Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Tome 71, n° 5 (2021), p. 2019-2045.

<http://aif.centre-mersenne.org/item/AIF_2021__71_5_2019_0>

Ann. Inst. Fourier, Grenoble
71, 5 (2021) 2019-2045

STABILITY AND HÖLDER REGULARITY OF SOLUTIONS TO COMPLEX MONGE–AMPÈRE EQUATIONS ON COMPACT HERMITIAN MANIFOLDS

by Chinh H. LU, Trong-Thuc PHUNG & Tât-Dat TÔ (*)

Abstract. — Let (X, ω) be a compact Hermitian manifold. We establish a stability result for solutions to complex Monge–Ampère equations with right-hand side in L^p, $p > 1$. Using this we prove that the solutions are Hölder continuous with the same exponent as in the Kähler case by Demailly–Dinew–Guedj–Kołodziej–Pham–Zeriahi. Our techniques also apply to the setting of big cohomology classes on compact Kähler manifolds.

Résumé. — Soit (X, ω) une variété Hermitienne compacte de dimension n. On établit la stabilité des solutions des équations de Monge–Ampère avec second membre dans L^p, $p > 1$. En utilisant ce résultat on montre que les solutions sont continues höldériennes avec le même exposant que celui obtenu dans le cas Kählérien par Demailly-Dinew–Guedj–Kołodziej–Pham–Zeriahi. Notre méthode s'applique également dans le contexte des classes de cohomologie sur une variété Kählérienne.

1. Introduction

One of the central problems in complex geometry is the existence of canonical metrics. On compact Kähler manifolds this question is intimately related to the study of complex Monge–Ampère equations. Culminating with Yau's work [56], which solves Calabi's conjecture, complex Monge–Ampère equations have been studied and generalized in several directions with many important geometric applications.

Keywords: Hermitian manifold, Complex Monge–Ampère equation, Stability, Comparison principle.
2010 *Mathematics Subject Classification:* 32W20, 32U05, 32Q15.
(*) C.H.Lu is supported by the CNRS project PEPS "Jeune chercheuse, jeune chercheur". T.T. Phung is supported by Ho Chi Minh City University of Technology under grant number T-KHUD-2020-32. T.D. Tô is partially supported by the IEA project PLUTOCHE.

An essential step in solving complex Monge–Ampère equations on compact manifolds is the uniform L^∞ estimate. In Yau's work [56], it was achieved via Moser iteration process. Twenty years later, Kołodziej [42] gave a novel proof using pluripotential theory which has been applied to many geometric situations. In the recent breakthrough of X.X. Chen and J. Cheng [10, 11, 12], pluripotential estimates of Kołodziej [42] and Błocki, see [4, 5], were used to obtain a uniform estimate along the continuity path introduced earlier by X.X. Chen [9].

In this paper we shall study complex Monge–Ampère equations on compact (non-Kähler) Hermitian manifolds (X,ω) of dimension n,

$$(\omega + dd^c u)^n = cf\omega^n, \tag{1.1}$$

where $0 \leqslant f \in L^p(X)$, for some $p > 1$, and c is a positive constant. Here, if nothing is stated, the L^p-norm is computed with respect to the volume form ω^n. Unlike the Kähler case, here we have an extra variable: the constant c which is in general not determined by X, ω. The non-degenerate case, i.e. when $0 < f$ is smooth, has been studied by Cherrier [14], Guan–Li [36] under restrictive conditions. The general case was recently proved by Tosatti and Weinkove [53]: there exists a unique constant $c = c_f > 0$ and a unique (modulo adding a constant) smooth function u with $\omega + dd^c u > 0$, solving (1.1).

In the last decade, pluripotential theory on compact Hermitian manifolds has been developed intensively by S. Dinew, S. Kołodziej, and N-C. Nguyên (see [29, 30, 46, 51]). The main difficulty in the Hermitian setting is that the comparison principle, which in the Kähler setting says that, for bounded ω-psh functions u, v,

$$\int_{\{u<v\}} \omega_v^n \leqslant \int_{\{u<v\}} \omega_u^n, \tag{1.2}$$

is missing. Nevertheless, a replacement for this, called the "modified comparison principle" was established in [46] which is a key tool in proving the existence of continuous solutions [46, Theorem 5.8]. The uniqueness of the constant c was later proved in [51].

Our first main result is a generalization of [38] to the Hermitian setting.

Theorem 1.1. — *Fix $0 \leqslant f, g \in L^p(X,\omega^n)$, $p > 1$ such that $\int_X f\omega^n > 0$ and $\int_X g\omega^n > 0$. Assume that u and v are bounded ω-psh functions on X satisfying*

$$(\omega + dd^c u)^n = e^u f\omega^n \quad \text{and} \quad (\omega + dd^c v)^n = e^v g\omega^n.$$

Then for some constant $C > 0$ depending on X, ω, n, p, an upper bound for $\|f\|_p, \|g\|_p$ and a positive lower bound for $\int_X f^{1/n}\omega^n$, $\int_X g^{1/n}\omega^n$, we have

$$\|u - v\|_{L^\infty(X)} \leqslant C \|f - g\|_p^{1/n}.$$

The proof of Theorem 1.1 goes along the same lines as in [38]. An immediate consequence of Theorem 1.1 is a stability estimate for the constant c:

Corollary 1.2. — *Assume that $0 \leqslant f, g \in L^p(X)$ for some $p > 1$. Then*

$$|c_f - c_g| \leqslant C \|f - g\|_p^{1/n},$$

where $C > 0$ is a constant depending on (X, ω, n, p), an upper bound for $\|f\|_p, \|g\|_p$ and a positive lower bound for $\|f\|_{1/n}$, $\|g\|_{1/n}$.

Using Theorem 1.1 we can greatly improve the stability exponent in [48, Theorem A] :

Theorem 1.3. — *Assume that u, v are ω-psh continuous solutions to*

$$(\omega + dd^c u)^n = f\omega^n\,;\ (\omega + dd^c v)^n = g\omega^n,\ \sup_X u = \sup_X v = 0,$$

where $f, g \in L^p(X), p > 1$ and $f \geqslant c_0 > 0$. Then

$$\|u - v\|_\infty \leqslant C \|f - g\|_p^{1/n},$$

where C depends on X, ω, n, p, c_0 and an upper bound for $\|f\|_p, \|g\|_p$, and a positive lower bound for $\|g\|_{1/n}$.

Compared to [48] the exponent is improved to be the same as in the Kähler case [31], but we still assume that $f \geqslant c_0 > 0$ for some positive constant c_0. It is interesting to know whether our techniques can be applied to treat more general right-hand sides considered in [49]. Improving the stability exponent is an interesting question because, at least, the stability estimate can be used to prove the Hölder continuity of solutions. Moreover, in the recent breakthrough of Chen–Donaldson–Sun [13] the Hölder continuity of solutions to degenerate complex Monge–Ampère equations was exploited.

If the comparison principle (1.2) holds on (X, ω) then many arguments from the Kähler case can be employed. In particular, Theorem 1.3 holds without the strict positivity condition. Our argument also applies to the more general case of big cohomology classes, improving a stability result of Guedj–Zeriahi [39]:

Theorem 1.4. — *Let (X, ω) be a compact Kähler manifold of dimension n. Fix a closed smooth real $(1,1)$-form θ whose cohomology class $\{\theta\}$ is big. Assume that $0 \leqslant f, g \in L^p(X, \omega^n)$ are such that $\int_X f\omega^n = \int_X g\omega^n$*

$= \mathrm{Vol}(\theta) = 1$. *If u and v are θ-psh functions with minimal singularities on X such that*

$$\theta_u^n = f\omega^n, \quad \theta_v^n = g\omega^n, \quad \sup_X u = \sup_X v = 0,$$

then, for some constant $C > 0$ depending on $(X, \omega, n, \theta, p)$ and an upper bound for $\|f\|_p, \|g\|_p$, we have

$$\sup_X |u - v| \leqslant C \, \|f - g\|_p^{1/n}.$$

Compared to [39, Theorem C], here we have improved the exponent from

$$\frac{1}{2^n(n+1)-1} \text{ to } \frac{1}{n}.$$

Note that one can replace the L^p norm by the L^1 norm and the exponent becomes slightly smaller (see [38, Remark 2.2]). The proof of Theorem 1.4 uses [38, Theorem A] and Kołodziej's techniques as in [43, Theorem 4.1]. The main point here is that using [38, Theorem A] we reduce the problem to the case in which the two densities f, g are very close to each other in the following sense: $e^{-\varepsilon} f \leqslant g \leqslant e^{\varepsilon} f$, for some small constant $\varepsilon > 0$. In case θ is additionally semipositive we get the same exponent as in [31]. Our arguments also apply to the setting of prescribed singularities, where instead of asking for u, v to have minimal singularities we ask u, v to have the same singularity type as a given model potential [19, 21, 22].

A classical use of such stability estimates is in proving Hölder continuity of solutions. Given $0 \leqslant f \in L^p$ and u a continuous solution to $\omega_u^n = f\omega^n$, it was proved by S. Kołodziej and N.C. Nguyên [48, Theorem B] that if $f \geqslant c_0 > 0$ then u is Hölder continuous. The strict positivity assumption was relaxed by the same authors recently in [47], but the exponent is not optimal. Also, due to the lack of uniqueness, the result in [47] does not give that all solutions are Hölder continuous. In the Kähler case, the Hölder continuity was first proved by Kołodziej [44] and improved by Demailly–Dinew–Guedj–Kołodziej–Pham–Zeriahi [26] using Demailly's approximation theorem [24]. Related questions on Hölder continuity of solutions to complex Monge–Ampère equations on compact Kähler manifolds have been studied by many authors. T.C. Dinh and V.A. Nguyên [32] proved that a probability measure μ admits a Hölder continuous solution φ_μ, i.e. φ_μ is ω-psh and $(\omega + dd^c\varphi_\mu)^n = \mu$, if and only if the super-potential associated to μ is Hölder continuous. The notion of super-potentials was introduced by T.C. Dinh and N. Sibony [34]. Using this notion, D.V. Vu has established in [55] a Hölder stability of families of Monge–Ampère measures of Hölder continuous potentials. He has also studied in [54] Hölder continuity of potentials of probability measures supported in real $\mathcal{C}^3$ submanifolds. The study of Hölder continuity of solutions to complex Monge–Ampère equations has many important applications in complex dynamics, we refer the reader to [33] for more details.

Using Theorem 1.1 we prove that any bounded solution to (1.1) is Hölder continuous with exponent in $(0, p_n)$. The constant p_n here is the same as the one obtained in the Kähler case in [26].

Theorem 1.5. — *Let (X, ω) be a compact Hermitian manifold of dimension n. Fix $0 \leqslant f \in L^p(X), p > 1$ with $\int_X f\omega^n > 0$. Then any solution u to $\omega_u^n = c_f f \omega^n$ is Hölder continuous with Hölder exponent in $(0, p_n)$, where $p_n = \frac{2}{nq+1}$.*

Here, q is the conjugate of p, i.e. $1/p + 1/q = 1$. The proof strictly follows [48] and [26] in which the stability estimate is used. The only difference is that we use Theorem 1.1 to construct the perturbation functions, allowing to avoid the technical assumption $f \geqslant c_0 > 0$. Interestingly, our method also increases the Hölder exponent by a factor n compared to [48, Theorem B].

In the last part of the paper we adapt the techniques of [3] to establish Hölder regularity of plurisubharmonic envelopes, see Theorem 4.3.

Organization of the paper

In Section 2 we collect several known tools in pluripotential theory on compact Hermitian manifolds. The stability results will be proved in Section 3, while Theorem 1.5 will be proved in Section 4.

Acknowledgements

We thank Văn-Dông Nguyên for reading the first version of this paper and giving many useful comments. We are indebted to Ahmed Zeriahi for his very important help concerning Lemma 4.4. We thank Vincent Guedj and the referee for many useful suggestions which helped to improve the presentation of the paper. We thank Ngoc-Cuong Nguyên for pointing out an error in the proof of Theorem 3.7 in a previous version of the paper.

2. Backgrounds

Fix (X,ω) a compact Hermitian manifold of dimension n. In this section we review some background material in pluripotential theory on compact Hermitian manifolds. For a detailed treatment we refer the reader to [30], [46, Section 1] and the recent surveys [29, 45].

A function $u : X \to \mathbb{R}\cup\{-\infty\}$ is quasi plurisubharmonic if locally it is the sum of a smooth and a psh function. We say that u is ω-psh if u is quasi-psh and $\omega + dd^c u \geqslant 0$ in the sense of currents. Here, $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$ are real differential operators so that $dd^c = 2i\partial\bar{\partial}$. We let $\mathrm{PSH}(X,\omega)$ denote the set of all ω-psh functions on X which are not identically $-\infty$. It follows from Demailly's approximation theorem [23] that any ω-psh function can be approximated from above by smooth strictly ω-psh functions.

For a bounded ω-psh function u, the complex Monge–Ampère operator ω_u^n is defined by the method of Bedford and Taylor [1]. It was proved in [46, Remark 5.7] that $\int_X \omega_u^n > 0$, if u is bounded.

The main difficulty in the Hermitian setting is that the total mass of the Monge–Ampère measure ω_u^n depends on the function u. This is why the comparison principle does not hold in general. It was proved in [46] that the following replacement for the comparison principle holds.

Theorem 2.1 (Modified comparison principle [46, Theorem 2.3]). — *Let $u, v \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. Fix $0 < \varepsilon < 1$ and set*

$$m_\varepsilon := \inf_X \left(u - (1-\varepsilon)v\right).$$

Then for all $0 < s < \frac{\varepsilon^3}{16B}$,

$$\int_{\{u < (1-\varepsilon)\, v + m_\varepsilon + s\}} \omega_{(1-\varepsilon)v}^n \leqslant \left(1 + \frac{Cs}{\varepsilon^n}\right) \int_{\{u < (1-\varepsilon)v + m_\varepsilon + s\}} \omega_u^n,$$

where $C > 0$ is a constant depending on n, B.

The constant B depends only on (X,ω,n), it is chosen so that

$$\begin{cases} -B\omega^2 \leqslant 2n dd^c\omega \leqslant B\omega^2 \\ -B\omega^3 \leqslant 4n^2 d\omega \wedge d^c\omega \leqslant B\omega^3 \end{cases}.$$

Note that the modified comparison principle is only valid on very small sublevel sets. This local analysis is suitable for proving the domination principle. The proof of this result is (implicitly) written in [51, Lemma 2.3]. In the Kähler case, the domination principle was proved by Dinew (see [6, Proposition A1]) using his uniqueness result [28] (see [17, Proposition 2.21], and [50] for a different proof using the envelope technique).

Proposition 2.2. — *If u, v are bounded ω-psh functions such that $\omega_u^n(u < v) = 0$ then $u \geqslant v$.*

Proof. — Assume by contradiction that $U := \{u < v\}$ is not empty and set $m_\varepsilon := \inf_X(u - (1-\varepsilon)v)$, for $\varepsilon \in [0, 1)$. Since v is bounded and $m_0 < 0$, we see that for $\varepsilon > 0$ small enough $m_\varepsilon < m_0/2 < 0$. Set $U(\varepsilon, s) := \{u < (1-\varepsilon)v + m_\varepsilon + s\}$. Then for $s > 0$ and $\varepsilon > 0$ small enough we have $U(\varepsilon, s) \subset U$. Hence by the modified comparison principle we have

$$\varepsilon^n \int_{U(\varepsilon,\, s)} \omega^n \leqslant \int_{U(\varepsilon,\, s)} \omega^n_{(1-\varepsilon)v} \leqslant \left(1 + \frac{Cs}{\varepsilon^n}\right) \int_{U(\varepsilon,\, s)} \omega_u^n = 0.$$

It follows that, for such choice of s, ε, $\int_{U(\varepsilon,\, s)} \omega^n = 0$, hence $U(\varepsilon, s) = \emptyset$ which is a contradiction. □

Using the modified comparison principle, it was proved in [51, Lemma 2.3] that subsolutions are smaller than supersolutions for L^p-density. The same proof applies to give the following:

Proposition 2.3 ([51]). — *Assume that u and v are bounded ω-psh functions such that*

$$\omega_u^n \geqslant e^{\lambda(u-v)} \omega_v^n,$$

for some constant $\lambda > 0$. Then $u \leqslant v$.

Yet another application of the modified comparison principle yields the following minimum principle:

Proposition 2.4 ([48, Proposition 2.5], [51, Corollary 2.4]). — *Assume that u and v are continuous ω-psh functions such that $\omega_u^n \leqslant c\omega_v^n$ on an open set $\Omega \subset X$. If $c < 1$ then $\Omega \neq X$ and*

$$\min_{\Omega}(u - v) = \min_{\partial \Omega}(u - v).$$

As shown in [46], given $0 \leqslant f \in L^p$ with $\int_X f\omega^n > 0$, there exist a unique constant $c_f > 0$ and $u \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ such that $\omega_u^n = c_f f \omega^n$. The density f is MA-admissible if $c_f = 1$. The total mass of an admissible density in L^p is uniformly controlled from below.

Proposition 2.5 ([48, Proposition 2.4], [49, Proposition 2.7]). — *Fix a constant $A_0 > 1$. Then there exists a constant $V_{\min} > 0$ depending on (X, ω, n, A_0) such that for any MA-admissible $0 \leqslant f \in L^p$ with $\|f\|_p \leqslant A_0$, we have*

$$\int_X f\omega^n \geqslant 2^{n+1} V_{\min}.$$

Although the total mass of ω_u^n depends on u, we can control the total mass of the Laplacian of u by using a Gauduchon metric.

Lemma 2.6. — *There exists a uniform constant $C > 0$ such that*

$$C^{-1} \leqslant \int_X \omega_u \wedge \omega^{n-1} \leqslant C, \ \forall\, u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X).$$

Proof. — Let G be a smooth function on X such that $dd^c(e^G\omega^{n-1}) = 0$. The existence of G follows from [35]. Using Stokes' theorem we then have

$$\int_X \omega_u \wedge \left(e^G \omega^{n-1}\right) = \int_X e^G \omega^n,$$

from which the estimates follow. □

3. Stability of solutions

3.1. On Hermitian manifolds

We first extend the elliptic stability theorem in [38] to the non-Kähler case.

Theorem 3.1. — *Fix $0 \leqslant f, g \in L^p(X,\omega^n)$, $p > 1$ such that $\int_X f\omega^n > 0$ and $\int_X g\omega^n > 0$. Assume that u, v are bounded ω-psh functions on X satisfying*

$$(\omega + dd^c u)^n = e^u f\omega^n \quad \text{and} \quad (\omega + dd^c v)^n = e^v g\omega^n.$$

Then for some constant $C > 0$ depending on X, ω, n, p, an upper bound for $\|f\|_p$, $\|g\|_p$ and a positive lower bound for $\|f\|_{1/n}$, $\|g\|_{1/n}$, we have

$$|u - v| \leqslant C \, \|f - g\|_p^{1/n}. \tag{3.1}$$

Remark 3.2. — As shown in [38, Remark 2.2] the L^p-norm can be replaced by the L^1-norm and the exponent becomes $1/(n+\varepsilon)$, where $\varepsilon > 0$ is arbitrarily small.

Proof. — The proof uses a perturbation argument due to Kołodziej [41]. By uniqueness, [51, Theorem 0.1], if $\|f - g\|_p = 0$ then $u = v$ and (3.1) holds for any C. Hence we can assume that $\|f - g\|_p > 0$.

Let φ be a bounded ω-psh function on X such that $\sup_X \varphi = 0$ and

$$(\omega + dd^c \varphi)^n = c_f f\omega^n,$$

where c_f is a constant. The existence of φ and c_f follows from [46, Theorem 5.8]. It follows from [48, Proposition 2.4] that $0 < c_f$ is uniformly bounded from below. To bound c_f from above we use the Gauduchon metric as in [51]. Let G be a smooth function on X such that $dd^c(e^G\omega^{n-1}) = 0$. It follows from the mixed Monge–Ampère inequality, [51, Lemma 1.9] that

$$(\omega + dd^c\varphi) \wedge e^G\omega^{n-1} \geqslant e^G(c_f f)^{1/n}\omega^n.$$

Integrating over X and using Stokes theorem we arrive at

$$\int_X e^G\omega^n \geqslant e^{\min_X G} \int_X (c_f f)^{1/n}\, \omega^n.$$

Thus $c_f > 0$ is uniformly bounded. The uniform a priori estimate in [46] also ensures that φ is uniformly bounded. Hence, for some uniform constant $C_1 > 0$ we have that

$$(\omega + dd^c\varphi)^n \geqslant e^{\varphi - C_1} f\omega^n\, ;\; (\omega + dd^c\varphi)^n \leqslant e^{\varphi + C_1} f\omega^n.$$

Combining this with [51, Lemma 2.3], we obtain $\varphi - C_1 \leqslant u \leqslant \varphi + C_1$, hence u is also uniformly bounded by a constant C_2 depending on the parameters in the statement of Theorem 1.1. By the same arguments as above, we see that $|v| \leqslant C_3$ for some uniform constant $C_3 > 0$.

Let ρ be the unique continuous ω-psh function on X, normalized by $\sup_X \rho = 0$, such that

$$(\omega + dd^c\rho)^n = c_h h\omega^n = c_h \left(\frac{|f-g|}{\|f-g\|_p} + 1 \right) \omega^n. \tag{3.2}$$

The existence of ρ follows from [46, Theorem 5.8]. It follows from [48, Lemma 2.1] that $c_h \leqslant 1$. Since $1 \leqslant \|h\|_p \leqslant 2$, it follows from [48, Proposition 2.4] that $c_h \geqslant c_1 > 0$ where c_1 is a uniform constant.

We now set $\varepsilon := e^{(\sup_X u - \ln c_1)/n}\|f-g\|_p^{1/n}$ and consider two cases. If $\varepsilon > 1/2$ then

$$\|f-g\|_p^{1/n} \geqslant \frac{c_1^{1/n}}{2} e^{-\sup_X u},$$

hence, for $C \geqslant 2(C_2 + C_3)c_1^{-1/n} e^{\sup_X u/n}$, we have

$$|u - v| \leqslant C_2 + C_3 \leqslant C\, \|f-g\|_p^{1/n}.$$

If $\varepsilon \leqslant 1/2$ we consider

$$\phi := (1-\varepsilon)\, u + \varepsilon\rho - K\varepsilon + n\log(1-\varepsilon),$$

where $K > 0$ is a constant to be specified later. The Monge–Ampère measure of ϕ is estimated as follows:

$$(\omega + dd^c\phi)^n \geqslant e^{u + n\log(1-\varepsilon)} f\omega^n + e^u|f-g|\omega^n \geqslant e^{u+n\log(1-\varepsilon)} g\omega^n.$$

If we choose $K = \sup_X(-u)$ then

$$(\omega + dd^c\phi)^n \geqslant e^{\phi} g\omega^n,$$

and Proposition 2.3 yields $\phi \leqslant v$, hence $u - v \leqslant C_4\varepsilon$. Reversing the role of u and v we obtain the result. □

Using Theorem 3.1 we will improve the stability exponent in [48]. We first prove the following refinement of [48, Lemma 3.4].

Lemma 3.3. — *Assume that $0 \leqslant f, g \in L^p(X)$ satisfy*

$$e^{-\varepsilon} f \leqslant g \leqslant e^{\varepsilon} f, \tag{3.3}$$

for some (small) positive constant ε. Let u and v be continuous ω-psh functions on X such that

$$\omega_u^n = f\omega^n,\ \omega_v^n = g\omega^n \text{ and } \sup_X u = \sup_X v = 0.$$

Fix $t_1 > t_0 := \inf_X(u - v)$. If $\int_{\{u-v<t_1\}} f\omega^n \leqslant V_{\min}$ then, for some uniform constant $C > 0$ depending on (X, ω, n, p), an upper bound C_p for $\|f\|_p$, and a positive lower bound for $\|f\|_{1/n}$, we have

$$t_1 - t_0 \leqslant C\varepsilon.$$

Here $V_{\min}$ is the constant in Proposition 2.5 corresponding to $A_0 := 2^n C_p$.

Proof. — Define

$$\widehat{f}(z) = \begin{cases} f(z), & \text{if } u(z) < v(z) + t_1, \\ \frac{1}{A} f(z), & \text{if } u(z) \geqslant v(z) + t_1, \end{cases}$$

where $A > 1$ is a uniform constant ensuring that $\int_X \widehat{f}\omega^n < 2V_{\min}$. Let $\widehat{c} > 0$ be a constant and $\widehat{u}$ be a continuous ω-psh function such that

$$(\omega + dd^c\widehat{u})^n = \widehat{c}\widehat{f}\omega^n, \quad \sup_X \widehat{u} = 0.$$

It follows from Proposition 2.5 and [51, Corollary 2.4] that $2^n \leqslant \widehat{c} \leqslant A$, hence by [46, Corollary 5.6], $\widehat{u}$ is uniformly bounded.

For $s \in (0, 1)$, define $\psi_s := (1 - s)v + s\widehat{u}$. By the mixed Monge–Ampère inequality [51, Lemma 1.9] we have

$$(\omega + dd^c\psi_s)^n \geqslant \left((1 - s)g^{1/n} + s\left(\widehat{c}f\right)^{1/n}\right)^n \omega^n$$

on $\Omega(t_1) := \{u < v + t_1\}$. By the assumption (3.3) and the inequality $a^{1/n} \geqslant a$, for $a \in (0, 1)$, we have

$$(\omega + dd^c\psi_s)^n \geqslant \left((1 - s)e^{-\varepsilon/n} f^{1/n} + s(2^n f)^{1/n}\right)^n \omega^n,$$

in $\Omega(t_1)$. Thus, for $s=\varepsilon$ we have $(\omega+dd^c\psi_s)^n \geqslant (1+\varepsilon^2/n)f\omega^n$ in $\Omega(t_1)$. As in [48, Lemma 3.4] we now invoke the minimum principle, Proposition 2.4, to obtain

$$\max_{\Omega(t_1)}(\psi_s - u) = \max_{\partial\Omega(t_1)} (\psi_s - u)\,.$$

But on $\partial\Omega(t_1)$ we have $u = v+t_1$, hence $\psi_s - u + t_1 \leqslant C_1 s$ on $\partial\Omega(t_1)$, where C_1 is a uniform constant. Let $x_0 \in X$ be such that $u(x_0)-v(x_0) = t_0$. Then $x_0 \in \Omega(t_1)$, hence $\psi_s(x_0) - u(x_0) \leqslant \max_{\partial\Omega(t_1)}(\psi_s - u)$. We then infer that $t_1 - t_0 \leqslant Cs$ as desired. □

Proposition 3.4. — *Assume that u is a continuous ω-psh function such that $\omega_u^n = f\omega^n$, where $0 \leqslant f \in L^p(X)$, $p>1$. Let $f_j > 0$ be a sequence of smooth densities converging to f in $L^p(X)$ and let u_j be a sequence of smooth ω-psh functions decreasing to u. Let v_j be the unique smooth ω-psh function such that*

$$\omega_{v_j}^n = e^{v_j - u_j} f_j \omega^n.$$

Then v_j converges uniformly to u.

Note that the smoothness of v_j follows from [14].

Proof. — Recall that, from [48, Remark 5.7] we have $\int_X f^{1/n}\omega^n > 0$. Set $F_j := e^{-u_j} f_j$ and $F := e^{-u} f$. By [46, Corollary 5.6], v_j is uniformly bounded. Hence $1/C \leqslant \int_X F_j^{1/n}$ and $\|F_j\|_p \leqslant C_1$, for a uniform constant C_1. Theorem 1.1 yields $|v_j - u| \leqslant C_2 \|F_j - F\|_p^{1/n}$, for a uniform constant C_2. Hence v_j uniformly converges to u. □

Theorem 3.5. — *Assume that u and v are ω-psh continuous solutions to*

$$(\omega + dd^c u)^n = f\omega^n\,,\; (\omega + dd^c v)^n = g\omega^n,\; \sup_X u = \sup_X v = 0,$$

where $f, g \in L^p(X), p>1$ and $f \geqslant c_0 > 0$. Then

$$\sup_X |u-v| \leqslant C\,\|f-g\|_p^{1/n}\,,$$

where C depends on X, ω, n, p, c_0, an upper bound for $\|f\|_p + \|g\|_p$, and a positive lower bound for $\|g\|_{1/n}$.

Proof. — For convenience we can assume that $\int_X \omega^n = 1$. We first assume that u, v are smooth and

$$e^{-\varepsilon} f \leqslant g \leqslant e^{\varepsilon} f, \tag{3.4}$$

for some small constant $\varepsilon > 0$. Then, following the proof of [48, Theorem A] we obtain

$$|u - v| \leqslant C\varepsilon,$$

for a uniform constant $C > 0$. The only difference compared to [48, Lemma 3.4] is that we can replace ε^α by ε (see Lemma 3.3). For convenience of the reader we briefly recall the arguments of [48].

We set $t_0 := \min_X(u-v)$, $\widehat{t}_0 := \max_X(u-v) > t_0$. Then $t_0 \leqslant 0$ and $\widehat{t}_0 \geqslant 0$. The goal is to prove that $\widehat{t}_0 - t_0 \leqslant C\varepsilon$. Set

$$t_1 := \sup\left\{ t \geqslant t_0\,;\ \int_{\{u\,<\,v+t\}} f\omega^n \leqslant V_{\min}/2 \right\},$$

$$\widehat{t}_1 := \inf\left\{ t \leqslant \widehat{t}_0\,;\ \int_{\{u\,>\,v+t\}} f\omega^n \leqslant V_{\min}/2 \right\}.$$

It follows from Lemma 3.3 that $t_1 \leqslant t_0 + C\varepsilon$. Since ε is small we infer that $\int_{\{v\,<\,u-t\}} g\omega^n \leqslant V_{\min}$, for all $\widehat{t}_1 < t \leqslant \widehat{t}_0$. It thus follows from Lemma 3.3 that $-\widehat{t}_1 + \widehat{t}_0 \leqslant C\varepsilon$. Hence it remains to prove that $\widehat{t}_1 - t_1 \leqslant C\varepsilon$. Set $s_1 := t_1 + \varepsilon$ and $\widehat{s}_1 := \widehat{t}_1 - \varepsilon$. We prove that $\widehat{s}_1 - s_1 \leqslant C\varepsilon$. By definition of t_1 and $\widehat{t}_1$ we have

$$\int_{\{u\,<\,v+s_1\}} f\omega^n \geqslant V_{\min}/2\,;\ \int_{\{u\,>\,v+\hat{s}_1\}} f\omega^n \geqslant V_{\min}/2.$$

We choose a uniform constant $\gamma > 0$ depending on $\|f\|_p$, p, and $V_{\min}$ such that, for all Borel set $E \subset X$,

$$\int_E f\omega^n \geqslant V_{\min}/2 \Longrightarrow \int_E \omega^n \geqslant \gamma.$$

The existence and uniformity of γ follow from the Hölder inequality.

We now use the main novelty of [48]: estimate of the Laplacian mass on small collars (which uses the assumption $f \geqslant c_0 > 0$). Define $s_0 := t_0$, $s_k := 2^{k-1}(s_1 - s_0) + s_0$, for $k \geqslant 2$.

If $\int_{\{u\,>\,v+s_N\}} f\omega^n \geqslant V_{\min}/2$, then $\int_{\{u\,>\,v+s_N\}} \omega^n \geqslant \gamma$ and [48, Proposition 3.8] applies, giving

$$\int_{\{s_0\,<\,u-v\,\leqslant\,s_N\}} \omega_u \wedge \omega^{n-1} \geqslant (N-1)Cc_0\gamma^4.$$

But the left hand side is uniformly bounded by a constant depending on (X,ω) (see Lemma 2.6). It thus follows that for N large enough we have $\int_{\{u\,>\,v+s_N\}} f\omega^n < V_{\min}/2$. By definition of $\widehat{t}_1$ we have $\widehat{t}_1 \leqslant s_N$. But $s_N - s_0 \leqslant 2^{N-1}C\varepsilon$, hence $\widehat{s}_1 - s_1 \leqslant C\varepsilon$ as desired. The first step is completed.

We next assume that u, v are smooth but we remove the assumption (3.4). Let w be the unique smooth ω-psh function such that

$$(\omega + dd^c w)^n = e^{w-v} f\omega^n =: h\omega^n.$$

The smoothness of w was proved by Cherrier [14]. Since v satisfies $\omega_v^n = e^{v-v} g\omega^n$, we can apply Theorem 1.1 with $F = e^{-v}f$ and $G = e^{-v}g$ and obtain

$$|w-v| \leqslant C_1 \|f-g\|_p^{1/n}, \tag{3.5}$$

where $C_1 > 0$ is a uniform constant.

We thus have $e^{-\varepsilon} f \leqslant h \leqslant e^{\varepsilon} f$, where $\varepsilon := C_1 \|f-g\|_p^{1/n}$. The previous step yields

$$\left| w - \sup_X w - u \right| \leqslant C_2 \varepsilon.$$

But from (3.5) we see that $|\sup_X w| \leqslant 2\varepsilon$, hence the result follows.

We now treat the general case. We approximate u, v as in Proposition 3.4. Let u_j, v_j be smooth ω-psh functions decreasing to u, v. Let f_j, g_j be smooth functions converging to f, g in L^p and $f_j \geqslant c_0/2$. Let φ_j, ψ_j be smooth ω-psh functions solving

$$(\omega + dd^c \varphi_j)^n = e^{\varphi_j - u_j} f_j \omega^n \,,\, (\omega + dd^c \psi_j)^n = e^{\psi_j - v_j} g_j \omega^n.$$

It follows from Proposition 3.4 that φ_j, ψ_j converge uniformly to u, v. For j large enough we have $F_j := e^{\varphi_j - u_j} f_j \geqslant c_0/4$. Set $G_j := e^{\psi_j - v_j} g_j$ and observe that $\|F_j\|_p, \|G_j\|_p$ are uniformly bounded. It thus follows from the second step that

$$|\varphi_j - \psi_j| \leqslant C \|F_j - G_j\|_p^{1/n},$$

where $C > 0$ is a uniform constant. Letting $j \to +\infty$ we arrive at the result. $\square$

Using the same ideas we prove a stability estimate for the MA-constant. Recall that (see [46, 48]) for each $0 \leqslant f \in L^p, p > 1$ with $\int_X f\omega^n > 0$ there exists a unique constant $c = c_f > 0$ such that the equation $\omega_u^n = c_f f \omega^n$ has a bounded weak solution in $\mathrm{PSH}(X, \omega)$.

Corollary 3.6. — *Assume that $0 \leqslant f, g \in L^p$ for some $p > 1$. Then*

$$|c_f - c_g| \leqslant C \|f-g\|_p^{1/n},$$

where $C > 0$ is a constant depending on (X, ω, n, p), an upper bound for $\|f\|_p, \|g\|_p$, and a positive lower bound for $\|f\|_{1/n}$, $\|g\|_{1/n}$.

Proof. — Let u be a continuous ω-psh function on X, normalized by $\sup_X u = 0$, such that $(\omega + dd^c u)^n = c_f f \omega^n$. By Lemma 2.6 and the mixed Monge-Ampère inequality [51, Lemma 1.9] we have that

$$(\omega + dd^c u) \wedge e^G \omega^{n-1} \geqslant c_f^{1/n} f^{1/n} e^G \omega^n,$$

where G is a smooth function such that $dd^c(e^G\omega^{n-1}) = 0$ (see [35]). Integrating on X we see that c_f is uniformly bounded from above. Proposition 2.5 then ensures that c_f is uniformly bounded from below. It follows from [51, Theorem 0.1] that there exists a unique continuous ω-psh function v such that

$$(\omega + dd^c v)^n = e^{v-u} c_f g\omega^n.$$

Theorem 1.1 yields $|v - u| \leqslant C_1 c_f \|f - g\|_p^{1/n}$, for a uniform constant C_1, hence

$$\left(1 - C_2 \|f - g\|_p^{1/n}\right) c_f g\omega^n \leqslant (\omega + dd^c v)^n \leqslant \left(1 + C_2 \|f - g\|_p^{1/n}\right) c_f g\omega^n,$$

for some uniform constant C_2. It thus follows from [48, Lemma 2.1] that

$$\left(1 - C_2 \|f - g\|_p^{1/n}\right) c_f \leqslant c_g \leqslant \left(1 + C_2 \|f - g\|_p^{1/n}\right) c_f,$$

yielding

$$|c_f - c_g| \leqslant C_2 \|f - g\|_p^{1/n},$$

and concluding the proof of Corollary 3.6. □

3.2. The case of big cohomology classes on Kähler manifolds

Using the idea of the proof of Theorem 1.1 we can also improve [39, Theorem C]. We first recall a few known facts on pluripotential theory in big cohomology classes. We refer the reader to [2, 8, 18, 19, 20, 21, 22] for more details.

We assume (only in this section) that ω is Kähler (i.e. $d\omega = 0$). Fix a closed smooth real $(1,1)$-form θ. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is θ-psh if it is quasi-psh and $\theta + dd^c u \geqslant 0$ in the sense of currents. We let $\mathrm{PSH}(X,\theta)$ denote the set of all θ-psh functions which are not identically $-\infty$. By elementary properties of psh functions one has $\mathrm{PSH}(X,\theta) \subset L^1(X)$. Here, if nothing is stated, $L^1(X)$ is $L^1(X,\omega^n)$. The De Rham cohomology class $\{\theta\}$ is big if $\mathrm{PSH}(X, \theta - \varepsilon\omega)$ is non-empty for some $\varepsilon > 0$.

We let V_θ denote the envelope:

$$V_\theta := \sup \{u \in \mathrm{PSH}(X,\theta) \mid u \leqslant 0\}.$$

There is a Zariski open set Ω, called the ample locus of $\{\theta\}$, on which V_θ is locally bounded. A θ-psh function u has minimal singularities if $u - V_\theta$ is globally bounded on X. For a θ-psh function u with minimal singularities the operator $(\theta + dd^c u)^n$ is well-defined as a positive Borel measure on Ω. One extends this measure trivially over X. The total mass of the resulting measure depends only on the cohomology class of θ and is called the volume of θ, denoted by $\mathrm{Vol}(\theta)$ (see [8, 7]). Given a θ-psh function u, the non-pluripolar Monge–Ampère measure of u is defined by

$$(\theta + dd^c u)^n := \lim_{j \to +\infty} \mathbf{1}_{\{u > V_\theta - j\}} \left(\theta + dd^c \max\left(u, V_\theta - j\right)\right)^n,$$

where the sequence of measures on the right-hand side is increasing in j. Note that $\int_X (\theta + dd^c u)^n \leqslant \mathrm{Vol}(\theta)$ and the equality holds if and only if $u \in \mathcal{E}(X, \theta)$.

It was proved in [8] that for all L^p-density ($p > 1$), $0 \leqslant f$ with $\int_X f\omega^n = \mathrm{Vol}(\theta)$, there exists a unique θ-psh function with minimal singularities u such that $\sup_X u = 0$ and $\theta_u^n = f\omega^n$.

We assume throughout this section the following normalization:

$$\int_X \omega^n = \mathrm{Vol}(\theta) = 1. \tag{3.6}$$

Theorem 3.7. — *Under the above setting, assume that $0 \leqslant f, g \in L^p(X, \omega^n)$, $p > 1$. If u, v are θ-psh functions with minimal singularities on X such that*

$$\theta_u^n = f\omega^n, \quad \theta_v^n = g\omega^n, \quad \sup_X u = \sup_X v = 0,$$

then for some constant $C > 0$ depending on $(X, \omega, n, \theta, p)$ and an upper bound for $\|f\|_p, \|g\|_p$ we have

$$\sup_X |u - v| \leqslant C \left\|f - g\right\|_p^{1/n}.$$

In the proof below we let $C_1, C_2, \ldots$ denote various uniform constants.

Proof. — By [8, Theorem 4.1], for some uniform constant $C_1 > 0$, we have

$$u \geqslant V_\theta - C_1 \quad \text{and} \quad v \geqslant V_\theta - C_1.$$

From this we get $v - C_1 \leqslant u \leqslant v + C_1$.

By the uniform version of Skoda's integrability theorem, see [40, Theorem 8.11], [57], and the Hölder inequality, there exists a small positive constant $a > 0$ such that $\int_X e^{-a\varphi} f\omega^n < +\infty$ for all $\varphi \in \mathrm{PSH}(X, \theta)$. By [21, Theorem 5.3] there exists a unique $w \in \mathcal{E}(X, \theta)$ such that

$$(\theta + dd^c w)^n = e^{a(w - v)} f\omega^n =: h\omega^n. \tag{3.7}$$

Observe that $u - C_1$ (respectively $u - C_1$) is a subsolution (respectively supersolution) to the above equation, hence by the comparison principle (see e.g. [19, Lemma 4.24]) we have $u - C_1 \leqslant w \leqslant u + C_1$.

We claim that $|w - v| \leqslant A_1 \|f - g\|_p^{1/n}$, for some uniform constant $A_1 > 0$.

We set $\varepsilon := e^{2aC_1/n}\|f-g\|_p^{1/n}$ and consider two cases. If $\varepsilon > 1/2$ then, by choosing $A_1 = 4C_1 e^{2aC_1/n}$ we have

$$|w-v| \leqslant 2C_1 \leqslant A_1 \|f-g\|_p^{1/n}.$$

Assume that $\varepsilon \leqslant 1/2$. By the Hölder inequality and the normalization (3.6) we can find a constant $b \geqslant 0$ such that

$$\int_X \left(\frac{|f-g|}{\|f-g\|_p} + b\right)\omega^n = \mathrm{Vol}(\theta).$$

Let $\rho \in \mathrm{PSH}(X,\theta)$ be the unique solution with minimal singularities to

$$(\theta + dd^c\rho)^n = \left(\frac{|f-g|}{\|f-g\|_p} + b\right)\omega^n, \quad \sup_X \rho = 0.$$

It follows from [8, Theorem 4.1] that $\rho \geqslant V_\theta - C_3$, hence $|\rho - w| \leqslant C_4$. We now show that for a suitable choice of $B > 0$, the function $\varphi := (1-\varepsilon)w + \varepsilon\rho - B\varepsilon$ is a subsolution to (3.7). Indeed,

$$\theta_\varphi^n \geqslant (1-\varepsilon)^n e^{a(w-v)} f\omega^n + e^{2aC_1}|f-g|\,\omega^n \geqslant e^{a(w-v)+n\log(1-\varepsilon)} g\omega^n.$$

For B large enough (depending on C_4, a) we have that $a\varphi \leqslant aw + n\log(1-\varepsilon)$, hence φ is a subsolution to $(\theta + dd^c\phi)^n = e^{a(\phi - v)} g\omega^n$. We thus have, by the comparison principle, that $\varphi \leqslant v$. Exchanging the role of v and w we finish the proof of the claim.

We next prove that $|w-u| \leqslant A_2\|f-g\|_p^{1/n}$, for some uniform constant $A_2 > 0$. Since $|w-v| \leqslant A_1\|f-g\|_p^{1/n}$ and $\sup_X v = 0$ it follows that $|\sup_X w| \leqslant A_1\|f-g\|_p^{1/n}$. It thus suffices to prove that $\mathrm{osc}_X(w-u) \leqslant A_2\|f-g\|_p^{1/n}$. Replacing w by $w+c$, for some constant c, we can assume that

$$\sup_X(w-u) = \sup_X(u-w) =: s \geqslant 0.$$

It is then enough to prove that $s \leqslant A_2\|f-g\|_p^{1/n}$. To do this we can assume that

$$2\int_{\{w<u\}} h\omega^n \leqslant \int_X h\omega^n$$

(otherwise we change the role of w and u). Note that

$$\theta_w^n = h\omega^n \ ; \ \theta_u^n = f\omega^n, \ 2^{-\delta}h \leqslant f \leqslant 2^\delta h,$$

where $\delta = C_5\|f-g\|_p^{1/n}$. Now, it suffices to prove that $u \leqslant w + A_2\|f-g\|_p^{1/n}$. Set $U := \{w < u\}$. Let ρ be the unique θ-psh function with minimal singularities such that

$$\theta_\rho^n = 2h\mathbf{1}_U\omega^n + b_1\omega^n, \quad \sup_X \rho = 0,$$

where $b_1 \geqslant 0$ is a normalization constant. It follows from [8, Theorem 4.1] that $|\rho - u| \leqslant C_6$, hence

$$V := \left\{ w < (1-\delta)u + \delta\,(\rho - C_6) \right\} \subset U.$$

On U, the Monge–Ampère measure $\theta^n_{(1-\delta)u+\delta\rho}$ can be estimated as follows, using the mixed Monge–Ampère inequalities (see [8, 27]),

$$\begin{aligned}\theta^n_{(1-\delta)u+\delta\rho} &\geqslant \left((1-\delta)f^{1/n} + \delta(2h)^{1/n}\right)^n \omega^n \\ &\geqslant \left((1-\delta)2^{-\delta/n} + 2^{1/n}\delta\right)^n h\omega^n.\end{aligned}$$

Using the inequality $2^x = e^{x\,\log 2} \geqslant 1 + x\log 2$ we have, for $\delta \in (0,1)$,

$$\begin{aligned}(1-\delta)2^{-\delta/n} + 2^{1/n}\delta &\geqslant (1-\delta)\left(1 - \frac{\delta\log 2}{n}\right) + \left(1 + \frac{\log 2}{n}\right)\delta \\ &= 1 + \frac{\delta^2\log 2}{n} =: 1+\gamma.\end{aligned}$$

We thus have

$$\theta^n_{(1-\delta)\,u+\delta\rho} \geqslant (1+\gamma)h\omega^n.$$

The comparison principle, see [8, Corollary 2.3], gives

$$\int_V (1+\gamma)h\omega^n \leqslant \int_V \theta^n_{(1-\delta)u+\delta\rho} \leqslant \int_V \theta^n_w = \int_V h\omega^n,$$

hence $\int_V h\omega^n = 0$. Using the domination principle, see [19, Corollary 3.10], we then infer $w \geqslant (1-\delta)u + \delta(\rho - C_6)$, hence $w - u \geqslant -C_6\|f-g\|_p^{1/n}$ which completes the proof. □

Remark 3.8. — In the Hermitian setting, if the comparison principle holds (which implies certain geometric conditions on X, see [15]), then the above proof can be applied.

4. Hölder continuity

4.1. Hölder regularity of solutions

THEOREM 4.1. — *Let (X,ω) be a compact Hermitian manifold of dimension n. Fix $0 \leqslant f \in L^p(X), p > 1$ with $\int_X f\omega^n > 0$. Then any solution u to $\omega_u^n = f\omega^n$ is Hölder continuous with Hölder exponent in $(0, p_n)$, where $p_n = 2/(nq+1)$.*

We note here that bounded solutions to (1.1) are automatically continuous. Indeed, let u be a bounded solution and v be a continuous ω-psh function such that $\omega_v^n = e^{v-u} f\omega^n$. The existence of v follows from [51]. By uniqueness $v = u$, hence u is continuous.

Proof. — Assume that u is a bounded ω-psh function solving

$$(\omega + dd^c u)^n = f\omega^n.$$

By adding a constant to u we can assume that $\inf_X u = 1$ and set $b := 2\sup_X u$. We will use the same notations as in [48, Section 4]. Fix $\alpha \in (0, p_n)$. We prove that u is Hölder continuous with exponent α by showing that $\rho_t u - u \leqslant ct^\alpha$, for t small enough (see [26, page 632], [37, Lemma 4.2] or Lemma 4.4 below). Here, following Demailly [23], $\rho_t(u)$ is defined by

$$\rho_t(u)(z) := \frac{1}{t^{2n}} \int_{T_z X} u\left(\mathrm{exph}_z(\zeta)\right) \rho\left(\frac{\|\zeta\|_\omega^2}{t^2}\right) dV_\omega(\zeta), \tag{4.1}$$

where $\zeta \mapsto \mathrm{exph}_z(\zeta)$ is the (formal) holomorphic part of the Taylor expansion of the exponential map of the Chern connection on the tangent bundle of X associated to ω, and ρ is a smoothing kernel defined by

$$\rho(t) := \begin{cases} \frac{\eta}{(1-t)^2} \exp\left(\frac{1}{t-1}\right), \text{ if } t \in [0,1], \\ 0, \text{ if } t > 1, \end{cases}$$

where $\eta > 0$ is a constant such that $\int_{\mathbb{C}^n} \rho(\|z\|^2) dV(z) = 1$. Here dV is the Lebesgue measure on $\mathbb{C}^n$.

Following [23] and [48], we define the Kiselman–Legendre transform:

$$U_{\delta,\,c} := \inf_{t\,\in\,[0,\,\delta]} \left(\rho_t(u) + K\left(t^2 - \delta^2\right) + K(t-\delta) - c\log(t/\delta)\right), \tag{4.2}$$

where $c > 0, \delta > 0$, and K is a positive (curvature) constant and as in [23] we choose K to ensure that $t \mapsto \rho_t(u) + Kt^2$ is increasing in t. In the following arguments we choose $c = \delta^\alpha$ and we write U_δ instead of $U_{\delta,\,c}$. It follows from [48, Lemma 4.1] that

$$\omega + dd^c U_\delta \geqslant -A\delta^\alpha \omega,$$

where $A > 0$ is a uniform curvature constant. Setting

$$u_\delta := \frac{1}{1 + 2A\delta^\alpha} U_\delta,$$

we then have $\omega + dd^c u_\delta \geqslant \gamma\omega$, for some positive constant γ. Note that by construction and by the choice of K, we have

$$\rho_\delta(u) + K\delta^2 \geqslant u, \quad \text{and} \quad \rho_\delta(u) \geqslant U_\delta.$$

Set $s := e^{-5Ab}$ and

$$E(\delta) := \{\rho_\delta(u) - u > Ab\delta^\alpha\}, \quad F(\delta) := \{\rho_{s\delta} u \geqslant u + 5Ab\delta^\alpha\}.$$

Up to decreasing δ we can assume that $2K\delta \leqslant Ab\delta^\alpha$. We claim that on $F(\delta)$ we have $U_\delta - u \geqslant 4Ab\delta^\alpha$. Indeed, since $t \mapsto \rho_t u + Kt^2$ is increasing and s is small, we have

$$\begin{aligned}\rho_t(u) + K\left(t^2 - \delta^2\right) + K(t-\delta) - \delta^\alpha \log(t/\delta)\\ \geqslant u - 2K\delta + 5Ab\delta^\alpha,\ \forall\, t \in [0, s\delta],\end{aligned}$$

and

$$\begin{aligned}\rho_t(u) + K\left(t^2 - \delta^2\right) + K(t-\delta) - \delta^\alpha \log(t/\delta)\\ \geqslant \rho_{s\delta} u - 2K\delta, \forall\, t \in [s\delta, \delta].\end{aligned}$$

It thus follows that on $F(\delta)$ we have $U_\delta \geqslant u + 4Ab\delta^\alpha$, as claimed.

Now we prove that the set $F(\delta)$ is empty for $\delta > 0$ small enough. It follows from [48, equation (4.9)] (which is a lemma in [26]) that

$$\int_X (\rho_t u - u)\omega^n \leqslant Ct^2.$$

Hence

$$\int_{E(\delta)} \omega^n \leqslant \frac{C}{Ab}\delta^{2-\alpha},$$

and an application of the Hölder inequality yields

$$\int_{E(\delta)} f\omega^n \leqslant C_1 \delta^\beta,$$

where $\beta := (2-\alpha)/q$, and q is the conjugate of p.

We let v be the unique continuous ω-psh function such that

$$(\omega + dd^c v)^n = e^{v-u} f \mathbf{1}_{X \setminus E(\delta)} \omega^n.$$

Theorem 1.1 yields, for each $\varepsilon > 0$, $|v - u| \leqslant C_3 \delta^{\beta/(n+\varepsilon)}$, where C_3 depends also on ε. Since $\alpha < p_n$ we can choose $\varepsilon > 0$ so small that $\beta/(n+\varepsilon) > \alpha$. Decreasing δ we can ensure that $|v - u| \leqslant Ab\delta^\alpha/2$. The choice of b ensures that

$$|u_\delta - U_\delta| \leqslant \frac{Ab\delta^\alpha}{2}.$$

Assume by contradiction that $F(\delta) \neq \emptyset$. On $F(\delta)$ we have

$$u_\delta - v = u_\delta - U_\delta + U_\delta - u + u - v \geqslant 3Ab\delta^\alpha,$$

while on $X \setminus E(\delta)$, we have

$$u_\delta - v = u_\delta - U_\delta + U_\delta - \rho_\delta u + \rho_\delta u - u \leqslant 2Ab\delta^\alpha.$$

It thus follows that $u_\delta - v$ attains its maximum over X at some point $z_0 \in E(\delta)$, contradicting the minimum principle (Proposition 2.4) since $\omega_v^n = 0 < \omega_{u_\delta}^n$ on $E(\delta)$. Hence, for δ small enough, $F(\delta)$ is empty. This completes the proof. □

4.2. Hölder regularity of plurisubharmonic envelopes

For a continuous function $f : X \to \mathbb{R}$ we define its ω-psh envelope by:

$$P_\omega(f) := \left(\sup \{\phi \,|\, \phi \in \mathrm{PSH}(X,\omega) \quad \text{and} \quad \phi \leqslant f\} \right)^*.$$

Is was proved in [52] (for the Kähler case) and in [16] (for the Hermitian case) that $P_\omega(f)$ belongs to $C^{1,1}(X)$ if f is smooth. If f is (Lipschitz) continuous then $P_\omega(f)$ is also (Lipschitz) continuous [16]. In this section we prove that $P_\omega(f)$ is Hölder continuous provided that f is Hölder continuous.

Lemma 4.2. — *If $f \in C^0(X)$ then $P_\omega(f) \in C^0(X)$.*

Proof. — Let $f_j \in C^\infty(X)$ be a sequence of smooth functions which converges uniformly to f. Since $P_\omega(f_j)$ is continuous and

$$\left\| P_\omega(f_j) - P_\omega(f) \right\|_{L^\infty(X)} \leqslant \left\| f_j - f \right\|_{L^\infty(X)},$$

we see that $P_\omega(f)$ is continuous. □

Theorem 4.3. — *Assume that $f \in C^{0,\alpha}(X)$ for some $\alpha \in (0,1)$. Then $P_\omega(f) \in C^{0,\alpha}(X)$.*

Proof. — It follows from Choquet's lemma and the definition of the psh envelope that there exists a sequence of ω-psh functions $(\phi^j)_{j\in\mathbb{N}}$ such that

$$P_\omega(f) = \left(\sup_j \phi^j \right)^*, \ \phi^j \leqslant f \quad \text{and} \quad \left\| \phi^j \right\|_\infty \leqslant C\left(\|f\|_\infty \right).$$

Replacing ϕ^j by $(\sup_{k\leqslant j} \phi^k)^*$ we can assume that $\phi^j \nearrow P_\omega(f)$. Since $P_\omega(f)$ is continuous on X we also have, by Dini's theorem, that ϕ^j converges uniformly to $P_\omega(f)$.

For a ω-psh function u we consider the convolution $\rho_t u$ defined as in (4.1) and the Kiselman–Legendre transform defined as in (4.2). Since $\phi^j \leqslant f$ and $f \in C^{0,\alpha}(X)$, we have

$$\rho_\delta \phi^j \leqslant \rho_\delta f \leqslant f + C\delta^\alpha \|f\|_{0,\alpha}, \tag{4.3}$$

where C depends only on X, ω.

We now use the Kiselman–Legendre transform $\Phi^j_{\delta,\,c} := \Phi_{\delta,\,c}(\phi^j)$. From (4.2), with $t=\delta$, we have that $\Phi^j_{\delta,\,c} \leqslant \rho_\delta \phi^j$. It follows from [48, Lemma 4.1] that

$$\omega + dd^c \Phi^j_{\delta,\,c} \geqslant -\left(Ac + 2K\delta\right)\omega,$$

where A is a positive curvature constant.

We now fix $c = (\delta^\alpha - 2K\delta)/A$ so that $Ac + 2K\delta = \delta^\alpha$. We have

$$\omega + dd^c \Phi^j_{\delta,\,c} \geqslant -\delta^\alpha \omega.$$

Setting

$$\varphi^j_\delta := \left(1-\delta^\alpha\right)\Phi^j_{\delta,\,c}$$

we then have $\omega + dd^c\varphi^j_\delta \geqslant \delta^{2\alpha}\omega$ and $\|\varphi^j_\delta - \Phi^j_{\delta,\,c}\| \leqslant C_0\delta^\alpha$, where C_0 depends on $\|\phi^j\|_\infty$. From (4.3) and the fact that $\Phi^j_{\delta,\,c} \leqslant \rho_\delta\phi^j$, we infer $\varphi^j_\delta - C_1\delta^\alpha \leqslant f$, where C_1 depends only on $|f\|_{0,\,\alpha}$, $\|\phi^j\|_\infty$ and A. Therefore we get

$$\varphi^j_\delta - C_1\delta^\alpha \leqslant P_\omega(f) \tag{4.4}$$

by the definition of $P_\omega(f)$ and the fact that φ^j_δ is ω-psh. This implies that

$$\Phi^j_{\delta,\,c} - P_\omega(f) \leqslant C_2\delta^\alpha, \tag{4.5}$$

where C_2 depending only C_1 and $\|\phi^j\|_\infty$. Since ϕ^j converges uniformly to $P_\omega(f)$ we infer

$$\Phi^j_{\delta,c} - \phi^j \leqslant 2C_2\delta^\alpha, \tag{4.6}$$

for j sufficiently large.

Following [47] we now use (4.6) to estimate $\rho_\delta\phi^j - P_\omega(f)$. For any $x \in X$, the minimum in the definition of $\Phi^j_{\delta,\,c}$ achieves at $t_0 = t_0(x,j)$. It follows from (4.6) that

$$\rho_{t_0}\phi^j + K(t_0-\delta) + K\left(t_0^2 - \delta^2\right) - c\log\left(t_0/\delta\right) - \phi^j \leqslant C_3\delta^\alpha, \tag{4.7}$$

where C_3 depends only on $\|f\|_{0,\,\alpha}$, $\|\phi^j\|_\infty$. Since $\rho_t\phi^j + Kt^2 + Kt - \phi^j \geqslant 0$, we have

$$c\log\frac{t_0}{\delta} \geqslant -C_4\delta^\alpha.$$

For δ small enough we have $c \geqslant \delta^\alpha/(2A)$, hence

$$t_0 \geqslant a\delta, \quad \text{for} \quad a = e^{-2AC_4}. \tag{4.8}$$

Since $\rho_t + Kt^2 + Kt$ is increasing in t and $t_0 \geqslant a\delta$, we infer

$$\begin{aligned}\rho_{a\delta}\phi^j + Ka\delta + K(a\delta)^2 - P_\omega(f) &\leqslant \rho_{t_0}\phi^j + Kt_0 + Kt_0^2 - P_\omega(f)\\ &\leqslant \Phi^j_{\delta,\,c} - P_\omega(f) - c\log a\\ &\leqslant C_5\delta^\alpha,\end{aligned}$$

where C_5 depends only on $\|f\|_{0,\alpha}$, $\|\phi^j\|_\infty$, K, A, and in the last line we have used (4.5). Since $\phi^j \nearrow P_\omega(f)$, we have that $\rho_\delta\phi^j$ converges to $\rho_\delta P_\omega(f)$ as $j \to \infty$. Therefore, letting j tend to ∞, and then replacing $a\delta$ by δ we get

$$\rho_\delta P_\omega(f) - C_6\delta^\alpha \leqslant P_\omega(f), \tag{4.9}$$

where C_6 depends only on $\|f\|_{0,\alpha}$, K and A. Invoking Lemma 4.4 below we conclude that $P_\omega(f) \in \mathrm{Lip}_\alpha(X)$. □

Lemma 4.4. — *Assume that u is a bounded ω-psh function on X such that $\rho_t u \leqslant u + C_0 t^\alpha$ for some positive constants C_0 and $0 < \alpha < 1$. Then $u \in \mathrm{Lip}_\alpha(X)$.*

The proof of the lemma was implicitly written in [26], [37]. We include it for completeness.

Proof. — We can assume that $u \leqslant 0$. Let d be the Riemann distance on X induced by the metric ω. Define

$$\tau(\delta) := \sup\left\{|u(x) - u(y)|\,|x, y \in X, d(x,y) \leqslant \delta\right\}, \delta > 0.$$

We assume by contradiction that $\limsup_{\delta\to 0^+} \delta^{-\alpha}\tau(\delta) = +\infty$. For each $\delta > 0$ we can find $x_\delta \in X$, $y_\delta \in X$ such that $d(x_\delta, y_\delta) \leqslant \delta$ and $\tau(\delta) = u(y_\delta) - u(x_\delta) > 0$. We can thus find $x_0 \in X$ and a sequence $\delta_j \searrow 0$ such that

$$\lim_{j\to+\infty} d\left(x_{\delta_j}, x_0\right) = \lim_{j\to+\infty} d\left(y_{\delta_j}, x_0\right) = 0$$

and

$$\lim_{j\to+\infty} \delta_j^{-\alpha}\left|u\left(x_{\delta_j}\right) - u\left(y_{\delta_j}\right)\right| = +\infty.$$

Let $B \subset X$ be a small ball around x_0 which will be identified with the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ via a biholomorphism. Up to adding a smooth function we can now view u as a psh function in $\mathbb{B}$ and $d(x,y) \simeq \|x-y\|$ for $x, y \in \mathbb{B}$. It follows from [24, Remark 4.6] that $\rho_r u(x_\delta) = u \star \rho_r(x_\delta) + O(r^2)$. Fix $b > 1$ so large that

$$(b+2)^\alpha\left(1 - \frac{b^{2n}}{(b+1)^{2n}}\right) < \frac{1}{2}.$$

Fix $\delta > 0$ so small that $2(b+1)\delta < 1$. For $\xi \in \mathbb{B}$ we denote (see [25, p. 32])

$$\mu_S(u;\xi,r) := \frac{1}{\sigma_{2n-1}}\int_{\mathbb{S}} u(\xi + rx)d\sigma(x),$$

$$\mu_B(u;\xi,r) := \frac{1}{V_{2n}r^{2n}}\int_{B(\xi,r)} u(x)dV(x).$$

Here, σ is the area measure of the unit sphere $\mathbb{S} = \partial\mathbb{B}$, $\sigma_{2n-1} = \sigma(S(0,1))$, $V_{2n} = \mathrm{Vol}(\mathbb{B})$. Note that $\mu_S \geqslant \mu_B$ and these are non-decreasing in r. By the mean value inequality we have that, for $r = (b+1)\delta$,

$$\begin{aligned}
&\mu_B\left(u; x_\delta, r\right)\\
&\quad= \frac{1}{V_{2n} r^{2n}} \int_{B(x_\delta,\, r)} u(x) dV(x)\\
&\quad= \frac{1}{V_{2n} r^{2n}} \left(\int_{B(y_\delta,\, b\delta)} u(x) dV(x) + \int_{B(x_\delta,\, r)\setminus B(y_\delta,\, b\delta)} u(x) dV(x) \right)\\
&\quad\geqslant \frac{b^{2n}}{(b+1)^{2n}} u\left(y_\delta\right) + \left(1 - \frac{b^{2n}}{(b+1)^{2n}}\right) \big(u(y_\delta) - \tau(r+\delta)\big)\\
&\quad= u(y_\delta) - \left(1 - \frac{b^{2n}}{(b+1)^{2n}}\right) \tau\big((b+2)\delta\big).
\end{aligned}$$

Since $\mu_S(u; x_\delta, t) - u(x_\delta) \geqslant 0$ and non decreasing in $t > 0$, we have

$$\begin{aligned}
u \star \rho_{2r}(x_\delta) - u(x_\delta) &= \sigma_{2n-1} \int_0^1 \big(\mu_S\left(u; x_\delta, 2tr\right) - u(x_\delta)\big) t^{2n-1} \rho(t) dt\\
&\geqslant \big(\mu_S\left(u; x_\delta, r\right) - u(x_\delta)\big) \sigma_{2n-1} \int_{\frac{1}{2}}^1 t^{2n-1} \rho(t) dt\\
&\geqslant C_2 \big(\mu_B\left(u; x_\delta, r\right) - u(x_\delta)\big)\\
&\geqslant C_2 \left(\tau(\delta) - \frac{1}{2} \frac{\tau((b+2)\delta)}{(b+2)^\alpha}\right).
\end{aligned}$$

Using these estimates and the assumption that $\rho_{2r} u(x_\delta) \leqslant u(x_\delta) + C_0 (2r)^\alpha$ we arrive at

$$\tau(\delta) - \frac{1}{2} \frac{\tau\big((b+2)\delta\big)}{(b+2)^\alpha} \leqslant C_5 \delta^\alpha.$$

We set $h(\delta) = \delta^{-\alpha} \tau(\delta)$. For $\delta > 0$ small enough, say $\delta \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$ fixed, and $c = b + 2$ we then have

$$h(\delta) \leqslant \frac{1}{2} h(c\delta) + C_5.$$

Applying this several times we obtain, for all $k \in \mathbb{N}$ with $c^{k-1}\delta \leqslant \varepsilon_0$,

$$h(\delta) \leqslant 2^{-k} h\left(c^k \delta\right) + 2C_5,\ k \in \mathbb{N}. \tag{4.10}$$

We are now ready to derive a contradiction. We set

$$C_6 := \sup_{\delta \,\in\, [\varepsilon_0,\, c\varepsilon_0]} h(\delta) < +\infty.$$

We have assumed that there exists a sequence $\delta_j \searrow 0$ such that $h(\delta_j) \to +\infty$. Take j so large that $\delta_j < \varepsilon_0$ and $h(\delta_j) > 2C_5 + C_6 + 1$. We choose $k \in \mathbb{N}$ such that

$$\frac{\log(\varepsilon_0/\delta_j)}{\log c} \leqslant k \leqslant \frac{\log(\varepsilon_0/\delta_j)}{\log c} + 1.$$

Then $c^k\delta_j \in [\varepsilon_0, c\varepsilon_0]$. From this and (4.10) we obtain $h(\delta_j) \leqslant 2^{-k}h(c^k\delta_j) + 2C_5 \leqslant 2C_5 + C_6$, a contradiction. □

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Manuscrit reçu le 24 mars 2020,
révisé le 8 août 2020,
accepté le 13 novembre 2020.

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