t_n\big|^2\Big]\Big)\\ &=\sum_{|n|\leqslant N}\langle n\rangle^{-2\alpha}\Big(\big|g^t_n(\omega)\big|^2-1\Big).\end{aligned}$$

Fix $(t,x) \in \mathbb{R}\times\mathbb{T}^3$. By using the independence of $\{g_n^t\}_{n\in\Lambda}$ with Λ as in (1.3) and Lemma 2.5, we have[15]

$$\begin{aligned}(3.12)\quad \|\mathrm{I}_M(t,x)-\mathrm{I}_N(t,x)\|_{L^2(\Omega)}^2 &\lesssim \sum_{\substack{|n_1|\leqslant M,\,|n_2|\leqslant M\\ \max(|n_1|,|n_2|)>N}} \frac{1}{\langle n_1+n_2\rangle^{-2s_2}\langle n_1\rangle^{2\alpha}\langle n_2\rangle^{2\alpha}}\\ &\leqslant CN^{-\delta},\end{aligned}$$

for some $\delta>0$, with C independent of $M \geqslant N \geqslant 1$ and $(t,x)\in\mathbb{R}\times\mathbb{T}^3$, provided that $4\alpha-2s_2>6$. Namely, $s_2<2(\alpha-\frac{3}{2})$. Similarly, by using the independence of $\{|g_n^t(\omega)|^2-1\}_{n\in\Lambda}$, we have

$$(3.13)\qquad \big\|\mathrm{II}_M(t,x)-\mathrm{II}_N(t,x)\big\|_{L^2(\Omega)}^2 \lesssim \sum_{N<|n|\leqslant M}\frac{1}{\langle n\rangle^{4\alpha}} \leqslant CN^{-\delta}$$

for some $\delta>0$, with C independent of $M \geqslant N \geqslant 1$ and $(t,x)\in\mathbb{R}\times\mathbb{T}^3$, provided $4\alpha>3$, which is guaranteed by the assumption $\alpha>1$. Therefore, from (3.11), (3.12), and (3.13), we obtain

$$\Big\|\langle\nabla\rangle^{s_2}\big(Z_{2,M}-Z_{2,N}\big)(t,x)\Big\|_{L^2(\Omega)} \leqslant CN^{-\delta}$$

for some $\delta>0$, with a constant C independent of $M \geqslant N \geqslant 1$ and $(t,x)\in\mathbb{R}\times\mathbb{T}^3$. By the Wiener chaos estimate (Lemma 2.6), we then obtain

$$(3.14)\qquad \Big\|\langle\nabla\rangle^{s_2}\big(Z_{2,M}-Z_{2,N}\big)(t,x)\Big\|_{L^p(\Omega)} \leqslant CN^{-\delta}p$$

for any finite $p\geqslant 2$. Then, arguing as in (3.10) with Minkowski's integral inequality, the estimate (3.8) for $j=2$ follows from (3.14).

Let us finally turn to (3.8) for $j=3$. Write

$$\langle\nabla\rangle^{s_3}Z_{3,N} = \mathrm{III}_N + \mathrm{IV}_N,$$

where

$$\mathrm{III}_N(t,x) := \sum_{\substack{|n_j|\leqslant N,\, j=1,2,3\\ (n_1+n_2)(n_1+n_3)(n_2+n_3)\neq 0}} \frac{g_{n_1}^t(\omega)g_{n_2}^t(\omega)g_{n_3}^t(\omega)}{\langle n_1+n_2+n_3\rangle^{-s_3}\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\alpha}\langle n_3\rangle^{\alpha}}e^{i(n_1+n_2+n_3)\cdot x}$$

(15) Strictly speaking, in applying Lemma 2.5 when $\alpha=\frac{3}{2}$, we need to replace 2α in the exponent by $2\alpha-\varepsilon$ for some small $\varepsilon>0$. This, however, does not affect the outcome since the condition on s_2 is open. The same comment applies to (3.15) below.

and by the inclusion-exclusion principle

$$\text{IV}_N(t,x) := 3 \sum_{|n|\leqslant N} \frac{|g_n^t(\omega)|^2 - \mathbb{E}\left[|g_n^t|^2\right]}{\langle n\rangle^{2\alpha}} \sum_{|m|\leqslant N} \frac{g_m^t(\omega)}{\langle m\rangle^{\alpha-s_3}} e^{im\cdot x} - 3 \sum_{|n|\leqslant N} \frac{|g_n^t(\omega)|^2 g_n^t(\omega)}{\langle n\rangle^{3\alpha-s_3}} e^{in\cdot x} + \left|g_0^t(\omega)\right|^2 g_0^t(\omega).$$

Proceeding as above with Lemma 2.5, we have

$$\begin{aligned} &\left\|\text{III}_M(t,x) - \text{III}_N(t,x)\right\|_{L^2(\Omega)}^2 \\ &\lesssim \sum_{\substack{|n_j|\leqslant M,\, j=1,2,3\\ \max(|n_1|,|n_2|,|n_3|)>N}} \frac{1}{\langle n_1+n_2+n_3\rangle^{-2s_3}\langle n_1\rangle^{2\alpha}\langle n_2\rangle^{2\alpha}\langle n_3\rangle^{2\alpha}} \\ &\leqslant CN^{-\delta} \end{aligned} \tag{3.15}$$

for some $\delta > 0$, with C independent of $M \geqslant N \geqslant 1$ and $(t,x) \in \mathbb{R}\times\mathbb{T}^3$, provided $6\alpha - 2s_3 > 9$ and $\alpha > 1$. See Remark 3.6. Namely, $s_3 < 3(\alpha - \frac{3}{2})$ and $\alpha > 1$. Then, by the Wiener chaos estimate (Lemma 2.6), we obtain

$$\left\|\text{III}_M(t,x) - \text{III}_N(t,x)\right\|_{L^p(\Omega)} \leqslant CN^{-\delta} p^{\frac{3}{2}} \tag{3.16}$$

for any finite $p \geqslant 2$.

Let us now estimate IV_N. By Lemma 2.6 and Hölder's inequality, we have

$$\begin{aligned} \left\|\text{IV}_N(t,x)\right\|_{L^p(\Omega)} &\lesssim p^{\frac{3}{2}} \sum_{|n|\leqslant N} \frac{1}{\langle n\rangle^{3\alpha-s_3}} \\ &+ \left\| \sum_{|n|\leqslant N} \frac{|g_n^t(\omega)|^2 - \mathbb{E}\left[|g_n^t|^2\right]}{\langle n\rangle^{2\alpha}} \right\|_{L^{2p}(\Omega)} \left\| \sum_{|m|\leqslant N} \frac{g_m^t(\omega)}{\langle m\rangle^{\alpha-s_3}} e^{im\cdot x} \right\|_{L^{2p}(\Omega)}. \end{aligned}$$

The first sum on the right-hand side is convergent if $3\alpha - s_3 > 3$. Note that this condition is guaranteed under (3.4). Both factors in the second term on the right hand-side can be treated by the arguments presented above. We therefore have the bounds:

$$\left\| \sum_{|n|\leqslant N} \frac{|g_n^t(\omega)|^2 - \mathbb{E}\left[|g_n^t|^2\right]}{\langle n\rangle^{2\alpha}} \right\|_{L^{2p}(\Omega)} \leqslant Cp \tag{3.17}$$

and

$$\left\| \sum_{|m|\leqslant N} \frac{g_m^t(\omega)}{\langle m\rangle^{\alpha-s_3}} e^{im\cdot x} \right\|_{L^{2p}(\Omega)} \leqslant Cp^{\frac{1}{2}} \tag{3.18}$$

for any finite $p \geqslant 2$, provided that $4\alpha > 3$ for (3.17) and $2\alpha - 2s_3 > 3$ for (3.18). Note that the second condition is guaranteed under (3.4) with $\alpha \leqslant \frac{3}{2}$. Then, by applying the Wiener chaos estimate (Lemma 2.6), this leads to

$$\left\| \mathrm{IV}_N(t,x) \right\|_{L^p(\Omega)} \leqslant Cp^{\frac{3}{2}}.$$

A similar argument yields

$$\left\| \mathrm{IV}_M(t,x) - \mathrm{IV}_N(t,x) \right\|_{L^p(\Omega)} \leqslant CN^{-\delta}p^{\frac{3}{2}} \tag{3.19}$$

for some $\delta > 0$. Then, arguing as in (3.10) with Minkowski's integral inequality, the estimate (3.8) for $j = 3$ follows from (3.16) and (3.19). This completes the proof of Lemma 3.3. □

Thanks to Lemma 3.3, we already know that the sequences $\{Z_{j,N}\}_{N\in\mathbb{N}}$, $j = 1, 2, 3$, converge in $L^p(\Omega; L^q([-T,T]; W^{s_j,r}(\mathbb{T}^3)))$ to some limits Z_j. It turns out that the quantitative properties (3.8) of the convergence allow us to upgrade these convergences to almost sure convergences. See the proof of Proposition 3.2 below. In order to obtain convergence in $C([-T,T]; W^{s_j,r}(\mathbb{T}^3))$, however, we need to establish a difference estimate at two different times. The following lemma will be useful in this context.

Lemma 3.4. — *Let $k \geqslant 1$ be an integer. Then, we can write*

$$\prod_{j=1}^{k} g_{n_j}^t - \prod_{j=1}^{k} g_{n_j}^\tau = \sum_{\ell} c_\ell(t,\tau,n_1,\cdots,n_k) \prod_{j=1}^{k} g_{n_j}^*, \tag{3.20}$$

where $g_{n_j}^*$ is either g_{n_j} or h_{n_j} and the sum in ℓ runs over all such possibilities. Furthermore, given any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|c_\ell(t,\tau,n_1,\cdots,n_k)| \leqslant C_\delta |t-\tau|^\delta \sum_{j=1}^{k} \langle n_j\rangle^\delta. \tag{3.21}$$

Proof. — From the definition (3.1) of g_n^t, a typical term in the sum defining the right-hand side of (3.20) is given by

$$\left(\prod_{j=1}^{k} H_j\big(t\,\langle n_j\rangle\big) - \prod_{j=1}^{k} H_j\big(\tau\,\langle n_j\rangle\big) \right) \prod_{j=1}^{k} g_{n_j}^*, \tag{3.22}$$

where $H_j(t\langle n_j\rangle) = \cos(t\langle n_j\rangle)$ (with $g^*_{n_j} = g_{n_j}$) or $\sin(t\langle n_j\rangle)$ (with $g^*_{n_j} = h_{n_j}$). By the mean value theorem and the boundedness of H_j, we have

$$\left|\prod_{j=1}^k H_j\big(t\langle n_j\rangle\big) - \prod_{j=1}^k H_j\big(\tau\langle n_j\rangle\big)\right| \lesssim |t-\tau| \sum_{j=1}^k \langle n_j\rangle. \tag{3.23}$$

We also have the trivial bound

$$\left|\prod_{j=1}^k H_j(t\langle n_j\rangle) - \prod_{j=1}^k H_j\big(\tau\langle n_j\rangle\big)\right| \leqslant 2. \tag{3.24}$$

By interpolating (3.23) and (3.24), we conclude that (3.22) satisfies the claimed bound (3.21). This completes the proof of Lemma 3.4. □

In view of Lemma 3.4, a slight modification of the proof of Lemma 3.3 yields the following statement.

Lemma 3.5. — *Let $1 < \alpha \leqslant \frac{3}{2}$ and s_j satisfies (3.4), $j = 1, 2, 3$. Then, given $2 \leqslant r \leqslant \infty$, there exists $\delta > 0$ such that the following estimates hold for $j = 1, 2, 3$:*

$$\big\|\langle\nabla\rangle^{s_j}\delta_h Z_{j,N}(t)\big\|_{L^p(\Omega;\, L^r(\mathbb{T}^3))} \leqslant C p^{\frac{j}{2}} |h|^\delta, \tag{3.25}$$

$$\Big\|\langle\nabla\rangle^{s_j}\big(\delta_h Z_{j,M}(t) - \delta_h Z_{j,N}(t)\big)\Big\|_{L^p(\Omega;\, L^r(\mathbb{T}^3))} \leqslant C N^{-\delta} p^{\frac{j}{2}} |h|^\delta, \tag{3.26}$$

for any $M \geqslant N \geqslant 1$, $t \in [-T, T]$, and $h \in \mathbb{R}$ such that $t+h \in [-T, T]$, where the constant C is independent of M, N, T, p, t, and h. Here, δ_h denotes the difference operator defined by

$$\delta_h Z_{j,N}(t) = Z_{j,N}(t+h) - Z_{j,N}(t).$$

In handling the renormalized pieces, we also need the following identity, which follows directly from (3.1):

$$\begin{aligned}
\Big(|g_n^t|^2 - \mathbb{E}\Big[|g_n^t|^2\Big]\Big) &- \Big(|g_n^\tau|^2 - \mathbb{E}\Big[|g_n^\tau|^2\Big]\Big)\\
&= \big(\cos^2\big(t\langle n\rangle\big) - \cos^2(\tau\langle n\rangle)\big)\Big(|g_n|^2 - 1\Big)\\
&\quad + \big(\sin^2\big(t\langle n\rangle\big) - \sin^2(\tau\langle n\rangle)\big)\big(|h_n|^2 - 1\big)\\
&\quad + 2\big(\cos\big(t\langle n\rangle\big)\sin\big(t\langle n\rangle\big) - \cos\big(\tau\langle n\rangle\big)\sin\big(\tau\langle n\rangle\big)\big)\cdot \operatorname{Re}(g_n\overline{h_n}).
\end{aligned}$$

The first two terms on the right-hand side can be treated exactly as in the renormalized pieces in the proof of Lemma 3.3, while the last term can be handled without any difficulty.

We conclude this section by presenting the proof of Proposition 3.2.

Proof of Proposition 3.2. — Fix $2 \leqslant q < \infty$ and $j = 1, 2$, or 3. Passing to the limit $N \to \infty$ in (3.7) of Lemma 3.3, we obtain that the limit Z_j of $Z_{j,N}$ satisfies

$$\Big\| \|Z_j\|_{L^q_T W^{s_j,\infty}_x} \Big\|_{L^p(\Omega)} \leqslant C T^{\frac{1}{q}} p^{\frac{j}{2}}$$

for any finite $p \geqslant 1$. Then, it follows from Chebyshev's inequality[16] that there exists a set $\Omega^{(1)}_{T,\infty}$ of complemental probability smaller than $C \exp(-c/T^{\frac{2}{jq}})$ such that

$$\|Z_j\|_{L^q_T W^{s_j,\infty}_x} \leqslant \frac{1}{2} T^{\frac{1}{2q}} \tag{3.27}$$

for any $\omega \in \Omega^{(1)}_{T,\infty}$. Similarly, given any $N \in \mathbb{N}$, it follows from (3.8) (with $M \to \infty$) that there exists a set $\Omega^{(1)}_{T,N}$ of complemental probability smaller than $C \exp(-cN^{\frac{2\delta}{j}}/T^{\frac{2}{jq}})$ such that

$$\|Z_j - Z_{j,N}\|_{L^q_T W^{s_j,\infty}_x} \leqslant \frac{1}{2} T^{\frac{1}{2q}} \tag{3.28}$$

for any $\omega \in \Omega^{(1)}_{T,N}$. Combining (3.27) and (3.28), we see that (3.5) holds for any $\omega \in \Omega^{(1)}_T$ defined by

$$\Omega^{(1)}_T := \bigcap_{N \in \mathbb{N} \cup \{\infty\}} \Omega^{(1)}_{T,N} \tag{3.29}$$

whose complemental probability is smaller than $C \exp(-c/T^{\frac{2}{jq}})$.

Lemma 3.3 shows that the sequence $\{Z_{j,N}\}_{N \in \mathbb{N}}$ converges in

$$L^p\Big(\Omega; L^q\left([-T,T]; W^{s_j,\infty}\left(\mathbb{T}^3\right)\right)\Big)$$

to the limit Z_j. A slight modification of the proof of Lemma 3.3 shows that, given $t \in \mathbb{R}$, the sequence $\{Z_{j,N}(t)\}_{N \in \mathbb{N}}$ converges to the limit $Z_j(t)$ in $L^p(\Omega; W^{s_j,\infty}(\mathbb{T}^3))$ with the uniform bound:

$$\|Z_{j,N}(t)\|_{L^p(\Omega; W^{s_j,\infty})} \leqslant C p^{\frac{j}{2}}.$$

We first upgrade this convergence to almost sure convergence. From (3.8) in Lemma 3.3 and Chebyshev's inequality, we obtain that

$$P\left(\omega \in \Omega : \|Z_j(t) - Z_{j,N}(t)\|_{W^{s_j,\infty}} \geqslant \frac{1}{k}\right) \leqslant e^{-cN^{\frac{2\delta}{j}} k^{-\frac{2}{j}}}$$

for $k \in \mathbb{N}$, where the positive constant c is independent of k and N. Noting that the right-hand side is summable in $N \in \mathbb{N}$, we can invoke the Borel–Cantelli lemma to conclude that there exists Ω_k of full probability such that

[16] See for example [52, Lemma 4.5] and [6, the proof of Lemma 3/2.2].

for each $\omega \in \Omega_k$, there exists $M = M(\omega) \geqslant 1$ such that for any $N \geqslant M$, we have

$$\big\|Z_j(t;\omega) - Z_{j,N}(t;\omega)\big\|_{W^{s_j,\infty}} < \frac{1}{k}.$$

Now, by setting $\Sigma = \bigcap_{k=1}^{\infty} \Omega_k$, we see that $P(\Sigma) = 1$ and that, for each $\omega \in \Sigma$, $Z_{j,N}(t)$ converges to $Z_j(t)$ in $W^{s_j,\infty}(\mathbb{T}^3)$. Note that the set Σ is dependent on the choice of $t \in \mathbb{R}$.

We now prove that $\{Z_{j,N}\}_{N\in\mathbb{N}}$ converges to Z_j almost surely in $C([-T,T];W^{s_j,\infty}(\mathbb{T}^3))$. Fix $t \in [-T,T]$ and $h \in \mathbb{R}$ (such that $t+h \in [-T,T]$). From (3.25), (3.26), the almost sure convergence of $Z_{j,N}(t)$ to $Z_j(t)$, and the dominated convergence theorem, we obtain

$$\big\|\delta_h Z_j(t)\big\|_{L^p(\Omega;\,W^{s_j,\infty})} \leqslant Cp^{\frac{j}{2}}|h|^{\delta}, \tag{3.30}$$

$$\big\|\delta_h Z_j(t) - \delta_h Z_{j,N}(t)\big\|_{L^p(\Omega;\,W^{s_j,\infty})} \leqslant CN^{-\delta}p^{\frac{j}{2}}|h|^{\delta} \tag{3.31}$$

for any $N \geqslant 1$. By choosing $p \gg 1$ sufficiently large such that $p\delta > 1$, it follows from Kolmogorov's continuity criterion [5] applied to (3.25) and (3.30) that $Z_{j,N}$, $N \in \mathbb{N}$, and Z_j are almost surely continuous with values in $W^{s_j,\infty}(\mathbb{T}^3)$.

In the following, we only consider $[0,T]$. Let $Y_N = Z_j - Z_{j,N}$ and choose $p \gg 1$ sufficiently large such that $p\delta > 2$. Then, with $\theta \in (0, \delta - \frac{1}{p})$, it follows from Chebyshev's inequality and (3.31) that

$$\begin{aligned}
&P\left(\sup_{N\in\mathbb{N}}\max_{j=1,\dots,2^\ell} N^{\frac{\delta}{2}}\left\|Y_N\left(\frac{j}{2^\ell}T\right) - Y_N\left(\frac{j-1}{2^\ell}T\right)\right\|_{W^{s_j,\infty}} \geqslant 2^{-\theta\ell}\right)\\
&= P\left(\bigcup_{N\in\mathbb{N}}\bigcup_{j=1}^{2^\ell}\left\{\left\|Y_N\left(\frac{j}{2^\ell}T\right) - Y_N\left(\frac{j-1}{2^\ell}T\right)\right\|_{W^{s_j,\infty}} \geqslant N^{-\frac{\delta}{2}}2^{-\theta\ell}\right\}\right)\\
&\leqslant \sum_{N=1}^{\infty}\sum_{j=1}^{2^\ell} P\left(\left\|Y_N\left(\frac{j}{2^\ell}T\right) - Y_N\left(\frac{j-1}{2^\ell}T\right)\right\|_{W^{s_j,\infty}} \geqslant N^{-\frac{\delta}{2}}2^{-\theta\ell}\right)\\
&\leqslant \sum_{N=1}^{\infty}\sum_{j=1}^{2^\ell} N^{\frac{p\delta}{2}}2^{p\theta\ell}\,\mathbb{E}\left[\left\|Y_N\left(\frac{j}{2^\ell}T\right) - Y_N\left(\frac{j-1}{2^\ell}T\right)\right\|_{W^{s_j,\infty}}^p\right]\\
&\leqslant C(p)\cdot 2^{(p(\theta-\delta)+1)\ell}\sum_{N=1}^{\infty} N^{-\frac{p\delta}{2}}\\
&\leqslant C(p)\cdot 2^{(p(\theta-\delta)+1)\ell},
\end{aligned}$$

where we used the fact that $p\delta > 2$ in the second to the last step. Note that $p(\theta - \delta) + 1 < 0$. Then, summing over $\ell \in \mathbb{N}$, we obtain

$$\sum_{\ell=0}^{\infty} P\left(\sup_{N\in\mathbb{N}} \max_{j=1,\dots,2^\ell} N^{\frac{\delta}{2}} \left\| Y_N\left(\frac{j}{2^\ell}T\right) - Y_N\left(\frac{j-1}{2^\ell}T\right)\right\|_{W^{s_j,\infty}} \geqslant 2^{-\theta\ell}\right) < \infty.$$

Hence, by the Borel–Cantelli lemma, there exists a set $\widetilde{\Sigma} \subset \Omega$ with $P(\widetilde{\Sigma}) = 1$ such that, for each $\omega \in \widetilde{\Sigma}$, we have

$$\sup_{N\in\mathbb{N}} \max_{j=1,\dots,2^\ell} N^{\frac{\delta}{2}} \left\| Y_N\left(\frac{j}{2^\ell}T;\omega\right) - Y_N\left(\frac{j-1}{2^\ell}T;\omega\right)\right\|_{W^{s_j,\infty}} \leqslant 2^{-\theta\ell}$$

for all $\ell \geqslant L = L(\omega)$. This in particular implies that there exists $C = C(\omega) > 0$ such that

$$\max_{j=1,\dots,2^\ell} \left\| Y_N\left(\frac{j}{2^\ell}T;\omega\right) - Y_N\left(\frac{j-1}{2^\ell}T;\omega\right)\right\|_{W^{s_j,\infty}} \leqslant C(\omega) N^{-\frac{\delta}{2}} 2^{-\theta\ell} \tag{3.32}$$

for any $\ell \geqslant 0$, uniformly in $N \in \mathbb{N}$.

Fix $t \in [0,T]$. By expressing t in the following binary expansion (dilated by T):

$$t = T\sum_{j=1}^{\infty} \frac{b_j}{2^j},$$

where $b_j \in \{0,1\}$, we set $t_\ell = T\sum_{j=1}^{\ell} \frac{b_j}{2^j}$ and $t_0 = 0$. Then, from (3.32) along with the continuity of Y_N with values in $W^{s_j,\infty}(\mathbb{T}^3)$, we have

$$\begin{aligned} \|Y_N(t;\omega)\|_{W^{s_j,\infty}} &\leqslant \sum_{\ell=1}^{\infty} \|Y_N(t_\ell;\omega) - Y_N(t_{\ell-1};\omega)\|_{W^{s_j,\infty}} + \|Y_N(0;\omega)\|_{W^{s_j,\infty}} \\ &\leqslant C(\omega) N^{-\frac{\delta}{2}} \sum_{\ell=1}^{\infty} 2^{-\theta\ell} + \|Y_N(0;\omega)\|_{W^{s_j,\infty}} \\ &\leqslant C'(\omega) N^{-\frac{\delta}{2}} + \|Y_N(0;\omega)\|_{W^{s_j,\infty}} \end{aligned} \tag{3.33}$$

for each $\omega \in \widetilde{\Sigma}$. Note that the right-hand side of (3.33) is independent of $t \in [0,T]$. Hence, by taking a supremum in $t \in [0,T]$, we obtain

$$\begin{aligned} &\|Z_j(\omega) - Z_{j,N}(\omega)\|_{C([0,T];W^{s_j,\infty})} \\ &\qquad \leqslant C'(\omega) N^{-\frac{\delta}{2}} + \|Y_N(0;\omega)\|_{W^{s_j,\infty}} \longrightarrow 0 \end{aligned}$$

as $N \to \infty$. Here, we used the almost sure convergence of $\{Z_{j,N}(0)\}_{N\in\mathbb{N}}$ to $Z_j(0)$ in $W^{s_j,\infty}(\mathbb{T}^3)$. This proves almost sure convergence of $\{Z_{j,N}\}_{N\in\mathbb{N}}$ in $C([-T,T];W^{s_j,\infty}(\mathbb{T}^3))$.

Lastly, it follows from Egoroff's theorem that, given $T>0$, there exists $\Omega_T^{(2)}$ of complemental probability smaller than $C\exp(-c/T^\kappa)$ such that the estimate (3.6) holds. Finally, by setting $\Omega_T = \Omega_T^{(1)} \cap \Omega_T^{(2)}$, where $\Omega_T^{(1)}$ is as in (3.29), we see that both (3.5) and (3.6) hold on Ω_T. This completes the proof of Proposition 3.2. □

Remark 3.6. — The restriction $\alpha > 1$ in Proposition 3.2 appears in making sense of the renormalized cubic power Z_3 and it reflects the well-known fact that Wick powers of degree $\geqslant 3$ for the three-dimensional Gaussian free field do not exist. See for example [22, Section 2.7].

4. On the second order stochastic term $z_{2,N}$

We first study the regularity and convergence properties of $z_{2,N}$ defined in (1.17). For notational convenience, we set

$$\begin{aligned} Z_{4,N} := z_{2,N} &= -\mathcal{L}^{-1}\left((z_{1,N})^3 - 3\sigma_N z_{1,N}\right) \\ &= -\mathcal{L}^{-1} Z_{3,N}. \end{aligned} \tag{4.1}$$

As a consequence of Proposition 3.2, we have the following statement.

Proposition 4.1. — *Let $1 < \alpha \leqslant \frac{3}{2}$ and set*

$$s_4 < 3\left(\alpha - \frac{3}{2}\right) + 1. \tag{4.2}$$

Then, given any $T>0$, $Z_{4,N}$ converges almost surely to some limit Z_4 in $C([-T,T];W^{s_4,\infty}(\mathbb{T}^3))$ as $N\to\infty$. Moreover, there exist positive constants C, c, κ, θ such that for every $T>0$, there exists a set Ω_T of complemental probability smaller than $C\exp(-c/T^\kappa)$ such that given $\varepsilon>0$, there exists $N_0 = N_0(T,\varepsilon)\in\mathbb{N}$ such that

$$\|Z_{4,N}\|_{C([-T,T];W^{s_4,\infty}(\mathbb{T}^3))} \leqslant T^\theta$$

and

$$\|Z_{4,M} - Z_{4,N}\|_{C([-T,T];W^{s_4,\infty}(\mathbb{T}^3))} < \varepsilon$$

for any $\omega\in\Omega_T$ and any $M \geqslant N \geqslant 1$, where we allow $N=\infty$ with the understanding that $Z_{4,\infty} = Z_4$.

Proof. — Given s_4 satisfying (4.2), choose $\varepsilon > 0$ sufficiently small such that

$$s_4 + 2\varepsilon < 3\left(\alpha - \frac{3}{2}\right) + 1. \tag{4.3}$$

By Sobolev's inequality, there exists finite $r \gg 1$ such that

$$\left\|Z_{4,\,N}\right\|_{C_T W_x^{s_4,\,\infty}} \lesssim \left\|Z_{4,\,N}\right\|_{C_T W_x^{s_4+\varepsilon,\,r}}. \tag{4.4}$$

Furthermore, by Lemma 2.3, there exists finite $q \gg 1$ such that

$$\begin{aligned}\left\|Z_{4,\,N}\right\|_{C_T W_x^{s_4+\varepsilon,\,r}} &\leqslant \left\|Z_{4,\,N}\right\|^{1-\frac{1}{q}}_{L^q_T W_x^{s_4+2\varepsilon,\,r}} \left\|Z_{4,\,N}\right\|^{\frac{1}{q}}_{W_T^{1,\,q} W_x^{s_4-1,\,r}} \\ &\lesssim \left\|Z_{4,\,N}\right\|_{L^q_T W_x^{s_4+2\varepsilon,\,r}} + \left\|\partial_t Z_{4,\,N}\right\|_{L^q_T W_x^{s_4-1,\,r}},\end{aligned} \tag{4.5}$$

where we applied Young's inequality in the second step. From (4.1) with (2.5), we have

$$\partial_t Z_{4,\,N} = -\int_0^t \cos\big((t-t')\langle\nabla\rangle\big) Z_{3,\,N}(t')dt'. \tag{4.6}$$

Hence, from (4.4), (4.5), and (4.6) with (4.1), we obtain

$$\left\|Z_{4,\,N}\right\|_{C_T W_x^{s_4,\,\infty}} \lesssim T^{1-\frac{1}{q}} \sum_{\beta \in \{-1,1\}} \left\|F_N^\beta(t,t')\right\|_{L^q_{t,\,t'}\left([-T,T]^2;\, W_x^{s_4-1+2\varepsilon,\,r}\right)}, \tag{4.7}$$

where F_N^β is given by

$$F_N^\beta(t,t') = e^{i\beta(t-t')\langle\nabla\rangle} Z_{3,\,N}(t').$$

Fix $(t,t',x) \in \mathbb{R}^2 \times \mathbb{T}$. Since the propagator $e^{i\beta(t-t')\langle\nabla\rangle}$ does not affect the computation done for $Z_{3,\,N}$ in the proof of Lemma 3.3, we obtain

$$\left\|F_N^\beta(t,t',x)\right\|_{L^p(\Omega)} \leqslant Cp^{\frac{3}{2}}, \tag{4.8}$$

uniformly in $(t,t',x) \in \mathbb{R}^2 \times \mathbb{T}$. Therefore, given finite $p \geqslant \max(q,r)$, from (4.7), Minkowski's integral inequality, and (4.8), we have

$$\begin{aligned}&\Big\|\left\|Z_{4,\,N}\right\|_{C_T W_x^{s_4,\,\infty}}\Big\|_{L^p(\Omega)} \\ &\qquad \lesssim T^{1-\frac{1}{q}} \sum_{\beta \in \{-1,1\}} \Big\|\left\|F_N^\beta(t,t',x)\right\|_{L^p(\Omega)}\Big\|_{L^q_{t,\,t'}\left([-T,T]^2;\, W_x^{s_4-1+2\varepsilon,\,r}\right)} \\ &\qquad \lesssim p^{\frac{3}{2}} T^{1-\frac{1}{q}}\end{aligned}$$

thanks to the regularity restriction (4.3). Then, the rest follows from proceeding as in the proof of Proposition 3.2 (in addition to Lemma 3.4, one should take into account the trivial continuity property in t of the time integration in the definition of $Z_{4,N}$). $\square$

We also need to study the following quintic stochastic term:

$$\begin{aligned} Z_{5,N} &:= \left\{(z_{1,N})^2 - \sigma_N\right\} z_{2,N} \\ &= -\left\{(z_{1,N})^2 - \sigma_N\right\} \cdot \mathcal{L}^{-1}\left((z_{1,N})^3 - 3\sigma_N z_{1,N}\right). \end{aligned} \tag{4.9}$$

We have the following statement.

Proposition 4.2. — *Let $1 < \alpha \leqslant \frac{3}{2}$ and set*

$$s_5 < \min\left(5\alpha - \frac{13}{2}, 2\left(\alpha - \frac{3}{2}\right)\right). \tag{4.10}$$

Then, given any $T > 0$, $Z_{5,N}$ converges almost surely to some limit Z_5 in $C([-T,T]; W^{s_5,\infty}(\mathbb{T}^3))$ as $N \to \infty$. Moreover, there exist positive constants C, c, κ, θ such that for every $T > 0$, there exists a set Ω_T of complemental probability smaller than $C\exp(-c/T^\kappa)$ such that given $\varepsilon > 0$, there exists $N_0 = N_0(T,\varepsilon) \in \mathbb{N}$ such that

$$\left\|Z_{5,N}\right\|_{C([-T,T];W^{s_5,\infty}(\mathbb{T}^3))} \leqslant T^\theta$$

and

$$\left\|Z_{5,M} - Z_{5,N}\right\|_{C([-T,T];W^{s_5,\infty}(\mathbb{T}^3))} < \varepsilon$$

for any $\omega \in \Omega_T$ and any $M \geqslant N \geqslant 1$, where we allow $N = \infty$ with the understanding that $Z_{5,\infty} = Z_5$.

Remark 4.3. — When $\alpha \geqslant \frac{7}{6}$ (which in particular includes the case $\alpha > \frac{5}{4}$), the regularity condition (4.10) reduces to $s_5 < 2(\alpha - \frac{3}{2})$.

Proof. — By the paraproduct decomposition (1.32), we have

$$\begin{aligned} Z_{5,N} &= Z_{2,N} z_{2,N} \\ &= Z_{2,N} \olessthan z_{2,N} + Z_{2,N} \ominus z_{2,N} + Z_{2,N} \ogreaterthan z_{2,N}. \end{aligned}$$

Note that $2(\alpha - \frac{3}{2}) \leqslant \min(0, 3(\alpha - \frac{3}{2}) + 1)$ for $\alpha \in (1, \frac{3}{2}]$. Then, from Lemma 2.1, we have

$$\left\|Z_{2,N} \olessthan z_{2,N}(t)\right\|_{W^{s_5,\infty}} \lesssim \left\|Z_{2,N}(t)\right\|_{W^{2\left(\alpha-\frac{3}{2}\right)-\varepsilon,\infty}} \left\|z_{2,N}(t)\right\|_{W^{3\left(\alpha-\frac{3}{2}\right)+1-\varepsilon,\infty}}$$

for small $\varepsilon > 0$, provided that s_5 satisfies

$$s_5 < 2\left(\alpha - \frac{3}{2}\right) + 3\left(\alpha - \frac{3}{2}\right) + 1 = 5\alpha - \frac{13}{2}. \tag{4.11}$$

Similarly, for s_5 satisfying (4.11), Lemma 2.1 yields

$$\left\|Z_{2,N} \ogreaterthan z_{2,N}(t)\right\|_{W^{s_5,\infty}} \lesssim \left\|Z_{2,N}(t)\right\|_{W^{2\left(\alpha-\frac{3}{2}\right)-\varepsilon,\infty}} \left\|z_{2,N}(t)\right\|_{W^{3\left(\alpha-\frac{3}{2}\right)+1-\varepsilon,\infty}}$$

for small $\varepsilon > 0$, provided that $3(\alpha - \frac{3}{2}) + 1 - \varepsilon < 0$ namely, $\alpha \leqslant \frac{7}{6}$. On the other hand, when $\alpha > \frac{7}{6}$, we see from Proposition 4.1 that $z_{2,N}$ has a spatial positive regularity (for each fixed t). In this case, we have

$$\left\| Z_{2,N} \circledcirc z_{2,N}(t) \right\|_{W^{s_5,\infty}} \lesssim \left\| Z_{2,N}(t) \right\|_{W^{2\left(\alpha - \frac{3}{2}\right)-\varepsilon,\infty}} \left\| z_{2,N}(t) \right\|_{L^\infty}$$

as long as

$$s_5 < 2\left(\alpha - \frac{3}{2}\right). \tag{4.12}$$

Note that the condition (4.12) is stronger than (4.11) when $\alpha > \frac{7}{6}$.

It remains to study the resonant product $z_{2,N} \circleddash Z_{2,N}$. When $\alpha > \frac{13}{10}$, we have

$$2\left(\alpha - \frac{3}{2}\right) + 3\left(\alpha - \frac{3}{2}\right) + 1 = 5\alpha - \frac{13}{2} > 0$$

and thus Lemma 2.1 yields

$$\left\| Z_{2,N} \circleddash z_{2,N}(t) \right\|_{W^{s_5,\infty}} \lesssim \left\| Z_{2,N}(t) \right\|_{W^{2\left(\alpha - \frac{3}{2}\right)-\varepsilon,\infty}} \left\| z_{2,N}(t) \right\|_{W^{3\left(\alpha - \frac{3}{2}\right)+1-\varepsilon,\infty}}$$

for s_5 satisfying (4.11). Next, we consider the case $1 < \alpha \leqslant \frac{13}{10}$. Using the independence of $\{g_n^t\}_{n \in \Lambda}$, we have

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{(t,x) \in \mathbb{R} \times \mathbb{T}^3} \mathbb{E}\left[\left| \langle \nabla \rangle^{s_5} \left(Z_{2,N} \circleddash z_{2,N} \right)(t,x) \right|^2 \right] & \\ \lesssim \sum_{n \in \mathbb{Z}^3} \langle n \rangle^{2s_5} \sum_{\substack{n = m_1 + m_2 \\ |m_1| \sim |m_2|}} A(m_1) B(m_2), & \end{aligned}$$

where $A(m_1)$ and $B(m_2)$ are given by

$$A(m_1) = \sum_{m \in \mathbb{Z}^3} \frac{1}{\langle m \rangle^{2\alpha} \langle m_1 - m \rangle^{2\alpha}}$$

and

$$B(m_2) = \frac{1}{\langle m_2 \rangle^2} \sum_{(n_1, n_2) \in \mathbb{Z}^6} \frac{1}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha} \langle m_2 - n_1 - n_2 \rangle^{2\alpha}}$$

In the following, we only consider the case $\alpha < \frac{3}{2}$. We clearly have the bound

$$A(m_1) \lesssim \frac{1}{\langle m_1 \rangle^{4\alpha - 3}},$$

provided that $\alpha > \frac{4}{3}$. Similarly, by Lemma 2.5, we have

$$B(m_2) \lesssim \frac{1}{\langle m_2 \rangle^{6\alpha - 4}},$$

provided that $\alpha > 1$. Hence, we obtain

$$\sum_{\substack{n = m_1 + m_2 \\ |m_1| \sim |m_2|}} A(m_1)B(m_2) \lesssim \sum_{\substack{m_1 \in \mathbb{Z}^3 \\ |m_1| \sim |n - m_1|}} \frac{1}{\langle m_1 \rangle^{4\alpha - 3} \langle n - m_1 \rangle^{6\alpha - 4}} \lesssim \frac{1}{\langle n \rangle^{10\alpha - 10}},$$

where we crucially used the resonant restriction $|m_1| \sim |n - m_1|$. Therefore, we obtain

$$\sup_{N \in \mathbb{N}} \sup_{(t,x) \in [0,T] \times \mathbb{T}^3} \mathbb{E}\left[\left| \langle \nabla \rangle^{s_5} \left(Z_{2,N} \circleddash z_{2,N} \right)(t,x) \right|^2 \right] \lesssim \sum_{n \in \mathbb{Z}^3} \frac{1}{\langle n \rangle^{-2s_5 + 10\alpha - 10}},$$

where the last sum is convergent, provided that s_5 satisfies (4.11). With this bound in hand, we can proceed as in the proof of Proposition 3.2 (with Lemmas 3.3 and 3.5). This completes the proof of Proposition 4.2. □

Remark 4.4. —

(i) When $\alpha > \frac{13}{10}$, we made sense of the resonant product $Z_{2,N} \circleddash z_{2,N}$ in a *deterministic* manner. Namely, we only used the almost sure regularity properties of $Z_{2,N}$ and $z_{2,N}$ but did not use the random structure of these terms in making sense of their resonant product. On the other hand, when $1 < \alpha \leqslant \frac{13}{10}$, the sum of the regularities of $Z_{2,N}$ and $z_{2,N}$ is negative and thus their resonant product does not make sense in a deterministic manner. This requires us to make sense of the resonant product $Z_{2,N} \circleddash z_{2,N}$ via a *probabilistic* argument. Hence, when $1 < \alpha \leqslant \frac{13}{10}$, we need to view the limit $Z_5^{\circleddash} = Z_{2,\infty} \circleddash z_{2,\infty}$ as part of a predefined enhanced data set, leading to a different interpretation of the equation for $w = u - z_1 - z_2$. See Subsection 1.3 for a further discussion. Lastly, we point out that the resulting regularity restriction (4.11) holds for both cases $\alpha > \frac{13}{10}$ and $1 < \alpha \leqslant \frac{13}{10}$.

(ii) When $\alpha = 1$, there is a logarithmically divergent contribution in taking a limit of $Z_{5,N}$ as $N \to \infty$. In this case, we need to introduce another renormalization, eliminating a quartic singularity. For a related argument in the parabolic setting, see [41].

5. Proof of Theorem 3

5.1. Setup

Recall that $u_N = z_{1,N} + z_{2,N} + w_N$, where w_N solves the equation (1.19). In Sections 3 and 4, we already established the necessary regularity and convergence properties of the sequences $\{z_{j,N}\}_{N \in \mathbb{N}}$, $j = 1, 2$. It remains to establish the convergence of the sequence $\{w_N\}_{N \in \mathbb{N}}$. This will be done by (i) first establishing multilinear estimates via a purely deterministic method and then (ii) applying the regularity and convergence properties of the relevant stochastic terms from Sections 3 and 4.

With (3.3) and (4.9), we can write the equation (1.19) as

$$\begin{cases} \mathcal{L} w_N + F_0 + F_1(w_N) + F_2(w_N) + F_3(w_N) = 0 \\ (w_N, \partial_t w_N)\,|_{t=0} = (0,0), \end{cases}$$

where the source term[17] is given by

$$F_0 = 3Z_{5,N} + 3z_{1,N}(z_{2,N})^2 + (z_{2,N})^3,$$

the linear term in w_N is given by

$$F_1(w_N) = 3Z_{2,N} w_N + 6 z_{1,N} z_{2,N} w_N + 3(z_{2,N})^2 w_N,$$

and the nonlinear terms in w_N are as follows:

$$F_2(w_N) = 3z_{1,N}(w_N)^2 + 3z_{2,N} w_N^2 \quad \text{and} \quad F_3(w_N) = w_N^3.$$

In the following, we study the Duhamel formulation for w_N:

$$w_N = \mathcal{L}^{-1}(F_0) + \mathcal{L}^{-1}(F_1(w_N)) + \mathcal{L}^{-1}(F_2(w_N)) + \mathcal{L}^{-1}(F_3(w_N)). \tag{5.1}$$

In the next three subsections, we first establish estimates for each individual term in the X_T-norm defined in (2.8). In Subsection 5.5, we then combine these estimates with the regularity and convergence properties of the relevant stochastic terms from Sections 3 and 4 and prove almost sure convergence of the sequence $\{w_N\}_{N \in \mathbb{N}}$. In the following, we fix $0 < T \leqslant 1$.

5.2. On the nonlinear terms in w_N

By the Strichartz estimate (2.7), we have

$$\left\| \mathcal{L}^{-1}\left(F_3(w_N)\right) \right\|_{X_T} \lesssim \left\| w_N^3 \right\|_{L^{\frac{4}{3}}_{T,x}} \leqslant \|w_N\|_{X_T}^3. \tag{5.2}$$

(17) Namely, the purely stochastic terms independent of the unknown w_N.

We now turn to the analysis of $\mathcal{L}^{-1}(F_2(w_N))$. By (2.7), we have

$$\left\|\mathcal{L}^{-1}\left(F_2(w_N)\right)\right\|_{X_T} \lesssim \left\|\langle\nabla\rangle^{-\frac{1}{2}}\left(z_{1,N}w_N^2 + z_{2,N}w_N^2\right)\right\|_{L^1_T L^2_x}. \tag{5.3}$$

In the following, we first establish an estimate for fixed $t \in [-T, T]$. Let $\sigma_1 > 0$. By Sobolev's inequality,

$$\left\|\langle\nabla\rangle^{-\frac{1}{2}}\left(z_{1,N}w_N^2\right)(t)\right\|_{L^2} \lesssim \left\|\langle\nabla\rangle^{-\sigma_1}\left(z_{1,N}w_N^2\right)(t)\right\|_{L^r},$$

provided that

$$\frac{1}{2} - \sigma_1 \geqslant \frac{3}{r} - \frac{3}{2}. \tag{5.4}$$

By Lemma 2.2 (ii), we have

$$\left\|\langle\nabla\rangle^{-\sigma_1}\left(z_{1,N}w_N^2\right)(t)\right\|_{L^r} \lesssim \left\|\langle\nabla\rangle^{-\sigma_1}z_{1,N}(t)\right\|_{L^p}\left\|\langle\nabla\rangle^{\sigma_1}\left(w_N^2\right)(t)\right\|_{L^q},$$

provided that $0 \leqslant \sigma_1 \leqslant 1$ and

$$\sigma_1 \geqslant \frac{3}{p} + \frac{3}{q} - \frac{3}{r}. \tag{5.5}$$

In the following, we will choose $p \gg 1$ such that $\sigma_1 > \frac{3}{q} - \frac{3}{r}$ guarantees (5.5). By Lemma 2.2 (i) and Sobolev's inequality, we have

$$\begin{aligned}\left\|\langle\nabla\rangle^{\sigma_1}\left(w_N^2\right)(t)\right\|_{L^q} &\lesssim \left\|\langle\nabla\rangle^{\sigma_1}w_N(t)\right\|_{L^{q_1}}\left\|w_N(t)\right\|_{L^4}\\ &\lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}w_N\right\|_{L^2}\left\|w_N(t)\right\|_{L^4},\end{aligned}$$

provided that

$$\frac{1}{q} = \frac{1}{4} + \frac{1}{q_1} \quad \text{and} \quad \frac{1}{2} - \sigma_1 \geqslant \frac{3}{2} - \frac{3}{q_1}. \tag{5.6}$$

In summary, if the conditions (5.4), (5.5), and (5.6) are satisfied, then we obtain the estimate

$$\begin{aligned}&\left\|\langle\nabla\rangle^{-\frac{1}{2}}\left(z_{1,N}w_N^2\right)(t)\right\|_{L^2}\\ &\qquad\lesssim \left\|\langle\nabla\rangle^{-\sigma_1}z_{1,N}(t)\right\|_{L^p}\left\|\langle\nabla\rangle^{\frac{1}{2}}(w_N)(t)\right\|_{L^2}\left\|w_N(t)\right\|_{L^4}.\end{aligned} \tag{5.7}$$

Let us now show that we may ensure (5.4), (5.5), and (5.6). Since $p \gg 1$, it suffices to ensure that

$$\sigma_1 > \frac{3}{q} - \frac{3}{r} = \frac{3}{4} + \frac{3}{q_1} - \frac{3}{r} \geqslant \frac{3}{4} + \frac{3}{2} - \left(\frac{1}{2} - \sigma_1\right) - \frac{3}{r} \geqslant 2\sigma_1 - \frac{1}{4}.$$

This shows that we can ensure (5.5) and (5.6) if $\sigma_1 < \frac{1}{4}$. In this case, by (5.7), we arrive at the bound:

$$\left\| \langle \nabla \rangle^{-\frac{1}{2}} \left(z_{1,N} w_N^2 \right) \right\|_{L^1_T L^2_x} \lesssim \left\| \langle \nabla \rangle^{-\sigma_1} z_{1,N} \right\|_{L^{\frac{4}{3}}_T L^p_x} \left\| \langle \nabla \rangle^{\frac{1}{2}} w_N \right\|_{L^\infty_T L^2_x} \| w_N \|_{L^4_T L^4_x} . \tag{5.8}$$

Therefore, from (5.3) and (5.8) with the definition (2.8) of the X_T-norm, we obtain

$$\left\| \mathcal{L}^{-1} \left(F_2(w_N) \right) \right\|_{X_T} \lesssim T^{\frac{1}{4}} \left(\| z_{1,N} \|_{L^2_T W^{s_1,\infty}_x} + \| z_{2,N} \|_{L^2_T W^{s_1,\infty}_x} \right) \| w_N \|^2_{X_T} , \tag{5.9}$$

provided that

$$s_1 = -\sigma_1 > -\frac{1}{4}. \tag{5.10}$$

5.3. On the linear terms in w_N

Let us next turn to the analysis of the terms linear in w_N. By the Strichartz estimate (2.7), we have

$$\left\| \mathcal{L}^{-1} \left(F_1(w_N) \right) \right\|_{X_T} \lesssim \left\| \langle \nabla \rangle^{-\frac{1}{2}} \left(Z_{2,N} w_N \right) \right\|_{L^1_T L^2_x} + \left\| \langle \nabla \rangle^{-\frac{1}{2}} \left(z_{1,N} z_{2,N} w_N \right) \right\|_{L^1_T L^2_x} + \left\| \left(z_{2,N} \right)^2 w_N \right\|_{L^{\frac{4}{3}}_{T,x}} . \tag{5.11}$$

We now evaluate each contribution on the right-hand side of (5.11). By Hölder's inequality, we have

$$\begin{aligned} \left\| \left(z_{2,N} \right)^2 w_N \right\|_{L^{\frac{4}{3}}_{,x}} &\leqslant T^{\frac{1}{4}} \| z_{2,,N} \|^2_{L^8_T L^4_x} \| w_N \|_{L^4_{T,x}} \\ &\leqslant T^{\frac{1}{4}} \| z_{2,N} \|^2_{L^8_T W^{s_4,\infty}_x} \| w_N \|_{X_T} , \end{aligned} \tag{5.12}$$

provided that

$$s_4 \geqslant 0. \tag{5.13}$$

By Lemma 2.2 (ii), we have

$$\begin{aligned}(5.14)\quad \left\|\langle\nabla\rangle^{-\frac{1}{2}}\left(Z_{2,N} w_N\right)\right\|_{L_T^1 L_x^2} &\lesssim T^{\frac{1}{2}}\left\|\langle\nabla\rangle^{-\frac{1}{2}} Z_{2,N}\right\|_{L_T^2 L_x^6}\left\|\langle\nabla\rangle^{\frac{1}{2}} w_N\right\|_{L_T^\infty L_x^2} \\ &\lesssim T^{\frac{1}{2}}\left\|Z_{2,N}\right\|_{L_T^2 W_x^{s_2,\infty}}\left\|w_N\right\|_{X_T},\end{aligned}$$

provided that

$$(5.15)\quad s_2 \geqslant -\frac{1}{2}.$$

Finally, by applying Lemma 2.2 (ii) twice, we obtain

$$\begin{aligned}(5.16)\quad \left\|\langle\nabla\rangle^{-\frac{1}{2}}\left(z_{1,N} z_{2,N} w_N\right)\right\|_{L_T^1 L_x^2} &\lesssim\left\|\langle\nabla\rangle^{s_1}\left(z_{1,N} z_{2,N}\right)\right\|_{L_T^1 L_x^6}\left\|\langle\nabla\rangle^{\frac{1}{2}} w_N\right\|_{L_T^\infty L_x^2} \\ &\lesssim T^{\frac{1}{2}}\left\|z_{1,N}\right\|_{L_T^4 W_x^{s_1,\infty}}\left\|z_{2,N}\right\|_{L_T^4 W_x^{s_4,\infty}}\left\|w_N\right\|_{X_T},\end{aligned}$$

provided that

$$(5.17)\quad \max\left(-\frac{1}{2},-s_4\right) \leqslant s_1 \leqslant 0.$$

Therefore, putting (5.11), (5.12), (5.14), and (5.16), we obtain

$$\begin{aligned}(5.18)\quad \left\|\mathcal{L}^{-1}\left(F_1\left(w_N\right)\right)\right\|_{X_T} \lesssim T^\theta\Big\{&\left\|Z_{2,N}\right\|_{L_T^2 W_x^{s_2,\infty}}+\left\|z_{1,N}\right\|_{L_T^4 W_x^{s_1,\infty}}\left\|z_{2,N}\right\|_{L_T^4 W_x^{s_4,\infty}} \\ &+\left\|z_{2,N}\right\|_{L_T^8 W_x^{s_4,\infty}}^2\Big\}\left\|w_N\right\|_{X_T}\end{aligned}$$

for some $\theta>0$ and s_1, s_2, and s_4 satisfying (5.13), (5.15), and (5.17).

5.4. On the source terms

We now estimate the contributions from the source terms. Let s_1 and s_4 satisfy (5.17). Then, by Lemma 2.2 (ii) with Hölder's inequality followed by Lemma 2.2 (i), we have

$$
\begin{aligned}
(5.19)\quad & \left\| \langle\nabla\rangle^{-\frac{1}{2}} \left(z_{1,N} \left(z_{2,N}\right)^2 \right) \right\|_{L^1_T L^2_x} \\
&\leqslant \left\| \langle\nabla\rangle^{s_1} \left(z_{1,N} \left(z_{2,N}\right)^2 \right) \right\|_{L^1_T L^2_x} \\
&\lesssim \left\| \langle\nabla\rangle^{s_1} z_{1,N} \right\|_{L^2_T L^4_x} \left\| \langle\nabla\rangle^{s_4} \left(\left(z_{2,N}\right)^2 \right) \right\|_{L^2_T L^4_x} \\
&\lesssim T^{\frac{1}{4}} \left\| \langle\nabla\rangle^{s_1} z_{1,N} \right\|_{L^4_{T,x}} \left\| \langle\nabla\rangle^{s_4} z_{2,N} \right\|^2_{L^4_T L^8_x} .
\end{aligned}
$$

Hence, from the Strichartz estimate (2.7) and (5.19), we obtain

$$
\begin{aligned}
(5.20)\quad & \left\| \mathcal{L}^{-1}(F_0) \right\|_{X_T} \\
&\lesssim \left\| \langle\nabla\rangle^{-\frac{1}{2}} Z_{5,N} \right\|_{L^1_T L^2_x} + \left\| \langle\nabla\rangle^{-\frac{1}{2}} \left(z_{1,N} \left(z_{2,N}\right)^2 \right) \right\|_{L^1_T L^2_x} + \left\| z_{2,N} \right\|^3_{L^4_T L^4_x} \\
&\lesssim T^{\theta} \Big\{ \left\| Z_{5,N} \right\|_{L^2_T W^{s_5,\infty}_x} + \left\| z_{1,N} \right\|_{L^4_T W^{s_1,\infty}_x} \left\| z_{2,N} \right\|^2_{L^4_T W^{s_4,\infty}_x} \\
&\qquad + \left\| z_{2,N} \right\|^3_{L^8_T W^{s_4,\infty}_x} \Big\}
\end{aligned}
$$

for some $\theta > 0$, provided that s_1 and s_4 satisfy (5.17) and that s_5 satisfies

$$
s_5 \geqslant -\frac{1}{2}. \tag{5.21}
$$

5.5. End of the proof

Let s_1, s_2, s_4, and s_5 satisfy (5.10), (5.13), (5.15), (5.17), and (5.21). Then, from (5.1), (5.2), (5.9), (5.18), and (5.20), we have

$$
\begin{aligned}
\|w_N\|_{X_T} &\leqslant CT^{\theta} A^{(1)}_N + CT^{\theta} A^{(2)}_N \|w_N\|_{X_T} \\
&\qquad + CT^{\theta} \left(\sum_{j=1}^{2} \|z_{j,N}\|_{L^2_T W^{s_j,\infty}_x} \right) \|w_N\|^2_{X_T} + C\|w_N\|^3_{X_T},
\end{aligned}
$$

where $A^{(1)}_N$ and $A^{(2)}_N$ are defined by

$$
\begin{aligned}
(5.22)\quad A^{(1)}_N &= \|Z_{5,N}\|_{L^2_T W^{s_5,\infty}_x} \\
&\qquad + \|z_{1,N}\|_{L^4_T W^{s_1,\infty}_x} \|z_{2,N}\|^2_{L^4_T W^{s_4,\infty}_x} + \|z_{2,N}\|^3_{L^8_T W^{s_4,\infty}_x}, \\
A^{(2)}_N &= \|Z_{2,N}\|_{L^2_T W^{s_2,\infty}_x} \\
&\qquad + \|z_{1,N}\|_{L^4_T W^{s_1,\infty}_x} \|z_{2,N}\|_{L^4_T W^{s_4,\infty}_x} + \|z_{2,N}\|^2_{L^8_T W^{s_4,\infty}_x} .
\end{aligned}
$$

Suppose that

$$R(T) := \sup_{N \in \mathbb{N}} \max \Big(\|z_{1,N}\|_{L^4_T W^{s_1,\infty}_x}, \|z_{2,N}\|_{L^8_T W^{s_4,\infty}_x}, \|Z_{2,N}\|_{L^2_T W^{s_2,\infty}_x}, \|Z_{5,N}\|_{L^2_T W^{s_5,\infty}_x} \Big) \leqslant T^{\theta_0} \tag{5.23}$$

for some $\theta_0 > 0$. Then, it follows from a standard continuity argument that there exists $T_0 > 0$ such that

$$\|w_N\|_{X_T} \leqslant C(R) T^{\theta}$$

for any $0 < T \leqslant T_0$, uniformly in $N \in \mathbb{N}$. Here, we used the fact that $(w, \partial_t w)|_{t=0} = (0,0)$.

Let $M \geqslant N \geqslant 1$. Note that F_j, $j = 0,1,2,3$, are multilinear in w_N and the stochastic terms $z_{1,N}$, $z_{2,N}$, $Z_{2,N}$, and $Z_{5,N}$. Then, by proceeding as in Subsections 5.2, 5.3, and 5.4, we also obtain the following difference estimate:

$$\begin{aligned} \|w_M - w_N\|_{X_T} &\leqslant CT^\theta B^{(1)}_{M,N} + CT^\theta B^{(2)}_{M,N} \|w_N\|_{X_T} \\ &\quad + CT^\theta A^{(2)}_N \|w_M - w_N\|_{X_T} \\ &\quad + CT^\theta \left(\sum_{j=1}^2 \|z_{j,M} - z_{j,N}\|_{L^2_T W^{s_j,\infty}_x} \right) \|w_M\|^2_{X_T} \\ &\quad + CT^\theta \left(\sum_{j=1}^2 \|z_{j,N}\|_{L^2_T W^{s_j,\infty}_x} \right) \\ &\qquad \times (\|w_M\|_{X_T} + \|w_N\|_{X_T}) \|w_M - w_N\|_{X_T} \\ &\quad + C \left(\|w_M\|^2_{X_T} + \|w_N\|^2_{X_T} \right) \|w_M - w_N\|_{X_T}, \end{aligned} \tag{5.24}$$

where $B^{(1)}_{M,N}$ and $B^{(2)}_{M,N}$ are defined by

$$\begin{aligned} B^{(1)}_{M,N} &= \|Z_{5,M} - Z_{5,N}\|_{L^2_T W^{s_5,\infty}_x} + \|z_{1,M} - z_{1,N}\|_{L^4_T W^{s_1,\infty}_x} \|z_{2,M}\|^2_{L^4_T W^{s_4,\infty}_x} \\ &\quad + \|z_{1,N}\|_{L^4_T W^{s_1,\infty}_x} \|z_{2,M} - z_{2,N}\|_{L^4_T W^{s_4,\infty}_x} \\ &\qquad \times \left(\|z_{2,M}\|_{L^4_T W^{s_4,\infty}_x} + \|z_{2,N}\|_{L^4_T W^{s_4,\infty}_x} \right) \\ &\quad + \left(\|z_{2,M}\|^2_{L^8_T W^{s_4,\infty}_x} + \|z_{2,N}\|^2_{L^8_T W^{s_4,\infty}_x} \right) \|z_{2,M} - z_{2,N}\|_{L^8_T W^{s_4,\infty}_x}, \\ B^{(2)}_{M,N} &= \|Z_{2,M} - Z_{2,N}\|_{L^2_T W^{s_2,\infty}_x} + \|z_{1,M} - z_{1,N}\|_{L^4_T W^{s_1,\infty}_x} \|z_{2,M}\|_{L^4_T W^{s_4,\infty}_x} \\ &\quad + \|z_{1,N}\|_{L^4_T W^{s_1,\infty}_x} \|z_{2,M} - z_{2,N}\|_{L^4_T W^{s_4,\infty}_x} \\ &\quad + \left(\|z_{2,M}\|_{L^8_T W^{s_4,\infty}_x} + \|z_{2,N}\|_{L^8_T W^{s_4,\infty}_x} \right) \|z_{2,M} - z_{2,N}\|_{L^8_T W^{s_4,\infty}_x}. \end{aligned}$$

In addition to the assumption (5.23), we now suppose that as $N \to \infty$, $z_{1,N}$, $z_{2,N}$, $Z_{2,N}$, and $Z_{5,N}$ converge to the limits z_1, z_2, Z_2, and Z_4 in $C([-T,T];W^{s,\infty}(\mathbb{T}^3))$ for $s = s_1, s_4, s_2$, and s_5, respectively. Then, from (5.24), we obtain

$$\|w_M - w_N\|_{X_T} \leqslant C(R)T^\theta \|w_M - w_N\|_{X_T} + o_{M,N\to\infty}(1)$$

Then, by possibly making $T_0 > 0$ smaller, we conclude that

$$\|w_N - w_M\|_{X_T} \longrightarrow 0$$

for any $0 < T \leqslant T_0$ as $M, N \to \infty$. This implies that w_N converges to some w in X_T as $N \to \infty$. Recalling the decomposition $u_N = z_{1,N} + z_{2,N} + w_N$, we conclude that u_N converges to $u = z_1 + z_2 + w$ in $C([-T,T];H^{s_1}(\mathbb{T}^3))$ as $N \to \infty$.

It remains to check that the assumption (5.23) and the assumption on the convergence of $z_{1,N}$, $z_{2,N}$, $Z_{2,N}$, and $Z_{5,N}$ hold true with large probability. By choosing $s_1 = \alpha - \frac{3}{2} - \varepsilon$, $s_2 = 2(\alpha - \frac{3}{2}) - \varepsilon$, $s_4 = 3(\alpha - \frac{3}{2}) + 1 - \varepsilon$, and $s_5 = 2(\alpha - \frac{3}{2}) - \varepsilon$ for some small $\varepsilon > 0$, it is easy to see that the conditions (5.10), (5.13), (5.15), (5.17), and (5.21) are satisfied for $\frac{5}{4} < \alpha \leqslant \frac{3}{2}$. (Note that the restriction $\alpha > \frac{5}{4}$ appears in (5.10), (5.15), (5.17), and (5.21).) Therefore, it follows from Proposition 3.2, 4.1, and 4.2 that there exists a set Ω_T of complemental probability smaller than $C\exp(-c/T^\kappa)$ such that the assumption (5.23) and the assumption on the convergence of $z_{1,N}$, $z_{2,N}$, $Z_{2,N}$, and $Z_{5,N}$ hold true on Ω_T, allowing us to prove the convergence of u_N to u in $C([-T,T];H^{s_1}(\mathbb{T}^3))$ as above. This completes the proof of Theorem 3.

6. On the triviality of the limiting dynamics without renormalization

6.1. Reformulation of the problem

Fix $1 \leqslant \alpha \leqslant \frac{3}{2}$ and a pair $(w_0, w_1) \in \mathcal{H}^{\frac{3}{4}}(\mathbb{T}^3)$. Let u_N be the solution to the (un-renormalized) NLW (1.1) with the following initial data:

$$(u_N, \partial_t u_N)\,|_{t=0} = (w_0, w_1) + \left(\widetilde{u}^\omega_{0,N}, \widetilde{u}^\omega_{1,N}\right), \tag{6.1}$$

where the random initial data $(\widetilde{u}^\omega_{0,N}, \widetilde{u}^\omega_{1,N})$ is given by (1.33) with $C_N > 0$ implicitly defined as in (1.40). In this section, we present the proof of Theorem 4 by reformulating the Cauchy problem for u_N as

$$\begin{cases} \mathcal{L}_N u_N + u_N^3 - C_N u_N = 0 \\ (u_N, \partial_t u_N)\,|_{t=0} = (w_0, w_1) + (\widetilde{u}^\omega_{0,N}, \widetilde{u}^\omega_{1,,N}), \end{cases} \tag{6.2}$$

where $\mathcal{L}_N = \partial_t^2 - \Delta + C_N$ as in (1.35).

Since C_N in (1.40) is implicitly defined, we first need to study the asymptotic behavior of C_N as $N \to \infty$.

Lemma 6.1. — *Let $1 \leqslant \alpha \leqslant \frac{3}{2}$. Then, for each $N \in \mathbb{N}$, there exists a unique number $C_N \geqslant 1$ satisfying the equation (1.40). Moreover, we have*

$$C_N = 3\sigma_N + R_N \tag{6.3}$$

for all sufficiently large $N \gg 1$, where $\sigma_N = \sum_{|n| \leqslant N} \langle n \rangle^{-2\alpha}$ is as in (3.2) and the error term R_N satisfies

$$|R_N| \sim \begin{cases} \log \log N, & \text{for } \alpha = \frac{3}{2}, \\ N^{\frac{1}{2}(3-2\alpha)^2}, & \text{for } 1 \leqslant \alpha < \frac{3}{2}. \end{cases}$$

In particular, we have $R_N = o(\sigma_N)$ as $N \to \infty$.

Proof. — Let C_N be as in (1.40). As C_N increases from 0 to ∞, the right-hand side of (1.40) decreases from ∞ to 0. Hence, for each $N \in \mathbb{N}$, there exists a unique solution $C_N > 0$ to (1.40).

Suppose that $C_N < 1$ for some $N \in \mathbb{N}$. Then, considering the contribution from $n = 0$ on the right-hand side of (1.40), we obtain $C_N \geqslant 3$, leading to a contradiction. Hence, we must have $C_N \geqslant 1$ for any $N \in \mathbb{N}$.

We first consider the case $1 \leqslant \alpha < \frac{3}{2}$. Since $C_N \geqslant 1$, it follows from (1.40) that $C_N \lesssim N^{3-2\alpha}$. Using this upper bound on C_N, we estimate the contribution from $|n| \sim N$:

$$C_N \gtrsim \sum_{|n| \leqslant N} \frac{1}{(N^{3-2\alpha} + |n|^2)\langle n \rangle^{2(\alpha-1)}} \gtrsim \sum_{|n| \sim N} \frac{1}{\langle n \rangle^{2\alpha}} \sim N^{3-2\alpha},$$

where we used the assumption $\alpha \geqslant 1$ in the second step. This shows that $C_N \sim N^{3-2\alpha}$. Using this asymptotic behavior with (3.2), we then obtain (6.3) with the error term R_N given by

$$R_N = 3 \sum_{|n| \leqslant N} \frac{1}{\langle n \rangle^{2(\alpha-1)}} \left(\frac{1}{(C_N + |n|^2)} - \frac{1}{\langle n \rangle^2} \right). \tag{6.4}$$

By separately estimating the contributions from $\{|n| \ll N^{\frac{3}{2}-\alpha}\}$ and $\{N^{\frac{3}{2}-\alpha} \leqslant |n| \leqslant N\}$, we have

$$|R_N| = 3 \sum_{|n| \leqslant N} \frac{1}{\langle n \rangle^{2\alpha}} \frac{C_N - 1}{(C_N + |n|^2)} \sim N^{\left(\frac{3}{2}-\alpha\right)(3-2\alpha)}.$$

Next, we consider the case $\alpha = \frac{3}{2}$. Proceeding as above, we immediately see that $C_N \sim \log N$. The contribution to R_N in (6.4) from $\{|n| \gtrsim \sqrt{\log N}\}$

is $O(1)$, while the contribution to R_N in (6.3) from $\{|n| \ll \sqrt{\log N}\}$ is $O(\log\log N)$. This completes the proof of Lemma 6.1. $\square$

6.2. On the Strichartz estimates with a parameter

In order to study the equation (6.2), we review the relevant Strichartz estimates for the Klein–Gordon operator with a general mass. Given $a \geqslant 1$, with a slight abuse of notation, define $\mathcal{L}_a$ by

$$\mathcal{L}_a := \partial_t^2 - \Delta + a.$$

Let $\mathcal{L}_a^{-1}$ be the Duhamel integral operator given by

$$\mathcal{L}_a^{-1} F(t) = \int_0^t \frac{\sin\left((t-t')\sqrt{a-\Delta}\right)}{\sqrt{a-\Delta}} F(t') dt'.$$

Namely, $u := \mathcal{L}_a^{-1}(F)$ is the solution to the following nonhomogeneous linear equation:

$$\begin{cases} \mathcal{L}_a u = F \\ (u, \partial_t u)\,|_{t=0} = (0,0). \end{cases}$$

Then, by making systematic modifications of the proof of Lemma 2.4 on $\mathbb{R}^3$ (see [53]) and applying the finite speed of propagation, we see that the same non-homogeneous Strichartz estimate as (2.7) holds, uniformly in $a \geqslant 1$:

$$\begin{aligned} &\left\|\mathcal{L}_a^{-1}(F)\right\|_{X_T} \\ &\qquad \lesssim \min\left(\|F\|_{L^1\left([-T,T];\, H^{-\frac{1}{2}}(\mathbb{T}^3)\right)}, \|F\|_{L^{\frac{4}{3}}([-T,T]\times\mathbb{T}^3)} \right) \end{aligned} \tag{6.5}$$

for any $0 < T \leqslant 1$, where the X_T-norm is defined in (2.8).

We also record the following lemma on the linear solution associated with $\mathcal{L}_a$, $a \geqslant 1$.

Lemma 6.2. — *Given $a \geqslant 1$, define $S_a(t)$ by*

$$S_a(t)(w_0, w_1) = \cos\left(t\sqrt{a-\Delta}\right) w_0 + \frac{\sin\left(t\sqrt{a-\Delta}\right)}{\sqrt{a-\Delta}} w_1.$$

Then, there exists $C > 0$ such that

$$\|S_a(t)(w_0,w_1)\|_{X_T} \leqslant C \,\|(w_0,w_1)\|_{\mathcal{H}^{\frac{3}{4}}} \tag{6.6}$$

for any $(w_0, w_1) \in \mathcal{H}^{\frac{3}{4}}(\mathbb{T}^3)$ and $0 < T \leqslant 1$, uniformly in $a \geqslant 1$. Moreover, $S_a(t)(w_0,w_1)$ tends to 0 in the space-time distributional sense as $a \to \infty$.

Proof. — The estimate (6.6) follows easily from Hölder's inequality in t and Sobolev's inequality in x along with the boundedness of $S_a(t)$ in $\mathcal{H}^{\frac{3}{4}}(\mathbb{T}^3)$. As for the second claim, we only consider $e^{it\sqrt{a-\Delta}}f$ for $f \in L^2(\mathbb{T}^3)$. Note that, for each fixed $n \in \mathbb{Z}^3$, $\sqrt{a+|n|^2}-\sqrt{a}$ tends to 0 as $a \to \infty$. Then, by the dominated convergence theorem (for the summation in $n \in \mathbb{Z}^3$) and the Riemann–Lebesgue lemma (for the integration in t), we have

$$\begin{aligned}\lim_{a\to\infty} \iint \left(e^{it\sqrt{a-\Delta}}f\right)(x)\overline{\phi(t,x)}\,dxdt &= \lim_{a\to\infty}\int e^{it\sqrt{a}}\left(\sum_{n\in\mathbb{Z}^3} e^{it\left(\sqrt{a+|n|^2}-\sqrt{a}\right)}\widehat{f}(n)\overline{\widehat{\phi}(t,n)}\right)dt\\ &= \lim_{a\to\infty}\int e^{it\sqrt{a}}\langle f,\,\phi(t)\rangle_{L^2_x}dt\\ &=0\end{aligned}$$

for any test function $\phi \in C^\infty(\mathbb{R}\times\mathbb{T}^3)$ with a compact support in t. □

Remark 6.3. — Let $a \geqslant 1$. Then, we have the following homogeneous Strichartz estimate:

$$\|S_a(t)(w_0,w_1)\|_{L^q([0,1];\,L^r(\mathbb{T}^3))} \leqslant C\,\|(w_0,w_1)\|_{H_a^{\frac{2}{q}}\times H_a^{\frac{2}{q}-1}(\mathbb{T}^3)} \tag{6.7}$$

for $2 < q \leqslant \infty$ and $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$, where the H^s_a-norm is defined by

$$\|f\|_{H^s_a} = \left(\sum_{n\in\mathbb{Z}^3}\left(a+|n|^2\right)^s\left|\widehat{f}(n)\right|^2\right)^{\frac{1}{2}}.$$

The proof of (6.7) follows from a straightforward modification of the standard homogeneous Strichartz estimate (i.e. $a=1$). For $s>0$, the H^s_a-norm diverges as $a\to\infty$ and hence the homogeneous Strichartz estimate (6.7) is not useful for our application.

6.3. Proof of Theorem 4

Let $\widetilde{z}_{1,N}$ and $\widetilde{\sigma}_N$ be as in (1.37) and (1.39). As in (3.3), (4.1), and (4.9), we define

$$\begin{aligned}&\widetilde{Z}_{1,N} := \widetilde{z}_{1,N}, \quad \widetilde{Z}_{2,N} := \left(\widetilde{z}_{1,N}\right)^2 - \widetilde{\sigma}_N,\\ &\widetilde{Z}_{3,N} := \left(\widetilde{z}_{1,N}\right)^3 - 3\widetilde{\sigma}_N\widetilde{z}_{1,N},\\ &\widetilde{Z}_{4,N} := \widetilde{z}_{2,N} := -\mathcal{L}_N^{-1}\left(\left(\widetilde{z}_{1,N}\right)^3 - 3\widetilde{\sigma}_N\widetilde{z}_{1,N}\right),\\ &\widetilde{Z}_{5,N} := \left\{\left(\widetilde{z}_{1,N}\right)^2 - \widetilde{\sigma}_N\right\}\widetilde{z}_{2,N},\end{aligned} \tag{6.8}$$

where $\mathcal{L}_N$ is as in (1.35). Then, by repeating the arguments in Sections 3 and 4, we see that the analogues of Propositions 3.2, 4.1, and 4.2 hold for $\widetilde{Z}_{j,N}$, $j = 1, \dots, 5$. In the following lemma, we summarize the regularity and convergence properties of these stochastic terms.

Lemma 6.4. — *Let $1 < \alpha < \frac{3}{2}$ and s_j, $j = 1, \dots, 5$, satisfy the regularity assumptions* (3.4), (4.2), *and* (4.10). *Fix $j = 1, \dots, 5$. Then, given any $T > 0$, $\widetilde{Z}_{j,N}$ converges almost surely to 0 in $C([-T,T]; W^{s_j,\infty}(\mathbb{T}^3))$ as $N \to \infty$. Moreover, given $2 \leqslant q < \infty$, there exist positive constants C, c, κ, θ and small $\delta > 0$ such that for every $T > 0$, there exists a set Ω_T of complemental probability smaller than $C\exp(-c/T^\kappa)$ such that*

$$\left\| \widetilde{Z}_{j,N} \right\|_{L^q([-T,T];\, W^{s_j,\infty}(\mathbb{T}^3))} \leqslant C_N^{-\delta} T^\theta \tag{6.9}$$

for any $\omega \in \Omega_T$ and any $N \geqslant 1$. In particular, for any $\omega \in \Omega_T$, $\widetilde{Z}_{j,N}$ tends to 0 in $L^q([-T,T]; W^{s_j,\infty}(\mathbb{T}^3))$ as $N \to \infty$.

When $\alpha = \frac{3}{2}$, the same result holds but only along a subsequence $\{N_k\}_{k\in\mathbb{N}}$.

Proof. — We only consider the case $j = 1$ since the other cases follow in a similar manner. Fix $N \in \mathbb{N}$. With $\langle n \rangle_N$ as in (1.34), let

$$g_n^{t,N}(\omega) := \cos(t\langle n \rangle_N)\, g_n(\omega) + \sin(t\langle n \rangle_N)\, h_n(\omega).$$

Then, from (1.38), we have

$$\langle \nabla \rangle^{s_1} \widetilde{Z}_{1,N} = \sum_{|n| \leqslant N} \frac{g_n^{t,N}(\omega)}{\langle n \rangle_N \langle n \rangle^{\alpha - 1 - s_1}} e^{in\cdot x}.$$

Let $q, r < \infty$. Then, proceeding as in (3.10) with $\langle n \rangle_N \geqslant \max(C_N^{\frac{1}{2}}, \langle n \rangle)$, we have

$$\begin{aligned}
\Big\| \|\langle \nabla \rangle^{s_1} Z_{1,N}\|_{L^q_T L^r_x} \Big\|_{L^p(\Omega)} &\leqslant \Big\| \|\langle \nabla \rangle^{s_1} Z_{1,N}(t,x)\|_{L^p(\Omega)} \Big\|_{L^q_T L^r_x} \\
&\lesssim T^{\frac{1}{q}} p^{\frac{1}{2}} \left(\sum_{|n| \leqslant N} \frac{1}{\langle n \rangle_N^2 \langle n \rangle^{2(\alpha - 1 - s_1)}} \right)^{\frac{1}{2}} \\
&\lesssim C_N^{-\delta_0} T^{\frac{1}{q}} p^{\frac{1}{2}}
\end{aligned}$$

for any $p \geqslant \max(q, r)$ and sufficiently small $\delta_0 > 0$ such that $2(\alpha - s_1 - 2\delta_0) > 3$. By Chebyshev's inequality, we then have

$$P\left(\left\|\widetilde{Z}_{j,N}\right\|_{L^q([-T,T];\,W^{s_j,\infty}(\mathbb{T}^3))} > C_N^{-\delta} T^\theta\right) \leqslant C \exp\left(-c\frac{C_N^{2(\delta_0-\delta)}}{T^{2\left(\frac{1}{q}-\theta\right)}}\right). \tag{6.10}$$

In view of Lemma 6.1 with (3.2), by choosing $\delta, \theta > 0$ sufficiently small, the right-hand side of (6.10) is summable over $N \in \mathbb{N}$, as long as $1 < \alpha < \frac{3}{2}$. Define Ω_T by

$$\Omega_T = \bigcap_{N \in \mathbb{N}} \left\{\omega \in \Omega : \left\|\widetilde{Z}_{j,N}\right\|_{L^q([-T,T];\,W^{s_j,\infty}(\mathbb{T}^3))} \leqslant C_N^{-\delta} T^\theta\right\}. \tag{6.11}$$

Then, we have $P(\Omega_T^c) \leqslant C\exp(-c/T^\kappa)$ and (6.9) holds for any $\omega \in \Omega_T$ and any $N \in \mathbb{N}$, when $1 < \alpha < \frac{3}{2}$. When $\alpha = \frac{3}{2}$, we need to choose a subsequence $\{N_k\}_{k\in\mathbb{N}}$ growing sufficiently fast such that the right-hand side of (6.10) is summable along this subsequence $\{N_k\}_{k\in\mathbb{N}}$. Then, we define Ω_T as in (6.11) but by taking an intersection over $\{N_k\}_{k\in\mathbb{N}}$. This yields (6.9) for any $\omega \in \Omega_T$ and any $N = N_k$, $k \in \mathbb{N}$, when $\alpha = \frac{3}{2}$.

The rest follows exactly as in the proofs of Lemma 3.5 and Propositions 3.2, 4.1, and 4.2. Lastly, in view of Fatou's lemma and the asymptotic behavior $C_N \to \infty$, we conclude from (6.9) that $\widetilde{Z}_{1,N}$ tends to 0 (along the subsequence $\{N_k\}_{k\in\mathbb{N}}$ when $\alpha = \frac{3}{2}$). □

With Lemma 6.4 in hand, we can proceed as in Section 5.[18] Namely, given $(w_0, w_1) \in \mathcal{H}^{\frac{3}{4}}(\mathbb{T}^3)$, let u_N be the solution to the (un-renormalized) NLW (1.1) with the initial data in (6.1):

$$(u_N, \partial_t u_N)\,|_{t=0} = (w_0, w_1) + \left(\widetilde{u}_{0,N}^\omega, \widetilde{u}_{1,N}^\omega\right),$$

where $(\widetilde{u}_{0,N}^\omega, \widetilde{u}_{1,N}^\omega)$ is the truncated random initial data defined in (1.33). Now, we write

$$\widetilde{u}_N = \widetilde{z}_{1,N} + \widetilde{z}_{2,N} + \widetilde{w}_N, \tag{6.12}$$

where $\widetilde{z}_{1,N}$ and $\widetilde{z}_{2,N}$ are as in (1.37) and (6.8), respectively. Recalling that u_N also satisfies (6.2), we see that $\widetilde{w}_N$ is the solution to

$$\begin{cases} \mathcal{L}_N \widetilde{w}_N + \widetilde{F}_0 + \widetilde{F}_1(\widetilde{w}_N) + \widetilde{F}_2(\widetilde{w}_N) + \widetilde{F}_3(\widetilde{w}_N) = 0 \\ (\widetilde{w}_N, \partial_t \widetilde{w}_N)\,|_{t=0} = (w_0, w_1), \end{cases} \quad s \tag{6.13}$$

[18] In the following, it is understood that when $\alpha = \frac{3}{2}$, we work on the subsequence $\{N_k\}_{k\in\mathbb{N}}$ from Lemma 6.4 instead of the whole natural numbers $N \in \mathbb{N}$.

where $\mathcal{L}_N$ is as in (1.35) and $\widetilde{F}_j$, $j = 0, \ldots, 3$, are given by

$$\begin{aligned}
\widetilde{F}_0 &= 3\widetilde{Z}_{5,N} + 3\widetilde{z}_{1,N}\left(\widetilde{z}_{2,N}\right)^2 + \left(\widetilde{z}_{2,N}\right)^3, \\
\widetilde{F}_1\left(\widetilde{w}_N\right) &= 3\widetilde{Z}_{2,N}\widetilde{w}_N + 6\widetilde{z}_{1,N}\widetilde{z}_{2,N}\widetilde{w}_N + 3\left(\widetilde{z}_{2,N}\right)^2\widetilde{w}_N, \\
\widetilde{F}_2\left(\widetilde{w}_N\right) &= 3\widetilde{z}_{1,N}\left(\widetilde{w}_N\right)^2 + 3\widetilde{z}_{2,N}\widetilde{w}_N^2, \\
\widetilde{F}_3\left(\widetilde{w}_N\right) &= \widetilde{w}_N^3.
\end{aligned}$$

Given $N \in \mathbb{N}$, define $S_N(t)$ by

$$S_N(t)(w_0, w_1) = \cos\left(t\sqrt{C_N - \Delta}\right) w_0 + \frac{\sin\left(t\sqrt{C_N - \Delta}\right)}{\sqrt{C_N - \Delta}} w_1.$$

Then, the Duhamel formulation of (6.13) is given by

$$\widetilde{w}_N = S_N(t)(w_0, w_1) + \mathcal{L}_N^{-1}\left(\widetilde{F}_0 + \widetilde{F}_1(\widetilde{w}_N) + \widetilde{F}_2(\widetilde{w}_N) + \widetilde{F}_3(\widetilde{w}_N)\right),$$

Define $\widetilde{A}_N^{(1)}$, $\widetilde{A}_N^{(2)}$, and $\widetilde{R}(T)$ by replacing $z_{j,N}$ and $Z_{j,N}$ in (5.22) and (5.23) with $\widetilde{z}_{j,N}$ and $\widetilde{Z}_{j,N}$. Then, by repeating the analysis in Section 5 with (6.5), we obtain

$$\begin{aligned}
\text{(6.14)} \quad \|\widetilde{w}_N\|_{L_T^\infty H_x^{\frac{1}{2}}} &\leqslant \|(w_0, w_1)\|_{\mathcal{H}^{\frac{1}{2}}} + CT^\theta \widetilde{A}_N^{(1)} + CT^\theta \widetilde{A}_N^{(2)} \|\widetilde{w}_N\|_{X_T} \\
&\quad + CT^\theta \left(\sum_{j=1}^{2} \|\widetilde{z}_{j,N}\|_{L_T^2 W_x^{s_j,\infty}}\right) \|\widetilde{w}_N\|_{X_T}^2 + C\|\widetilde{w}_N\|_{L_{T,x}^4}^3
\end{aligned}$$

and

$$\begin{aligned}
\text{(6.15)} \quad \|\widetilde{w}_N\|_{L_{T,vx}^4} &\leqslant \|S_N(t)(w_0, w_1)\|_{L_{T,x}^4} \\
&\quad + CT^\theta \widetilde{A}_N^{(1)} + CT^\theta \widetilde{A}_N^{(2)} \|\widetilde{w}_N\|_{X_T} \\
&\quad + CT^\theta \left(\sum_{j=1}^{2} \|\widetilde{z}_{j,N}\|_{L_T^2 W_x^{s_j,\infty}}\right) \|\widetilde{w}_N\|_{X_T}^2 + C\|\widetilde{w}_N\|_{L_{T,x}^4}^3,
\end{aligned}$$

where the constants are independent of $N \in \mathbb{N}$, thanks to the uniform Strichartz estimate (6.5). By Hölder's inequality in t and Sobolev's inequality in x (as in the proof of Lemma 6.2), we have

$$\text{(6.16)} \qquad \|S_N(t)(w_0, w_1)\|_{L_{T,x}^4} \lesssim T^{\frac{1}{4}} \|(w_0, w_1)\|_{\mathcal{H}^{\frac{3}{4}}},$$

uniformly in $N \in \mathbb{N}$. Then, it follows from (6.14), (6.15), and (6.16) that there exists small $T_1 > 0$ depending on $\widetilde{R}(T)$ such that

$$\begin{aligned} &\|\widetilde{w}_N\|_{L^\infty_T H^{\frac{1}{2}}_x} \leqslant 2\,\|(w_0, w_1)\|_{H^{\frac{1}{2}}_x}, \\ &\|\widetilde{w}_N\|_{L^4_{T,x}} \leqslant \left(1 + C\left(\widetilde{R}(T)\right)\right) T^\theta \end{aligned} \tag{6.17}$$

for any $0 < T \leqslant T_1$, uniformly in $N \in \mathbb{N}$. It follows from Lemma 6.4 that for each small $0 < T \leqslant T_1$, there exists a set Ω_T of complemental probability smaller than $C\exp(-c/T^\kappa)$ such that

$$\widetilde{R}(T) \leqslant C_N^{-\delta} T^\theta. \tag{6.18}$$

In the following, we fix $\omega \in \Omega_T$ and show that $\widetilde{w}_N$ tends to 0 as a space-time distribution as $N \to \infty$. From Lemma 6.4 with (6.17) and (6.18), we see that

$$\mathcal{L}_N^{-1}\left(\widetilde{F}_0 + \widetilde{F}_1\left(\widetilde{w}_N\right) + \widetilde{F}_2(\widetilde{w}_N)\right) \longrightarrow 0 \tag{6.19}$$

in X_T as $N \to \infty$. On the other hand, by Sobolev's inequality (with $\delta > 0$ sufficiently small) and Lemma 6.1, we have

$$\begin{aligned} &\left\|\mathcal{L}_N^{-1}\left(\widetilde{F}_3(\widetilde{w}_N)\right)\right\|_{L^\infty_T L^2_x} \\ &\qquad \leqslant C_N^{-\delta}\left\|\widetilde{w}_N^3\right\|_{L^1_T H^{-1+2\delta}_x} \lesssim C_N^{-\delta}\left\|\widetilde{w}_N\right\|^3_{L^4_{T,x}} \longrightarrow 0 \end{aligned} \tag{6.20}$$

as $N \to \infty$. Therefore from Lemmas 6.1 and 6.2 with (6.19) and (6.20), we conclude that $\widetilde{w}_N$ tends to 0 in the space-time distributional sense.

Finally, from the decomposition (6.12), Lemma 6.4, and the convergence property of $\widetilde{w}_N$ discussed above, we conclude that, for each $\omega \in \Omega_T$, $\widetilde{u}_N$ converges to 0 as space-time distributions on $[-T, T] \times \mathbb{T}^3$ as $N \to \infty$. This completes the proof of Theorem 4.

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