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EFFECTIVE OPERATORS FOR ROBIN EIGENVALUES IN DOMAINS WITH CORNERS

by Magda KHALILE, Thomas OURMIÈRES-BONAFOS & Konstantin PANKRASHKIN

ABSTRACT. — We study the eigenvalues of the Laplacian with a strong attractive Robin boundary condition in curvilinear polygons. It was known from previous works that the asymptotics of several first eigenvalues is essentially determined by the corner openings, while only rough estimates were available for the next eigenvalues. Under some geometric assumptions, we go beyond the critical eigenvalue number and give a precise asymptotics of any individual eigenvalue by establishing a link with an effective Schrödinger-type operator on the boundary of the domain with boundary conditions at the corners.

RÉSUMÉ. — Nous étudions les valeurs propres du laplacien avec une condition de Robin fortement attractive dans des polygones curvilignes. Grâce à de précédents travaux, on sait que le comportement asymptotique de quelques premières valeurs propres est essentiellement déterminé par les ouvertures des coins, alors que seules quelques estimées grossières sont disponibles pour les valeurs propres suivantes. Sous certaines hypothèses géométriques, nous allons au-delà du nombre critique de valeurs propres et nous donnons un développement asymptotique précis pour chaque valeur propre individuelle en établissant un lien avec un opérateur effectif de type Schrödinger agissant sur le bord du domaine et muni de conditions aux limites aux coins.

1. Introduction

1.1. Problem setting and previous results

Given a domain $\Omega \subset \mathbb{R}^d$, $d \ge 2$, with a suitably regular boundary $\partial \Omega$ and a parameter $\alpha > 0$, we denote by R^{Ω}_{α} the Laplacian in $L^{2}(\Omega)$ with the

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Robin condition $\partial u/\partial \nu = \alpha u$ at the boundary, where ν is the outer unit normal. The operator is rigorously defined using its quadratic form

$$H^1(\Omega) \ni u \mapsto \int_{\Omega} |\nabla u|^2 \, \mathrm{d}x - \alpha \int_{\partial \Omega} u^2 \, \mathrm{d}s$$

with ds being the (d-1)-dimensional Hausdorff measure, provided that the form is lower semibounded and closed. The spectral properties of the operator R^{Ω}_{α} have attracted a lot of attention during the last years, and a recent review of various results and open problems can be found in the paper [12] by Bucur, Freitas, Kennedy. In the present paper we will be interested in the behavior of the eigenvalues $E_n(R^{\Omega}_{\alpha})$ in the asymptotic regime $\alpha \to +\infty$. Let us recall some available results in this direction.

It seems that the study of the above asymptotic regime was first proposed by Lacey, Ockedon, Sabina [42] when considering a reaction-diffusion system, and Giorgi and Smits [24, 25] obtained a number of estimates with links to the theory of enhanced surface superconductivity. Remark that for bounded Lipschitz domains Ω it follows from the general theory of Sobolev spaces that there exists C > 0 with $E_1(R_{\alpha}^{\Omega}) \ge -C\alpha^2$ for large α (see Lemma 2.7 below). Lacey, Ockedon, Sabina in [42] conjectured that under suitable regularity assumptions on Ω the lower bound can be upgraded to an asymptotics

(1.1)
$$E_1(R^{\Omega}_{\alpha}) \sim -C_{\Omega} \alpha^2,$$

with some $C_{\Omega} > 0$, and they have shown that $C_{\Omega} = 1$ for C^4 smooth domains. Levitin and Parnovski in [43] have shown the asymptotics (1.1) for piecewise smooth domains satisfying the interior cone condition, and they have shown that the constant C_{Ω} is explicitly determined through the spectra of model Robin Laplacians by

(1.2)
$$(-C_{\Omega}) = \inf_{x \in \partial \Omega} \inf \operatorname{spec}(R_1^{T_x}),$$

where T_x is the tangent cone to Ω at x and spec stands for the spectrum of the operator. Bruneau and Popoff in [10] gave an improved remainder estimate under the slightly stronger assumption that Ω is a so-called corner domain. We also mention the recent paper [39] by Kovařík and Pankrashkin on non-Lipschitz domains, for which the eigenvalue behavior is completely different.

More precise estimates are available for smooth domains. The lower bound by Lou and Zhu [45] and the upper bound due to Daners and Kennedy [15] imply that if Ω is a bounded C^1 domain, then for each fixed $n \in \mathbb{N}$ one has $E_n(R^{\Omega}_{\alpha}) \sim -\alpha^2$. It seems that a more precise asymptotics was first obtained by Pankrashkin in [52]: it was shown that if $\Omega \subset \mathbb{R}^2$ is bounded with a C^3 boundary, then $E_1(R^{\Omega}_{\alpha}) = -\alpha^2 - H_*\alpha + \mathcal{O}(\alpha^{\frac{2}{3}})$, where H_* is the maximum of the curvature of the boundary. Exner, Minakov and Parnovski in [18] show that the asymptotics

(1.3)
$$E_n(R^{\Omega}_{\alpha}) = -\alpha^2 - H_*\alpha + \mathcal{O}(\alpha^{\frac{2}{3}})$$

holds for any fixed $n \in \mathbb{N}$, and then Exner and Minakov [17] obtained similar results for a class of non-compact domains. Helffer and Kachmar [30] obtained a complete asymptotic expansion for eigenvalues under the additional assumption that the curvature of the boundary admits a single nondegenerate maximum. Pankrashkin and Popoff in [56] started the study of the multidimensional case: if $\Omega \subset \mathbb{R}^d$ is a C^3 domain, then the asymptotics (1.3) holds with $H_* := \max H$ and H is defined as the sum of the principal curvatures at the boundary, i.e. H = (d-1) times the mean curvature. An analog of the asymptotics (1.3) for the first eigenvalue of Robin p-Laplacians was obtained by Kovařík and Pankrashkin in [38]. Among possible applications of the asymptotics (1.3) one may mention various optimization issues concerning the eigenvalues of R^{Ω}_{α} . It was conjectured by Bareket [5] that among the domains Ω of fixed volume, for any $\alpha > 0$ the quantity $E_1(R^{\Omega}_{\alpha})$ is maximized by the balls. In this most general form, the conjecture was disproved by Freitas and Krejčiřík [23], but an additional analysis shows that the conjecture may hold in a weaker form under additional restrictions on the geometry of Ω , we refer to the papers by Antunes, Freitas, Krejčiřík [1], Bandle and Wagner [4], Bucur, Ferone, Nitsch, Trombetti [11], Ferone, Nitsch, Trombetti [20], Trani [62] and Savo [61] for domains on manifolds. As noted by Pankrashkin and Popoff in [56], if the ball is the maximizer of $E_1(R^{\Omega}_{\alpha})$ for all $\alpha > 0$ in some class of smooth domains Ω , then it is also the minimizer for the maximum mean curvature H_* in the same class of domains, and this observation leads to some new inequalities for H_* , see e.g. Ferone, Nitsch, Trombetti [21], and it was used to construct a number of counterexamples, for example, the asymptotics (1.3)was used by Krejčiřík and Lotoreichik [40, 41] in the study of isoperimetric inequalities for Robin laplacians in exterior domains.

In [57] Pankrashkin and Popoff proposed an effective operator to study the eigenvalues of R^{Ω}_{α} . Namely, it was shown for C^3 domains Ω , either bounded or with a controllable behavior at infinity, that for any fixed $n \in \mathbb{N}$ one has the asymptotics

(1.4)
$$E_n(R^{\Omega}_{\alpha}) = -\alpha^2 + E_n(L_{\alpha}) + \mathcal{O}(1),$$

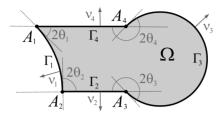


Figure 1.1. An example of a curvilinear polygon Ω with four vertices and sides of constant curvature. The vertices A_1 and A_2 are convex, and the vertices A_3 and A_4 are concave. One has $H_1 < 0$, $H_3 > 0$ and $H_2 = H_4 = 0$.

where L_{α} is the Schrödinger operator in $L^2(\partial\Omega)$ acting as $L_{\alpha} = -\Delta_{\partial\Omega} - \alpha H$ with $\Delta_{\partial\Omega}$ being the Laplace–Beltrami operator on $\partial\Omega$. Kachmar, Keraval, Raymond [35] and Helffer, Kachmar, Raymond [31] have shown that the same effective operator appears in other spectral questions for R^{Ω}_{α} , e.g. the Weyl asymptotics and the tunneling effect for R^{Ω}_{α} are also controlled by those for L_{α} at the leading orders. Pankrashkin [54] and Bruneau, Pankrashkin, Popoff [9] used the effective operator in order to study the accumulation of eigenvalues for Robin Laplacians on some non-compact domains.

We also mention some related papers going slightly beyond the initial problem setting. Colorado and García-Melián [14] obtained some results in the same spirit for Laplacians with the boundary condition $\partial u/\partial \nu = \alpha p u$ for variable functions p and $\alpha \to +\infty$. Filinovskii in [22] obtained the estimate lim $\inf_{\alpha\to+\infty} \alpha^{-1} \partial E_1(R^{\Omega}_{\alpha})/\partial \alpha \leq -1$. Helffer and Pankrashkin [32] studied the exponential splitting between the first two eigenvalues of R^{Ω}_{α} in a domain Ω with two congruent corners. Cakoni, Chaulet and Haddar [13] have shown that, in a sense, the only finite accumulation points of the eigenvalues of R^{Ω}_{α} for large positive α are the Dirichlet Laplacian eigenvalues of Ω .

1.2. Main results

In the present paper, we would like to combine the existing results and techniques in order to study the eigenvalues of R^{Ω}_{α} for the case of $\Omega \subset \mathbb{R}^2$ being a *curvilinear polygon* and to better understand the role of corners in the spectral properties. A complete definition of curvilinear polygons will be given later in the text (Subsection 5.1), and for the moment we restrict

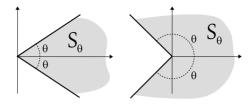


Figure 1.2. The infinite sector S_{θ} for $\theta < \pi/2$ (left) and $\theta > \pi/2$ (right).

ourselves to a less formal intuitive definition: one says that a bounded planar domain Ω is a curvilinear polygon if its boundary is smooth except near M points (vertices) A_1, \ldots, A_M , and if Γ_{j-1} and Γ_j are two smooth pieces of boundary meeting at A_j , then the half-angle θ_j between them (measured inside Ω) is non-degenerate and non-trivial, i.e. $\theta_j \notin \{0, \pi/2, \pi\}$. We say that a vertex A_j is convex if $\theta_j < \pi/2$, otherwise it is called concave. Furthermore, let H_j be the curvature defined on Γ_j , with the convention that $H_j \ge 0$ for convex domain, and ℓ_j denotes the length of Γ_j . We refer to Figure 1.1 for an illustration.

Using the general result (1.2) one is reduced first to the study of Robin Laplacians in all possible tangent sectors, which have a simple structure in two dimensions. Namely, consider the infinite planar sectors $S_{\theta} := \{(x_1, x_2) : |\arg(x_1 + ix_2)| < \theta\} \subset \mathbb{R}^2$, see Figure 1.2, then the tangent sector to Ω at A_j is a rotated copy of S_{θ_j} , while at all other points the tangent sectors are isometric to $S_{\frac{\pi}{2}}$, which is just the half-plane. Denote by T_{θ} the Laplacian in S_{θ} with the normalized Robin boundary condition $\partial u / \partial \nu = u$. Its spectral properties were studied in detail by Khalile and Pankrashkin [37] and are summarized below in Proposition 2.11. For the current presentation we remark that the essential spectrum is always $[-1, +\infty)$, and, in addition, it has $\kappa(\theta) < \infty$ discrete eigenvalues $\mathcal{E}_1(\theta), \ldots, \mathcal{E}_{\kappa(\theta)}(\theta)$, while $\kappa(\theta) = 0$ for $\theta \ge \pi/2$ (i.e. there are no discrete eigenvalues at all if the sector is concave), and $\mathcal{E}_1(\theta) = -1/\sin^2 \theta$ for $\theta < \pi/2$. Furthermore, one has $\kappa(\theta) = 1$ for $\frac{\pi}{6} \le \theta < \frac{\pi}{2}$. Hence, with Ω we associate the following objects:

$$K := \kappa(\theta_1) + \dots + \kappa(\theta_M),$$

 $\mathcal{E} := \text{the disjoint union of } \{\mathcal{E}_n(\theta_j), \ n = 1, \dots, \kappa(\theta_j)\}, \ j \in \{1, \dots, M\},\$

 $\mathcal{E}_n :=$ the *n*th element of \mathcal{E} when numbered in the non-decreasing order.

Khalile in [36] gives an improved version of (1.2) for curvilinear polygons, namely, for each $n \in \{1, \ldots, K\}$ one has $E_n(R^{\Omega}_{\alpha}) = \mathcal{E}_n \alpha^2 + \mathcal{O}(\alpha^{\frac{4}{3}})$, while the remainder estimate can be improved for polygons with straight sides, and $E_{K+n}(R^{\Omega}_{\alpha}) \sim -\alpha^2$ for each $n \in \mathbb{N}$. (We remark that paper [36] was

in turn motivated by the earlier work by Bonnaillie–Noël and Dauge [8] on magnetic Neumann Laplacians in corner domains.) Therefore, the behavior of the first K eigenvalues at the leading order is determined by the corners only, so one might call them *corner-induced*. In the present work we would like to understand in greater detail the asymptotics of the higher eigenvalues $E_{K+n}(R^{\Omega}_{\alpha})$ with a fixed $n \in \mathbb{N}$, which will be referred to as sideinduced. As the main term $(-\alpha^2)$ in the asymptotics is the same as in the smooth case, one might expect that their behavior should take into account the geometry of the boundary away from the corners, so that a kind of an effective Schrödinger-type operator may appear by analogy with (1.4). On the other hand, one might expect that the corners should contribute to the effective operator: due to the singularities at the vertices, some boundary conditions might be needed in order to make the effective operator selfadjoint. It seems that the only result obtained in this direction is the one by Pankrashkin [53]: if Ω is the exterior of a convex polygon with side lengths ℓ_j , then for any fixed n one has $E_n(R^{\Omega}_{\alpha}) = -\alpha^2 + E_n(\bigoplus_j D_j) + \mathcal{O}(\alpha^{-\frac{1}{2}})$ as $\alpha \to +\infty$, where D_i is the Dirichlet Laplacian on $(0, \ell_i)$. Remark that this result is in agreement with what precedes: as all the corners are concave, one simply has K = 0. We are going to obtain a result in the same spirit for a more general case, in particular, by allowing the presence of convex corners.

Our analysis will be based on the notion of *non-resonant* convex vertex (it will be seen from the proof that concave vertices are much easier to deal with), which is formulated in terms of a model Robin eigenvalue problem on a truncated sector. Namely, for $\theta \in (0, \pi/2)$ and r > 0 let A_r^{\pm} be the two points lying on the two boundary rays of the sector S_{θ} at the distance r > 0from the origin O, and let B_r be the intersection point of the straight lines passing through A_r^{\pm} perpendicular to the boundary, see Figure 1.3. Denote by \mathcal{S}^r_{θ} the quadrangle $OA^+_r B_r A^-_r$ and by N^r_{θ} the Laplacian $u \mapsto -\Delta u$ in \mathcal{S}^r_{θ} with the Robin boundary condition $\partial u/\partial \nu = u$ at OA^{\pm}_r and the Neumann boundary condition at $A_r^{\pm}B_r$. Using rather standard methods one sees that the first $\kappa(\theta)$ eigenvalues of N_{θ}^r converge to those of T_{θ} as $r \to +\infty$ (Lemma 3.6), and the non-resonance condition is a hypothesis on the behavior of the next eigenvalue. We say that a half-angle θ is nonresonant if for some C > 0 one has $E_{\kappa(\theta)+1}(N_{\theta}^r) \ge -1 + C/r^2$ for large r. One shows in Proposition 3.10, using a combination of a separation of variables with a monotonicity argument that all half-angles $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ are non-resonant.

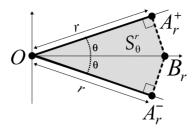


Figure 1.3. The quadrangle \mathcal{S}^r_{θ} .

In order to concentrate on the contribution of the corners, let us discuss first the case when Ω is a polygon with straight edges. We denote

 $D_j :=$ the Dirichlet Laplacian on $(0, \ell_j)$.

Our main result reads as follows:

THEOREM 1.1. — Let $\Omega \subset \mathbb{R}^2$ be a polygon with M vertices, halfangles θ_j and side lengths ℓ_j . Assume that each θ_j is either concave or non-resonant, then for any fixed $n \in \mathbb{N}$ and $\alpha \to +\infty$ there holds

$$E_{K+n}(R_{\alpha}^{\Omega}) = -\alpha^{2} + E_{n}\left(\bigoplus_{j=1}^{M} D_{j}\right) + \mathcal{O}\left(\frac{\log \alpha}{\sqrt{\alpha}}\right).$$

As it will be seen in the proof, using the Dirichlet–Neumann bracketing and the non-resonance condition, it is quite elementary to obtain the twosided estimate

$$-\alpha^{2} + E_{n}\left(\bigoplus_{j=1}^{M} N_{j}\right) + \mathcal{O}\left(\frac{\log \alpha}{\alpha}\right) \leqslant E_{K+n}(R_{\alpha}^{\Omega}),$$
$$E_{K+n}(R_{\alpha}^{\Omega}) \leqslant -\alpha^{2} + E_{n}\left(\bigoplus_{j=1}^{M} D_{j}\right) + \mathcal{O}\left(\frac{\log \alpha}{\alpha}\right),$$

where N_j is the Neumann laplacian on $(0, \ell_j)$, and one easily sees that the difference between the lower and upper bounds is of order 1. It takes then the most efforts to close this gap and to show that it is the upper bound which gives the main term of the eigenvalue asymptotics, and this is the main contribution of the present paper.

Using the above observation that all obtuse angles θ are non-resonant with $\kappa(\theta) = 1$, one arrives at the following corollary:

COROLLARY 1.2. — Let $\Omega \subset \mathbb{R}^2$ be a polygon with M vertices, halfangles θ_j and sides of length ℓ_j . Assume that $\theta_j \ge \pi/4$ for all j, then for any $n \in \mathbb{N}$ and $\alpha \to +\infty$ there holds

$$E_{K+n}(R^{\Omega}_{\alpha}) = -\alpha^2 + E_n\left(\bigoplus_{j=1}^M D_j\right) + \mathcal{O}\left(\frac{\log\alpha}{\sqrt{\alpha}}\right),$$

where K is the number of convex vertices.

It is an important point that a different eigenvalue asymptotics can arise if no condition is imposed on the corners. In order to see it, remark first that in the situation of Theorem 1.1 one has

(1.5)
$$\lim_{\alpha \to +\infty} \left(E_{K+1}(R^{\Omega}_{\alpha}) + \alpha^2 \right) = E_1 \left(\bigoplus_{j=1}^M D_j \right) > 0.$$

On the other hand, the computations by McCartin [46] for an explicit configuration (which we review in Subsection 6.1) give the following result

PROPOSITION 1.3. — Let Ω be an equilateral triangle of side length $\ell > 0$. Then for $\alpha \to +\infty$ there holds $E_n(R^{\Omega}) = -4\alpha^2 + o(1)$ for $n \in \{1, 2, 3\}$ and

(1.6)
$$E_{3+n}(R^{\Omega}) = -\alpha^2 + E_n(L) + o(1) \text{ for any fixed } n \in \mathbb{N},$$

where L is the Laplacian on $(0, 3\ell)$ with the periodic boundary condition.

For the equilateral triangle one has indeed K = 3, while $E_1(L) = 0$. (1.6) implies $\lim_{\alpha \to +\infty} (E_{K+1}(R_{\alpha}^{\Omega}) + \alpha^2) = 0$, which contradicts (1.5). This means that the half-angle $\theta = \frac{\pi}{6}$ is resonant (i.e. it does not satisfy the above non-resonance condition). We remark that the non-resonance condition we use is strictly adapted to our proof method and is not supposed to be optimal, but we are not aware of any suitable alternative. In fact our choice is strongly motivated by some recent studies of Laplacians in domains collapsing on graphs, and some analogies with waveguides and possible reformulations of the non-resonance condition are discussed in Subsection 6.3.

For the case of curvilinear polygons, a number of additional difficulties arise due to the presence of non-trivial curvatures on the sides, and we were not able to study the most general case in the present text (the most important technical obstacles are discussed in Subsection 6.2). Nevertheless, we were able to consider two important cases.

First, we consider the case when the maximum curvature is not attained at the corners.

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THEOREM 1.4. — Denote $H_{j,*} := \max_{s \in [0,\ell_j]} H_j(s)$, $H_* := \max_j H_{j,*}$, and assume that

for all $j \in \{1, \ldots, M\}$ there holds $H_j(0) \neq H_*$ and $H_j(\ell_j) \neq H_*$,

and that each corner of Ω is either concave or non-resonant. Then for each $n \in \mathbb{N}$ and $\alpha \to +\infty$ one has

(1.7)
$$E_{K+n}(R^{\Omega}_{\alpha}) = -\alpha^{2} + E_{n}\left(\bigoplus_{j:H_{j,*}=H_{*}}(D_{j}-\alpha H_{j})\right) + \mathcal{O}(1),$$
$$= -\alpha^{2} + E_{n}\left(\bigoplus_{j:H_{j,*}=H_{*}}(N_{j}-\alpha H_{j})\right) + \mathcal{O}(1).$$

In fact, the proof of Theorem 1.4 appears to be less involved than the one of Theorem 1.1: the main ingredient is that the eigenvalues of $D_j - \alpha H_j$ are exponentially close to those of $N_j - \alpha H_j$, which is a simple consequence of Agmon-type estimates, hence, the contribution of the boundary conditions to be imposed at the vertices is very small (as the eigenfunctions are concentrated near the set on which the curvature attains its maximal value). The remainder $\mathcal{O}(1)$ is the same as for the effective operator in (1.4) obtained for smooth domains. Furthermore, under suitable geometric assumptions a complete asymptotic expansion can be obtained, see Subsection 5.4. We remark that the non-resonance condition is still used in the proof of Theorem 1.4, and we have no intuition on what kind of asymptotics can be expected without additional conditions on the corners.

The second important case we were able to study is as follows:

(1.8) the curvatures H_i are constant, and we denote $H_* := \max H_i$;

i.e. each side is either a line segment or a circle arc. We explicitly mention that H_j can be different for different j. Then we obtain the following result, which is in the same spirit as Theorem 1.1:

THEOREM 1.5. — Assume that all corners are concave or non-resonant and that (1.8) is satisfied, then for any fixed $n \in \mathbb{N}$ and $\alpha \to +\infty$ one has the asymptotics

(1.9)
$$E_{K+n}(R^{\Omega}_{\alpha}) = -\alpha^2 - H_*\alpha - \frac{1}{2}H^2_* + E_n\left(\bigoplus_{j:H_j=H_*} D_j\right) + \mathcal{O}\left(\frac{\log\alpha}{\sqrt{\alpha}}\right).$$

Remark that (1.9) can be formally viewed as a particular case of the asymptotics (1.7) as the terms $-\frac{1}{2}H_*^2 + \mathcal{O}(\frac{\log \alpha}{\sqrt{\alpha}})$ in (1.9) can be viewed as a resolution of the remainder $\mathcal{O}(1)$ in (1.7). The presence of the new term $\frac{1}{2}H_*^2$ was not observed in earlier papers on Robin eigenvalues.

The text is organized as follows. In Section 2 we recall basic tools from the functional analysis (min-max based eigenvalue estimates, distance between subspaces, Sobolev trace theorems) and study or recall the spectral properties of some model operators (Robin Laplacians on intervals and infinite sectors). Section 3 is devoted to the study of Robin Laplacians in convex sectors truncated in a special way: we obtain some estimates for the eigenvalues and decay estimate for the eigenfunctions, then we introduce the new notion of non-resonant angle and show that it is satisfied by the obtuse angles. In Section 4 we prove Theorem 1.1, i.e. the case of polygons with straight sides. We first decompose the polygon into vertex neighborhoods and side neighborhoods, and apply Dirichlet-Neumann bracketing in order to give first a rough eigenvalue estimate in terms of the direct sum of operators in each part. This approach appears to be sufficient for the upper bound. The proof of the lower bound is much more involved and represents the main contribution of the paper. Our approach is based on the construction of an identification operator between functions in Ω and functions on the boundary satisfying the Dirichlet boundary condition at the vertices. This machinery was initially proposed by Post [58] for the analysis of thin branching domains, and it was already used by Pankrashkin [53] to study the Robin Laplacians in the exterior of convex polygons. The difference with the present case comes from the fact that we want to obtain estimates on the K + n eigenvalue, namely the identification only applies to the orthogonal complement of the K first eigenfunctions of R^{Ω}_{α} . The strategy consists in proving that the lowest eigenspaces of the polygon are close in a suitable sense to the ones of the vertex neighborhoods. The non-resonance condition is then used to analyze their orthogonal complement: it allows us to obtain a control on the trace of the eigenfunctions at the boundary of the vertex neighborhoods which gives a necessary input for the eigenvalue estimates. We explicitly remark that our analysis is not based on the construction of quasimodes for the operators in play, but on a construction of test functions which are not in the operator domains. In particular, we do not see any sufficiently direct way to obtain a complete asymptotic expansion for the eigenvalues.

In Section 5 we discuss the case of curvilinear polygons. We still need a special decomposition of the domain into pieces of a very special form as well as the existence of some diffeomorphisms and special cut-off functions. The procedure is summarized at the beginning of the section while a complete justification of the geometric constructions is given in Appendix A, as we are not aware of suitable constructions in the existing literature. As in

the case of straight polygons we then estimate the portion of the operator in each piece of the domain, which gives the sought upper bound and some lower bound. We prove Theorem 1.4 in Subsection 5.4 by showing that the lower and upper bounds are close enough (when compared with the order of the eigenvalue) under the geometric condition imposed. In Subsection 5.5 we then prove Theorem 1.5. With preparation in the preceding subsection, the proof scheme is almost identical to the one of Theorem 1.1, and differences are mostly of a technical nature.

Finally, in Section 6 we discuss possible extensions of the results, in particular, we show that some angles do not satisfy the non-resonance condition and give a different eigenvalue asymptotics, and we explain some links between our study and the spectral analysis of waveguides. As already mentioned, Appendix A contains some geometric constructions in curvilinear sectors, and we believe that they can be of use for other problems involving differential operators in domains with corners.

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2. Preliminaries

2.1. Notation

For $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ we will use the length $|x| = \sqrt{x_1^2 + x_2^2}$, the scalar product $x \cdot y = x_1y_1 + x_2y_2$ and the wedge product $x \wedge y = x_1y_2 - x_2y_1$. In this paper we only deal with real-valued operators, so we prefer to work with real Hilbert spaces in order to have a simpler writing. Let \mathcal{H} be a Hilbert space and $u, v \in \mathcal{H}$, then we denote by $\langle u, v \rangle_{\mathcal{H}}$ the scalar product of u and v. It will be sometimes shortened

to $\langle u, v \rangle$ if there is no ambiguity in the choice of the Hilbert space, and the same applies to the associated norm $\|\cdot\|_{\mathcal{H}}$. For a self-adjoint operator $(A, \mathcal{D}(A))$ in \mathcal{H} , with $\mathcal{D}(A)$ being the operator domain, we denote by $\operatorname{spec}(A)$, $\operatorname{spec}_{\operatorname{disc}}(A)$ and $\operatorname{spec}_{\operatorname{ess}}(A)$ the spectrum of A, its discrete spectrum and its essential spectrum, respectively. For $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, by $E_n(A)$ we denote the *n*th discrete eigenvalue of A (if it exists) when enumerated in the non-decreasing order counting the multiplicities. If the operator A is semibounded from below, then $\mathcal{Q}(A)$ denotes the domain of its sesquilinear form, and the value of the sesquilinear form on two vectors $u, v \in \mathcal{Q}(A)$ will be denoted by A[u, v].

2.2. Min-max principle and its consequences

Let \mathcal{H} be an infinite-dimensional Hilbert space and A be a lower semibounded self-adjoint operator in \mathcal{H} , with $A \ge -c$ for some $c \in \mathbb{R}$. Recall that $\mathcal{Q}(A)$ equipped with the scalar product

 $\mathcal{Q}(A) \times \mathcal{Q}(A) \ni (u, v) \mapsto A[u, v] + (c+1)\langle u, v \rangle_{\mathcal{H}}$

is a Hilbert space. The following result, giving a variational characterization of eigenvalues (usually referred to as the *min-max* principle), is a standard tool in the spectral theory of self-adjoint operators, see e.g. [60, Section XIII.1] or [7, Section 10.2]:

PROPOSITION 2.1. — Let $\Sigma := \inf \operatorname{spec}_{\operatorname{ess}}(A)$ if $\operatorname{spec}_{\operatorname{ess}}(A) \neq \emptyset$, otherwise set $\Sigma := +\infty$. Let $n \in \mathbb{N}$ and D be a dense subspace of $\mathcal{Q}(A)$. Define the *n*th Rayleigh quotient $\Lambda_n(A)$ of A by

$$\Lambda_n(A) := \inf_{G \subset D: \dim G = n} \sup_{u \in G \setminus \{0\}} \frac{A[u, u]}{\|u\|_{\mathcal{H}}^2},$$

then one and only one of the following two assertions is true:

- $\Lambda_n(A) < \Sigma$ and $E_n(A) = \Lambda_n(A)$.
- $\Lambda_n(A) = \Sigma$ and $\Lambda_m(A) = \Lambda_n(A)$ for all $m \ge n$.

The following corollary is also well known, see e.g. [7, Section 10.2, Theorem 5]:

COROLLARY 2.2. — Let A and B be lower semibounded self-adjoint operators in an infinite-dimensional Hilbert space \mathcal{H} . Assume that there exists $d \in \mathbb{N}$ and a d-dimensional subspace D such that $\mathcal{Q}(A) = \mathcal{Q}(B) \oplus D$ and that A[u, u] = B[u, u] for all $u \in \mathcal{Q}(B)$, then $\Lambda_n(B) \leq \Lambda_{n+d}(A)$ for all $n \in \mathbb{N}$. Furthermore, the following min-max-based eigenvalue estimate will be of use to compare the eigenvalues of operators acting in different spaces. It was introduced and used by Exner and Post in [19, Lemma 2.1] as well as by Post in [58, Lemma 2.2]:

PROPOSITION 2.3. — Let \mathcal{H} and \mathcal{H}' be infinite-dimensional Hilbert spaces, B be a non-negative self-adjoint operator with a compact resolvent in \mathcal{H} and B' be a lower semibounded self-adjoint operator in \mathcal{H}' . Pick $n \in \mathbb{N}$ and assume that there exists a linear map $J : \mathcal{Q}(B) \to \mathcal{Q}(B')$ and constants $\varepsilon_1 \ge 0$ and $\varepsilon_2 \ge 0$ such that $\varepsilon_1 < 1/(1 + E_n(B))$ and that for any $u \in \mathcal{Q}(B)$ one has

$$\|u\|_{\mathcal{H}}^2 - \|Ju\|_{\mathcal{H}'}^2 \leqslant \varepsilon_1 \left(B[u,u] + \|u\|_{\mathcal{H}}^2 \right),$$
$$B'[Ju,Ju] - B[u,u] \leqslant \varepsilon_2 \left(B[u,u] + \|u\|_{\mathcal{H}}^2 \right),$$
$$\text{then } \Lambda_n(B') \leqslant E_n(B) + \frac{\left(E_n(B)\varepsilon_1 + \varepsilon_2 \right) \left(1 + E_n(B) \right)}{1 - \left(1 + E_n(B) \right) \varepsilon_1}.$$

2.3. Distance between closed subspaces

We will use the well-known notion of a distance between two closed subspaces:

DEFINITION 2.4. — Let E and F be closed subspaces of a Hilbert space \mathcal{H} and denote by P_E and P_F the orthogonal projectors in \mathcal{H} on E and F respectively. The distance d(E, F) between E and F is defined by

$$d(E,F) := \sup_{x \in E, \ x \neq 0} \frac{\|x - P_F x\|}{\|x\|} \equiv \|P_E - P_F P_E\| \equiv \|P_E - P_E P_F\|.$$

One easily sees that the distance is not symmetric, i.e. $d(E, F) \neq d(F, E)$ in general, but the triangular inequality is satisfied, i.e. $d(E, G) \leq d(E, F) + d(F, G)$ for any closed subspaces E, F, G. Furthermore, we will need the following result due to Helffer and Sjöstrand [33, Proposition 2.5] allowing to estimate the distance between two subspaces in a special case.

PROPOSITION 2.5. — Let A be a self-adjoint operator in a Hilbert space \mathcal{H} and $I \subset \mathbb{R}$ be a compact interval. For some $n \in \mathbb{N}$ let $\mu_1, \ldots, \mu_n \in I$ and $\psi_1, \ldots, \psi_n \in \mathcal{D}(A)$ be linearly independent vectors, then we denote

$$\varepsilon := \max_{j \in \{1, \dots, n\}} \left\| (A - \mu_j) \psi_j \right\|, \quad \eta := \frac{1}{2} \operatorname{dist} \left(I, (\operatorname{spec} A) \setminus I \right)$$

 $\lambda :=$ the smallest eigenvalue of the Gram matrix $(\langle \psi_j, \psi_k \rangle)_{i,k \in \{1,\dots,n\}}$.

If $\eta > 0$, then the distance d(E, F) between the subspaces

 $E := \operatorname{span}\{\psi_1, \dots, \psi_n\},\$

F := the spectral subspace associated with A and I

satisfies $d(E, F) \leq \frac{\varepsilon}{\eta} \sqrt{\frac{n}{\lambda}}$.

2.4. Laplacians with mixed boundary conditions and a trace estimate

In what follows we will deal with numerous Laplacians with various combinations of boundary conditions. In order to simplify the writing, we introduce the following definition:

DEFINITION 2.6 (Laplacians with mixed boundary conditions). — Let $U \subset \mathbb{R}^d$ be an open set and Γ_D , Γ_N , Γ_R be disjoint subsets of ∂U such that $\overline{\Gamma_D \cup \Gamma_N \cup \Gamma_R} = \partial U$. In addition, let $\alpha \in \mathbb{R}$, then by the Laplacian in Ω with Dirichlet condition at Γ_D , Neumann condition at Γ_N and α -Robin condition at Γ_R we mean the self-adjoint operator A in $L^2(U)$ with

$$A[u, u] = \int_{U} |\nabla u|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_R} |u|^2 \, \mathrm{d}s,$$
$$\mathcal{Q}(A) = \left\{ u \in H^1(U) : u = 0 \text{ at } \Gamma_D \right\},$$

where ds is the (d-1)-dimensional Hausdorff measure on ∂U , provided that the above expression defines a closed semibounded from below sesquilinear form (which is the case for bounded Lipschitz domains U). Informally, the operator A acts then as $u \mapsto -\Delta u$ on suitably regular functions u in U satisfying u = 0 at Γ_D , $\partial_{\nu} u = 0$ at Γ_N , $\partial_{\nu} u = \alpha u$ at Γ_R , where ∂_{ν} stands for the outer normal derivative.

We will need a variant of the Sobolev trace inequality on scaled domains.

LEMMA 2.7. — Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain, then there exists c > 0 such that

$$\int_{\partial(tU)} f^2 \, \mathrm{d}s \leqslant c \left(t\varepsilon \int_{tU} |\nabla f|^2 \, \mathrm{d}x + \frac{1}{t\varepsilon} \int_{tU} f^2 \, \mathrm{d}x \right)$$

for all t > 0, $f \in H^1(tU)$, $\varepsilon \in (0, 1]$. In particular, if for $\alpha > 0$ one denotes by R^{tU}_{α} the Laplacian in tU with α -Robin condition at the whole boundary, then there exists C > 0 such that $R^{tU}_{\alpha} \ge -C\alpha^2$ for αt sufficiently large.

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Proof. — The standard trace inequality, see e.g. Grisvard [27, Theorem 1.5.1.10], implies that there exists c > 0 such that

(2.1)
$$\int_{\partial U} u^2 \, \mathrm{d}s \leqslant c \left(\varepsilon \int_U |\nabla u|^2 \, \mathrm{d}x + \frac{1}{\varepsilon} \int_U u^2 \, \mathrm{d}x \right)$$
for all $u \in H^1(U)$ and $\varepsilon \in (0, 1]$.

For $f \in L^2(tU)$ denote by $f_t \in L^2(U)$ the function given by $f_t(x) = f(tx)$, then $f \in H^1(tU)$ if and only if $f_t \in H^1(U)$. Using (2.1) we see that

(2.2)
$$\int_{\partial U} f_t^2 \, \mathrm{d}s \leqslant c \left(\varepsilon \int_U |\nabla f_t|^2 \, \mathrm{d}x + \frac{1}{\varepsilon} \int_U f_t^2 \, \mathrm{d}x \right)$$
for all $f \in H^1(tU), \ \varepsilon \in (0, 1].$

and using the change of variables x = y/t one easily obtains

$$\int_{\partial U} f_t^2 \, \mathrm{d}s = \int_{\partial U} f(tx)^2 \, \mathrm{d}s = t^{1-d} \int_{\partial(tU)} f(y)^2 \, \mathrm{d}s,$$
$$\int_U |\nabla f_t|^2 \, \mathrm{d}x = \int_U t^2 |(\nabla f)(tx)|^2 \, \mathrm{d}x = t^{2-d} \int_{tU} |\nabla f(y)|^2 \, \mathrm{d}y,$$
$$\int_U f_t^2 \, \mathrm{d}x = \int_U f(tx)^2 \, \mathrm{d}x = t^{-d} \int_{tU} f(y)^2 \, \mathrm{d}y.$$

The substitution of these three equalities into (2.2) gives the desired trace inequality. Furthermore, for all $f \in H^1(tU)$ and $\varepsilon \in (0,1]$ one has

$$\begin{aligned} R^{tU}_{\alpha}[f,f] &= \int_{tU} |\nabla f|^2 \, \mathrm{d}x - \alpha \int_{\partial(tU)} f^2 \, \mathrm{d}s \\ &\geqslant (1 - c\alpha t\varepsilon) \int_{tU} |\nabla f|^2 \, \mathrm{d}x - \frac{c\alpha}{t\varepsilon} \int_{tU} f^2 \, \mathrm{d}x. \end{aligned}$$

ence, taking $\varepsilon := 1/(c\alpha t)$ we arrive at $R^{tU}_{t} \geqslant -c^2 \alpha^2$.

Hence, taking $\varepsilon := 1/(c\alpha t)$ we arrive at $R^{tU}_{\alpha} \ge -c^2 \alpha^2$.

2.5. One-dimensional model operators

Let us recall some eigenvalue estimates for Laplacians on finite intervals with a combination of boundary conditions.

PROPOSITION 2.8. — For $\delta > 0$ and $\alpha > 0$, let L_D be the Laplacian on $(0, \delta)$ with α -Robin condition at 0 and the Dirichlet boundary condition at δ , then for $\alpha \delta \to +\infty$ there holds $E_1(L_D) = -\alpha^2 (1 + \mathcal{O}(e^{-\delta \alpha}))$ and $E_2(L_D) \ge 0.$

The result is obtained by direct computations, details can be found e.g. in [52, Lemma 4].

PROPOSITION 2.9. — For $\delta > 0$, $\alpha > 0$ and $\beta \ge 0$, let L_N denote the Laplacian on $(0, \delta)$ with α -Robin condition at 0 and β -Robin condition at δ , then for $\alpha\delta \to +\infty$ and $\beta\delta \to 0^+$ one has $E_1(L_N) = -\alpha^2 (1 + \mathcal{O}(e^{-\alpha\delta}))$ and $E_2(L_N) \ge 1/\delta^2$.

Proof. — The estimate for the first eigenvalue was already obtained by direct computations e.g. in [52, Lemma 3]. To study the second eigenvalue, let B_{β} be the Laplacian on $(0, \delta)$ with the Dirichlet boundary condition at 0 and β -Robin condition at δ , then the sesquilinear form of B_{β} is a restriction of the sesquilinear form of L_N , and $\mathcal{Q}(L_N) = H^1(0, \delta)$ only differs from $\mathcal{Q}(B_{\beta}) = \{f \in H^1(0, \delta) : f(0) = 0\}$ by a one-dimensional subspace. It follows by Corollary 2.2 that $E_2(L_N) \ge E_1(B_{\beta})$. Now, let us obtain a lower bound for B_{β} . For $\beta = 0$ one obtains simply the Laplacian on $(0, \delta)$ with the Dirichlet boundary condition at 0 and the Neumann boundary condition at δ , and $E_1(B_0) = \pi^2/(4\delta^2)$. By Lemma 2.7 there is c > 0 such that

$$f(\delta)^2 \leqslant c \left(\delta \int_0^{\delta} (f')^2 \, \mathrm{d}t + \frac{1}{\delta} \int_0^{\delta} f^2 \, \mathrm{d}t \right) \text{ for all } f \in H^1(0,\delta).$$

It follows that for $f \in \mathcal{Q}(B_{\beta}) \equiv \mathcal{Q}(B_0)$ one has

$$B_{\beta}[f,f] = \int_0^{\delta} (f')^2 \,\mathrm{d}t - \beta f(\delta)^2 \ge (1 - c\beta\delta) \int_0^{\delta} (f')^2 \,\mathrm{d}t - \frac{c\beta}{\delta} \int_0^{\delta} f^2 \,\mathrm{d}t,$$

and the min-max principle implies that for $\beta \delta \to 0^+$ one has

$$E_1(B_{\delta}) \ge (1 - c\beta\delta)E_1(B_0) - \frac{c\beta\delta}{\delta^2} = \frac{(1 - c\beta\delta)\pi^2 - 4c\beta\delta}{4\delta^2} \ge \frac{1}{\delta^2}. \quad \Box$$

2.6. Robin Laplacians in infinite sectors

Now, let us recall some basic facts on Robin laplacians in infinite sectors.

DEFINITION 2.10. — For $\theta \in (0, \pi)$ denote by S_{θ} the following infinite sector of opening angle 2θ :

$$\mathcal{S}_{\theta} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : -\theta < \arg(x_1 + \mathrm{i}x_2) < \theta \right\}, \quad 0 < \theta < \pi,$$

see Figure 1.2 in the introduction. For $\theta = \pi/2$ one obtains simply the half-plane $\mathbb{R}_+ \times \mathbb{R}$.

The following proposition summarizes the basic properties of the associated Robin Laplacians proved in the paper [37] by Khalile and Pankrashkin: PROPOSITION 2.11. — For $\theta \in (0, \pi)$ and $\alpha > 0$, let $T_{\theta, \alpha}$ be the Laplacian on S_{θ} with α -Robin condition at the whole boundary, then:

- the operator $T_{\theta,\alpha}$ is well-defined, lower semibounded and is unitarily equivalent to $\alpha^2 T_{\theta,1}$ for all $\theta \in (0,\pi)$ and $\alpha > 0$,
- spec_{ess} $(T_{\theta,\alpha}) = [-\alpha^2, +\infty)$ for all $\theta \in (0,\pi)$ and $\alpha > 0$,
- the discrete spectrum of $T_{\theta,\alpha}$ is non-empty if and only if $\theta < \frac{\pi}{2}$, in particular,

$$E_1(T_{\theta,\alpha}) = -\alpha^2 / \sin^2 \theta \text{ for } \theta \in \left(0, \frac{\pi}{2}\right).$$

• if one denotes

$$\kappa(\theta) :=$$
 the number of discrete eigenvalues of $T_{\theta,\alpha}$,

which is independent of α , then

- $\kappa(\theta) < +\infty$ and $\theta \mapsto \kappa(\theta)$ is non-increasing with $\kappa(0^+) = +\infty$, • for all $\frac{\pi}{6} \leq \theta < \frac{\pi}{2}$ one has $\kappa(\theta) = 1$,
- there exist b > 0 and B > 0 such that if $n \in \{1, \ldots, \kappa(\theta)\}$ and $\psi_{n,\alpha}$ is an eigenfunction of $T_{\theta,\alpha}$ for the *n*th eigenvalue, then for any $\alpha > 0$ one has the Agmon-type decay estimate

(2.3)
$$\int_{\mathcal{S}_{\theta}} e^{b\alpha|x|} \left(\frac{1}{\alpha^2} \left| \nabla \psi_{n,\alpha}(x) \right|^2 + \psi_{n,\alpha}^2(x) \right) \mathrm{d}x \leqslant B \|\psi_{n,\alpha}\|_{L^2(\mathcal{S}_{\theta})}^2.$$

Remark that the above properties of $T_{\theta,\alpha}$ are also of relevance for Steklovtype eigenvalue problems in domains with corners, see e.g. the papers by Ivrii [34] and Levitin, Parnovski, Polterovich, Sher [44]. For a subsequent use we give a special name to the eigenvalues of the above operator with $\alpha = 1$:

$$\mathcal{E}_n(\theta) := E_n(T_{\theta,1}) \text{ for } \theta \in \left(0, \frac{\pi}{2}\right) \text{ and } n \in \{1, \dots, \kappa(\theta)\}.$$

Remark that due to Proposition 2.11 one has $\mathcal{E}_1(\theta) = -1/\sin^2 \theta$ and

$$\mathcal{E}_n(\theta) < -1, \quad E_n(T_{\theta,\alpha}) = \mathcal{E}_n(\theta) \,\alpha^2 < -\alpha^2,$$

for $\theta \in \left(0, \frac{\pi}{2}\right), \ n \in \{1, \dots, \kappa(\theta)\}, \ \alpha > 0.$

3. Analysis in truncated convex sectors

3.1. Robin Laplacians in truncated convex sectors

Recall that the infinite sectors S_{θ} are defined above in Definition 2.10. Let us introduce their truncated versions. DEFINITION 3.1 (Truncated convex sector S_{θ}^{r}). — Let $\theta \in (0, \frac{\pi}{2})$ and r > 0. Consider the points

$$A_r^{\pm} = r(\cos\theta, \pm\sin\theta) \in \partial S_{\theta}, \quad B_r = r(1/\cos\theta, 0) \in S_{\theta}.$$

and denote by S_{θ}^{r} the interior of the quadrangle $OA_{r}^{+}B_{r}A_{r}^{-}$ (remark that the sides $B_{r}A_{r}^{\pm}$ are orthogonal to ∂S_{θ} at A_{r}^{\pm} , see Figure 3.1). We will distinguish between two parts of the boundary of S_{θ}^{r} , namely, we set

$$\partial_* \mathcal{S}^r_{\theta} := \partial \mathcal{S}^r_{\theta} \cap \partial \mathcal{S}_{\theta} := \text{polygonal chain } A^+_r O A^-_r,$$
$$\partial_{\text{ext}} \mathcal{S}^r_{\theta} := \partial \mathcal{S}^r_{\theta} \setminus \partial_* \mathcal{S}^r_{\theta} := \text{polygonal chain } A^+_r B_r A^-_r$$

In what follows we will need some properties of three operators associated with S_{θ}^{r} , namely, for $\theta \in (0, \frac{\pi}{2})$, $\alpha > 0$ and r > 0 we introduce:

- $D^{r}_{\theta,\alpha} := \text{the Laplacian in } \mathcal{S}^{r}_{\theta} \text{ with } \alpha \text{-Robin condition at } \partial_{*}\mathcal{S}^{r}_{\theta} \text{ and } Dirichlet \text{ condition at } \partial_{\text{ext}}\mathcal{S}^{r}_{\theta},$
- (3.1) $N_{\theta,\alpha}^r := \text{the Laplacian in } \mathcal{S}_{\theta}^r \text{ with } \alpha \text{-Robin condition at } \partial_* \mathcal{S}_{\theta}^r \text{ and } Neumann \text{ condition at } \partial_{\text{ext}} \mathcal{S}_{\theta}^r,$
 - $R^r_{\theta,\alpha} :=$ the Laplacian in \mathcal{S}^r_{θ} with α -Robin condition at the whole boundary.

We remark that $N_{\theta,\alpha}^r$ will play a key role in the subsequent considerations (in particular, see Subsection 3.3), while the other two operators will be used mostly for auxiliary constructions. The following properties of the three operators are easily established by a standard routine computation:

LEMMA 3.2. — For t, r > 0 denote by Ξ_t the unitary operators (dilations) $\Xi_t : L^2(\mathcal{S}^{tr}_{\theta}) \to L^2(\mathcal{S}^r_{\theta}), \ (\Xi_t u)(x) = t u(tx)$. Let $X^r_{\theta,\alpha}$ be any of the three operators $D^r_{\theta,\alpha}, N^r_{\theta,\alpha}, R^r_{\theta,\alpha}$, then $X^r_{\theta,t\alpha}\Xi_t = t^2 \Xi_t X^{tr}_{\theta,\alpha}$, which then gives the eigenvalue identities

(3.2)
$$E_n(X_{\theta,\alpha}^r) = \alpha^2 E_n(X_{\theta,1}^{\alpha r}) \text{ for all } n \in \mathbb{N}.$$

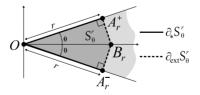


Figure 3.1. The truncated sector S^r_{θ} is shaded (see Definition 3.1). The part of the boundary $\partial_* S^r_{\theta}$ is indicated by the thick solid line, and the part of the boundary $\partial_{\text{ext}} S^r_{\theta}$ is shown as the thick dashed line.

Let us show that in a suitable asymptotic regime the lowest eigenvalues of the Robin–Dirichlet Laplacians $D_{\theta,\alpha}^r$ are close to the Robin eigenvalues of the associated infinite sectors:

LEMMA 3.3. — For some c > 0 one has

$$E_n(D_{\theta,\alpha}^r) = E_n(T_{\theta,\alpha}) + \mathcal{O}(\alpha^2 e^{-c\alpha r}) \equiv \alpha^2 \left(\mathcal{E}_n(\theta) + \mathcal{O}(e^{-c\alpha r}) \right)$$

for $n \in \{1, \ldots, \kappa(\theta)\}$, and $E_{\kappa(\theta)+1}(D^r_{\theta,\alpha}) \ge -\alpha^2$ as $\alpha r \to +\infty$.

Proof. — The result is quite standard and is based on the fact that the Robin eigenfunctions of the infinite sectors satisfy an Agmon-type estimate at infinity, but we provide a proof for the sake of completeness. In view of the above scaling (3.2) it is sufficient to study the case $\alpha = 1$ and $r \to +\infty$. Recall that

$$D_{\theta,1}^{r}[u,u] = \int_{\mathcal{S}_{\theta}^{r}} |\nabla u|^{2} \,\mathrm{d}x - \int_{\partial_{*}\mathcal{S}_{\theta}^{r}} u^{2} \,\mathrm{d}s,$$
$$\mathcal{Q}(D_{\theta,1}^{r}) = \left\{ H^{1}(\mathcal{S}_{\theta}^{r}) : u = 0 \text{ on } \partial_{\mathrm{ext}}\mathcal{S}_{\theta}^{r} \right\},$$
$$T_{\theta,1}[u,u] = \int_{\mathcal{S}_{\theta}} |\nabla u|^{2} \,\mathrm{d}x - \int_{\partial \mathcal{S}_{\theta}} u^{2} \,\mathrm{d}s, \quad \mathcal{Q}(T_{\theta,1}) = H^{1}(\mathcal{S}_{\theta}).$$

The min-max principle gives $E_n(D_{\theta,1}^r) \ge \Lambda_n(T_{\theta,1})$ for any r > 0 and $n \in \mathbb{N}$. For $n \in \{1, \ldots, \kappa(\theta)\}$ one has $\Lambda_n(T_{\theta,1}) = E_n(T_{\theta,1}) \equiv \mathcal{E}_n(\theta)$, while $\Lambda_{\kappa(\theta)+1}(T_{\theta,1}) = \inf \operatorname{spec}_{\operatorname{ess}} T_{\theta,1} = -1$. This proves the required lower bounds.

To prove the upper bound, let us pick $n \in \{1, \ldots, \kappa(\theta)\}$ and let ψ_j , $j = 1, \ldots, n$, be eigenfunctions of the operator $T_{\theta,1}$ in the infinite sector corresponding to the *n* first eigenvalues and chosen to form an orthonormal family, i.e.

$$\langle \psi_j, \psi_k \rangle_{L^2(\mathcal{S}_{\theta})} = \delta_{j,k}, \quad T_{\theta,1}[\psi_j, \psi_k] = \mathcal{E}_j(\theta) \,\delta_{j,k}, \quad j,k = 1, \dots, n.$$

Let $\chi_0, \chi_1 : \mathbb{R} \to [0, 1]$ be smooth functions such that $\chi_0 = 1$ in $\left(-\infty, \frac{1}{2}\right]$, $\chi_0 = 0$ in $[1, \infty)$ and $\chi_0^2 + \chi_1^2 = 1$. We define $\chi_j^r : \mathbb{R}^2 \to \mathbb{R}$ by $\chi_j^r(x) = \chi_j(|x|/r), j = 0, 1$, and $\psi_j^r : S_\theta \to \mathbb{R}$ by $\psi_j^r := \chi_0^r \psi_j, j = 1, \ldots, n$, and keep the same symbols for the restrictions of these functions to S_θ^r . Remark that the functions ψ_j^r belong to $H^1(S_\theta^r)$ and vanish at $\partial_{\text{ext}}S_\theta^r$, i.e. they belong to $\mathcal{Q}(D_{\theta,1}^r)$ and can be used to estimate the Rayleigh quotients. Let us now use the Agmon-type estimate (2.3) with suitable b > 0 and B > 0 for the eigenfunctions ψ_j . Denote

$$C_{j,k}^r := \int_{\mathcal{S}_{\theta}} (\chi_1^r)^2 \psi_j \psi_k \, \mathrm{d}x,$$

then $|C_{j,k}^r| \leqslant \frac{1}{2} \left(C_{j,j}^r + C_{k,k}^r \right)$ and

$$C_{j,j}^r = \int_{\mathcal{S}_{\theta}} (\chi_1^r)^2 \psi_j^2 \, \mathrm{d}x \leqslant \int_{\mathcal{S}_{\theta}: |x| > r/2} \psi_j^2 \, \mathrm{d}x$$
$$\leqslant e^{-\frac{br}{2}} \int_{\mathcal{S}_{\theta}: |x| > r/2} e^{b|x|} \psi_j^2 \, \mathrm{d}x \leqslant B e^{-\frac{br}{2}}.$$

Therefore, for large r one has $C_{j,k}^r = \mathcal{O}(e^{-cr})$ with $c := \frac{1}{2}b$ and

$$\langle \psi_j^r, \psi_k^r \rangle_{L^2(\mathcal{S}_\theta^r)} = \langle \psi_j, \psi_k \rangle_{L^2(\mathcal{S}_\theta)} - C_{j,k}^r = \delta_{j,k} + \mathcal{O}(e^{-cr}).$$

In particular, for large r the functions ψ_j^r are linearly independent. Using similar estimates we obtain

$$\int_{\mathcal{S}_{\theta}} \nabla (\chi_1^r \psi_j)^2 \, \mathrm{d}x = \mathcal{O}(e^{-cr}),$$
$$\int_{\mathcal{S}_{\theta}^r} \nabla \psi_j^r \cdot \nabla \psi_k^r \, \mathrm{d}x = \int_{\mathcal{S}_{\theta}} \nabla \psi_j \cdot \nabla \psi_k \, \mathrm{d}x + \mathcal{O}(e^{-cr}).$$

To estimate the quantities

$$G_{j,k}^r := \int_{\partial \mathcal{S}_\theta} (\chi_1^r)^2 \psi_j \, \psi_k \, \mathrm{d}s$$

we remark again that $|G_{j,k}^r| \leq \frac{1}{2} (G_{j,j}^r + G_{k,k}^r)$, and using $\chi_1^r \psi_j$ as a test function in the inequality $T_{\theta,1} \geq -(\sin \theta)^{-2}$ for the Robin Laplacian in the sector we obtain

$$G_{j,j}^r = \int_{\partial \mathcal{S}_{\theta}} (\chi_1^r)^2 \psi_j^2 \, \mathrm{d}s \leqslant \int_{\mathcal{S}_{\theta}} \nabla (\chi_1^r \psi_j)^2 \, \mathrm{d}x + \frac{1}{\sin^2 \theta} \int_{\mathcal{S}_{\theta}} \chi_1^r \psi_j^2 \, \mathrm{d}x = \mathcal{O}(e^{-cr}),$$

which implies $G_{j,k}^r = \mathcal{O}(e^{-cr})$ and

$$\int_{\partial_* S^r_{\theta}} \psi_j^r \psi_k^r \, \mathrm{d}s = \int_{\partial S_{\theta}} \psi_j \, \psi_k \, \mathrm{d}s - G^r_{j,k} = \int_{\partial S_{\theta}} \psi_j \, \psi_k \, \mathrm{d}s + \mathcal{O}(e^{-cr}).$$

Denote $L_r := \operatorname{span}(\psi_1^r, \ldots, \psi_n^r)$, which is an *n*-dimensional subspace of $\mathcal{Q}(D_{\theta,1}^r)$ for large *r*. For any function ψ of the form

$$\psi = \xi_1 \psi_1^r + \dots + \xi_n \psi_n^r \in L_r, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

one has, due to the preceding estimates, $\|\psi\|_{L^2(\mathcal{S}^r_{\theta})}^2 = |\xi|^2 (1 + \mathcal{O}(e^{-cr}))$ and

$$D_{\theta,1}^{r}[\psi,\psi] = \sum_{j,k=1}^{n} \left(T_{\theta,1}[\psi_{j},\psi_{k}] + \mathcal{O}(e^{-cr}) \right) \xi_{j} \xi_{k}$$
$$= \sum_{j,k=1}^{n} \left(\mathcal{E}_{j}(\theta) \,\delta_{j,k} + \mathcal{O}(e^{-cr}) \right) \xi_{j} \xi_{k} \leqslant \left(\mathcal{E}_{n}(\theta) + \mathcal{O}(e^{-cr}) \right) |\xi|^{2},$$

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and an application of the min-max principle gives

$$E_n(D_{\theta,1}^r) \leqslant \sup_{\psi \in L_r, \ \psi \neq 0} \frac{D_{\theta,1}^r[\psi, \psi]}{\|\psi\|_{L^2(\mathcal{S}_{\theta}^r)}^2} \leqslant \mathcal{E}_n(\theta) + \mathcal{O}(e^{-cr}).$$

In order to obtain an analogous result on the behavior of the first $\kappa(\theta)$ eigenvalues of $N_{\theta,\alpha}^r$ we need a preliminary estimate.

DEFINITION 3.4. — For $\theta \in (0, \frac{\pi}{2})$ and $0 < \rho < r$ denote $\mathcal{P}_{\theta}^{r,\rho} := \mathcal{S}_{\theta}^{r} \setminus \overline{\mathcal{S}_{\theta}^{\rho}} \equiv the \text{ hexagon } B_{\rho}A_{\rho}^{+}A_{r}^{+}B_{r}A_{r}^{-}A_{\rho}^{-},$

where one uses the same notation as in the definition of S^r_{θ} , see Figures 3.1 and 3.2(a). We again split the boundary of $\mathcal{P}^{r,\rho}_{\theta}$ into two parts by setting

$$\partial_* \mathcal{P}^{r,\rho}_{\theta} := \partial \mathcal{P}^{r,\rho}_{\theta} \cap \partial \mathcal{S}_{\theta} := \text{the union of the segments } [A^{\pm}_{\rho}, A^{\pm}_{r}],$$
$$\partial_{\text{ext}} \mathcal{P}^{r,\rho}_{\theta} := \partial \mathcal{P}^{r,\rho}_{\theta} \setminus \partial_* \mathcal{P}^{r,\rho}_{\theta}.$$

LEMMA 3.5. — Let $P_{\theta,\alpha}^{r,\rho}$ denote the Laplacian in $\mathcal{P}_{\theta}^{r,\rho}$ with α -Robin condition at $\partial_* \mathcal{P}_{\theta}^{r,\rho}$ and the Neumann boundary condition at $\partial_{\mathrm{ext}} \mathcal{P}_{\theta}^{r,\rho}$, then $E_1(P_{\theta,\alpha}^{r,\rho}) \ge -\alpha^2 (1 + \mathcal{O}(e^{-\alpha\rho\tan\theta}))$ as $\alpha\rho \to +\infty$.

Proof. — Denote for shortness $P := P_{\theta,\alpha}^{r,\rho}$. Let us decompose the polygon $\mathcal{P}_{\theta}^{r,\rho}$ as shown in Figure 3.2 (b). Namely, let C^{\pm} be orthogonal projection of B_{ρ} on the segment $[A_r^{\pm}, B_r]$ and let U be the domain obtained from $\mathcal{P}_{\theta}^{r,\rho}$ by taking out the segments $[B_{\rho}, C^{\pm}]$, then U is the disjoint union of two rectangles $\Pi^{\pm} := B_{\rho}A_{\rho}^{\pm}A_{r}^{\pm}C^{\pm}$ and the quadrangle $\Pi^{0} := B_{\rho}C^{+}B_{r}C^{-}$. Let Λ be the Laplacian in U with α -Robin condition on $\partial_{*}\mathcal{P}_{\theta}^{r,\rho} \subset \partial U$ and Neumann condition at the remaining boundary, then the min-max principle implies then $E_{1}(P) \geq E_{1}(\Lambda)$. Hence, it is sufficient to show the sought lower bound for $E_{1}(\Lambda)$.

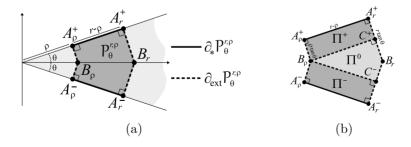


Figure 3.2. (a) Polygon $\mathcal{P}_{\theta}^{r,\rho}$, see Definition 3.4. (b) Decomposition of $\mathcal{P}_{\theta}^{r,\rho}$ for the proof of Lemma 3.5. The solid/dashed lines correspond to Robin/Neumann boundary conditions.

The operator Λ is the direct sum $\Lambda_0 \oplus \Lambda_+ \oplus \Lambda_-$ with Λ_j acting in $L^2(\Pi^j)$. Namely, Λ_0 is just the Neumann Laplacian in Π^0 , and, therefore, $E_1(\Lambda_0) = 0$. Furthermore, Λ^{\pm} are Laplacians in the rectangles Π^{\pm} with α -Robin condition on the sides $A_{\rho}^{\pm}A_{r}^{\pm}$ and Neumann condition at the remaining boundary. Therefore, they admit a separation of variables and are both unitary equivalent to $L_N \otimes 1 + 1 \otimes T$, where the operator L_N is the Laplacian on $(0, \rho \tan \theta)$ with α -Robin condition at 0 and Neumann condition at $\rho \tan \theta$ and T is the Neumann Laplacian on $(0, r - \rho)$. Therefore, $E_1(\Lambda^{\pm}) = E_1(L_N) + E_1(T) = E_1(L_N)$. The operator L_N is covered by Proposition 2.9 (with $\beta = 0$), and $E_1(L_N) = -\alpha^2 (1 + \mathcal{O}(e^{-\alpha \rho \tan \theta})) < 0$ for $\alpha \rho \to +\infty$. Therefore, for $\alpha \rho \to +\infty$ one has

$$E_1(P) \ge E_1(\Lambda) = E_1(\Lambda_0 \oplus \Lambda_+ \oplus \Lambda_-) = E_1(\Lambda^+)$$
$$= E_1(L_N) = -\alpha^2 (1 + \mathcal{O}(e^{-\alpha\rho \tan \theta})). \quad \Box$$

LEMMA 3.6. — For $\alpha r \to +\infty$ there holds

$$E_n(N_{\theta,\alpha}^r) = \left[\mathcal{E}_n(\theta) + \mathcal{O}\left(\frac{1}{(\alpha r)^2}\right) \right] \alpha^2, \quad n \in \{1, \dots, \kappa(\theta)\},$$
$$E_{\kappa(\theta)+1}(N_{\theta,\alpha}^r) \ge -\alpha^2 + o(\alpha^2).$$

Proof. — The min-max principle shows that the eigenvalues of $N_{\theta,\alpha}^r$ are bounded from above by the respective eigenvalues of $D_{\theta,\alpha}^r$. Hence, the upper bound for $E_n(N_{\theta,\alpha}^r)$ follows from the upper bound for the eigenvalues of $D_{\theta,\alpha}^r$ obtained in Lemma 3.3 above.

Let us pass to the proof of the lower bound. Let $\chi_0, \chi_1 : \mathbb{R} \to [0, 1]$ be smooth functions such that $\chi_0 = 1$ in $(-\infty, \frac{1}{2}], \chi_0 = 0$ in $[1, \infty)$ and $\chi_0^2 + \chi_1^2 = 1$. We define $\chi_j^r : \mathbb{R}^2 \to \mathbb{R}$ by $\chi_j^r(x) = \chi_j(|x|/r), j = 0, 1$. Recall that

$$N^{r}_{\theta,\alpha}[u,u] = \int_{\mathcal{S}^{r}_{\theta}} |\nabla u|^{2} \,\mathrm{d}x - \alpha \int_{\partial_{*}\mathcal{S}^{r}_{\theta}} u^{2} \,\mathrm{d}s, \quad \mathcal{Q}(N^{r}_{\theta,\alpha}) = H^{1}(\mathcal{S}^{r}_{\theta}),$$

and by direct computation for any $u \in \mathcal{Q}(N_{\theta,\alpha}^r)$ one has

$$(3.3) \quad N^{r}_{\theta,\alpha}[u,u] = N^{r}_{\theta,\alpha}[\chi^{r}_{0}u,\chi^{r}_{0}u] + N^{r}_{\theta,\alpha}[\chi^{r}_{1}u,\chi^{r}_{1}u] - \int_{\mathcal{S}^{r}_{\theta}} (|\nabla\chi^{r}_{0}|^{2} + |\nabla\chi^{r}_{1}|^{2})u^{2} \,\mathrm{d}x$$

$$\geqslant N^{r}_{\theta,\alpha}[\chi^{r}_{0}u,\chi^{r}_{0}u] + N^{r}_{\theta,\alpha}[\chi^{r}_{1}u,\chi^{r}_{1}u] - \frac{a}{r^{2}} \|u\|^{2}_{L^{2}(\mathcal{S}^{r}_{\theta})},$$

$$a := \|\chi'_{0}\|^{2}_{\infty} + \|\chi'_{1}\|^{2}_{\infty}.$$

One has $\chi_0^r u \in H^1(\mathcal{S}_{\theta}^r)$ and $\chi_0^r u = 0$ at $\partial_{\text{ext}} \mathcal{S}_{\theta}^r$. At the same time, the function $\chi_1^r u$ vanishes inside the disk $|x| \leq \frac{1}{2}r$ and, hence, is supported in

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the quadrangle $\mathcal{P}_{\theta}^{r,\rho}$ with $\rho := \frac{1}{2}r\cos\theta$ and belongs to $H^1(\mathcal{P}_{\theta}^{r,\rho})$. Therefore, one has $\chi_0^r u \in \mathcal{Q}(D_{\theta,\alpha}^r)$ and $\chi_1^r u \in \mathcal{Q}(P_{\theta,\alpha}^{r,\rho})$, and the inequality (3.3) rewrites as $N_{\theta,\alpha}^r[u,u] \ge D_{\theta,\alpha}^r[\chi_0^r u,\chi_0^r u] + P_{\theta,\alpha}^{r,\rho}[\chi_1^r u,\chi_1^r u] - (a/r^2) \|u\|_{L^2(\mathcal{S}_{\theta}^r)}^2$, and we recall that $\|u\|_{L^2(\mathcal{S}_{\theta}^r)}^2 = \|\chi_0^r u\|_{L^2(\mathcal{S}_{\theta}^r)}^2 + \|\chi_0^r u\|_{L^2(\mathcal{P}_{\theta}^{r,\rho})}^2$. By the minmax principle,

(3.4)
$$E_n(N^r_{\theta,\alpha}) \ge E_n(D^r_{\theta,\alpha} \oplus P^{r,\rho}_{\theta,\alpha}) - a/r^2, \quad r > 0, \ n \in \mathbb{N}.$$

Now let us pick $n \in \{1, \ldots, \kappa(\theta)\}$ and consider the regime $\alpha r \to +\infty$. Then one also has $\alpha \rho \to +\infty$, and the estimate of Lemma 3.5 for the first eigenvalue of $P_{\theta,\alpha}^{r,\rho}$ gives $E_1(P_{\theta,\alpha}^{r,\rho}) \ge -\alpha^2 + o(\alpha^2)$. On the other hand, the estimate of Lemma 3.3 for the first eigenvalues of $D_{\theta,\alpha}^r$ shows that $E_n(D_{\theta,\alpha}^R) = \alpha^2 (\mathcal{E}_n(\theta) + \mathcal{O}(e^{-c\alpha r}))$ with some c > 0, which is below $-\alpha^2 + o(\alpha^2)$ due to the inequality $\mathcal{E}_n(\theta) < -1$. Hence, $E_n(D_{\theta,\alpha}^r \oplus P_{\theta,\alpha}^{r,\rho}) = E_n(D_{\theta,\alpha}^r) = \alpha^2 (\mathcal{E}_n(\theta) + \mathcal{O}(e^{-c\alpha r}))$. Substituting this last estimate into the inequality (3.4) one arrives to

$$E_n(N_{\theta,\alpha}^r) \ge \left[\mathcal{E}_n(\theta) + \mathcal{O}(e^{-c\alpha r}) - \frac{a}{(\alpha r)^2}\right] \alpha^2 = \left[\mathcal{E}_n(\theta) + \mathcal{O}\left(\frac{1}{(\alpha r)^2}\right)\right] \alpha^2.$$

Using (3.4) for $n = \kappa(\theta) + 1$ we have

$$E_{\kappa(\theta)+1}(N_{\theta,\alpha}^r) \ge \min\left\{E_{\kappa(\theta)+1}(D_{\theta,\alpha}^r), E_1(P_{\theta,\alpha}^{r,\rho})\right\} - a/r^2.$$

By assumption one has $1/r = o(\alpha)$. In addition, $E_1(P_{\theta,\alpha}^{r,\rho}) \ge -\alpha^2 + o(\alpha^2)$, while $E_{\kappa(\theta)+1}(D_{\theta,\alpha}^r) \ge -\alpha^2$ by Lemma 3.3, which concludes the proof. \Box

Let us give a rough estimate for the first eigenvalue of $R^r_{\theta,\alpha}$ (it will be improved later).

LEMMA 3.7. — For some c > 0 there holds $R^r_{\theta,\alpha} \ge -c\alpha^2$ as $\alpha r \to +\infty$.

Proof. — Due to the scaling $S^r_{\theta} = rS^1_{\theta}$ the estimate follows from Lemma 2.7.

3.2. Eigenfunctions of the Robin–Neumann Laplacians

We will need an Agmon-type decay estimate for the first $\kappa(\theta)$ eigenfunctions of $N_{\theta,\alpha}^r$, which is established in the following lemma:

LEMMA 3.8. — There exist c > 0 and C > 0 such that if $n \in \{1, \ldots, \kappa(\theta)\}$ and $\psi_{\theta,\alpha}^{r,n}$ is an eigenfunction of $N_{\theta,\alpha}^r$ for the *n*th eigenvalue, then for $\alpha r \to +\infty$ there holds

$$\int_{\mathcal{S}_{\theta}^{r}} e^{c\alpha|x|} \left(\frac{1}{\alpha^{2}} |\nabla \psi_{\theta,\alpha}^{r,n}|^{2} + |\psi_{\theta,\alpha}^{r,n}|^{2}\right) \mathrm{d}x \leqslant C \left\|\psi_{\theta,\alpha}^{r,n}\right\|_{L^{2}(\mathcal{S}_{\theta}^{r})}^{2}.$$

Proof. — Denote for shortness

$$N := N_{\theta,\alpha}^r, \quad \psi := \psi_{\theta,\alpha}^{r,n} \quad \text{and} \quad \mathcal{E} := \mathcal{E}_{\kappa(\theta)}(\theta) < -1.$$

For b > 0 to be chosen later let us consider the function $\phi : S^r_{\theta} \ni x \mapsto b|x| \in \mathbb{R}$, then $|\nabla \phi| = b$, and a standard computation gives

$$N[e^{\alpha\phi}\psi, e^{\alpha\phi}\psi] = \int_{\mathcal{S}_{\theta}^{r}} e^{2\alpha\phi} (E_{n}(N) + b^{2}\alpha^{2})\psi^{2} \,\mathrm{d}x.$$

For $\alpha r \to +\infty$ one has $E_n(N) = (\mathcal{E}_n(\theta) + o(1))\alpha^2$ by Lemma 3.6. Therefore, for an arbitrary $\varepsilon > 0$ there holds $E_n(N) \leq (\mathcal{E} + \varepsilon)\alpha^2$, and

(3.5)
$$N[e^{\alpha\phi}\psi, e^{\alpha\phi}\psi] \leq (\mathcal{E} + b^2 + \varepsilon)\alpha^2 \int_{\mathcal{S}^r_{\theta}} e^{2\alpha\phi}\psi^2 \,\mathrm{d}x.$$

On the other hand, let us pick $\eta \in (0, 1)$ whose exact value will be chosen later, and set $\rho := L/\alpha$ with a value L > 0 to be chosen later as well, then

$$\begin{split} N \big[e^{\alpha \phi} \psi, e^{\alpha \phi} \psi \big] &\equiv \int_{\mathcal{S}_{\theta}^{r}} \big| \nabla (e^{\alpha \phi} \psi) \big|^{2} \, \mathrm{d}x - \alpha \int_{\partial_{*} \mathcal{S}_{\theta}^{r}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}s \\ &= \eta \int_{\mathcal{S}_{\theta}^{r}} \big| \nabla (e^{\alpha \phi} \psi) \big|^{2} \, \mathrm{d}x \\ &+ (1 - \eta) \bigg[\int_{\mathcal{S}_{\theta}^{\rho}} \big| \nabla (e^{\alpha \phi} \psi) \big|^{2} \, \mathrm{d}x - \frac{\alpha}{1 - \eta} \int_{\partial_{*} \mathcal{S}_{\theta}^{\rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}s \\ &+ \int_{\mathcal{S}_{\theta}^{r} \setminus \overline{\mathcal{S}_{\theta}^{\rho}}} \big| \nabla (e^{\alpha \phi} \psi) \big|^{2} \, \mathrm{d}x - \frac{\alpha}{1 - \eta} \int_{\partial_{*} \mathcal{S}_{\theta}^{r} \setminus \partial_{*} \mathcal{S}_{\theta}^{\rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}s \bigg] \\ &= \eta \int_{\mathcal{S}_{\theta}^{r}} \big| \nabla (e^{\alpha \phi} \psi) \big|^{2} \, \mathrm{d}x \\ &+ (1 - \eta) \Big(N_{\theta, \frac{\alpha}{1 - \eta}}^{\rho} \big[e^{\alpha \phi} \psi, e^{\alpha \phi} \psi \big] + P_{\theta, \frac{\alpha}{1 - \eta}}^{r, \rho} \big[e^{\alpha \phi} \psi, e^{\alpha \phi} \psi \big] \Big) \\ &\geqslant \eta \int_{\mathcal{S}_{\theta}^{r}} \big| \nabla (e^{\alpha \phi} \psi) \big|^{2} \, \mathrm{d}x + (1 - \eta) E_{1} \big(N_{\theta, \frac{\alpha}{1 - \eta}}^{\rho} \big) \int_{\mathcal{S}_{\theta}^{\rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x \\ &+ (1 - \eta) E_{1} \big(P_{\theta, \frac{\alpha}{1 - \eta}}^{r, \rho} \big) \int_{\mathcal{P}_{\theta}^{r, \rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x. \end{split}$$

By applying Lemma 3.6 for $N^{\rho}_{\theta,\frac{\alpha}{1-\eta}}$ and Lemma 3.5 to $P^{r,\rho}_{\theta,\frac{\alpha}{1-\eta}}$ we see that the constant L in the definition of ρ can be chosen sufficiently large to have, for large α ,

$$E_1\left(N_{\theta,\frac{\alpha}{1-\eta}}^{\rho}\right) \geqslant \left(\mathcal{E}_1(\theta) - \varepsilon\right) \frac{\alpha^2}{(1-\eta)^2}, \quad E_1\left(P_{\theta,\frac{\alpha}{1-\eta}}^{r,\rho}\right) \geqslant -\frac{(1+\varepsilon)\alpha^2}{(1-\eta)^2},$$

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and the substitution into the preceding inequality gives

$$\begin{split} N \Big[e^{\alpha \phi} \psi, e^{\alpha \phi} \psi \Big] \geqslant \eta \int_{\mathcal{S}_{\theta}^{r}} \left| \nabla (e^{\alpha \phi} \psi) \right|^{2} \, \mathrm{d}x + \frac{\mathcal{E}_{1}(\theta) - \varepsilon}{1 - \eta} \, \alpha^{2} \int_{\mathcal{S}_{\theta}^{\theta}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x \\ &- \frac{1 + \varepsilon}{1 - \eta} \, \alpha^{2} \int_{\mathcal{P}_{\theta}^{r, \rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x. \end{split}$$

Recall that $\mathcal{P}_{\theta}^{r,\rho} = \mathcal{S}_{\theta}^r \setminus \overline{\mathcal{S}_{\theta}^{\rho}}$ and substitute the last inequality into (3.5), then

$$\begin{split} \eta \int_{\mathcal{S}_{\theta}^{r}} \left| \nabla (e^{\alpha \phi} \psi) \right|^{2} \mathrm{d}x &+ \frac{\mathcal{E}_{1}(\theta) - \varepsilon}{1 - \eta} \, \alpha^{2} \int_{\mathcal{S}_{\theta}^{\rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x - \frac{1 + \varepsilon}{1 - \eta} \, \alpha^{2} \int_{\mathcal{P}_{\theta}^{r, \rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x \\ &\leqslant (\mathcal{E} + b^{2} + \varepsilon) \alpha^{2} \int_{\mathcal{P}_{\theta}^{r, \rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x + (\mathcal{E} + b^{2} + \varepsilon) \alpha^{2} \int_{\mathcal{S}_{\theta}^{\rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x, \end{split}$$

which we rewrite as

$$(3.6) \quad \eta \int_{\mathcal{S}_{\theta}^{r}} \left| \nabla(e^{\alpha \phi} \psi) \right|^{2} \mathrm{d}x + a_{0} \alpha^{2} \int_{\mathcal{S}_{\theta}^{r}} \sqrt{\mathcal{S}_{\theta}^{\rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x \leqslant b_{0} \alpha^{2} \int_{\mathcal{S}_{\theta}^{\rho}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x,$$
$$a_{0} := -\mathcal{E} - b^{2} - \varepsilon - \frac{1 + \varepsilon}{1 - \eta} = \frac{-\mathcal{E} - 1 + (\eta b^{2} - b^{2} + \eta \mathcal{E} - 2\varepsilon + \varepsilon \eta)}{1 - \eta},$$
$$b_{0} := \mathcal{E} + b^{2} + \varepsilon - \frac{\mathcal{E}_{1}(\theta) - \varepsilon}{1 - \eta} = \frac{\mathcal{E} - \mathcal{E}_{1}(\theta) - \eta \mathcal{E} + 2\varepsilon - \varepsilon \eta}{1 - \eta} + b^{2}.$$

Due to $\mathcal{E}_1(\theta) \leq \mathcal{E} < -1$, for any b > 0 one can choose $\varepsilon > 0$ and $\eta > 0$ sufficiently small to have $a_0 > 0$ and $b_0 > 0$. For $x \in S^{\rho}_{\theta}$ one has $|x| < \rho/\cos\theta$, hence, $\alpha\phi(x) \leq \alpha bL/(\alpha\cos\theta) = bL/\cos\theta$, and (3.6) takes the form

$$\begin{split} \eta \int_{\mathcal{S}_{\theta}^{r}} \left| \nabla(e^{\alpha \phi} \psi) \right|^{2} \mathrm{d}x + a_{0} \alpha^{2} \int_{\mathcal{S}_{\theta}^{r} \setminus \overline{\mathcal{S}_{\theta}^{\rho}}} e^{2\alpha \phi} \psi^{2} \, \mathrm{d}x \leqslant A \alpha^{2} \int_{\mathcal{S}_{\theta}^{\rho}} \psi^{2} \, \mathrm{d}x, \\ A := b_{0} e^{2bL/\cos\theta}, \end{split}$$

and then

$$(3.7) \qquad \int_{\mathcal{S}_{\theta}^{r}} \left| \nabla(e^{\alpha\phi}\psi) \right|^{2} \mathrm{d}x + 2b^{2}\alpha^{2} \int_{\mathcal{S}_{\theta}^{r}} \left| e^{2\alpha\phi}\psi \right|^{2} \mathrm{d}x \\ = \frac{1}{\eta} \eta \int_{\mathcal{S}_{\theta}^{r}} \left| \nabla(e^{\alpha\phi}\psi) \right|^{2} \mathrm{d}x + \frac{2b^{2}}{a_{0}}a_{0}\alpha^{2} \int_{\mathcal{S}_{\theta}^{r}} \sqrt{\mathcal{S}_{\theta}^{\rho}} e^{2\alpha\phi}\psi^{2} \,\mathrm{d}x + 2b^{2}\alpha^{2} \int_{\mathcal{S}_{\theta}^{\rho}}\psi^{2} \,\mathrm{d}x \\ \leqslant \left(\frac{1}{\eta}A + \frac{2b^{2}}{a_{0}}A + 2b^{2}\right)\alpha^{2} \int_{\mathcal{S}_{\theta}^{\rho}}\psi^{2} \,\mathrm{d}x =: A_{0}\alpha^{2} \int_{\mathcal{S}_{\theta}^{\rho}}\psi^{2} \,\mathrm{d}x \leqslant A_{0}\alpha^{2} \|\psi\|_{L^{2}(\mathcal{S}_{\theta}^{r})}^{2}.$$

Using $2xy \leqslant x^2 + y^2$ and $xy \leqslant \frac{1}{4}x^2 + y^2$ for $x, y \in \mathbb{R}$, we estimate

$$\begin{split} \left|\nabla(e^{\alpha\phi}\psi)\right|^2 &\geqslant |e^{\alpha\phi}\nabla\psi|^2 + b^2\alpha^2 |e^{\alpha\phi}\psi|^2 - 2\left|e^{\alpha\phi}\nabla\psi\right| \ \left|b\alpha e^{\alpha\phi}\psi\right| \\ &\geqslant \frac{1}{2}|e^{\alpha\phi}\nabla\psi|^2 - b^2\alpha^2 |e^{\alpha\phi}\psi|^2. \end{split}$$

The substitution into (3.7) gives

$$\int_{\mathcal{S}_{\theta}^{r}} e^{2b\alpha|x|} \left(\frac{1}{2} |\nabla\psi|^{2} + b^{2}\alpha^{2}\psi^{2}\right) \,\mathrm{d}x \leqslant A_{0}\alpha^{2} \|\psi\|_{L^{2}(\S_{\theta}^{r})}^{2}.$$

and one arrives to the claim by taking c := 2b and $C := A_0(2 + 1/b^2)$. \Box

3.3. Non-resonant sectors

Recall that the Robin–Neumann Laplacians $N_{\theta,\alpha}^r$ in the truncated convex sectors S_{θ}^r are defined in (3.1), and that due to the asymptotics of Lemma 3.6 their first $\kappa(\theta)$ eigenvalues are, in a sense, close to the first $\kappa(\theta)$ eigenvalues of the Robin Laplacian $T_{\theta,\alpha}$ in the associated infinite sectors S_{θ} in the regime $\alpha r \to +\infty$. For the subsequent study we will use the notion of a non-resonant angle, which involves a hypothesis on the behavior of the next eigenvalue of $N_{\theta,\alpha}^r$ in the same asymptotic regime. Namely, we will use the following definition:

DEFINITION 3.9. — A half-angle $\theta \in (0, \frac{\pi}{2})$ is called non-resonant if there is C > 0 such that

$$E_{\kappa(\theta)+1}(N_{\theta,\alpha}^r) \ge -\alpha^2 + C/r^2$$
 as $\alpha > 0$ is fixed and r is large.

By the scaling (3.2), the property only depends on θ and can be equivalently reformulated as

$$E_{\kappa(\theta)+1}(N_{\theta,\alpha}^r) \ge -\alpha^2 + C/r^2$$
 as αr is large.

We show in Proposition 3.10 that the non-resonance property is satisfied by an explicit wide range of half-angles, which is a key point for the whole analysis (we remark that there exist half-angles which *do not* satisfy the non-resonance property as it will be seen in Subsection 6.1).

PROPOSITION 3.10. — All half-angles θ with $\frac{\pi}{4} \leq \theta < \frac{\pi}{2}$ are non-resonant.

Proof. — We prove the result first for $\theta = \pi/4$ (Step 1) by rather direct computations, and then use a kind of monotonicity to extend it to other half-angles in the range indicated (Step 2). Without loss of generality we

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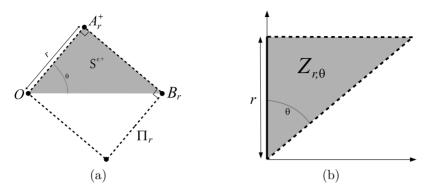


Figure 3.3. Constructions for the proof of Proposition 3.10. (a) The completion of the triangle $S_{\theta}^{r,+}$ (shaded) to a rectangle Π_r (surrounded by the dashed line). (b) The triangle $Z_{r,\theta}$ is a rotated copy of $S_{\theta}^{r,+}$. The solid/dashed lines correspond to Robin/Neumann boundary conditions.

set $\alpha = 1$ and remove the dependence on α from the notation and write N_{θ}^{r} instead of $N_{\theta,1}^{r}$.

Step 1: $\theta = \pi/4$. — By Proposition 2.11 we have $\kappa(\pi/4) = 1$, so we need to prove that there exists a constant C > 0 satisfying

(3.8)
$$E_2(N^r_{\pi}) \ge -1 + C/r^2$$
 as r is large.

Remark that $S_{\frac{r}{4}}^r$ is simply a square of side length r, and $N_{\frac{r}{4}}^r$ is the Laplacian with 1-Robin boundary condition on two neighboring sides and Neumann condition on the other two sides. Hence, on can separate the variables: using the one-dimensional Laplacians L_N on (0, r) with 1-Robin condition at 0 and Neumann condition at r one has $N_{\frac{r}{4}}^r = L_N \otimes 1 + 1 \otimes L_N$, and then $E_2(N_{\frac{r}{4}}^r) = E_1(L_N) + E_2(L_N)$. Using Proposition 2.9 one has $E_2(N_{\frac{r}{4}}^r) \geq -1 + 1/r^2 + \mathcal{O}(e^{-r})$ as $r \to +\infty$, which gives the sought inequality (3.8). Hence, the claim is proved for $\theta = \pi/4$.

Step 2: extension to $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$. — We still have $\kappa(\theta) = 1$ by Proposition 2.11, hence, we need to show that there exists C > 0 such that

(3.9)
$$E_2(N_{\theta}^r) \ge -1 + C/r^2 \text{ as } r \text{ is large.}$$

Using the symmetry with respect to the axis Ox_1 one easily sees that N_{θ}^r is unitarily equivalent to $T_{\theta}^{r,D} \oplus T_{\theta}^{r,N}$, where $T_{\theta}^{r,D/N}$ stand for the Laplacians in

$$\mathcal{S}^{r,+}_{\theta} := \mathcal{S}^r_{\theta} \cap \{(x_1, x_2) : x_2 > 0\} = \text{triangle } OA^+_r B_r$$

with 1-Robin condition at OA_r^+ , Neumann condition at $A_r^+B_r$ and the Dirichlet/Neumann boundary condition at OB_r (we refer to Figures 3.1) and 3.3(a) for an illustration). Let us study first the Dirichlet part $T_{\theta}^{r,D}$. Let Π_r be the rectangle constructed on the vectors OA_r^+ and $A_r^+B_r$, see Figure 3.3(a), then $S_{\theta}^{r,+} \subset \Pi_r$. Using the standard Dirichlet bracketing we obtain $E_n(T^{r,D}_{\theta}) \ge E_n(Q_r)$ for any $n \in \mathbb{N}$, where Q_r is the Laplacian in Π_r with 1-Robin condition at OA_r^+ , Neumann condition at $A_r^+B_r$ and the Dirichlet boundary condition at the remaining part of the boundary. Remark that $|A_r^+B_r| = r \tan \theta$, and the operator Q_r admits then a separation of variables and is unitarily equivalent to $L_D \otimes 1 + 1 \otimes D_r$, where D_r is the Laplacian on (0, r) with the Dirichlet boundary condition at 0 and the Neumann boundary condition at r, and L_D is the one-dimensional Laplacian on the interval $(0, r \tan \theta)$ with 1-Robin condition at 0 and Dirichlet condition on the other end. Therefore, $E_1(T_{\theta}^{r,D}) =$ $E_1(L_D) + E_1(D_r) = E_1(L_D) + \pi^2/(4r^2)$. Due to Proposition 2.8 we have $E_1(L_D) = -1 + \mathcal{O}(e^{-r \tan \theta})$, therefore, $E_1(T_{\theta}^{r,D}) \ge -1 + C_D/r^2$ for large r with any fixed $C_D \in (0, \pi^2/4)$. Therefore, the sought estimate (3.9) becomes equivalent to the existence of $C_N > 0$ for which there holds

(3.10)
$$E_2(T_{\theta}^{r,N}) \ge -1 + C_N/r^2 \text{ as } r \to +\infty,$$

which we already know to hold for $\theta = \frac{\pi}{4}$. In order to study $T_{\theta}^{r,N}$ we apply a rotation bringing the triangle $S_{\theta}^{r,+}$ onto the triangle $Z_{r,\theta} := \{(x_1, x_2) : 0 < x_1 \operatorname{cotan} \theta < x_2 < r\}$, so that $T_{\theta}^{r,N}$ becomes unitary equivalent to the Laplacian $Q_{r,\theta}$ in $L^2(Z_{r,\theta})$ with 1-Robin condition along the axis Ox_2 and Neumann condition at the remaining boundary, and $E_n(T_{\theta}^{r,N}) = E_n(Q_{r,\theta})$ for any $n \in \mathbb{N}$, and one easily sees that

$$Q_{r,\theta}[u,u] = \int_{Z_{r,\theta}} |\nabla u|^2 \,\mathrm{d}x - \int_0^r u(0,x_2)^2 \,\mathrm{d}x_2, \quad \mathcal{Q}(Q_{r,\theta}) = H^1(Z_{r,\theta}),$$

see Figure 3.3(b) for an illustration. Using the unitary transform

$$V: L^2(Z_{r\tan\theta,\frac{\pi}{4}}) \to L^2(Z_{r,\theta}), \quad (Vu)(x_1, x_2) = \sqrt{\tan\theta} \, u(x_1, x_2 \tan\theta),$$

which satisfies $V(H^1(Z_{r \tan \theta, \frac{\pi}{4}})) = H^1(Z_{r,\theta})$, we obtain, with $u_j := \partial u / \partial x_j$,

$$Q_{R,\theta}[Vu, Vu] = \tan\theta \int_{Z_{R,\theta}} \left(u_1(x_1, x_2 \tan\theta)^2 + \tan^2\theta \, u_2(x_1, x_2 \tan\theta)^2 \right) \mathrm{d}x$$
$$-\tan\theta \int_0^r u(0, x_2 \tan\theta)^2 \, \mathrm{d}x_2$$

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$$\begin{split} &= \int_{Z_{r \tan \theta, \frac{\pi}{4}}} \left(u_1(x_1, x_2)^2 + \tan^2 \theta \, u_2(x_1, x_2)^2 \right) \mathrm{d}x \\ &\quad - \alpha \int_0^{r \tan \theta} u(0, x_2)^2 \, \mathrm{d}x_2 \\ &= Q_{r \tan \theta, \frac{\pi}{4}}[u, u] + (\tan^2 \theta - 1) \int_{Z_{r \tan \theta, \frac{\pi}{4}}} u_2^2 \, \mathrm{d}x. \end{split}$$

For $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ we have $\tan \theta \ge 1$, hence, $Q_{r,\theta}[Vu, Vu] \ge Q_{r \tan \theta, \frac{\pi}{4}}[u, u]$ for all $u \in H^1(Z_{r,\theta})$, and by the min-max principle we have

$$E_n(T_{\theta}^{r,N}) = E_n(Q_{r,\theta}) \ge E_n(Q_{r\tan\theta,\frac{\pi}{4}}) = E_n(T_{\frac{\pi}{4}}^{r\tan\theta,N}).$$

It was already shown in Step 1 that for some C > 0 we have $E_2(T_{\frac{r}{4}}^{r \tan \theta, N}) \ge -1 + C/(r \tan \theta)^2$ for large r, so the substitution into the preceding inequality gives the sought estimate (3.10) with $C_N = C \operatorname{cotan}^2 \theta$.

Now we state some consequences of the non-resonance condition, which will provide important components for the subsequent asymptotic analysis:

LEMMA 3.11. — Assume that θ is non-resonant and denote by \mathcal{L} the subspace spanned by the eigenfunctions corresponding to the first $\kappa(\theta)$ eigenvalues of $N_{\theta,\alpha}^r$. Then there exists b > 0 such that for $\alpha r \to +\infty$ there holds

$$(3.11) \qquad \|v\|_{L^2(\mathcal{S}^r_{\theta})}^2 \leqslant b \, r^2 \big(N^r_{\theta,\alpha}[v,v] + \alpha^2 \|v\|_{L^2(\mathcal{S}^r_{\theta})}^2 \big), \qquad v \in H^1(\mathcal{S}^r_{\theta}) \cap \mathcal{L}^{\perp},$$

(3.12)
$$\int_{\partial_{\text{ext}} \mathcal{S}^r_{\theta}} v^2 \, \mathrm{d}s \leqslant b\alpha r^2 \left(N^r_{\theta,\alpha}[v,v] + \alpha^2 \|v\|^2_{L^2(\mathcal{S}^r_{\theta})} \right), \quad v \in H^1(\mathcal{S}^r_{\theta}) \cap \mathcal{L}^{\perp}$$

Proof. — The norm estimate (3.11) directly follows from the definition of a non-resonant half-angle (Definition 3.9) with the help of the spectral theorem. For (3.12), recall that by Lemma 2.7 one can find $c_0 > 0$ such that $E_1(R^r_{\theta,\alpha}) \ge -c_0 \alpha^2$ for large αr . Due to

$$\mathcal{Q}(N^r_{\theta,\alpha}) = \mathcal{Q}(R^r_{\theta,\alpha}) = H^1(\mathcal{S}^r_{\theta}), \quad R^r_{\theta,\alpha}[v,v] = N^r_{\theta,\alpha}[v,v] - \alpha \int_{\partial_{\text{ext}}\mathcal{S}^r_{\theta}} v^2 \, \mathrm{d}s,$$

the preceding inequality for $E_1(R_{\theta,\alpha}^r)$ takes the form

$$\int_{\partial_{\text{ext}} \mathcal{S}^r_{\theta}} v^2 \, \mathrm{d}s \leqslant \frac{1}{\alpha} N^r_{\theta,\alpha}[v,v] + c_0 \alpha \|v\|^2_{L^2(\mathcal{S}^r_{\theta})} \text{ for all } v \in H^1(\mathcal{S}^r_{\theta}).$$

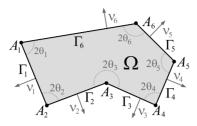


Figure 4.1. An example of a polygon Ω with six vertices. The vertex A_3 is concave, the other vertices are convex, and $\mathcal{J}_{cvx} = \{1, 2, 4, 5, 6\}$.

Assume in addition that $v \perp \mathcal{L}$, then one bounds from above the second term on the right-hand side using (3.11), which gives

$$\int_{\partial_{\text{ext}} \mathcal{S}_{\theta}^{r}} v^{2} \, \mathrm{d}s \leqslant \frac{1}{\alpha} N_{\theta,\alpha}^{r}[v,v] + c_{0} b \, \alpha r^{2} \left(N_{\theta,\alpha}^{r}[v,v] + \alpha^{2} \|v\|_{L^{2}(\mathcal{S}_{\theta}^{r})}^{2} \right)$$
$$= \left(\frac{1}{\alpha} + c_{0} b \, \alpha r^{2} \right) N_{\theta,\alpha}^{r}[v,v] + c_{0} b \, \alpha r^{2} \alpha^{2} \|v\|_{L^{2}(\mathcal{S}_{\theta}^{r})}^{2}$$
$$\leqslant \left(\frac{1}{\alpha} + c_{0} b \, \alpha r^{2} \right) \left(N_{\theta,\alpha}^{r}[v,v] + \alpha^{2} \|v\|_{L^{2}(\mathcal{S}_{\theta}^{r})}^{2} \right).$$

It remains to estimate, for $\alpha r \to +\infty$,

$$\frac{1}{\alpha} + c_0 b \,\alpha r^2 = \frac{1 + c_0 b \,(\alpha r)^2}{\alpha} \leqslant \frac{2c_0 b \,(\alpha r)^2}{\alpha} = 2c_0 b \alpha r^2. \qquad \Box$$

4. Robin eigenvalues in polygons: Proof of Theorem 1.1

4.1. Decomposition of a polygon

In this section we assume that Ω is a polygon with straight edges. We assume that Ω has M vertices A_1, \ldots, A_M , and for the notation convenience we identify $A_0 \equiv A_M$ and $A_{M+1} \equiv A_1$, and the same cyclic numbering convention will be applied to other related objects. We denote by ℓ_j the length of the side $\Gamma_j := [A_j, A_{j+1}], j = 1, \ldots, M$, and introduce the maps

$$\gamma_j : [0, \ell_j] \ni t \mapsto A_j + \frac{A_{j+1} - A_j}{\ell_j} t \in \mathbb{R}^2$$

providing an arc-length parametrization of Γ_j with $\gamma_j(0) = A_j$ and $\gamma_j(\ell_j) = A_{j+1}$. In addition, for $t \in (0, \ell_j)$ by $\nu_j(t)$ we denote the *outer* unit normal to $\partial\Omega$ at the point $\gamma_j(t)$ of Γ_j .

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By $\theta_j \in [0, \pi]$ we denote the half-angle of Ω at the vertex A_j , i.e. θ_j is the half of the angle between Γ_{j-1} and Γ_j when measured inside Ω . Our assumption is that there are neither zero angles nor artificial vertices, i.e. that $\theta_j \notin \{0, \frac{\pi}{2}, \pi\}$ for all $j = 1, \ldots, M$. One says that a vertex A_j is convex if $\theta_j < \frac{\pi}{2}$, otherwise it will be called *concave*. We denote

$$\mathcal{J}_{\mathrm{cvx}} := \{ j : A_j \text{ is convex} \}.$$

We refer to Figure 4.1 for an illustration.

For small $\delta > 0$ denote

$$\Omega_{\delta} = \left\{ x \in \Omega : \operatorname{dist}(x, \partial \Omega) < \delta \right\}, \quad \Omega_{\delta}^{c} := \Omega \setminus \overline{\Omega_{\delta}},$$

and Ω_{δ} will be further decomposed near each vertex. The construction is different for convex and concave vertices.

• Let A_j be a convex vertex, then there exists a unique point $Y_{j,\delta} \in \Omega$ such that $\operatorname{dist}(Y_{j,\delta}, \Gamma_{j-1}) = \operatorname{dist}(Y_{j,\delta}, \Gamma_j) = \delta$. Denote $\lambda_j := \cot \theta_j$, then the points

$$A_{j,\delta}^- := \gamma_{j-1}(\ell_j - \lambda_j \delta), \quad A_{j,\delta}^+ := \gamma_j(\lambda_j \delta)$$

are exactly the orthogonal projections of $Y_{j,\delta}$ on Γ_{j-1} and Γ_j , respectively. We denote the interior of the quadrangle $A_j A_{j,\delta}^- Y_{j,\delta} A_{j,\delta}^+$ by $V_{j,\delta}$, and, in turn, we decompose the boundary of $V_{j,\delta}$ into the following parts:

$$\partial_* V_{j,\delta} := \partial V_{j,\delta} \cap \partial \Omega, \quad \partial_{\text{ext}} V_{j,\delta} := \partial V_{j,\delta} \setminus \partial_* V_{j,\delta}, \quad \partial_{\text{out}} V_{j,\delta} := \emptyset.$$

• Let A_j be a concave vertex. Let L_j^- be the half-line emanating from A_j which is orthogonal to Γ_{j-1} and directed inside Ω . By L_j^+ we denote the half-line emanating from A_j , orthogonal to Γ_j at A_j and

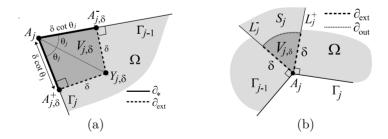


Figure 4.2. The construction of the neighborhoods $V_{j,\delta}$: (a) convex vertex, (b) concave vertex. The partial boundary $\partial_* V_{j,\delta}$ is shown with the thick solid line, the part $\partial_{\text{ext}} V_{j,\delta}$ is indicated with the thick dashed line, and the part $\partial_{\text{out}} V_{j,\delta}$ with the gray dotted line.

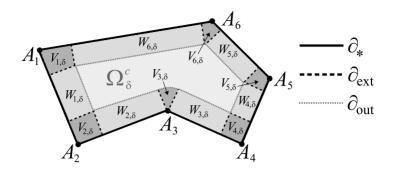


Figure 4.3. Decomposition of a polygon.

directed inside Ω . Denote by S_j the infinite sector bounded by $L_j^$ and L_j^+ which lies inside Ω near A_j . Then we set

$$V_{j,\delta} := S_j \cap B(A_j, \delta), \quad \lambda_j := 0,$$

where B(a, r) is the disk of radius r centered at a. We decompose the boundary of $V_{j,\delta}$ as follows:

$$\partial_* V_{j,\delta} := \emptyset, \quad \partial_{\text{out}} V_{j,\delta} := \partial V_{j,\delta} \cap \partial \Omega^c_{\delta}, \quad \partial_{\text{ext}} V_{j,\delta} := \partial V_{j,\delta} \setminus \partial_{\text{out}} V_{j,\delta}.$$

Remark that the numbers λ_j represent a kind of "length deficiency": the length of $\partial_* V_{j,\delta}$ is equal to $2\lambda_j \delta$ for both convex and concave vertices.

The set $\mathcal{W}_{\delta} := \Omega_{\delta} \setminus \overline{\bigcup_{j=1}^{M} V_{j,\delta}}$ is then the union of M disjoint thin rectangles. Namely, denote

(4.1)
$$I_{j,\delta} := (\lambda_j \delta, \ell_j - \lambda_{j+1} \delta), \quad \Pi_{j,\delta} := I_{j,\delta} \times (0,\delta), W_{j,\delta} := \Phi_j(\Pi_{j,\delta}), \quad \Phi_j(s,t) := \gamma_j(s) - t\nu_j(s),$$

then $\mathcal{W}_{\delta} = \bigcup_{j=1}^{M} W_{j,\delta}$, and $W_{j,\delta} \cap W_{k,\delta} = \emptyset$ for $j \neq k$. We decompose the boundary of each rectangle $W_{j,\delta}$ as follows:

$$\partial_* W_{j,\delta} := \partial W_{j,\delta} \cap \partial \Omega, \quad \partial_{\mathrm{out}} W_{j,\delta} := \partial W_{j,\delta} \cap \partial \Omega^c_{\delta},$$
$$\partial_{\mathrm{ext}} W_{j,\delta} := \partial W_{j,\delta} \setminus \Big(\partial_* W_{j,\delta} \cup \partial_{\mathrm{out}} W_{j,\delta}\Big).$$

By construction we have the equality

(4.2)
$$\bigcup_{j=1}^{M} \partial_{\text{ext}} V_{j,\delta} = \bigcup_{j=1}^{M} \partial_{\text{ext}} W_{j,\delta}$$

which will be of importance later. We refer to Figure 4.3 for an illustration of the above decomposition.

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4.2. First estimates for side-induced eigenvalues

With each $j \in \{1, \ldots, M\}$ we associate the corresponding number $\kappa(\theta_j)$ of discrete eigenvalues of the Robin Laplacians in the infinite sector of aperture $2\theta_j$ (see Section 2.6) and set

$$K := \kappa(\theta_1) + \dots + \kappa(\theta_M) \equiv \sum_{j \in \mathcal{J}_{\text{cvx}}} \kappa(\theta_j),$$

 $\mathcal{E} :=$ the disjoint union of $\{\mathcal{E}_n(\theta_j), n = 1, \dots, \kappa(\theta_j)\}$ for $j \in \mathcal{J}_{cvx}$,

 $\mathcal{E}_n :=$ the *n*th element of \mathcal{E} when numbered in the non-decreasing order,

(see Subsection 2.6 for a detailed notation). For what follows we introduce several operators:

- $N_j^V :=$ the Laplacian in $V_{j,\delta}$ with the α -Robin boundary condition at $\partial_* V_{j,\delta}$ and the Neumann boundary condition at the rest of the boundary,
- $D_j^V :=$ the Laplacian in $V_{j,\delta}$ with the α -Robin boundary condition at $\partial_* V_{j,\delta}$ and the Dirichlet boundary condition at the rest of the boundary.

We remark that for concave vertices A_j , the respective operators $(N/D)_j^V$ are just the Neumann/Dirichlet Laplacians in $V_{j,\delta}$ due to $\partial_* V_{j,\delta} = \emptyset$. Furthermore, denote

- $N_j^W :=$ the Laplacian in $W_{j,\delta}$ with the α -Robin boundary condition at $\partial_* W_{j,\delta}$ and the Neumann boundary condition at the rest of the boundary,
- $D_j^W :=$ the Laplacian in $W_{j,\delta}$ with the α -Robin boundary condition at $\partial_* W_{j,\delta}$ and the Dirichlet boundary condition at the rest of the boundary.

Finally, introduce

 $N^c :=$ the Neumann Laplacian in Ω^c_{δ} .

One easily sees that D_j^V and N_j^V with $j \in \mathcal{J}_{cvx}$ are covered by the analysis of Section 3, and the behavior of the first $\kappa(\theta_j)$ eigenvalues for $\alpha \delta \to +\infty$ is given in Lemmas 3.3 and 3.6, respectively. On the other hand, for $j \notin \mathcal{J}_{cvx}$ one has $D_j^V \ge 0$, $N_j^V \ge 0$, and $N^c \ge 0$. For the rest of the section we assume that

(4.3) all convex vertices are non-resonant,

then, in addition, we have a lower bound for the $(\kappa(\theta_j) + 1)$ -th eigenvalue of each N_j^V with $j \in \mathcal{J}_{cvx}$ due to Definition 3.9.

In the subsequent constructions we choose $\delta > 0$ depending on α in such a way that

(4.4)
$$\delta \to 0^+, \quad \alpha \delta \to +\infty \text{ for } \alpha \to +\infty,$$

and summarize the preceding observations as follows:

LEMMA 4.1. — With some c > 0 and $c_0 > 0$ there holds, for $\alpha \to +\infty$,

$$E_n\left(\bigoplus_{j=1}^M N_j^V\right) = \mathcal{E}_n \,\alpha^2 + \mathcal{O}(1/\delta^2), \qquad \text{for } n = 1, \dots, K,$$
$$E_n\left(\bigoplus_{j=1}^M D_j^V\right) = \mathcal{E}_n \,\alpha^2 + \mathcal{O}(\alpha^2 e^{-c\alpha\delta}), \qquad \text{for } n = 1, \dots, K,$$
$$E_{K+1}\left(\bigoplus_{j=1}^M D_j^V\right) \ge E_{K+1}\left(\bigoplus_{j=1}^M N_j^V\right) \ge -\alpha^2 + c_0/\delta^2.$$

In what follows we are going to use several one-dimensional operators. We denote

$$D_j := \text{the Dirichlet Laplacian on } (0, \ell_j),$$

$$D_{j,\delta} := \text{the Dirichlet Laplacian on } I_{j,\delta},$$

$$N_j := \text{the Neumann Laplacian on } (0, \ell_j),$$

$$N_{j,\delta} := \text{the Neumann Laplacian on } I_{j,\delta},$$

and remark that for each fixed $n \in \mathbb{N}$ one has $E_n(D_{j,\delta}) = E_n(D_j) + \mathcal{O}(\delta)$ and $E_n(N_{j,\delta}) = E_n(N_j) + \mathcal{O}(\delta)$. We start with a simple estimate for the eigenvalues of R_{α}^{Ω} :

PROPOSITION 4.2. — There holds, with some c > 0,

(4.5)
$$E_n(R^{\Omega}_{\alpha}) = \mathcal{E}_n \, \alpha^2 + \mathcal{O}(1/\delta^2 + \alpha^2 e^{-c\alpha\delta}), \quad n \in \{1, \dots, K\}.$$

In addition, for any $n \in \mathbb{N}$ there holds

(4.6)
$$-\alpha^{2} + E_{n}\left(\bigoplus_{j=1}^{M} N_{j}\right) + \mathcal{O}(\delta + \alpha^{2}e^{-\alpha\delta}) \leq E_{K+n}(R_{\alpha}^{\Omega})$$
$$\leq -\alpha^{2} + E_{n}\left(\bigoplus_{j=1}^{M} D_{j}\right) + \mathcal{O}(\delta + \alpha^{2}e^{-\alpha\delta}).$$

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Proof. — Due to the standard Dirichlet–Neumann bracketing, for any $n \in \mathbb{N}$ one has

$$(4.7) \quad E_n\left(N^c \oplus \left(\bigoplus_{j=1}^M N_j^V\right) \oplus \left(\bigoplus_{j=1}^M N_j^W\right)\right) \leqslant E_n(R_\alpha^\Omega)$$
$$\leqslant E_n\left(\bigoplus_{j=1}^M D_j^V \oplus \left(\bigoplus_{j=1}^M D_j^W\right)\right).$$

The operators N_j^W and D_j^W admits a separation of variables: if one denotes $L_{N/D}$ the Laplacian on $(0, \delta)$ with α -Robin condition at 0 and Neumann/Dirichlet condition at δ , then one has the unitary equivalences $N_j^W \simeq N_{j,\delta} \otimes 1 + 1 \otimes L_N$ and $D_j^W \simeq D_{j,\delta} \otimes 1 + 1 \otimes L_D$. For each fixed $n \in \mathbb{N}$ one has $E_n(N_{j,\delta}) = \mathcal{O}(1)$ and $E_n(D_{j,\delta}) = \mathcal{O}(1)$. On the other hand, by Propositions 2.8 and 2.9 we have $E_1(L_{N/D}) = -\alpha^2 + \mathcal{O}(\alpha^2 e^{-\alpha\delta})$ and $E_2(L_{N/D}) \ge 0$. Therefore, $E_n(N_j^W) = E_1(L_N) + E_n(N_{j,\delta})$ and $E_n(D_j^W) = E_1(L_D) + E_n(D_{j,\delta})$, and then

$$E_n\left(\bigoplus_{j=1}^M N_j^W\right) = E_1(L_N) + E_n\left(\bigoplus_{j=1}^M N_{j,\delta}\right)$$
$$= -\alpha^2 + E_n\left(\bigoplus_{j=1}^M N_j\right) + \mathcal{O}(\delta + \alpha^2 e^{-\alpha\delta}),$$
$$E_n\left(\bigoplus_{j=1}^M D_j^W\right) = E_1(L_D) + E_n\left(\bigoplus_{j=1}^M D_{j,\delta}\right)$$
$$= -\alpha^2 + E_n\left(\bigoplus_{j=1}^M D_j\right) + \mathcal{O}(\delta + \alpha^2 e^{-\alpha\delta}).$$

In view of the estimates of Lemma 4.1 one has then

$$E_{K}\left(N^{c} \oplus \left(\bigoplus_{j=1}^{M} N_{j}^{V}\right)\right) \leqslant E_{n}\left(\bigoplus_{j=1}^{M} N_{j}^{W}\right) \leqslant E_{K+1}\left(N^{c} \oplus \left(\bigoplus_{j=1}^{M} N_{j}^{V}\right)\right),$$
$$E_{K}\left(\left(\bigoplus_{j=1}^{M} D_{j}^{V}\right)\right) \leqslant E_{n}\left(\bigoplus_{j=1}^{M} D_{j}^{W}\right) \leqslant E_{K+1}\left(\left(\bigoplus_{j=1}^{M} D_{j}^{V}\right)\right).$$

Therefore, for each $n \in \{1, \ldots, K\}$ one has

$$E_n\left(N^c \oplus \left(\bigoplus_{j=1}^M N_j^V\right) \oplus \left(\bigoplus_{j=1}^M N_j^W\right)\right) = E_n\left(\bigoplus_{j=1}^M N_j^V\right)$$
$$= \mathcal{E}_n \,\alpha^2 + \mathcal{O}(1/\delta^2),$$
$$E_n\left(\left(\bigoplus_{j=1}^M D_j^V\right) \oplus \left(\bigoplus_{j=1}^M D_j^W\right)\right) = E_n\left(\bigoplus_{j=1}^M D_j^V\right)$$
$$= \mathcal{E}_n \,\alpha^2 + \mathcal{O}(\alpha^2 e^{-c\alpha\delta}),$$

and (4.7) reads as $\mathcal{E}_n \alpha^2 + \mathcal{O}(1/\delta^2) \leq E_n(R_\alpha^\Omega) \leq \mathcal{E}_n \alpha^2 + \mathcal{O}(e^{-c\alpha\delta})$ and gives (4.5). In order to obtain (4.6) we remark that for each $n \in \mathbb{N}$ one has

$$E_{K+n}\left(N^{c} \oplus \left(\bigoplus_{j=1}^{M} N_{j}^{V}\right) \oplus \left(\bigoplus_{j=1}^{M} N_{j}^{W}\right)\right) = E_{n}\left(\bigoplus_{j=1}^{M} N_{j}^{W}\right)$$
$$= -\alpha^{2} + E_{n}\left(\bigoplus_{j=1}^{M} N_{j}\right) + \mathcal{O}(\delta + \alpha^{2}e^{-\alpha\delta}),$$
$$E_{K+n}\left(\bigoplus_{j=1}^{M} D_{j}^{V} \oplus \left(\bigoplus_{j=1}^{M} D_{j}^{W}\right)\right) = E_{n}\left(\bigoplus_{j=1}^{M} D_{j}^{W}\right)$$
$$= -\alpha^{2} + E_{n}\left(\bigoplus_{j=1}^{M} D_{j}\right) + \mathcal{O}(\delta + \alpha^{2}e^{-\alpha\delta}),$$

and it remains to use these bounds on the both sides of (4.7).

By taking $\delta := b \log \alpha / \alpha$ with a sufficiently large b one then obtains:

COROLLARY 4.3. — There holds

(4.8)
$$E_n(R^{\Omega}_{\alpha}) = \mathcal{E}_n \, \alpha^2 + o(\alpha^2) \text{ for } n \in \{1, \dots, K\}.$$

In addition, for any $n \in \mathbb{N}$ there holds

(4.9)
$$E_{K+n}(R^{\Omega}_{\alpha}) = -\alpha^{2} + \mathcal{O}(1),$$
$$E_{K+n}(R^{\Omega}_{\alpha}) \leqslant -\alpha^{2} + E_{n}\left(\bigoplus_{j=1}^{M} D_{j}\right) + \mathcal{O}\left(\frac{\log \alpha}{\alpha}\right).$$

Remark that (4.8) is only given for completeness (and as a preparation for the analysis of the curvilinear case): the remainder is not optimal and can be improved to $\mathcal{O}(e^{-c\alpha})$ with a suitable c > 0 by using more advanced methods as shown by Khalile [36].

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4.3. Lower bound for side-induced eigenvalues

It remains to obtain a more precise lower bound for $E_{K+n}(R^{\Omega}_{\alpha})$. This is the most involved part of the whole analysis, and it will be done in the present subsection with the help of the Proposition 2.3 by constructing a suitable identification map. All estimates of this subsection are for α and δ in the asymptotic regime (4.4). Introduce some additional objects:

- L := the subspace of $L^2(\Omega)$ spanned by the first K eigenfunctions of R^{Ω}_{α} ,
- $L_j :=$ the subspace of $L^2(V_{j,\delta})$ spanned by the first $\kappa(\theta_j)$ eigenfunctions of N_j^V , with $j \in \mathcal{J}_{cvx}$,
- $$\begin{split} \sigma_j: L^2(\Omega) &\to L^2(V_{j,\delta}) \text{ the operator of restriction}, \\ (\sigma_j u)(x) &= u(x) \text{ for } x \in V_{j,\delta}, \end{split}$$

then the adjoint operators $\sigma_j^* : L^2(V_{j,\delta}) \to L^2(\Omega)$ are the operators of extension by zero. Recall that the distance d(E, F) between subspaces E and F was discussed in Subsection 2.3.

LEMMA 4.4. — For $j \in \mathcal{J}_{cvx}$ one has $d(\sigma_j^* L_j, L) = \mathcal{O}(e^{-c\alpha\delta})$ with some fixed c > 0.

Proof. — During the proof we denote $\Lambda_j := \sigma_j^* L_j \subset L^2(\Omega)$, and for $v \in L^2(V_{j,\delta})$ we denote $v_* := \sigma_j^* v \in L^2(\Omega)$.

Let 0 < a < b < 1. Consider a C^{∞} function $\varphi : \mathbb{R} \to [0,1]$ with $\varphi(t) = 1$ for $t \leq a$ and $\varphi(t) = 0$ for $t \geq b$. Introduce $\varphi_{\delta} : \Omega \to \mathbb{R}$ by $\varphi_{\delta}(x) = \varphi(|x - A_j|/(\delta \cot \theta_j))$, which clearly satisfies (as α is large, hence, δ is small):

- $0 \leq \varphi_{\delta} \leq 1$, and for all $\beta \in \mathbb{N}^2$ with $1 \leq |\beta| \leq 2$ there holds $\|\partial^{\beta}\varphi_{\delta}\|_{\infty} \leq C\delta^{-|\beta|}$,
- $\varphi_{\delta} = 1$ in $V_{j,a\delta}$, and $\varphi_{\delta} = 0$ in $\Omega \setminus \overline{V_{j,b\delta}}$,
- the normal derivative of φ_{δ} at $\partial \Omega$ is zero,

where C > 0 is some fixed constant. Denote

$$\varphi_{\delta}\Lambda_j := \left\{\varphi_{\delta}v_*: v_* \in \Lambda_j\right\} \subset L^2(\Omega),$$

then

(4.10)
$$d(\Lambda_j, L) \leq d(\Lambda_j, \varphi_{\delta} \Lambda_j) + d(\varphi_{\delta} \Lambda_j, L).$$

The first term on the right-hand side can be easily estimated by applying directly the definition of the distance. Namely, due to the Agmon-type estimate for the first $\kappa(\theta_j)$ eigenfunctions of N_j^V (Lemma 3.8), with some $b_0 > 0$ and B > 0 there holds

$$\int_{V_{j,\delta}} e^{b_0 \alpha |x - A_j|} \left(\frac{1}{\alpha^2} |\nabla v|^2 + v^2 \right) \mathrm{d}x \leqslant B \|v\|_{L^2(V_{j,\delta})}^2, \quad v \in L_j$$

Writing

$$\int_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}} \left(\frac{1}{\alpha^2} |\nabla v|^2 + v^2\right) \mathrm{d}x$$
$$= \int_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}} e^{b_0\alpha|x-A_j|} \cdot e^{b_0\alpha|x-A_j|} \left(\frac{1}{\alpha^2} |\nabla v|^2 + v_*^2\right) \mathrm{d}x$$

we obtain the following upper bound

$$\begin{split} \int_{V_{j,\delta} \setminus \overline{V_{j,a\delta}}} \left(\frac{1}{\alpha^2} |\nabla v|^2 + v^2 \right) \mathrm{d}x \\ &\leqslant e^{-b_0 \alpha a \delta} \int_{V_{j,\delta} \setminus \overline{V_{j,a\delta}}} e^{b_0 \alpha |x - A_j|} \left(\frac{1}{\alpha^2} |\nabla v|^2 + v_*^2 \right) \mathrm{d}x \\ &\leqslant e^{-b_0 \alpha a \delta} \int_{V_{j,\delta}} e^{b_0 \alpha |x - A_j|} \left(\frac{1}{\alpha^2} |\nabla v|^2 + v_*^2 \right) \mathrm{d}x. \end{split}$$

This finally gives

(4.11)
$$\int_{V_{j,\delta}\setminus\overline{V_{j,a\delta}}} \left(\frac{1}{\alpha^2} |\nabla v|^2 + v^2\right) \mathrm{d}x \leqslant Be^{-2c\alpha\delta} \|v\|_{L^2(V_{j,\delta})}^2, \quad c := b_0 a/2.$$

Therefore, for any $v_* \in \Lambda_j$ we have

$$\begin{aligned} \left\| v_* - \varphi_{\delta} v_* \right\|_{L^2(\Omega)}^2 &= \int_{\Omega} (1 - \varphi_{\delta})^2 v_*^2 \, \mathrm{d}x \leqslant \int_{\Omega \setminus \overline{V_{j,a\delta}}} v_*^2 \, \mathrm{d}x \\ &\equiv \int_{V_{j,\delta} \setminus \overline{V_{j,a\delta}}} v^2 \, \mathrm{d}x \leqslant B e^{-2c\alpha\delta} \|v\|_{L^2(V_{j,\delta})}^2 \equiv B e^{-2c\alpha\delta} \|v_*\|_{L^2(\Omega)}^2. \end{aligned}$$

Denote by P_j the orthogonal projector on $\varphi_{\delta}\Lambda_j$ in $L^2(\Omega)$, then for any $u \in L^2(\Omega)$ we have by definition $||u - P_j u|| = \inf_{\phi \in \varphi_{\delta}\Lambda_j} ||u - \phi||$. Therefore, for any non-zero $v_* \in \Lambda_j$ we have

(4.12)
$$\frac{\|v_* - P_j v_*\|}{\|v_*\|} \leqslant \frac{\|v_* - \varphi_\delta v_*\|}{\|v_*\|} \equiv \frac{\|(1 - \varphi_\delta) v_*\|}{\|v_*\|} \leqslant \sqrt{B} e^{-c\alpha\delta},$$
$$d(\Lambda_j, \varphi_\delta \Lambda_j) = \sup_{v_* \in \Lambda_j, \, v_* \neq 0} \frac{\|v_* - P_j v_*\|}{\|v_*\|} \leqslant \sqrt{B} e^{-c\alpha\delta}.$$

Now we need an estimate for the second term on the right-hand side of (4.10), which will be obtained with the help of Proposition 2.5. Namely,

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let v^n with $n \in \{1, \ldots, \kappa(\theta_j)\}$ be the eigenfunctions of N_j^V for the eigenvalues $E_n := E_n(N_j^V)$ forming an orthonormal basis of L_j , then, in particular,

$$-\Delta v^n = E_n v^n \text{ in } V_{j,\delta}, \quad \frac{\partial v^n}{\partial \nu} = \alpha v^n \text{ at } \partial_* V_{j,\delta} \subset \partial \Omega,$$

where ν the outer unit normal. Consider the functions $\psi_n := \varphi_{\delta} v_*^n$, then using the above properties of φ_{δ} we have

$$\Delta \psi_n = \left((\Delta \varphi_{\delta}) v^n + 2 \nabla \varphi_{\delta} \cdot \nabla v^n + \varphi_{\delta} \Delta v^n \right)_* \in L^2(\Omega),$$

$$\frac{\partial \psi_n}{\partial \nu} = \frac{\partial \varphi_{\delta}}{\partial \nu} v^n + \varphi_{\delta} \frac{\partial v^n}{\partial \nu} = \varphi_{\delta} \frac{\partial v^n}{\partial \nu} = \alpha \varphi_{\delta} v^n = \alpha \psi_n \text{ on } \partial\Omega,$$

which shows that ψ_n belong to the domain of R^{Ω}_{α} . We represent now

$$(R^{\Omega}_{\alpha} - E_n)\psi_n = (-\Delta - E_n)\psi_n = \left(-(\Delta\varphi_{\delta})v^n - 2\nabla\varphi_{\delta}\cdot\nabla v^n\right)_*$$

and note that the supports of $\nabla \varphi_{\delta}$ and $\Delta \varphi_{\delta}$ are contained in $V_{j,b\delta} \setminus \overline{V_{j,a\delta}}$. Therefore, with the help of (4.11) we can estimate

$$\int_{\Omega} \left| (\Delta \varphi_{\delta}) v_*^n \right|^2 \mathrm{d}x \leqslant \frac{C^2}{\delta^4} \int_{V_{j,b\delta} \setminus \overline{V_{j,a\delta}}} (v^n)^2 \mathrm{d}x$$
$$\leqslant \frac{BC^2}{\delta^4} e^{-2c\alpha\delta} \|v^n\|_{L^2(V_{j,\delta})}^2 \equiv \frac{BC^2}{\delta^4} e^{-2c\alpha\delta},$$
$$\int_{\Omega} \left| \nabla \varphi_{\delta} \cdot \nabla v_*^n \right|^2 \mathrm{d}x \leqslant \int_{\Omega} |\nabla \varphi_{\delta}|^2 |\nabla v_*^n|^2 \mathrm{d}x \leqslant \frac{C^2}{\delta^2} \int_{V_{j,b\delta} \setminus \overline{V_{j,a\delta}}} (\nabla v^n)^2 \mathrm{d}x$$
$$\leqslant \frac{BC^2 \alpha^2}{\delta^2} e^{-2c\alpha\delta} \|v^n\|_{L^2(V_{j,\delta})}^2 \equiv \frac{BC^2 \alpha^2}{\delta^2} e^{-2c\alpha\delta},$$

and by noting that $1/\delta^2 = o(\alpha/\delta)$ we have

$$\left\| (R_{\alpha}^{\Omega} - E_n)\psi_n \right\|_{L^2(\Omega)} = \mathcal{O}((\alpha/\delta) e^{-c\alpha\delta}).$$

Let us estimate the Gram matrix G of (ψ_n) . We have, using Cauchy–Schwarz and (4.11),

$$\begin{split} \left| \langle \psi_k, \psi_n \rangle_{L^2(\Omega)} - \langle v_*^k, v_*^n \rangle_{L^2(\Omega)} \right| \\ &= \left| \int_{\Omega} (\varphi_{\delta}^2 - 1) v_*^k v_*^n \, \mathrm{d}x \right| \leqslant \int_{V_{j,\delta} \setminus \overline{V_{j,a\delta}}} |v^k \, v^n| \, \mathrm{d}x \\ &\leqslant \frac{1}{2} \bigg(\int_{V_{j,\delta} \setminus \overline{V_{j,a\delta}}} (v^k)^2 \, \mathrm{d}x + \int_{V_{j,\delta} \setminus \overline{V_{j,a\delta}}} (v^n)^2 \, \mathrm{d}x \bigg) \leqslant B e^{-2c\alpha\delta}. \end{split}$$

Therefore, we have $\langle \psi_k, \psi_n \rangle_{L^2(\Omega)} = \delta_{k,n} + \mathcal{O}(e^{-2c\alpha\delta})$, and the lowest eigenvalue λ of G is estimated as $\lambda = 1 + \mathcal{O}(e^{-2c\alpha\delta})$.

Finally let $h := (-\mathcal{E}_K - 1)/2$, then the interval $I := ((\mathcal{E}_1 - h)\alpha^2, (\mathcal{E}_K + h)\alpha^2)$ contains all the above eigenvalues E_n due to Lemma 4.1, and it also

contains the first K eigenvalues of R^{Ω}_{α} and satisfies $\operatorname{dist}(I, \operatorname{spec}(R^{\Omega}_{\alpha}) \setminus I) \geq \frac{1}{4} h \alpha^2$ by (4.8). Therefore, we are exactly in the situation of Proposition 2.5 with the parameters

$$E = \varphi_{\delta} \Lambda_j, \quad F = L, \quad \varepsilon = \mathcal{O}\left(\frac{\alpha}{\delta} e^{-c\alpha\delta}\right), \quad \eta \ge \frac{1}{8}h\alpha^2, \quad \lambda = 1 + \mathcal{O}(e^{-2c\alpha\delta}),$$

which gives $d(\varphi_{\delta}\Lambda_j, L) = \mathcal{O}(e^{-c\alpha\delta}/(\alpha\delta))$. By combining this last inequality with (4.12) in the initial triangular inequality (4.10) one arrives at the conclusion.

LEMMA 4.5. — There exist b > 0 and c > 0 such that for each $j = 1, \ldots, M$ there holds

(4.13)
$$\|\sigma_{j}u\|_{L^{2}(V_{j,\delta})}^{2} \leq b\delta^{2} \Big(N_{j}^{V}[\sigma_{j}u,\sigma_{j}u] + \alpha^{2} \|\sigma_{j}u\|_{L^{2}(V_{j,\delta})}^{2} \Big) + b\alpha^{2}\delta^{2}e^{-c\alpha\delta} \|u\|_{L^{2}(\Omega)}^{2},$$

(4.14)
$$\int_{\partial_{\text{ext}}V_{j,\delta}} (\sigma_j u)^2 \,\mathrm{d}s \leqslant b\alpha \delta^2 \Big(N_j^V [\sigma_j u, \sigma_j u] + \alpha^2 \|\sigma_j u\|_{L^2(V_{j,\delta})}^2 \Big) \\ + b\alpha^3 \delta^2 e^{-c\alpha\delta} \|u\|_{L^2(\Omega)}^2$$

as $u \in H^1(\Omega)$ with $u \perp L$.

Proof. — Remark that the sought inequalities look quite similar to those in Lemma 3.11. The novelty is that we do not assume $\sigma_j u \perp L_j$ (in this case the result would follow directly) but just $u \perp L$. The main technical ingredient of the proof below is to show that the orthogonal projection of $\sigma_j u$ onto L_j is sufficiently small and absorbed by the last summands in the above inequalities (4.13) and (4.14). This will be achieved using the distance estimate of Lemma 4.4.

Assume first that $j \in \mathcal{J}_{cvx}$. Let P be the orthogonal projector on L in $L^2(\Omega)$ and P_j be the orthogonal projector on L_j in $L^2(V_{j,\delta})$. Consider the following functions of $L^2(V_{j,\delta})$:

$$u^V := \sigma_j u, \quad v_0 := P_j u^V, \quad v := (1 - P_j) u^V.$$

Due to $u \perp L$ we have u = (1 - P)u, hence,

$$\|v_0\|_{L^2(V_{j,\delta})} = \|\sigma_j^* v_0\|_{L^2(\Omega)} = \|\sigma_j^* P_j \sigma_j (1-P)u\|_{L^2(\Omega)}$$
$$\leqslant \|\sigma_j^* P_j \sigma_j (1-P)\| \|u\|_{L^2(\Omega)}.$$

The operator $\Pi_j := \sigma_j^* P_j \sigma_j$ is exactly the orthogonal projector on $\sigma_j^* L_j$ in $L^2(\Omega)$, and by Lemma 4.4 one has $\|\sigma_j^* P_j \sigma_j (1-P)\| = \|\Pi_j - \Pi_j P\| =$

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 $d(\sigma_j^*L_j, L) = \mathcal{O}(e^{-c\alpha\delta})$ with some c > 0. Then for some b > 0 one has

(4.15)
$$\|v_0\|_{L^2(V_{j,\delta})} \leq b e^{-c\alpha\delta} \|u\|_{L^2(\Omega)}$$

As P_j is a spectral projector for N_j^V , one has $N_j^V[u^V, u^V] = N_j^V[v_0, v_0] + N_i^V[v, v]$, and due to the spectral theorem we have the inequalities

$$E_1(N_j^V) \|v_0\|_{L^2(V_{j,\delta})}^2 \leqslant N_j^V[v_0, v_0] \leqslant E_{\kappa(\theta_j)}(N_j^V) \|v_0\|_{L^2(V_{j,\delta})}^2.$$

By Lemma 3.6 we have $E_n(N_j^V) = \mathcal{O}(\alpha^2)$ for $n = 1, \ldots, \kappa(\theta_j)$ and using (4.15) one arrives at

(4.16)
$$\begin{cases} \left| N_{j}^{V}[v_{0}, v_{0}] \right| \leqslant a_{0} \alpha^{2} e^{-2c\alpha\delta} \|u\|_{L^{2}(\Omega)}^{2}, \\ N_{j}^{V}[v, v] \leqslant N_{j}^{V}[u^{V}, u^{V}] + a_{0} \alpha^{2} e^{-2c\alpha\delta} \|u\|_{L^{2}(\Omega)}^{2} \end{cases}$$

As $v \perp L_j$, one can apply the trace and norm estimate for non-resonant truncated sectors (Lemma 3.11). Using first the norm estimate one has, with some $c_1 > 0$,

$$\|v\|_{L^{2}(V_{j,\delta})}^{2} \leq c_{1}\delta^{2} \Big(N_{j}^{V}[v,v] + \alpha^{2} \|v\|_{L^{2}(V_{j,\delta})}^{2} \Big),$$

and by using (4.15), (4.16) and the inequality $||v||^2_{L^2(V_{j,\delta})} \leq ||u^V||^2_{L^2(V_{j,\delta})}$ we have

$$\begin{split} \|u^{V}\|_{L^{2}(V_{j,\delta})}^{2} &= \|v\|_{L^{2}(V_{j,\delta})}^{2} + \|v_{0}\|_{L^{2}(V_{j,\delta})}^{2} \\ &\leqslant c_{1}\delta^{2} \left(N_{j}^{V}[v,v] + \alpha^{2} \|v\|_{L^{2}(V_{j,\delta})}^{2}\right) + b^{2}e^{-2c\alpha\delta} \|u\|_{L^{2}(\Omega)}^{2} \\ &\leqslant c_{1}\delta^{2} \left(N_{j}^{V}[u^{V},u^{V}] + \alpha^{2} \|u^{V}\|_{L^{2}(V_{j,\delta})}^{2}\right) \\ &\quad + (a_{0}c_{1}\alpha^{2}\delta^{2}e^{-2c\alpha\delta} + b^{2}e^{-2c\alpha\delta}) \|u\|_{L^{2}(\Omega)}^{2} \\ &\leqslant c_{1}\delta^{2} \left(N_{j}^{V}[u^{V},u^{V}] + \alpha^{2} \|u^{V}\|_{L^{2}(V_{j,\delta})}^{2}\right) \\ &\quad + b_{0}\alpha^{2}\delta^{2}e^{-2c\alpha\delta} \|u\|_{L^{2}(\Omega)}^{2} \end{split}$$

with a sufficiently large $b_0 > 0$, which proves (4.13). Furthermore, using first the trace estimate of Lemma 3.11 and then (4.16) we have, with some $c_2 > 0$,

(4.17)

$$\int_{\partial_{\text{ext}}V_{j,\delta}} v^2 \, \mathrm{d}s \leqslant c_1 \alpha \delta^2 \left(N_j^V[v,v] + \alpha^2 \|v\|_{L^2(V_{j,\delta})}^2 \right) \\
\leqslant c_1 \alpha \delta^2 \left(N_j^V[u^V,u^V] + \alpha^2 \|u^V\|_{L^2(V_{j,\delta})}^2 \right) \\
+ c_2 \alpha^3 \delta^2 e^{-2c\alpha\delta} \|u\|_{L^2(\Omega)}^2,$$

with $c_2 := c_1 a_0$. Let R_j^V be the Laplacian in $V_{j,\delta}$ with α -Robin condition at the whole boundary, then $E_1(R_j^V) \ge -c_0 \alpha^2$ with some $c_0 > 0$ (see Lemma 2.7), i.e.

$$R_j^V[f,f] \equiv N_j^V[f,f] - \alpha \int_{\partial_{\text{ext}}V_{j,\delta}} f^2 \, \mathrm{d}s \ge -c_0 \alpha^2 \|f\|_{L^2(V_{j,\delta})}^2,$$
$$\int_{\partial_{\text{ext}}V_{j,\delta}} f^2 \, \mathrm{d}s \le \frac{1}{\alpha} \left(N_j^V[f,f] + c_0 \alpha^2 \|f\|_{L^2(V_{j,\delta})}^2 \right) \text{ for all } f \in H^1(V_{j,\delta}).$$

Using this inequality for $f := v_0$ and then applying (4.15) and (4.16) on both terms on the right-hand side we arrive at

(4.18)
$$\int_{\partial_{\text{ext}} V_{j,\delta}} v_0^2 \, \mathrm{d}s \leqslant c_3 \alpha e^{-2c\alpha\delta} \|u\|_{L^2(\Omega)}^2$$

with some $c_3 > 0$. Finally,

$$\int_{\partial_{\text{ext}}V_{j,\delta}} (u^V)^2 \,\mathrm{d}s \equiv \int_{\partial_{\text{ext}}V_{j,\delta}} (v+v_0)^2 \,\mathrm{d}s \leqslant 2 \int_{\partial_{\text{ext}}V_{j,\delta}} v^2 \,\mathrm{d}s + 2 \int_{\partial_{\text{ext}}V_{j,\delta}} v_0^2 \,\mathrm{d}s,$$

and by estimating the two terms on the right-hand side by (4.17) and (4.18) one arrives at (4.14).

Now assume that $j \notin \mathcal{J}_{cvx}$, then $N_j^V \ge 0$ is just the Neumann Laplacian in $V_{j,\delta}$. In particular, for large α one has the obvious estimate

$$\|\sigma_{j}u\|_{L^{2}(V_{j,\delta})}^{2} \leqslant \frac{1}{\alpha^{2}} \Big(N_{j}^{V}[\sigma_{j}u,\sigma_{j}u] + \alpha^{2} \|\sigma_{j}u\|_{L^{2}(V_{j,\delta})}^{2} \Big),$$

implying (4.13) due to $1/\alpha^2 = o(\delta^2)$. To obtain (4.14) consider the Laplacian R_j^V in $V_{j,\delta}$ with the α -Robin boundary condition at the whole boundary, then $R_j^V \ge -c_4 \alpha^2$ with some $c_4 > 0$ by Lemma 2.7, and for all $f \in H^1(V_{j,\delta})$ we have

$$\int_{\partial V_{j,\delta}} f^2 \,\mathrm{d}s \leqslant \frac{1}{\alpha} \bigg(\int_{V_{j,\delta}} |\nabla f|^2 \,\mathrm{d}x + c_4 \alpha^2 \int_{V_{j,\delta}} f^2 \,\mathrm{d}x \bigg)$$
$$\equiv \frac{1}{\alpha} \bigg(N_j^V[f,f] + c_4 \alpha^2 \|f\|_{L^2(V_{j,\delta})}^2 \bigg).$$

The terms on the right-hand side are non-negative, so for $c_5 := \max\{1, c_4\}$ one has

$$\int_{\partial V_{j,\delta}} f^2 \,\mathrm{d}s \leqslant \frac{c_5}{\alpha} \left(N_j^V[f,f] + \alpha^2 \|f\|_{L^2(V_{j,\delta})}^2 \right) \text{ for all } f \in H^1(V_{j,\delta}).$$

Using this estimate for $f := \sigma_j u$ and noting that $1/\alpha = o(\alpha \delta^2)$ we arrive at (4.14). Remark that the last summands in (4.13) and (4.14) appear for convex vertices only.

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Now we have collected all technical ingredients for the proof of the main estimate:

PROPOSITION 4.6. — For any fixed $n \in \mathbb{N}$ one has $E_{K+n}(R^{\Omega}_{\alpha}) \geq -\alpha^2 + E_n(\bigoplus_{j=1}^M D_j) + \mathcal{O}(\frac{\log \alpha}{\sqrt{\alpha}})$ as $\alpha \to +\infty$.

Proof. — Consider the Hilbert spaces

 $\mathcal{H} :=$ the orthogonal complement of L in $L^2(\Omega)$, $\mathcal{H}' := \bigoplus_{j=1}^M L^2(I_{j,\delta}).$

During the proof for $u \in \mathcal{H}$ we denote $||u|| := ||u||_{L^2(\Omega)}$ and

$v_j :=$ the restriction of u to $V_{j,\delta}$,	$ v_j := v_j _{L^2(V_{j,\delta})},$
$w_j :=$ the restriction of u to $W_{j,\delta}$,	$ w_j := w_j _{L^2(W_{j,\delta})},$
$u_c :=$ the restriction of u to Ω^c_{δ} ,	$ u_c := u_c _{L^2(\Omega^c_\delta)},$

and remark that due to the preceding constructions and the equality (4.2) we have

(4.19)
$$\sum_{j=1}^{M} \int_{\partial_{\text{ext}} V_{j,\delta}} v_j^2 \, \mathrm{d}s = \sum_{j=1}^{M} \int_{\partial_{\text{ext}} W_{j,\delta}} w_j^2 \, \mathrm{d}s$$

Applying Lemma 4.5 we obtain, with some b > 0 and c > 0, the inequalities

$$\|v_j\|^2 \leqslant b\delta^2 \left(N_j^V[v_j, v_j] + \alpha^2 \|v_j\|^2 \right) + b\alpha^2 \delta^2 e^{-c\alpha\delta} \|u\|^2,$$
$$\int_{\partial_{\text{ext}} V_{j,\delta}} v_j^2 \, \mathrm{d}s \leqslant b\alpha\delta^2 \left(N_j^V[v_j, v_j] + \alpha^2 \|v_j\|^2 \right) + b\alpha^3 \delta^2 e^{-c\alpha\delta} \|u\|^2.$$

Now recall that each N_j^W admit a separation of variables, $N_j^W \simeq L_N \otimes 1 + 1 \otimes N_{j,\delta}$, where L_N is the Laplacian on $(0, \delta)$ with α -Robin condition at 0 and Neumann condition at δ . Denote by ψ a normalized eigenfunction for the first eigenvalue of L_N , consider the maps

(4.20)
$$P_j: \mathcal{H} \to L^2(I_{j,\delta}), \quad (P_j u)(s) := \int_0^\delta \psi(t) \, w_j \left(\Phi_j(s,t) \right) \mathrm{d}t,$$

with Φ_j defined in (4.1), and denote $\widetilde{w}_j := w_j \circ \Phi_j$ and $z_j := \widetilde{w}_j - (P_j u) \otimes \psi$, then one has the pointwise orthogonality

$$\int_0^\delta \psi(t) z_j(\,\cdot\,,t) \,\mathrm{d}t = 0.$$

Using the standard change of variables and then the spectral theorem for ${\cal L}_N$ one obtains

$$\begin{split} N^{W}[w_{j}, w_{j}] &= \int_{W_{j,\delta}} |\nabla w_{j}|^{2} \,\mathrm{d}x - \alpha \int_{\partial_{*}W_{j,\delta}} w^{2} \,\mathrm{d}s \\ &= \int_{I_{j,\delta}} \int_{0}^{\delta} \left(\left(\partial_{s} \widetilde{w}_{j}\right)^{2} + \left(\partial_{t} \widetilde{w}_{j}\right)^{2} \right) \mathrm{d}t \,\mathrm{d}s - \alpha \int_{I_{j,\delta}} \widetilde{w}_{j}(s,0)^{2} \,\mathrm{d}s \\ &= \int_{I_{j,\delta}} \left(\left(P_{j}u\right)'(s)\right)^{2} \,\mathrm{d}s + \int_{I_{j,\delta}} \int_{0}^{\delta} \left(\partial_{s}z_{j}\right)^{2} \,\mathrm{d}t \,\mathrm{d}s \\ &+ \int_{I_{j,\delta}} \left(\int_{0}^{\delta} \left(\partial_{t} \widetilde{w}_{j}\right)^{2} \,\mathrm{d}t - \alpha \widetilde{w}_{j}(s,0)^{2} \right) \,\mathrm{d}s \\ &\geqslant \int_{I_{j,\delta}} \left((P_{j}u)'(s)\right)^{2} \,\mathrm{d}s + \int_{I_{j,\delta}} \int_{0}^{\delta} \left(\partial_{s}z_{j}\right)^{2} \,\mathrm{d}t \,\mathrm{d}s \\ &+ E_{1}(L_{N}) \|P_{j}u\|^{2} + E_{2}(L_{N}) \|z_{j}\|^{2}. \end{split}$$

Using Proposition 2.9 we estimate $E_1(L) \ge -\alpha^2 - b_1 \alpha^2 e^{-c\alpha\delta}$ and $E_2(L_N) \ge 0$, which leads to

$$N_{j}^{W}[w_{j}, w_{j}] \ge -\alpha^{2} \|P_{j}u\|^{2} + \|(P_{j}u)'\|^{2} - b_{1}\alpha^{2}e^{-c\alpha\delta}\|P_{j}u\|^{2}.$$

Now let us set $\delta := (c' \log \alpha)/\alpha$ with $c' \ge 3/c$, then the conditions (4.4) for the choice of δ are satisfied, and $\alpha^2 e^{-c\alpha\delta} = o(\delta)$, which implies $\alpha^2 \delta^2 e^{-c\alpha\delta} = o(\delta^3)$ and $\alpha^3 \delta^2 e^{-c\alpha\delta} = o(\alpha\delta^3)$. This simplifies the remainders in the above inequalities, and one can pick a sufficiently large a > 0,

(4.21)
$$\|v_j\|^2 \leqslant \frac{a\log^2 \alpha}{\alpha^2} \left(N_j^V[v_j, v_j] + \alpha^2 \|v_j\|^2 \right) + \frac{a\log^3 \alpha}{\alpha^3} \|u\|^2,$$

(4.22)
$$\int_{\partial_{\text{ext}}V_{j,\delta}} v_j^2 \,\mathrm{d}s \leqslant \frac{a\log^2\alpha}{\alpha} \Big(N_j^V[v_j, v_j] + \alpha^2 \|v_j\|^2 \Big) + \frac{a\log^3\alpha}{\alpha^2} \|u\|^2,$$

(4.23)
$$N_j^W[w_j, w_j] \ge -\alpha^2 \|P_j u\|^2 + \|(P_j u)'\|^2 - \frac{a \log \alpha}{\alpha} \|P_j u\|^2.$$

Consider the self-adjoint operators

$$B := R_{\alpha}^{\Omega} + \alpha^{2} + \frac{(M+a)\log\alpha}{\alpha} \text{ viewed as an operator in } \mathcal{H},$$
$$B' := \bigoplus_{j=1}^{M} D_{j,\delta} \text{ in } \mathcal{H}',$$

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with $\mathcal{Q}(B) = H^1(\Omega) \cap \mathcal{H}$ and $\mathcal{Q}(B') = \bigoplus_{j=1}^M H^1_0(I_{j,\delta})$. Recall that

$$B[u, u] = \int_{\Omega} |\nabla u|^{2} dx - \alpha \int_{\partial \Omega} u^{2} ds + \alpha^{2} \int_{\Omega} u^{2} dx + \frac{(M+a) \log \alpha}{\alpha} \int_{\Omega} u^{2} dx = \sum_{j=1}^{M} \left(\int_{V_{j,\delta}} |\nabla u|^{2} dx - \alpha \int_{\partial_{*}V_{j,\delta}} u^{2} ds + \alpha^{2} \int_{V_{j,\delta}} u^{2} dx \right) + \sum_{j=1}^{M} \left(\int_{W_{j,\delta}} |\nabla u|^{2} dx - \alpha \int_{\partial_{*}W_{j,\delta}} u^{2} ds + \alpha^{2} \int_{W_{j,\delta}} u^{2} dx \right) + \int_{\Omega_{\delta}^{c}} |\nabla u|^{2} dx + \alpha^{2} \int_{\Omega_{\delta}^{c}} u^{2} dx + \frac{(M+a) \log \alpha}{\alpha} \int_{\Omega} u^{2} dx \geq \sum_{j=1}^{M} \left(N_{j}^{V} [v_{j}, v_{j}] + \alpha^{2} ||v_{j}||^{2} \right) + \sum_{j=1}^{M} \left(N_{j}^{W} [w_{j}, w_{j}] + \alpha^{2} ||w_{j}||^{2} \right) + \alpha^{2} ||u_{c}||^{2} + \frac{(M+a) \log \alpha}{\alpha} ||u||^{2}.$$

Using (4.21) and (4.22) we obtain

(4.25)
$$\sum_{j=1}^{M} \left(N_{j}^{V}[v_{j}, v_{j}] + \alpha^{2} \|v_{j}\|^{2} \right)$$
$$\geq \frac{\alpha^{2}}{a \log^{2} \alpha} \frac{1}{2} \left(\sum_{j=1}^{M} \|v_{j}\|^{2} + \frac{1}{\alpha} \sum_{j=1}^{M} \int_{\partial_{ext} V_{j,\delta}} v_{j}^{2} \, \mathrm{d}s \right) - \frac{M \log \alpha}{\alpha} \|u\|^{2}.$$

On the other hand, with the help of (4.23) we estimate

(4.26)
$$\sum_{j=1}^{M} \left(N_{j}^{W}[w_{j}, w_{j}] + \alpha^{2} ||w_{j}||^{2} \right)$$
$$\geq \sum_{j=1}^{M} \left\| (P_{j}u)' \right\|^{2} - \frac{a \log \alpha}{\alpha} ||u||^{2} + \alpha^{2} \sum_{j=1}^{M} \left(||w_{j}||^{2} - ||P_{j}u||^{2} \right),$$

where we use $\sum_j \|P_j u\|^2 \leqslant \sum_j \|w_j\|^2 \leqslant \|u\|^2$. Using (4.25) and (4.26) in (4.24) we arrive at

$$(4.27) \quad B[u,u] \ge \frac{\alpha^2}{2a\log^2 \alpha} \sum_{j=1}^M \|v_j\|^2 + \frac{\alpha}{2a\log^2 \alpha} \sum_{j=1}^M \int_{\partial_{ext}V_{j,\delta}} v_j^2 \, \mathrm{d}s + \sum_{j=1}^M \left\| (P_j u)' \right\|^2 + \alpha^2 \sum_{j=1}^M \left(\|w_j\|^2 - \|P_j u\|^2 \right) + \alpha^2 \|u_c\|^2.$$

Each term on the right-hand side is non-negative and, hence, the left-hand side is an upper bound for each term on the right-hand side. It also implies that B is positive and then

$$E_1(B) \equiv E_{K+1}(R_{\alpha}^{\Omega}) + \alpha^2 + \frac{(M+a)\log\alpha}{\alpha} \ge 0.$$

By (4.9), for any fixed $n \in \mathbb{N}$ there is $\mu_n > 0$ independent of α such that

(4.28)
$$0 \leq E_n(B) \leq \mu_n, \quad (1 + E_n(B))^{-1} \geq (1 + \mu_n)^{-1}.$$

In order to construct a suitable identification map $J : \mathcal{Q}(B) \to \mathcal{Q}(B')$ we pick functions $\rho_j^{\pm} \in C^1([0, \ell_j])$ such that

$$\rho_j^+ = \begin{cases} 1 & \text{in a neighborhood of } 0, \\ 0 & \text{in a neighborhood of } \ell_j, \end{cases} \quad j \in \mathcal{J}_*,$$
$$\rho_j^- = \begin{cases} 0 & \text{in a neighborhood of } 0, \\ 1 & \text{in a neighborhood of } \ell_j, \end{cases} \quad j \in \mathcal{J}_*,$$

and then choose a constant $\rho_0 > 0$ such that

(4.29)
$$\|\rho_j^{\pm}\|_{L^{\infty}(0,\ell_j)} + \|(\rho_j^{\pm})'\|_{L^{\infty}(0,\ell_j)} \leq \rho_0 \text{ for all } j \in \mathcal{J}_*.$$

Recall that $I_{j,\delta} := (\lambda_j \delta, \ell_j - \lambda_{j+1} \delta) =: (\iota_j, \tau_j)$, hence,

$$\rho_j^+(\iota_j) = 1, \quad \rho_j^+(\tau_j) = 0, \quad \rho_j^-(\iota_j) = 0, \quad \rho_j^-(\tau_j) = 1,$$

as α is sufficiently large. Therefore, the map

$$J : \mathcal{Q}(B) \to \mathcal{Q}(B') \equiv \bigoplus_{j=1}^{M} H_0^1(I_{j,\delta}), \quad Ju = (J_j u),$$
$$(J_j u)(s) := (P_j u)(s) - (P_j u)(\iota_j)\rho_j^+(s) - (P_j u)(\tau_j)\rho_j^-(s)$$

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is well-defined. We estimate, using the Cauchy–Schwarz inequality,

$$(4.30) |P_{j}u(\iota_{j})|^{2} + |P_{j}u(\tau_{j})|^{2}$$

$$= \left(\int_{0}^{\delta}\psi(t)w_{j}(\Phi_{j}(\iota_{j},t))dt\right)^{2} + \left(\int_{0}^{\delta}\psi(t)w_{j}(\Phi_{j}(\tau_{j},t))dt\right)^{2}$$

$$\leqslant \int_{0}^{\delta}w_{j}(\Phi_{j}(\iota_{j},t))^{2}dt + \int_{0}^{\delta}w_{j}(\Phi_{j}(\tau_{j},t))^{2}dt \equiv \int_{\partial_{ext}W_{j,\delta}}w_{j}^{2}ds.$$

Recall the inequality $(x+y)^2 \ge (1-\varepsilon)x^2 - y^2/\varepsilon$ valid for any $x,y\in\mathbb{R}$ and $\varepsilon>0$. Then

$$\begin{split} \|J_{j}u\|^{2} &= \int_{I_{j,\delta}} \left| (P_{j}u)(s) - (P_{j}u)(\iota_{j})\rho_{j}^{+}(s) - (P_{j}u)(\tau_{j})\rho_{j}^{-}(s) \right|^{2} \mathrm{d}s \\ &\geqslant (1-\varepsilon) \int_{I_{j,\delta}} \left| (P_{j}u)(s) \right|^{2} \mathrm{d}s \\ &- \frac{1}{\varepsilon} \int_{I_{j,\delta}} \left| (P_{j}u)(\iota_{j})\rho_{j}^{+}(s) + (P_{j}u)(\tau_{j})\rho_{j}^{-}(s) \right|^{2} \mathrm{d}s, \end{split}$$

and, using (4.30) and the constant ρ_0 from (4.29) we have

$$\begin{split} \int_{I_{j,\delta}} \left| (P_{j}u)(\iota_{j})\rho_{j}^{+}(s) + (P_{j}u)(\tau_{j})\rho_{j}^{-}(s) \right|^{2} \mathrm{d}s \\ &\leqslant 2\ell_{j}\rho_{0}^{2} \left(\left| P_{j}u(\iota_{j}) \right|^{2} + \left| P_{j}u(\tau_{j}) \right|^{2} \right) \leqslant 2\ell_{j}\rho_{0}^{2} \int_{\partial_{\mathrm{ext}}W_{j,\delta}} w_{j}^{2} \mathrm{d}s, \\ &\| J_{j}u\|^{2} \geqslant (1-\varepsilon) \| P_{j}u\|^{2} - \frac{2\ell\rho_{0}^{2}}{\varepsilon} \int_{\partial_{\mathrm{ext}}W_{j,\delta}} w_{j}^{2} \mathrm{d}s, \quad \ell := \max \ell_{j}. \end{split}$$

Therefore, using (4.19) we arrive at

$$\begin{split} \|u\|^{2} - \|Ju\|^{2} &= \sum_{j=1}^{M} \|v_{j}\|^{2} + \sum_{j=1}^{M} \|w_{j}\|^{2} + \|u_{c}\|^{2} - \sum_{j=1}^{M} \|J_{j}u\|^{2} \\ &\leqslant \sum_{j=1}^{M} \|v_{j}\|^{2} + \sum_{j=1}^{M} \|w_{j}\|^{2} + \|u_{c}\|^{2} \\ &- (1-\varepsilon) \sum_{j=1}^{M} \|P_{j}u\|^{2} + \frac{2\ell\rho_{0}^{2}}{\varepsilon} \sum_{j=1}^{M} \int_{\partial_{ext}W_{j,\delta}} w_{j}^{2} \, ds \\ &= \sum_{j=1}^{M} \|v_{j}\|^{2} + \sum_{j=1}^{M} \left(\|w_{j}\|^{2} - \|P_{j}u\|^{2}\right) \\ &+ \varepsilon \sum_{j=1}^{M} \|P_{j}u\|^{2} + \frac{2\ell\rho_{0}^{2}}{\varepsilon} \sum_{j=1}^{M} \int_{\partial_{ext}V_{j,\delta}} v_{j}^{2} \, ds + \|u_{c}\|^{2} \end{split}$$

$$\leq \sum_{j=1}^{M} \|v_{j}\|^{2} + \sum_{j=1}^{M} \left(\|w_{j}\|^{2} - \|P_{j}u\|^{2} \right) \\ + \varepsilon \|u\|^{2} + \frac{2\ell\rho_{0}^{2}}{\varepsilon} \sum_{j=1}^{M} \int_{\partial_{ext}V_{j,\delta}} v_{j}^{2} \, \mathrm{d}s + \|u_{c}\|^{2} \, \mathrm{d}s$$

Using (4.27) we obtain an upper bound for all terms on the right-hand side except $\varepsilon ||u||^2$:

$$||u||^2 - ||Ju||^2 \leqslant \left(\frac{2a\log^2\alpha}{\alpha^2} + \frac{2}{\alpha^2} + \frac{4\ell\rho_0^2 a\log^2\alpha}{\varepsilon\alpha}\right)B[u,u] + \varepsilon ||u||^2.$$

Taking $\varepsilon := \log \alpha / \sqrt{\alpha}$ and choosing $c_1 > 0$ sufficiently large we obtain

(4.31)
$$||u||^2 - ||Ju||^2 \leq \frac{c_1 \log \alpha}{\sqrt{\alpha}} \Big(B[u, u] + ||u||^2 \Big).$$

To study the difference B'[Ju, Ju] - B[u, u] recall that $B'[Ju, Ju] = \sum_{j=1}^{M} ||(J_j u)'||^2$. Using the inequality $(x+y)^2 \leq (1+\varepsilon)x^2 + 2y^2/\varepsilon$ valid for all $x, y \in \mathbb{R}$ and $\varepsilon \in (0, 1)$ we estimate

$$\begin{split} \left\| (J_{j}u)' \right\|^{2} &= \int_{I_{j,\delta}} \left| (P_{j}u)'(s) - (P_{j}u)(\iota_{j})(\rho_{j}^{+})'(s) - (P_{j}u)(\tau_{j})(\rho_{j}^{-})'(s) \right|^{2} \mathrm{d}s \\ &\leq (1+\varepsilon) \int_{I_{j,\delta}} \left| (P_{j}u)'(s) \right|^{2} \mathrm{d}s \\ &+ \frac{2}{\varepsilon} \int_{I_{j,\delta}} \left| (P_{j}u)(\iota_{j})(\rho_{j}^{+})'(s) + (P_{j}u)(\tau_{j})(\rho_{j}^{-})'(s) \right|^{2} \mathrm{d}s. \end{split}$$

Using (4.30) for the last term and the constant ρ_0 from (4.29) we have

$$\int_{I_{j,\delta}} \left| (P_j u)(\iota_j)(\rho_j^+)'(s) + (P_j u)(\tau_j)(\rho_j^-)'(s) \right|^2 \mathrm{d}s \leqslant 2\ell\rho_0^2 \int_{\partial_{\mathrm{ext}}W_{j,\delta}} w_j^2 \,\mathrm{d}s$$
$$B'[Ju, Ju] \leqslant (1+\varepsilon) \sum_{j=1}^M \left\| (P_j u)' \right\|^2 + \frac{4\ell\rho_0^2}{\varepsilon} \sum_{j=1}^M \int_{\partial_{\mathrm{ext}}W_{j,\delta}} w_j^2 \,\mathrm{d}s$$

Recall that due to (4.27) we have $B[u, u] \ge \sum_{j=1}^{M} \left\| (P_j u)' \right\|^2$ and

$$B[u,u] \geqslant \frac{\alpha}{2a\log^2 \alpha} \sum_{j=1}^M \int_{\partial_{\text{ext}} V_{j,\delta}} v_j^2 \, \mathrm{d}s \equiv \frac{\alpha}{2a\log^2 \alpha} \sum_{j=1}^M \int_{\partial_{\text{ext}} W_{j,\delta}} w_j^2 \, \mathrm{d}s,$$

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where we have used (4.19). Therefore,

$$B'[Ju, Ju] - B[u, u] \leq \varepsilon \sum_{j=1}^{M} \left\| (P_{j}u)' \right\|^{2} + \frac{4\ell\rho_{0}^{2}}{\varepsilon} \sum_{j=1}^{M} \int_{\partial_{ext}W_{j,\delta}} w_{j}^{2} ds$$
$$\leq \left(\varepsilon + \frac{4\ell\rho_{0}^{2}}{\varepsilon} \cdot \frac{2a\log^{2}\alpha}{\alpha}\right) B[u, u].$$

Therefore, by setting $\varepsilon = \log \alpha / \sqrt{\alpha}$ and by choosing $c_2 > 0$ sufficiently large we arrive at

$$(4.32) \quad B'[Ju, Ju] - B[u, u] \leqslant \frac{c_2 \log \alpha}{\sqrt{\alpha}} B[u, u] \leqslant \frac{c_2 \log \alpha}{\sqrt{\alpha}} \left(B[u, u] + \|u\|^2 \right).$$

With the inequalities (4.31) and (4.32) at hand, we are in the situation of Proposition 2.3 with $\varepsilon_j := c_j \log \alpha / \sqrt{\alpha}, \ j \in \{1, 2\}$. Remark that for each fixed *n* the assumption $\varepsilon_1 < 1/(1+E_n(B))$ is satisfied due to (4.28). Hence, for each fixed *n* we have

$$(4.33) \quad E_n\left(\bigoplus_{j=1}^M D_{j,\delta}\right) \equiv E_n(B')$$
$$\leqslant E_n(B) + \frac{\log\alpha}{\sqrt{\alpha}} \cdot \frac{\left(c_1 E_n(B) + c_2\right)\left(1 + E_n(B)\right)}{1 - c_1\left(1 + E_n(B)\right)\log\alpha/\sqrt{\alpha}}.$$

By (4.28) we have $E_n(B) = \mathcal{O}(1)$ for each fixed *n*, and the substitution into (4.33) gives

$$E_{K+n}(R^{\Omega}_{\alpha}) \ge -\alpha^2 + E_n\left(\bigoplus_{j=1}^M D_{j,\delta}\right) + \mathcal{O}\left(\frac{\log \alpha}{\sqrt{\alpha}}\right),$$

and it remains to note that for fixed n and j one has

$$E_n(D_{j,\delta}) = E_n(D_j) + \mathcal{O}(\delta) = E_n(D_j) + \mathcal{O}\left(\frac{\log \alpha}{\alpha}\right) = E_n(D_j) + o\left(\frac{\log \alpha}{\sqrt{\alpha}}\right),$$

which implies $E_n(\bigoplus_{j=1}^M D_{j,\delta}) = E_n(\bigoplus_{j=1}^M D_j) + \mathcal{O}(\frac{\log \alpha}{\sqrt{\alpha}})$. This concludes the proof of Proposition 4.6.

5. Robin eigenvalues in curvilinear polygons

If one tries to adapt the preceding proof scheme to curvilinear polygons, a number of points require more attention:

- (a) The initial construction of the vertex neighborhood $V_{j,\delta}$ become more technical: the shape of these neighborhoods cannot be chosen at random, as the subsequent analysis need the presence of two straight sides to which the side neighborhoods $W_{j,\delta}$ are glued.
- (b) A suitable control of eigenvalues of Robin–Dirichlet/Neumann Laplacians in $V_{j,\delta}$ is needed. Remark that these neighborhoods are not truncated sectors anymore, but curvilinear polygons. Hence, a suitable analog of the non-resonance condition is needed. This can be achieved using a suitable diffeomorphism between $V_{j,\delta}$ and truncated convex sectors.
- (c) We need an analog of the radial cut-off functions φ_{δ} to prove an analog of Lemma 4.4 for the curvilinear case. The cut-off functions are needed to satisfy the Neumann boundary condition at $\partial\Omega$, in order to ensure that the truncated eigenfunction are still in the domain of the Robin laplacian. This is important for the constructions, as it allows one to apply Proposition 2.5 to estimate the distance between two subspaces.
- (d) The analysis of the Robin laplacians in side neighborhoods $W_{j,\delta}$ become more involved, as a non-trivial curvature contribution appears.
- (e) The lower bound for the eigenvalues in Proposition 4.6 is in part based on the fact that the individual eigenvalues of the operators Band B' are bounded for large α . This is an important point when using Proposition 2.3: if the eigenvalues of B become large, it becomes difficult to satisfy the initial assumption on $\varepsilon_1(1 + E_1(B)) < 1$, see the discussion in Subsection 6.2.

We remark that the points (a)–(c) are purely geometric, and can be of importance for the analysis of other problems in curvilinear polygons. We are not aware of any suitable construction in the literature (due to the very specific shape of the vertex neighborhoods), and we have decided to give a self-contained discussion in Appendix A, to which we refer in the main text. In the present text we were not able to overcome completely the difficulties mentioned under (d) and (e), and we concentrate ourselves on two special but important cases.

5.1. Decomposition of curvilinear polygons

Let us describe more precisely the class of domains Ω we are going to deal with as well as its decomposition into pieces of special shape. Once the

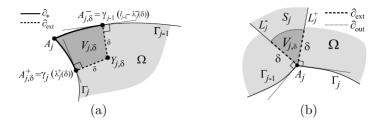


Figure 5.1. The construction of the neighborhoods $V_{j,\delta}$: (a) convex vertex, (b) concave vertex. The partial boundary $\partial_* V_{j,\delta}$ is shown with the thick solid line, the part $\partial_{\text{ext}} V_{j,\delta}$ is indicated with the thick dashed line, and the part $\partial_{\text{out}} V_{j,\delta}$ with the gray dotted line.

geometric justifications has been made, see Appendix A, the latter differs only in minor details from the case of straight polygons. A bounded domain $\Omega \subset \mathbb{R}^2$ will be called a *curvilinear polygon* with $M \ge 1$ vertices if there exist $A_1, \ldots, A_M \in \mathbb{R}^2$ and $\ell_1, \ldots, \ell_M > 0$ such that:

• there are injective C^3 maps $\gamma_j : \mathbb{R} \to \mathbb{R}^2$ with $|\gamma'_j| = 1$ such that

$$\gamma_j(0) = A_j, \quad \gamma_j(\ell_j) = A_{j+1}, \quad j \in \{1, \dots, M\},$$

where we identify $A_0 \equiv A_M$ and $A_{M+1} \equiv A_1$, and the same numbering convention applies to the finite arcs $\Gamma_j := \gamma_j ((0, \ell_j))$ which we assume mutually disjoint and such that $\Gamma := \partial \Omega = \bigcup_{j=1}^M \overline{\Gamma_j}$.

- The orientation of each γ_j is assumed to be chosen in such a way that if $\nu_j(t)$ is the outer unit normal to $\partial\Omega$ at a point $\gamma_j(t)$, then $\nu_j(s) \wedge \gamma'_j(s) = 1$, i.e. $\nu_j(s)$ is obtained by rotating the tangent vector $\gamma'_j(t)$ by $\frac{\pi}{2}$ in the clockwise direction, and the curvature $H_j(t)$ of Γ_j at the point $\gamma_j(s)$ is defined by $\nu'_j(s) = H_j(t) \gamma'_j(s)$.
- By $\theta_j \in [0, \pi]$ we denote the half-angle of the boundary at a vertex A_j , i.e. the number $\theta_j \in [0, \pi]$ is characterized by the conditions

$$\cos(2\theta_j) = \gamma'_{j-1}(\ell_{j-1}) \cdot (-\gamma'_j(0)), \quad \sin(2\theta_j) = \gamma'_j(0) \wedge (-\gamma'_{j-1}(\ell_{j-1})).$$

Our assumption is that there are neither zero angles nor artificial vertices, i.e. $\theta_j \notin \{0, \frac{\pi}{2}, \pi\}$ for $j = 1, \ldots, M$.

The above points $A_j \in \partial \Omega$ will be called the vertices of Ω . Furthermore, one says that A_j is a convex vertex if $\theta_j < \frac{\pi}{2}$ and is a concave one otherwise, and we denote

$$\mathcal{J}_{\mathrm{cvx}} := \{ j : A_j \text{ is convex} \}.$$

We refer to Figure 1.1 in the introduction for an illustration, and in that case one has $\mathcal{J}_{cvx} = \{1, 2\}$. Let us now proceed with a special decomposition

of Ω . For small $\delta > 0$, denote

$$\Omega_{\delta} = \left\{ x \in \Omega : \operatorname{dist}(x, \partial \Omega) < \delta \right\}, \quad \Omega_{\delta}^{c} := \Omega \setminus \overline{\Omega_{\delta}}.$$

We further decompose Ω_{δ} near each vertex as follows:

• Let A_j be a convex vertex. The following constructions are consequences of Lemma A.1 and are illustrated in Figure 5.1 (a). For sufficiently small δ there exists a unique point $Y_{j,\delta} \in \Omega$ such that $\operatorname{dist}(Y_{j,\delta}, \Gamma_{j-1}) = \operatorname{dist}(Y_{j,\delta}, \Gamma_j) = \delta$, and there are uniquely defined numbers $\lambda_i^{\pm}(\delta) > 0$ such that the points

$$A_{j,\delta}^- := \gamma_{j-1} \left(\ell_j - \lambda_j^-(\delta) \right), \quad A_{j,\delta}^+ := \gamma_j \left(\lambda_j^+(\delta) \right)$$

satisfy $|Y_{j,\delta} - A_{j,\delta}^-| = |Y_{j,\delta} - A_{j,\delta}^+| = \delta$. The quantities $\lambda_j^{\pm}(\delta) > 0$ satisfy

(5.1)
$$\lambda_j^{\pm}(\delta) = \delta \operatorname{cotan} \theta_j + \mathcal{O}(\delta^2) \quad \text{for } \delta \to 0^+$$

We denote by $V_{j,\delta}$ the curvilinear quadrangle whose boundary consists of the arcs $\gamma_{j-1}([\ell_j - \lambda_j^-(\delta), \ell_j])$, $\gamma_j([0, \lambda_j^+(\delta)])$ and the segments $A_{j,\delta}^{\pm}Y_{j,\delta}$, and we decompose its boundary into the following parts:

$$\partial_* V_{j,\delta} := \partial V_{j,\delta} \cap \partial \Omega, \quad \partial_{\text{ext}} V_{j,\delta} := \partial V_{j,\delta} \setminus \partial_* V_{j,\delta}, \quad \partial_{\text{out}} V_{j,\delta} := \emptyset.$$

• Let A_j be a concave vertex. The constructions are illustrated in Figure 5.1 (b). Let L_j^- be the half-line emanating from A_j , orthogonal to Γ_{j-1} at A_j and directed inside Ω . By L_j^+ we denote the half-line emanating from A_j , orthogonal to Γ_j at A_j and directed inside Ω . Denote by S_j the infinite sector bounded by L_j^- and L_j^+ which lies inside Ω near A_j . Then we set

$$V_{j,\delta} := S_j \cap B(A_j, \delta), \quad \lambda_j^{\pm}(\delta) := 0,$$

and decompose its boundary as follows:

 $\partial_* V_{j,\delta} := \emptyset, \quad \partial_{\mathrm{out}} V_{j,\delta} := \partial V_{j,\delta} \cap \partial \Omega^c_{\delta}, \quad \partial_{\mathrm{ext}} V_{j,\delta} := \partial V_{j,\delta} \setminus \partial_{\mathrm{out}} V_{j,\delta}.$

The "length deficiency" $\lambda_j^+(\delta) + \lambda_j^-(\delta)$ is exactly the length of $\partial_* V_{j,\delta}$ for both convex and concave vertices.

The set $W_{\delta} := \Omega_{\delta} \setminus \overline{\bigcup_{j=1}^{M} V_{j,\delta}}$ is the union of M disjoint curvilinear rectangles: if one denotes

$$I_{j,\delta} := \left(\lambda_j^+(\delta), \ell_j - \lambda_{j+1}^-(\delta)\right), \quad \Pi_{j,\delta} := I_{j,\delta} \times (0,\delta),$$
$$W_{j,\delta} := \Phi_j(\Pi_{j,\delta}), \quad \Phi_j(s,t) := \gamma_j(s) - t\nu_j(s),$$

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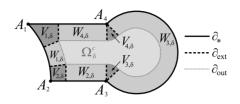


Figure 5.2. Decomposition of a curvilinear polygon.

then $W_{\delta} = \bigcup_{j=1}^{M} W_{j,\delta}$. We decompose the boundary of each $W_{j,\delta}$ as follows:

$$\begin{aligned} \partial_* W_{j,\delta} &:= \partial W_{j,\delta} \cap \partial \Omega, \quad \partial_{\text{out}} W_{j,\delta} &:= \partial W_{j,\delta} \cap \partial \Omega_{\delta}^c \\ \partial_{\text{ext}} W_{j,\delta} &:= \partial W_{j,\delta} \setminus \left(\partial_* W_{j,\delta} \cup \partial_{\text{out}} W_{j,\delta} \right). \end{aligned}$$

The resulting decomposition of Ω is illustrated in Figure 5.2, and we always have

(5.2)
$$\bigcup_{j=1}^{M} \partial_{\text{ext}} V_{j,\delta} = \bigcup_{j=1}^{M} \partial_{\text{ext}} W_{j,\delta}.$$

5.2. Some estimates for curvilinear neighborhoods

With each $j \in \{1, \ldots, M\}$ we associate the corresponding number $\kappa(\theta_j)$ of discrete eigenvalues of the Robin Laplacians in the infinite sector of aperture $2\theta_i$ (see Section 2.6) and set

$$K := \kappa(\theta_1) + \dots + \kappa(\theta_M) \equiv \sum_{j \in \mathcal{J}_{\text{cvx}}} \kappa(\theta_j),$$

 $\mathcal{E} :=$ the disjoint union of $\{\mathcal{E}_n(\theta_j), n = 1, \dots, \kappa(\theta_j)\}$ for $j \in \mathcal{J}_{cvx}$,

 $\mathcal{E}_n :=$ the $n \mathrm{th}$ element of $\mathcal E$ when numbered in the non-decreasing order,

(see Subsection 2.6 for a detailed notation). For what follows we introduce several operators:

- $N_j^V :=$ the Laplacian in $V_{j,\delta}$ with α -Robin condition at $\partial_* V_{j,\delta}$ and Neumann condition at the rest of the boundary,
- $D_j^V :=$ the Laplacian in $V_{j,\delta}$ with α -Robin condition at $\partial_* V_{j,\delta}$ and Dirichlet condition at the rest of the boundary.
- $R_j^V :=$ the Laplacian in $V_{j,\delta}$ with α -Robin condition at the whole boundary.

We remark that for concave vertices A_j , the respective operators $(N/D)_j^V$ are just the Neumann/Dirichlet Laplacians in $V_{j,\delta}$ due to $\partial_* V_{j,\delta} = \emptyset$. Furthermore, denote

- $N_j^W :=$ the Laplacian in $W_{j,\delta}$ with α -Robin condition at $\partial_* W_{j,\delta}$ and Neumann condition at the rest of the boundary,
- $D_j^W :=$ the Laplacian in $W_{j,\delta}$ with α -Robin condition at $\partial_* W_{j,\delta}$ and Dirichlet condition at the rest of the boundary,

Finally, introduce

 $N_c :=$ the Neumann Laplacian in Ω^c_{δ} .

Due to Lemma A.3 (Appendix A), for each $j \in \mathcal{J}_{cvx}$ one can find a bi-Lipschitz map Ψ_j between a neighborhood of the origin and a neighborhood of A_j , a rotation Z_j with $\Psi'_j(x) = Z_j + \mathcal{O}(|x|)$ for $x \to 0$ and a C^2 smooth function r_j defined near 0 with r(0) = 0 and $r'(0) = \cot a \theta_j$ such that for all sufficiently small $\delta > 0$ one has

$$\Psi_j(\mathcal{S}_{\theta_j}^{r(\delta)}) = V_{j,t}, \quad \Psi_j(\partial_*\mathcal{S}_{\theta_j}^{r(\delta)}) = \partial_*V_{j,\delta}, \quad \Phi_j(\partial_{\mathrm{ext}}\mathcal{S}_{\theta_j}^{r(\delta)}) = \partial_{\mathrm{ext}}V_{j,\delta}$$

Hence, for $u \in H^1(V_{j,\delta})$ one can use $u \circ \Psi_j \in H^1(\mathcal{S}^{r(\delta)}_{\theta_j})$ as test functions in truncated sectors, which implies in the standard way, see e.g. [10, Lemma 3.3], the following estimates for the eigenvalues of N_i^V, D_j^V and R_i^V :

LEMMA 5.1. — There exist a > 0, $a_0 > 0$, $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, $\alpha > 0$, $n \in \mathbb{N}$, $j \in \mathcal{J}_{cvx}$ there holds

$$(1-a_0\delta)E_n(N_{\theta_j,\alpha(1+a\delta)}^{r(\delta)}) \leqslant E_n(N_j^V) \leqslant (1+a_0\delta)E_n(N_{\theta,\alpha(1-a\delta)}^{r(\delta)}),$$

$$(1-a_0\delta)E_n(D_{\theta_j,\alpha(1+a\delta)}^{r(\delta)}) \leqslant E_n(D_j^V) \leqslant (1+a_0\delta)E_n(D_{\theta,\alpha(1-a\delta)}^{r(\delta)}),$$

$$(1-a_0\delta)E_n(R_{\theta_j,\alpha(1+a\delta)}^{r(\delta)}) \leqslant E_n(R_j^V) \leqslant (1+a_0\delta)E_n(R_{\theta,\alpha(1-a\delta)}^{r(\delta)}).$$

Here we recall that the operators $(D/N/R)_{\theta,\alpha}^r$ in truncated sectors S_{θ}^r were defined in (3.1). Using the estimates of Lemmas 3.3 and 3.6 for the eigenvalues in truncated sectors and by literally repeating the proof to obtain an Agmon estimate in Lemma 3.8, we arrive then at first estimates for $(D/N/R)_i^V$:

COROLLARY 5.2. — There is b > 0 such that for $\delta \to 0^+$ and $\alpha \delta \to +\infty$ one has

$$E_n(D_j^V) = \mathcal{E}_n(\theta_j) \,\alpha^2 + \mathcal{O}(\alpha^2 \delta + \alpha^2 e^{-b\alpha\delta}), \qquad n \in \{1, \dots, \kappa(\theta)\},\$$

$$E_n(N_j^V) = \mathcal{E}_n(\theta_j) \,\alpha^2 + \mathcal{O}(\alpha^2 \delta + 1/\delta^2), \qquad n \in \{1, \dots, \kappa(\theta)\},\$$

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and $E_{\kappa(\theta_j)+1}(D_j^V) \ge E_{\kappa(\theta_j)+1}(N_j^V) \ge -\alpha^2 + o(\alpha^2)$. In addition, there exist c > 0 and C > 0 such that if $n \in \{1, \ldots, \kappa(\theta_j)\}$ and ψ_n is an eigenfunction of N_j^V for the *n*th eigenvalue, then for $\delta \to 0^+$ and $\alpha \delta \to +\infty$ there holds

$$\int_{V_{j,\delta}} e^{c\alpha|x|} \left(\frac{1}{\alpha^2} \left| \nabla \psi_n(x) \right|^2 + \left| \psi_n(x) \right|^2 \right) \, \mathrm{d}x \leqslant C \, \|\psi_n\|_{L^2(V_{j,\delta})}^2.$$

There exists a > 0 such that $E_1(R_j^V) \ge -a\alpha^2$ for $\delta \to 0^+$ and $\alpha \delta \to +\infty$.

Like in the case of straight polygons, we introduce the following sub-spaces:

- $$\begin{split} L := \text{the subspace of } L^2(\Omega) \text{ spanned by the first } K \\ \text{eigenfunctions of } R^\Omega_\alpha, \end{split}$$
- $L_j := \text{the subspace of } L^2(V_{j,\delta}) \text{ spanned by the first } \kappa(\theta_j)$ eigenfunctions of N_j^V , with $j \in \mathcal{J}_{\text{cvx}}$,

$$\sigma_j: L^2(\Omega) \to L^2(V_{j,\delta}) \text{ the operator of restriction,}$$
$$(\sigma_j u)(x) = u(x) \text{ for } x \in V_{j,\delta},$$

then the adjoint operators $\sigma_j^* : L^2(V_{j,\delta}) \to L^2(\Omega)$ are the operators of extension by zero. The following distance estimate will again be of importance:

LEMMA 5.3. — Let $j \in \mathcal{J}_{cvx}$, then in the limit $\delta \to 0^+$ and $\alpha \delta \to +\infty$ there holds $d(\sigma_j^*L_j, L) = \mathcal{O}(e^{-c\alpha\delta})$ with some fixed c > 0.

Proof. — Pick 0 < a < b < 1, then due to Lemma A.5 one can find smooth cut-off functions $\varphi_{\delta} \in C^2(\overline{\Omega})$ with the following properties:

- $0 \leq \varphi_{\delta} \leq 1$, and for all $\beta \in \mathbb{N}^2$ with $1 \leq |\beta| \leq 2$ there holds $\|\partial^{\beta}\varphi_{\delta}\|_{\infty} \leq C\delta^{-|\beta|}$,
- $\varphi_{\delta} = 1$ in $V_{j,a\delta}$, and $\varphi_{\delta} = 0$ in $\Omega \setminus \overline{V_{j,b\delta}}$,
- the normal derivative of φ_{δ} at $\partial \Omega$ is zero,

where C > 0 is some fixed constant. By Corollary A.4 one can find $a_0 > 0$ such that $|x - A_j| > a_0 \delta$ for $x \in V_{j,\delta} \setminus \overline{V_{j,a\delta}}$. As the normal derivative of φ_{δ} at $\partial \Omega$ is zero, it follows that for any $v \in \mathcal{D}(N_j^V)$ we have then $\varphi_{\delta} v \in \mathcal{D}(R_{\alpha}^{\Omega})$, and the proof works literally as for the straight case (Lemma 4.4), as all other necessary components are contained in Corollary 5.2.

Let us now give some first estimates for the eigenvalues of N_j^W . Recall that H_j stands for the curvature of the *j*th side of Ω . We denote

(5.3)
$$H_{j,*} := \max_{s \in [0,\ell_j]} H_j(s), \quad H_* := \max_{j \in \{1,\dots,M\}} H_{j,*}, \quad \mathcal{J}_* := \{j : H_{j,*} = H_*\}.$$

As in the case of straight polygons, let us consider the one-dimensional operators

$$D_j := \text{the Dirichlet Laplacian on } (0, \ell_j),$$

$$D_{j,\delta} := \text{the Dirichlet Laplacian on } I_{j,\delta},$$

$$N_j := \text{the Neumann Laplacian on } (0, \ell_j),$$

$$N_{j,\delta} := \text{the Neumann Laplacian on } I_{j,\delta}.$$

A literal repetition of the constructions of [57, Section 6] gives the following result:

PROPOSITION 5.4. — For any $j \in \{1, \ldots, M\}$, $n \in \mathbb{N}$ and $\delta := \alpha^{-\kappa}$ with $\kappa \in [\frac{2}{3}, 1]$, for $\alpha \to +\infty$ there holds

(5.4)
$$E_n(N_j^W) = -\alpha^2 + E_n(N_{j,\delta} - \alpha H_j) + \mathcal{O}(1),$$
$$E_n(D_j^W) = -\alpha^2 + E_n(D_{j,\delta} - \alpha H_j) + \mathcal{O}(1).$$

In particular,

(5.5)
$$E_n(N_{j,\delta} - \alpha H_j) = -H_{j,*}\alpha + o(\alpha),$$
$$E_n(D_{j,\delta} - \alpha H_j) = -H_{j,*}\alpha + o(\alpha),$$

(5.6)
$$E_n(N_j^W) = -\alpha^2 - H_{j,*}\alpha + o(\alpha),$$
$$E_n(D_i^W) = -\alpha^2 - H_{j,*}\alpha + o(\alpha).$$

The preceding estimates work for all half-angles angle θ_j . For the rest of the section we assume that

(5.7) θ_j is non-resonant for all $j \in \mathcal{J}_{cvx}$.

5.3. Estimates for non-resonant convex sectors

Let us pick any $j \in \mathcal{J}_{cvx}$, which is then non-resonant by assumption. The estimates for N_j^V given in this subsection will be of crucial importance for the subsequent analysis. They slightly differ from the respective estimates for the straight case (Subsection 3.3), as some more parameters will be needed later. For the rest of the section we assume that δ is chosen depending on α such that

(5.8)
$$\alpha\delta \to +\infty, \quad \delta \to 0^+, \quad \alpha^2\delta^3 \to 0^+ \quad \text{as } \alpha \to +\infty.$$

An exact choice of δ will be made at a later stage.

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COROLLARY 5.5. — For any $A \in \mathbb{R}$ there exists c > 0 such that under the assumption (5.8) there holds $E_{\kappa(\theta_j)+1}(D_j^V) \ge E_{\kappa(\theta_j)+1}(N_j^V) \ge -\alpha^2 + A\alpha + c/\delta^2$.

Proof. — By Lemma 5.1 one has $E_n(N_j^V) \ge (1 - a_0 \delta) E_n(N_{\theta_j,\alpha(1+a\delta)}^{r(\delta)})$. As θ_j is non-resonant, with some C > 0 we have $E_{\kappa(\theta_j)+1}(N_{\theta_j,\alpha}^r) \ge -\alpha^2 + C/r^2$ as αr is large. In the asymptotic regime under consideration we have $\alpha(1+a\delta)r_j(\delta) \sim \alpha\delta \operatorname{cotan} \theta_j \to +\infty$, hence,

$$E_n(N_{\theta_j,\alpha(1+a\delta)}^{r(\delta)}) \ge -\alpha^2(1+a\delta)^2 + \frac{C}{r(\delta)^2}$$
$$\ge -\alpha^2 - 3a\alpha^2\delta + \frac{C_0}{\delta^2}, \quad C_0 := \frac{C\tan^2\theta_j}{2},$$
$$(1-a_0\delta)E_n(N_{\theta_j,\alpha(1+a\delta)}^{r(\delta)}) \ge (1-a_0\delta)\left(-\alpha^2 - 3a\alpha^2\delta + \frac{C_0}{\delta^2}\right)$$
$$\ge -\alpha^2 - 3a\alpha^2\delta + \frac{C_0}{\delta^2} - \frac{a_0C_0}{\delta}$$
$$\ge -\alpha^2 - 3a\alpha^2\delta + \frac{C_0}{2\delta^2}$$
$$= -\alpha^2 + A\alpha + \frac{1}{\delta^2}\left(\frac{1}{2}C_0 - 3a\alpha^2\delta^3 - A\alpha\delta^2\right)$$

For $\alpha^2 \delta^3 \to 0^+$ one has $\alpha \delta^2 = \alpha^2 \delta^3/(\alpha \delta) \to 0^+$, and for any fixed $c \in (0, C_0/2)$ there holds

$$E_{\kappa(\theta_j)+1}(N_j^V) \ge -\alpha^2 + A\alpha + c/\delta^2.$$

The inequality $E_{\kappa(\theta_j)+1}(D_j^V) \ge E_{\kappa(\theta_j)+1}(N_j^V)$ follows from the min-max principle. \Box

Proceeding almost literally as in the straight case (Lemma 3.11) one puts the preceding assertion into the following special form:

COROLLARY 5.6. — For any $A \in \mathbb{R}$ there exists b > 0 such that under the assumption (5.8) there following inequalities holds for any $u \in$ $H^1(V_{j,\delta}) \cap L^{\perp}_j$:

(5.9)
$$\|v\|_{L^{2}(V_{j,\delta})}^{2} \leq b \,\delta^{2} \Big(N_{j}^{V}[v,v] + (\alpha^{2} - A\alpha) \|v\|_{L^{2}(V_{j,\delta})}^{2} \Big),$$

(5.10)
$$\int_{\partial_{\text{ext}}V_{\delta}} v^2 \,\mathrm{d}s \leqslant b\alpha \delta^2 \Big(N_j^V[v,v] + (\alpha^2 - A\alpha) \|v\|_{L^2(V_{j,\delta})}^2 \Big).$$

By combining the eigenvalues and eigenfunction estimates of Corollary 5.6 with the distance estimate of Lemma 5.3 like in the proof of Lemma 4.5 we arrive then to the following estimate:

LEMMA 5.7. — For any $A \in \mathbb{R}$ one can find b > 0 and c > 0 such that under the assumption (5.8) there holds, for any $j = 1, \ldots, M$, and any $u \in H^1(\Omega)$ with $u \perp L$,

$$\begin{aligned} \|\sigma_{j}u\|_{L^{2}(V_{j,\delta})}^{2} \leqslant b\delta^{2} \Big(N_{j}^{V}[\sigma_{j}u,\sigma_{j}u] + (\alpha^{2} - A\alpha)\|\sigma_{j}u\|_{L^{2}(V_{j,\delta})}^{2}\Big) \\ &+ b\alpha^{2}\delta^{2}e^{-c\alpha\delta}\|u\|_{L^{2}(\Omega)}^{2}, \\ \int_{\partial_{\text{ext}}V_{j,\delta}} (\sigma_{j}u)^{2} \,\mathrm{d}s \leqslant b\alpha\delta^{2} \Big(N_{j}^{V}[\sigma_{j}u,\sigma_{j}u] + (\alpha^{2} - A\alpha)\|\sigma_{j}u\|_{L^{2}(V_{j,\delta})}^{2}\Big) \\ &+ b\alpha^{3}\delta^{2}e^{-c\alpha\delta}\|u\|_{L^{2}(\Omega)}^{2}. \end{aligned}$$

We are now able to obtain an analog of Proposition 4.2 for curvilinear polygons.

COROLLARY 5.8. — For any fixed $n \in \mathbb{N}$ and $\alpha \to +\infty$ there holds

(5.11)
$$E_n\left(\bigoplus_{j=1}^M N_j^W\right) \leqslant E_{K+n}(R_\alpha^\Omega) \leqslant E_n\left(\bigoplus_{j=1}^M D_j^W\right),$$

in particular,

$$-\alpha^{2} + E_{n}\left(\bigoplus_{j\in\mathcal{J}_{*}}(N_{j,\delta} - \alpha H_{j})\right) + \mathcal{O}(1) \leqslant E_{K+n}(R_{\alpha}^{\Omega})$$
$$\leqslant -\alpha^{2} + E_{n}\left(\bigoplus_{j\in\mathcal{J}_{*}}(D_{j,\delta} - \alpha H_{j})\right) + \mathcal{O}(1).$$

Proof. — The standard Dirichlet–Neumann bracketing gives

$$(5.12) \quad E_n\left(N^c \oplus \left(\bigoplus_{j=1}^M N_j^V\right) \oplus \left(\bigoplus_{j=1}^M N_j^W\right)\right) \leqslant E_n(R_\alpha^\Omega)$$
$$\leqslant E_n\left(\left(\bigoplus_{j=1}^M D_j^V\right) \oplus \left(\bigoplus_{j=1}^M D_j^W\right)\right).$$

In the asymptotic regime (5.8), thanks to Corollary 5.2, we have

$$E_n\left(\bigoplus_{j=1}^M D_j^V\right) = \mathcal{E}_n \alpha^2 + o(\alpha^2),$$
$$E_n\left(\bigoplus_{j=1}^M N_j^V\right) = \mathcal{E}_n \alpha^2 + o(\alpha^2) \text{ for } n \in \{1, \dots, K\},$$

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while for any A > 0 and some c > 0 one has, by Corollary 5.5,

$$E_{K+1}\left(\bigoplus_{j=1}^{M} D_{j}^{V}\right) \ge E_{K+1}\left(\bigoplus_{j=1}^{M} N_{j}^{V}\right) \ge -\alpha^{2} + A\alpha + c/\delta^{2}$$

By Proposition 5.4, for each $n \in \mathbb{N}$ we have $E_n(D_j^W) = -\alpha^2 + \mathcal{O}(\alpha)$ and $E_n(N_j^W) = -\alpha^2 + \mathcal{O}(\alpha)$, hence

$$E_{K}\left(\bigoplus_{j=1}^{M} N_{j}^{V}\right) \leqslant E_{n}(N_{j}^{W}) = -\alpha^{2} + \mathcal{O}(\alpha) \leqslant E_{K+1}\left(\bigoplus_{j=1}^{M} N_{j}^{V}\right),$$
$$E_{K}\left(\bigoplus_{j=1}^{M} D_{j}^{V}\right) \leqslant E_{n}(D_{j}^{W}) = -\alpha^{2} + \mathcal{O}(\alpha) \leqslant E_{K+1}\left(\bigoplus_{j=1}^{M} D_{j}^{V}\right).$$

It follows that for any $n \in \mathbb{N}$ one has

$$E_{K+n}\left(\left(\bigoplus_{j=1}^{M} D_{j}^{V}\right) \oplus \left(\bigoplus_{j=1}^{M} D_{j}^{W}\right)\right) = E_{n}\left(\bigoplus_{j=1}^{M} D_{j}^{W}\right),$$
$$E_{K+n}\left(N^{c} \oplus \left(\bigoplus_{j=1}^{M} N_{j}^{V}\right) \oplus \left(\bigoplus_{j=1}^{M} N_{j}^{W}\right)\right) = E_{n}\left(\bigoplus_{j=1}^{M} N_{j}^{W}\right),$$

and (5.12) implies (5.11). Now it is sufficient to apply Proposition 5.4 to each of the operators in the direct sums. In particular, due to (5.5) only $j \in \mathcal{J}_*$ contribute to the asymptotics of the individual eigenvalues.

5.4. Curvatures taking their maxima away from corners: Proof of Theorem 1.4

Remark that the estimate of Corollary 5.8 only gives a rough asymptotics in general, as there is a discrepancy between the lower and upper bounds due to the different boundary conditions (Neumann/Dirichlet). In the particular case of constant curvatures one obtains the same asymptotics for each individual eigenvalue. We remark nevertheless that the discrepancy can be very small under suitable geometric assumptions. The considerations of the present subsection rely on some general and well-known ideas of the semiclassical analysis (see e.g. Helffer's monograph [29]), so we only give a sketch of the proofs.

Namely, in the present subsection we consider the case when the maximum curvature H_* is not attained at any corner:

(5.13) for all $j \in \{1, \ldots, M\}$ there holds $H_j(0) \neq H_*$ and $H_j(\ell_j) \neq H_j$.

The analysis of both $E_n(N_{j,\delta} - \alpha H_j)$ and $E_n(D_{j,\delta} - \alpha H_j)$ is covered by the standard framework of the semiclassical analysis: the eigenfunctions for the lowest eigenvalues are exponentially localized near the set at which H_j takes its maximum value, and the boundary conditions only influence the eigenvalue asymptotics in exponentially small terms, see [29, Section 3]. One obtains then the following assertion:

PROPOSITION 5.9. — Under the assumption (5.13), there exists c > 0 such that for each $j \in \mathcal{J}_*$ and each $n \in \mathbb{N}$ the following estimates hold in the asymptotic regime (5.8):

$$E_n(D_{j,\delta} - \alpha H_j) = E_n(D_j - \alpha H_j) + \mathcal{O}(e^{-c\sqrt{\alpha}}),$$
$$E_n(N_{j,\delta} - \alpha H_j) = E_n(N_j - \alpha H_j) + \mathcal{O}(e^{-c\sqrt{\alpha}}),$$
$$E_n(D_j - \alpha H_j) - E_n(N_j - \alpha H_j) = \mathcal{O}(e^{-c\sqrt{\alpha}}).$$

By combining Corollary 5.8 with Proposition 5.9 one obtains then the following main result:

PROPOSITION 5.10. — Let Ω be a curvilinear polygon whose half-angles satisfy (5.7) and the side curvatures satisfy (5.13). Then for any fixed $n \in \mathbb{N}$ one has

$$E_{K+n}(R^{\Omega}_{\alpha}) = -\alpha^{2} + E_{n}\left(\bigoplus_{j\in\mathcal{J}_{*}}(D_{j}-\alpha H_{j})\right) + \mathcal{O}(1)$$
$$= -\alpha^{2} + E_{n}\left(\bigoplus_{j\in\mathcal{J}_{*}}(N_{j}-\alpha H_{j})\right) + \mathcal{O}(1).$$

We finally note that the analysis can be made more precise under additional geometric assumptions. In particular, the construction of Helffer– Kachmar [30] can be easily adapted in order to obtain the following result:

PROPOSITION 5.11. — Let Ω be a curvilinear polygon whose half-angles satisfy (5.7). Assume that there exists a unique $k \in \{1, \ldots, M\}$ and a unique $s_* \in (0, \ell_k)$ such that $H_k(s_*) = H_*$ and $h_* := -H''_j(s_*) > 0$ and that γ_k is C^{∞} in a neighborhood of s_* , then, for any $n \in \mathbb{N}$ there exists a sequence $(\beta_{i,n})_{i\geq 0}$ such that for any $m \in \mathbb{N}$ there holds

$$E_{K+n}(R^{\Omega}_{\alpha}) = -\alpha^2 - H_*\alpha + (2n-1)\sqrt{\frac{h_*}{2}}\sqrt{\alpha} + \sum_{i=0}^m \beta_{i,n}\alpha^{-\frac{i}{2}} + o(\alpha^{-\frac{m}{2}}).$$

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For the proof one remarks first that due to (5.6) one has

$$E_n\left(\bigoplus_{j=1}^M N_j^W\right) = E_n(N_k^W), \quad E_n\left(\bigoplus_{j=1}^M D_j^W\right) = E_n(D_k^W),$$

and then $E_n(N_k^W) \leq E_{K+n}(R_{\alpha}^{\Omega}) \leq E_n(D_k^W)$ by (5.11). The eigenvalues of N_k^W and D_k^W are then analyzed literally as in [30, Theorem 1.1], since all the analysis is done in a small neighborhood of $\gamma_k(s_*)$. Remark that the first three terms can also be deduced directly from Proposition 5.10 by applying the standard WKB analysis to the 1D operators $N_k - \alpha H_k$ and $D_k - \alpha H_k$, [16, Section 3].

5.5. Constant curvatures: Proof of Theorem 1.5

The main assumption in the present subsection is as follows:

(5.14) each function H_j is constant, and $H_* := \max_{j=1}^{M} H_j$,

and we recall that the we still assume the non-resonance condition (5.7), and that in the beginning of the section we introduced the diffeomorphisms Φ_j by $\Phi_j(s,t) := \gamma_j(s) - t\nu_j(s)$ and the open sets

$$W_{j,\delta} := \Phi_j(\Pi_{j,\delta}), \quad \Pi_{j,\delta} = I_{j,\delta} \times (0,\delta).$$

We will need some constructions in curvilinear coordinates in $W_{j,\delta}$. In order to avoid the use of special functions we prefer to use tubular coordinates instead of polar coordinates. The following lemma is obtained by direct computations using the standard change of variables.

LEMMA 5.12. — Consider the unitary transform

(5.15) $G_j: L^2(W_{j,\delta}) \to L^2(\Pi_{j,\delta}), \quad (G_j u)(s,t) = (1 - tH_j)^{\frac{1}{2}} u (\Phi_j(s,t)),$

then $u \in H^1(W_{j,\delta})$ if and only if $g := G_j u \in H^1(\Pi_{j,\delta})$, and there exists b > 0 such that for sufficiently small δ , all u and g as above and all $\alpha > 0$ one has the two-sided estimate

$$\begin{split} B_{-}[g,g] &\leqslant \int_{W_{j,\delta}} |\nabla u|^2 \, \mathrm{d}x - \alpha \int_{\partial_* W_{j,\delta}} u^2 \, \mathrm{d}s \leqslant B_{+}[g,g], \\ B_{\pm}[g,g] &:= \int_{I_{j,\delta}} \int_0^{\delta} \left[\left(1 \pm b\delta\right) \left(\frac{\partial g}{\partial s}\right)^2 + \left(\frac{\partial g}{\partial t}\right)^2 - \left(\frac{H_j^2}{4} \mp b\delta\right) g^2 \right] \mathrm{d}t \, \mathrm{d}s \\ &- \int_{I_{j,\delta}} \left(\alpha + \frac{H_j}{2}\right) g(s,0)^2 \, \mathrm{d}s \pm b \int_{I_{j,\delta}} g(s,\delta)^2 \, \mathrm{d}s \end{split}$$

The above change of variables will be now used for some constructions involving $(D/N)_i^W$. We start with an eigenvalue estimate for D_i^W :

LEMMA 5.13. — One can find b > 0 such that for $\delta \to 0^+$ and $\alpha \delta \to +\infty$ there holds

$$E_n(D_j^W) \leqslant -\alpha^2 - \alpha H_j - \frac{H_j^2}{2} + (1 + b\delta)E_n(D_j) + b(\delta + \alpha^2 e^{-\alpha\delta}), \quad n \in \mathbb{N}.$$

Proof. — Due to Lemma 5.12, for some $b_0 > 0$ one has $E_n(D^W) \leq E_n(B_+)$, where B_+ is the self-adjoint operator in $H^1(\Pi_{j,\delta})$ with

$$\mathcal{Q}(B_{+}) = \left\{ g \in H^{1}\left(I_{j,\delta} \times (0,\delta)\right) : \frac{g(\cdot,\delta) = 0}{g(\iota,\cdot)} = 0 \text{ for each } \iota \in \partial I_{j,\delta} \right\},\$$
$$B_{+}[g,g] = \int_{I_{j,\delta}} \int_{0}^{\delta} \left[\left(1 + b_{0}\delta\right)(g_{s})^{2} + (g_{t})^{2} - \left(\frac{H_{j}^{2}}{4} - b_{0}\delta\right)g^{2} \right] \mathrm{d}t \,\mathrm{d}s$$
$$- \int_{I_{j,\delta}} \left(\alpha + \frac{H_{j}}{2}\right)g(s,0)^{2} \,\mathrm{d}s,$$

where we have set $g_s := \partial g/\partial s$ and $g_t := \partial g/\partial t$. As H_j is constant, the operator B_+ admits a separation of variables and is unitarily equivalent to

$$C_{+} := (1 + b_0 \delta) D_{j,\delta} \otimes 1 + 1 \otimes L_D - (H_j^2 / 4 - b_0 \delta),$$

where L_D is the Laplacian on $(0, \delta)$ with $(\alpha + H_j/2)$ -Robin condition at 0 and Dirichlet condition at δ , so using Proposition 2.8 with some $b_1 > 0$ we have $E_1(L_D) = -(\alpha + H_j/2)^2 + b_1 \alpha^2 e^{-\alpha\delta}$ and $E_2(L_D) \ge 0$. Therefore, for each fixed $n \in \mathbb{N}$ due to $E_n(D_{j,\delta}) = \mathcal{O}(1)$ we have

$$E_n(D_j^W) \leq E_n(B_+) = E_n(C_+)$$

= $E_1(L_D) + (1 + b_0\delta)E_n(D_{j,\delta}) - \left(\frac{H_j^2}{4} - b_0\delta\right),$
 $\leq -\alpha^2 - \alpha H_j - \frac{H_j^2}{2} + (1 + b_0\delta)E_n(D_{j,\delta}) + b_0\delta + b_1\alpha^2 e^{\alpha\delta}.$

One arrives at the sought result by taking $b := \max\{b_0, b_1\}$ and by noting that $E_n(D_{j,\delta}) = E_n(D_j) + \mathcal{O}(\delta)$ for any fixed $n \in \mathbb{N}$.

By applying the estimate of Lemma 5.13 to each operator in the righthand side of (5.11) with $\delta := (3 \log \alpha)/\alpha$ so that (5.8) is satisfied, we arrive to an improved upper bound for $E_{K+n}(R_{\alpha}^{\Omega})$:

PROPOSITION 5.14. — Under the assumptions (5.7) and (5.14), for any fixed $n \in \mathbb{N}$ there holds

$$E_{K+n}(R^{\Omega}_{\alpha}) \leqslant -\alpha^2 - H_*\alpha - \frac{H_*^2}{2} + E_n\left(\bigoplus_{j\in\mathcal{J}_*} D_j\right) + \mathcal{O}\left(\frac{\log\alpha}{\alpha}\right) \quad \text{as } \alpha \to +\infty.$$

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Now we will bring the Robin–Neumann Laplacian N_j^W to a special form to use during subsequent proofs:

LEMMA 5.15. — Let G_j be defined by (5.15). There are functions $\psi_j \in L^2(0,\delta)$ with $\|\psi\|_{L^2(0,\delta)}^2 = 1$ such that if one defines the map

(5.16)
$$P_j: L^2(W_{j,\delta}) \to L^2(I_{j,\delta}), \quad (P_j u)(s) := \int_0^\delta \psi_j(t)(G_j u)(s,t) \, \mathrm{d}t,$$

then one has for $\delta \to 0^+$ and $\alpha \delta \to +\infty$, with some b > 0,

(5.17)
$$N_{j}^{W}[u, u] \ge -\left(\alpha^{2} + \alpha H_{j} + \frac{H_{j}^{2}}{2}\right) \|P_{j}u\|_{L^{2}(I_{j,\delta})}^{2} + \left(1 - b\delta\right) \|(P_{j}u)'\|_{L^{2}(I_{j,\delta})}^{2} - b\left(\delta + \alpha^{2}e^{-\alpha\delta}\right) \|P_{j}u\|_{L^{2}(I_{j,\delta})}^{2}$$

for all $u \in H^{1}(W_{j,\delta})$.

In particular,

(5.18)
$$E_1(N_j^W) \ge -\left(\alpha^2 + \alpha H_j + \frac{H_j^2}{2}\right) + \mathcal{O}(\delta + \alpha^2 e^{-c\alpha\delta}).$$

Proof. — We drop the index j in the notation. Denote $g := Gu \in L^2(\Pi_{\delta})$, then due to the standard change of variables (Lemma 5.12) one can find $b_0 > 0$ and $\beta > 0$ to have, for all $u \in H^1(W_{\delta})$,

$$\begin{split} N^{W}[u,u] \geqslant B_{-}[g,g] \\ &:= \int_{I_{\delta}} \int_{0}^{\delta} \left[\left(1 - b_{0}\delta\right)g_{s}^{2} + g_{t}^{2} - \left(\frac{H^{2}}{4} + b_{0}\delta\right)g^{2} \right] \mathrm{d}t \,\mathrm{d}s \\ &- \int_{I_{\delta}} \left(\alpha + \frac{H}{2}\right)g(s,0)^{2} \,\mathrm{d}s - \beta \int_{I_{\delta}} g(s,\delta)^{2} \,\mathrm{d}s, \end{split}$$

Denote by L_N the one-dimensional Laplacian in $(0, \delta)$ with the $(\alpha + H/2)$ -Robin boundary condition at 0 and the β -Robin boundary condition at δ , and let ψ be its eigenfunction for the first eigenvalue, normalized by $\|\psi\|_{L^2(0,\delta)} = 1$. With this choice of ψ , define the map P as in (5.16). For shortness we denote f := Pu and define $z \in L^2(\Pi_{\delta})$ by z(s,t) := $g(s,t) - f(s)\psi(t)$, then, with $z_s := \partial z/\partial s$, we have the identities

(5.19)
$$\int_{0}^{\delta} \psi(t) z(\cdot, t) \, \mathrm{d}t = 0, \quad \int_{0}^{\delta} \psi(t) z_{s}(\cdot, t) \, \mathrm{d}t = 0, \\ \|u\|_{L^{2}(W_{\delta})}^{2} = \|g\|_{L^{2}(\Pi_{\delta})}^{2} = \|f\|_{L^{2}(I_{\delta})}^{2} + \|z\|_{L^{2}(\Pi_{\delta})}^{2},$$

and due to the spectral theorem for the operator L_N there holds

$$\begin{split} \int_{I_{\delta}} \int_{0}^{\delta} g_{t}^{2} \, \mathrm{d}t \, \mathrm{d}s &- \int_{I_{\delta}} \left(\alpha + \frac{H}{2} \right) g(s,0)^{2} \, \mathrm{d}s - \beta \int_{I_{\delta}} g(s,\delta)^{2} \, \mathrm{d}s \\ &\geqslant \int_{I_{\delta}} \int_{0}^{\delta} \left(E_{1}(L_{N}) f(s)^{2} \psi(t)^{2} + E_{2}(L_{N}) z(s,t)^{2} \right) \, \mathrm{d}t \, \mathrm{d}s \\ &= E_{1}(L_{N}) \|f\|_{L^{2}(I_{\delta})}^{2} + E_{2}(L_{N}) \|z\|_{L^{2}(\Pi_{\delta})}^{2}; \end{split}$$

an using the second equality in (5.19) we also have

$$\int_{I_{\delta}} \int_{0}^{\delta} g_{s}^{2} \, \mathrm{d}t \, \mathrm{d}s = \|f'\|_{L^{2}(I_{\delta})}^{2} + \|z_{s}\|_{L^{2}(\Pi_{\delta})}^{2} \ge \|f'\|_{L^{2}(I_{\delta})}^{2}$$

Therefore,

$$B_{-}[g,g] \ge (1-b_{0}\delta) \|f'\|_{L^{2}(I_{\delta})}^{2} + \left(E_{1}(L_{N}) - \frac{H^{2}}{4} - b_{0}\delta\right) \|f\|_{L^{2}(I_{\delta})}^{2} + \left(E_{2}(L_{N}) - \frac{H^{2}}{4} - b_{0}\delta\right) \|z\|_{L^{2}(\Pi_{\delta})}^{2}.$$

Using Proposition 2.9 in order to estimate the eigenvalues of L_N one has then, with a suitable $a_0 > 0$,

$$E_{1}(L_{N}) - \frac{H^{2}}{4} - b_{0}\delta = -\left(\alpha + \frac{H}{2}\right)^{2} - a_{0}\alpha^{2}e^{-\alpha\delta} - \frac{H^{2}}{4} - b_{0}\delta$$

$$\geqslant -\alpha^{2} - \alpha H - \frac{H^{2}}{2} - a_{1}\left(\delta + \alpha^{2}e^{-\alpha\delta}\right),$$

$$a_{1} := \max\{a_{0}, b_{0}\},$$

$$E_{2}(L_{N}) - \frac{H^{2}}{4} - b_{0}\delta \geqslant \frac{1}{\delta^{2}} - \frac{H^{2}}{4} - b_{0}\delta \geqslant 0,$$

and then

$$B_{-}[g,g] \ge (1-b_0\delta) \|f'\|_{L^2(I_{\delta})}^2 - \left(\alpha^2 + \alpha H + \frac{H^2}{2}\right) \|f\|_{L^2(I_{\delta})}^2 - a_1 \left(\delta + \alpha^2 e^{-\alpha\delta}\right) \|f\|_{L^2(I_{\delta})}^2.$$

Hence, one arrives at the sought inequality (5.17) by taking $b := \max\{b_0, a_1\}$. To prove the lower bound (5.18) it remains to use the inequality $||f||_{L^2(I_{\delta})} \leq ||u||_{L^2(W_{\delta})}$.

As in the straight case we will obtain the sought lower bound for the eigenvalues $E_{K+n}(R^{\Omega}_{\alpha})$ with the help of the Proposition 2.3 by constructing a suitable identification map. Nevertheless, the construction involves a number of new components, so we prefer to give a sketch.

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PROPOSITION 5.16. — Under the assumptions (5.7) and (5.14) and for any fixed $n \in \mathbb{N}$ one has, as $\alpha \to +\infty$,

$$E_{K+n}(R^{\Omega}_{\alpha}) \ge -\alpha^2 - H_*\alpha - \frac{H_*^2}{2} + E_n\left(\bigoplus_{j\in\mathcal{J}_*} D_j\right) + \mathcal{O}\left(\frac{\log\alpha}{\sqrt{\alpha}}\right).$$

Proof. — With the above preparations and with a suitable redefinition of the main objects, the proof becomes almost identical to the one of Proposition 4.6. We are not giving all details, but just introducing the main objects and identifying the main steps.

Assume first that δ satisfies (5.8) and consider the Hilbert spaces

$$\mathcal{H} := \text{ the orthogonal complement of } L \text{ in } L^2(\Omega), \quad \mathcal{H}' := \bigoplus_{j \in \mathcal{J}_*} L^2(I_{j,\delta}).$$

During the proof for $u \in \mathcal{H}$ we denote

$$\begin{aligned} v_j &:= \text{the restriction of } u \text{ to } V_{j,\delta}, & \|v_j\| &:= \|v_j\|_{L^2(V_{j,\delta})}, \\ w_j &:= \text{the restriction of } u \text{ to } W_{j,\delta}, & \|w_j\| &:= \|w_j\|_{L^2(W_{j,\delta})}, \\ u_c &:= \text{the restriction of } u \text{ to } \Omega^c_{\delta}, & \|u_c\| &:= \|u_c\|_{L^2(\Omega^c_{\delta})}, \end{aligned}$$

and remark that due to the constructions and the equality (4.2) we have

(5.20)
$$\sum_{j=1}^{M} \int_{\partial_{\text{ext}} V_{j,\delta}} v_j^2 \, \mathrm{d}s = \sum_{j=1}^{M} \int_{\partial_{\text{ext}} W_{j,\delta}} w_j^2 \, \mathrm{d}s$$

Applying Lemma 5.7 with $A := -H_j$ we obtain, with some b > 0 and c > 0, the inequalities

$$\|v_j\|^2 \leqslant b\delta^2 \left(N_j^V[v_j, v_j] + (\alpha^2 + H_j\alpha)\|v_j\|^2\right) + b\alpha^2 \delta^2 e^{-c\alpha\delta} \|u\|^2,$$
$$\int_{\partial_{\text{ext}} V_{j,\delta}} v_j^2 \, \mathrm{d}s \leqslant b\alpha\delta^2 \left(N_j^V[v_j, v_j] + (\alpha^2 + H_j\alpha)\|v_j\|^2\right) + b\alpha^3 \delta^2 e^{-c\alpha\delta} \|u\|^2.$$

Furthermore, by applying Lemma 5.15 to each $W_{j,\delta}$ we conclude that there are functions $\psi_j \in L^2(0,\delta)$ with $\|\psi_j\|_{L^2(0,\delta)}^2 = 1$ such that if one defines

$$P_j: \mathcal{H} \to L^2(I_{j,\delta}), \quad (P_j u)(s) := \int_0^\delta \psi_j(t) \sqrt{1 - H_j t} \, w_j\left(\Phi_j(s,t)\right) \mathrm{d}t,$$

then, with some $b_1 > 0$,

$$N_{j}^{W}[w_{j}, w_{j}] \ge -\left(\alpha^{2} + \alpha H_{j} + \frac{H_{j}^{2}}{2}\right) \|P_{j}u\|^{2} + (1 - b_{1}\delta) \|(P_{j}u)'\|^{2} - b_{1}(\delta + \alpha^{2}e^{-c\alpha\delta}) \|P_{j}u\|^{2},$$

and we recall that, using the Cauchy–Schwarz inequality and $\|\psi_j\|_{L^2(0,\delta)} = 1$,

(5.21)
$$||P_{j}u||^{2} = \int_{I_{j,\delta}} \left(\int_{0}^{\delta} \psi(t) \sqrt{1 - H_{j}t} w_{j}(\Phi_{j}(s,t)) dt \right)^{2} ds$$

$$\leq \int_{I_{j,\delta}} \int_{0}^{\delta} (1 - H_{j}t) w_{j}(\Phi_{j}(s,t))^{2} dt ds = \int_{W_{j,\delta}} w_{j}^{2} dx = ||w_{j}||^{2}.$$

Now let us set $\delta := (c' \log \alpha)/\alpha$ with $c' \ge 3/c$, then the conditions (5.8) for the choice of δ are satisfied, and $\alpha^2 e^{-c\alpha\delta} = o(\delta)$, which implies $\alpha^2 \delta^2 e^{-c\alpha\delta} = o(\delta^3)$ and $\alpha^3 \delta^2 e^{-c\alpha\delta} = o(\alpha\delta^3)$. This simplifies the remainders in the above inequalities, and one can pick a sufficiently large a > 0 such that, for the same choice of ψ_i ,

$$\|v_{j}\|^{2} \leqslant \frac{a \log^{2} \alpha}{\alpha^{2}} \left(N_{j}^{V}[v_{j}, v_{j}] + (\alpha^{2} + H_{j}\alpha) \|v_{j}\|^{2} \right) + \frac{a \log^{3} \alpha}{\alpha^{3}} \|u\|^{2},$$

$$\int_{\partial_{\text{ext}}V_{j,\delta}} v_{j}^{2} \, \mathrm{d}s \leqslant \frac{a \log^{2} \alpha}{\alpha} \left(N_{j}^{V}[v_{j}, v_{j}] + (\alpha^{2} + H_{j}\alpha) \|v_{j}\|^{2} \right) + \frac{a \log^{3} \alpha}{\alpha^{2}} \|u\|^{2},$$

$$N_{j}^{W}[w_{j}, w_{j}] \geqslant - \left(\alpha^{2} + \alpha H_{j} + \frac{H_{j}^{2}}{2} \right) \|P_{j}u\|^{2} + \left(1 - \frac{a \log \alpha}{\alpha} \right) \|(P_{j}u)'\|^{2}$$

$$- \frac{a \log \alpha}{\alpha} \|P_{j}u\|^{2}.$$

Consider the self-adjoint operators

$$B := R_{\alpha}^{\Omega} + \left(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\right) + \frac{(M+a)\log\alpha}{\alpha}$$

viewed as an operator in \mathcal{H}

viewed as an operator in \mathcal{H} ,

$$B' := \bigoplus_{j \in \mathcal{J}_*} D_{j,\delta} \text{ in } \mathcal{H}',$$

with $\mathcal{Q}(B) = H^1(\Omega) \cap \mathcal{H}$ and $\mathcal{Q}(B') = \bigoplus_{j \in \mathcal{J}_*} H^1_0(I_{j,\delta})$. By combining the preceding estimates as in the proof of Proposition 4.6 one arrives at the estimate

(5.22)
$$B[u,u] \ge \frac{\alpha^2}{2a\log^2 \alpha} \sum_{j=1}^M \|v_j\|^2 + \frac{\alpha}{2a\log^2 \alpha} \sum_{j=1}^M \int_{\partial_{ext}V_{j,\delta}} v_j^2 \, \mathrm{d}s + a_0 \alpha \sum_{j \notin \mathcal{J}_*} \|P_j u\|^2 + \left(1 - \frac{a\log\alpha}{\alpha}\right) \sum_{j \in \mathcal{J}_*} \|(P_j u)'\|^2 + \frac{\alpha^2}{2} \sum_{j=1}^M \left(\|w_j\|^2 - \|P_j u\|^2\right) + \frac{\alpha^2}{2} \|u_c\|^2;$$

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the new summand $\sum_{j \notin \mathcal{J}_*} \|P_j u\|^2$ is due to the modified definition of B'. Each term on the right-hand side is non-negative and, hence, the left-hand side is an upper bound for each term on the right-hand side. It also implies that B is positive and then

$$E_1(B) \equiv E_{K+1}(R_{\alpha}^{\Omega}) + \left(\alpha^2 + \alpha H_* + \frac{H_*^2}{2}\right) + \frac{(M+a)\log\alpha}{\alpha} \ge 0.$$

By combining with Proposition 5.14 we see that for any fixed $n \in \mathbb{N}$ one can choose $\lambda_n > 0$ which is independent of α and such that

(5.23)
$$0 \leqslant E_n(B) \leqslant \lambda_n, \quad \left(1 + E_n(B)\right)^{-1} \geqslant (1 + \lambda_n)^{-1}.$$

In order to construct a suitable identification map $J : \mathcal{Q}(B) \to \mathcal{Q}(B')$ we pick functions $\rho_j^{\pm} \in C^1([0, \ell_j])$ such that

$$\begin{split} \rho_j^+ &= \begin{cases} 1 & \text{in a neighborhood of } 0, \\ 0 & \text{in a neighborhood of } \ell_j, \end{cases} \quad j \in \mathcal{J}_*, \\ \rho_j^- &= \begin{cases} 0 & \text{in a neighborhood of } 0, \\ 1 & \text{in a neighborhood of } \ell_j, \end{cases} \quad j \in \mathcal{J}_*, \end{split}$$

and then choose a constant $\rho_0 > 0$ such that

(5.24)
$$\|\rho_j^{\pm}\|_{L^{\infty}(0,\ell_j)} + \|(\rho_j^{\pm})'\|_{L^{\infty}(0,\ell_j)} \leq \rho_0 \text{ for all } j \in \mathcal{J}_*.$$

We have $I_{j,\delta} := (\lambda_j^+(\delta), \ell_j - \lambda_{j+1}^-(\delta)) =: (\iota_j, \tau_j)$, and that due to $\lambda_j^\pm(\delta) = \mathcal{O}(\delta)$ we have $\iota_j = \mathcal{O}(\delta)$ and $\tau_j = \ell_j + \mathcal{O}(\delta)$, hence,

$$\rho^+(\iota_j) = 1, \quad \rho^+(\tau_j) = 0, \quad \rho^-(\iota_j) = 0, \quad \rho^-(\tau_j) = 1$$

as α is sufficiently large. Therefore, the following map is well-defined:

$$J: \mathcal{Q}(B) \to \mathcal{Q}(B') \equiv \bigoplus_{j \in \mathcal{J}_*} H^1_0(I_{j,\delta}), \quad Ju = (J_j u),$$
$$(J_j u)(s) := (P_j u)(s) - (P_j u)(\iota_j)\rho_j^+(s) - (P_j u)(\tau_j)\rho_j^-(s)$$

For large α one has $1-H_jt \leq 2$ for $t \in (0, \delta)$, therefore, by Cauchy–Schwarz,

$$(5.25) |P_{j}u(\iota_{j})|^{2} + |P_{j}u(\tau_{j})|^{2}$$

$$= \left(\int_{0}^{\delta} \psi_{j}(t)\sqrt{1 - H_{j}t} w_{j}\left(\Phi_{j}(\iota_{j}, t)\right) dt\right)^{2}$$

$$+ \left(\int_{0}^{\delta} \psi_{j}(t)\sqrt{1 - H_{j}t} w_{j}\left(\Phi_{j}(\tau_{j}, t)\right) dt\right)^{2}$$

$$\leqslant \int_{0}^{\delta} (1 - H_{j}t)w_{j}\left(\Phi_{j}(\iota_{j}, t)\right)^{2} dt + \int_{0}^{\delta} (1 - H_{j}t)w_{j}\left(\Phi_{j}(\tau_{j}, t)\right)^{2} dt$$

$$\leqslant 2\left(\int_{0}^{\delta} w_{j}\left(\Phi_{j}(\iota_{j}, t)\right)^{2} dt + \int_{0}^{\delta} w_{j}\left(\Phi_{j}(\tau_{j}, t)\right)^{2} dt\right)$$

$$\equiv 2\int_{\partial_{ext}W_{j,\delta}} w_{j}^{2} ds.$$

Using $(x+y)^2 \ge (1-\varepsilon)x^2 - y^2/\varepsilon$ for any $x, y \in \mathbb{R}$ and $\varepsilon > 0$ we estimate

$$\begin{split} \|J_{j}u\|^{2} &= \int_{I_{j,\delta}} \left| (P_{j}u)(s) - (P_{j}u)(\iota_{j})\rho_{j}^{+}(s) - (P_{j}u)(\tau_{j})\rho_{j}^{-}(s) \right|^{2} \mathrm{d}s \\ &\geqslant (1-\varepsilon) \int_{I_{j,\delta}} \left| (P_{j}u)(s) \right|^{2} \mathrm{d}s \\ &- \frac{1}{\varepsilon} \int_{I_{j,\delta}} \left| (P_{j}u)(\iota_{j})\rho_{j}^{+}(s) + (P_{j}u)(\tau_{j})\rho_{j}^{-}(s) \right|^{2} \mathrm{d}s. \end{split}$$

Using (5.25) and the constant ρ_0 from (5.24) we have

$$\begin{split} &\int_{I_{j,\delta}} \left| (P_j u)(\iota_j) \rho_j^+(s) + (P_j u)(\tau_j) \rho_j^-(s) \right|^2 \mathrm{d}s \leqslant 4\ell_j \rho_0^2 \int_{\partial_{\mathrm{ext}} W_{j,\delta}} w_j^2 \, \mathrm{d}s, \\ &\|J_j u\|^2 \geqslant (1-\varepsilon) \|P_j u\|^2 - \frac{4\ell\rho_0^2}{\varepsilon} \int_{\partial_{\mathrm{ext}} W_{j,\delta}} w_j^2 \, \mathrm{d}s, \quad \ell := \max_{j \in \mathcal{J}_*} \ell_j. \end{split}$$

Therefore, using (5.22),

$$\begin{aligned} \|u\|^{2} - \|Ju\|^{2} &\leq \sum_{j=1}^{M} \|v_{j}\|^{2} + \sum_{j=1}^{M} \left(\|w_{j}\|^{2} - \|P_{j}u\|^{2}\right) + \sum_{j \notin \mathcal{J}_{*}} \|P_{j}u\|^{2} \\ &+ \varepsilon \|u\|^{2} + \frac{4\ell\rho_{0}^{2}}{\varepsilon} \sum_{j=1}^{M} \int_{\partial_{ext}V_{j,\delta}} v_{j}^{2} \, \mathrm{d}s + \|u_{c}\|^{2} \\ &\leq \left(\frac{2a\log^{2}\alpha}{\alpha^{2}} + \frac{4}{\alpha^{2}} + \frac{1}{a_{0}\alpha} + \frac{8\ell\rho_{0}^{2}a\log^{2}\alpha}{\varepsilon\alpha}\right) B[u, u] + \varepsilon \|u\|^{2}.\end{aligned}$$

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Taking $\varepsilon := \log \alpha / \sqrt{\alpha}$ and choosing $c_1 > 0$ sufficiently large we obtain

(5.26)
$$||u||^2 - ||Ju||^2 \leq \frac{c_1 \log \alpha}{\sqrt{\alpha}} \Big(B[u, u] + ||u||^2 \Big).$$

To study the difference B'[Ju, Ju] - B[u, u] recall that $B'[Ju, Ju] = \sum_{j \in \mathcal{J}_*} ||(J_j u)'||^2$. Using the elementary inequality $(x + y)^2 \leq (1 + \varepsilon)x^2 + 2y^2/\varepsilon$ valid for all $x, y \in \mathbb{R}$ and $\varepsilon \in (0, 1)$ we estimate

$$\begin{split} \left\| (J_{j}u)' \right\|^{2} &= \int_{I_{j,\delta}} \left| (P_{j}u)'(s) - (P_{j}u)(\iota_{j})(\rho_{j}^{+})'(s) - (P_{j}u)(\tau_{j})(\rho_{j}^{-})'(s) \right|^{2} \mathrm{d}s \\ &\leq (1+\varepsilon) \int_{I_{j,\delta}} \left| (P_{j}u)'(s) \right|^{2} \mathrm{d}s \\ &+ \frac{2}{\varepsilon} \int_{I_{j,\delta}} \left| (P_{j}u)(\iota_{j})(\rho_{j}^{+})'(s) + (P_{j}u)(\tau_{j})(\rho_{j}^{-})'(s) \right|^{2} \mathrm{d}s. \end{split}$$

Using the estimate (5.25) for the last term and the constant ρ_0 from (5.24) we have

$$\begin{split} \int_{I_{j,\delta}} \left| (P_{j}u)(\iota_{j})(\rho_{j}^{+})'(s) + (P_{j}u)(\tau_{j})(\rho_{j}^{-})'(s) \right|^{2} \mathrm{d}s &\leq 4\ell\rho_{0}^{2} \int_{\partial_{\mathrm{ext}}W_{j,\delta}} w_{j}^{2} \,\mathrm{d}s, \\ B'[Ju, Ju] &\leq (1+\varepsilon) \sum_{j \in \mathcal{J}_{*}} \left\| (P_{j}u)' \right\|^{2} + \frac{8\ell\rho_{0}^{2}}{\varepsilon} \sum_{j \in \mathcal{J}_{*}} \int_{\partial_{\mathrm{ext}}W_{j,\delta}} w_{j}^{2} \,\mathrm{d}s \\ &\leq (1+\varepsilon) \sum_{j \in \mathcal{J}_{*}} \left\| (P_{j}u)' \right\|^{2} + \frac{8\ell\rho_{0}^{2}}{\varepsilon} \sum_{j=1}^{M} \int_{\partial_{\mathrm{ext}}W_{j,\delta}} w_{j}^{2} \,\mathrm{d}s. \end{split}$$

Recall that due to (5.22) we have

$$B[u, u] \ge \left(1 - \frac{a \log \alpha}{\alpha}\right) \sum_{j \in \mathcal{J}_*} \left\| (P_j u)' \right\|^2,$$

$$B[u, u] \ge \frac{\alpha}{2a \log^2 \alpha} \sum_{j=1}^M \int_{\partial_{\text{ext}} V_{j,\delta}} v_j^2 \, \mathrm{d}s \equiv \frac{\alpha}{2a \log^2 \alpha} \sum_{j=1}^M \int_{\partial_{\text{ext}} W_{j,\delta}} w_j^2 \, \mathrm{d}s,$$

where we used (5.20) on the last step. Therefore,

$$B'[Ju, Ju] - B[u, u] \leq \left(\varepsilon + \frac{a\log\alpha}{\alpha}\right) \sum_{j \in \mathcal{J}_*} \left\| (P_j u)' \right\|^2 \\ + \frac{8\ell\rho_0^2}{\varepsilon} \sum_{j=1}^M \int_{\partial_{ext} W_{j,\delta}} w_j^2 \, \mathrm{d}s \\ \leq \left(\frac{\varepsilon + \frac{a\log\alpha}{\alpha}}{1 - \frac{a\log\alpha}{\alpha}} + \frac{8\ell\rho_0^2}{\varepsilon} \cdot \frac{2a\log^2\alpha}{\alpha}\right) B[u, u].$$

By setting $\varepsilon = \log \alpha / \sqrt{\alpha}$ and choosing $c_2 > 0$ sufficiently large we arrive at

(5.27)
$$B'[Ju, Ju] - B[u, u] \leq \frac{c_2 \log \alpha}{\sqrt{\alpha}} B[u, u] \leq \frac{c_2 \log \alpha}{\sqrt{\alpha}} \left(B[u, u] + \|u\|^2 \right).$$

In virtue of (5.26) and (5.27) we can apply Proposition 2.3. Remark that for each fixed n the assumption $\varepsilon_1 < 1/(1 + E_n(B))$ is satisfied due to (5.23). Hence, for each fixed n there holds

$$E_n\left(\bigoplus_{j\in\mathcal{J}_*}D_{j,\delta}\right) \equiv E_n(B') \leqslant E_n(B) + \frac{\log\alpha}{\sqrt{\alpha}} \frac{\left(c_1E_n(B) + c_2\right)\left(1 + E_n(B)\right)}{1 - c_1\left(1 + E_n(B)\right)\log\alpha/\sqrt{\alpha}}.$$

By (5.23) we have $E_n(B) = \mathcal{O}(1)$ for each fixed *n*, and the preceding inequality implies

$$E_{K+n}(R^{\Omega}_{\alpha}) \ge -\alpha^2 - \alpha H_* - \frac{H^2_*}{2} + E_n\left(\bigoplus_{j\in\mathcal{J}_*} D_{j,\delta}\right) + \mathcal{O}\left(\frac{\log\alpha}{\sqrt{\alpha}}\right).$$

It remains to remark that $E_n(\bigoplus_{j\in\mathcal{J}_*} D_{j,\delta}) = E_n(\bigoplus_{j\in\mathcal{J}_*} D_j) + \mathcal{O}(\delta)$, while $\delta = (c'\log\alpha)/\alpha = o(\frac{\log\alpha}{\sqrt{\alpha}})$.

The combination of Propositions 5.14 and 5.16 gives Theorem 1.5.

6. Concluding remarks

6.1. Resonant angles: equilateral triangle

As already mentioned in the introduction, we are going to show that there are some angles which do not satisfy the non-resonance condition. This will be done in an indirect way. First, remark that if Ω is a convex polygon (with straight sides) with non-resonant vertices, K corner-induced eigenvalues, and side lengths ℓ_i , then

(6.1)
$$\lim_{\alpha \to +\infty} \left(E_{K+1}(R_{\alpha}^{\Omega}) + \alpha^2 \right) = E_1\left(\bigoplus_{j=1}^M D_j\right) \equiv \pi^2/\ell^2 > 0,$$

where D_j is the Dirichlet Laplacian on $(0, \ell_j)$ and $\ell := \max \ell_j$. Let us show that this can be violated for some particular polygons Ω and lead to a different eigenvalue asymptotics.

The paper by McCartin [46] contains a detailed analysis of the operator R^{Ω}_{α} for the case when Ω is an equilateral triangle using a separation of variables in a suitably chosen coordinate system. To be more precise, we assume that the side length of the triangle is 1. Let us give a short account

of the results of [46] concerning the behavior of the eigenvalues as $\alpha \to +\infty$ (which corresponds to $\sigma \to -\infty$ in the reference).

One constructs first a complete orthogonal system of eigenfunctions, noted $T_s^{m,n}$ with $n \ge m \ge 0$ and $T_a^{m,n}$ with $n > m \ge 0$ and $m, n \in \mathbb{N} \cup \{0\}$, and $R_{\alpha}^{\Omega} T_{\circ}^{m,n} = E_{\alpha}(m,n) T_{\circ}^{m,n}$ for $\circ \in \{s,a\}$, i.e. $T_s^{m,n}$ and $T_a^{m,n}$ share the same eigenvalue for n > m. It is then shown that $E_{\alpha}(m,n) \ge 0$ for $m \ge 2$, therefore, only $m \in \{0,1\}$ contribute to the negative spectrum. One shows then the following asymptotics for $\alpha \to +\infty$ (we cite the respective equations in Subsection 7.2 of [46]):

$$E_{\alpha}(0,0) = -4\alpha^2 + o(1),$$
 Eq. (37),

$$E_{\alpha}(0,1) = -4\alpha^2 + o(1),$$
 Eq. (50),

$$E_{\alpha}(0,n) = -\alpha^2 + \frac{4}{27} \left[\frac{\pi}{r} \left(n - \frac{3}{2} \right) \right]^2 + o(1) \text{ for } n \ge 2, \qquad \text{Eq. (53)},$$

$$E_{\alpha}(1,1) = -\alpha^2 + o(1),$$
 Eq. (67),

$$E_{\alpha}(1,n) = -\alpha^{2} + \frac{4}{27} \left[\frac{\pi}{r} (n-1) \right]^{2} + o(1) \text{ for } n \ge 2, \qquad \text{Eq. (80)},$$

where $r := 1/(2\sqrt{3})$ is the inradius. The eigenvalues $E_{\alpha}(0,0)$ and $E_{\alpha}(0,1)$ (twice) are corner-induced: the half-angle at each corner is $\pi/6$, and $\kappa(\pi/6) = 1$ (see Subsection 2.6), hence K = 3 (we remark that a more precise remainder for the first three eigenvalues was obtained in [32]). Furthermore, by inspecting the above expressions and by taking into account the multiplicities one sees that for any fixed $n \in \mathbb{N}$ one has the asymptotics $E_{K+n}(R_{\alpha}^{\Omega}) = -\alpha^2 + z_n + o(1)$, where z_n is the *n*th element (when enumerated in the non-decreasing order) of the multiset $Z := \{(2\pi m/3)^2 : m \in \mathbb{Z}\}$. In particular, one has $z_1 = 0$ and $E_{K+1}(R^{\Omega}) = -\alpha^2 + o(1)$, which is in contradiction to (6.1). Hence, the half-angle $\pi/6$ is resonant. In fact, in the above multiset Z one easily recognizes the spectrum of the Laplacian on a circle of length 3, i.e. on the three sides of the triangles glued to each other without any obstacle at the vertices. This operator can be then viewed as the effective operator on the boundary.

We remark that the text of the paper [46] is included into McCartin's book [47] as Chapter 7, but due to a typesetting error some of the important formulas are missing on page 105 of [47], which complicates the understanding of the eigenvalue asymptotics. An interested reader should better refer to the original paper [46] for full details.

6.2. Variable curvature

By analogy with the works on smooth domains, see e.g. [57] one might expect the following asymptotics to be valid for general curvilinear polygons (i.e. without assuming that H_i are constant): if all corners are concave or convex non-resonant, then $E_{K+n}(R^{\Omega}_{\alpha}) = -\alpha^2 + E_n(\bigoplus_j (D_j - \alpha H_j)) + r(\alpha)$ with a suitable error term $r(\alpha)$. Some steps of the above scheme are still easily transferable, but the whole machinery appears to fail when trying to prove the lower bound. The main obstacles, when projected to the proof of Proposition 5.16, are that the eigenvalues of the comparison operator B' = $\bigoplus_i (D_j - \alpha H_j) + \alpha H_* + \rho(\alpha)$ with suitably chosen constants $\rho(\alpha)$ and $H_* :=$ $\max_j \max H_j$, may become infinitely large for large α , and much smaller value of ε_i are needed to satisfy the initial assumption of Proposition 2.3 and to have a non-trivial resulting estimate. In a sense, the machinery we use implicitly aims at showing that the eigenfunctions are suitably small near vertices by controlling their norms and traces using the values in the rest of the domain (Lemma 4.5 and 5.7). For non-constant curvatures, the eigenfunctions are localized near the points of maximal curvature, similarly as in the smooth case [30]. In particular, if the curvature takes its maximum at one of the corners, then the respective eigenfunctions should be localized near the corner, so the strategy of showing that it asymptotically vanishes at the corners (which then gives an effective operator with the Dirichlet boundary conditions) becomes contradictory. One might expect that a more precise analysis in this case can be done under explicit hypotheses on the curvatures (e.g. an isolated maximum at a corner) by showing first some semiclassical localization properties for the eigenfunctions, which might be a task of a higher complexity.

6.3. Resonance and non-resonance conditions

Our non-resonance condition introduced in Definition 3.9 and used in the proof is a slightly naive adaptation of a condition appearing in the spectral analysis of Laplacians on domain collapsing onto a graph. The topic is presented in a systematic way e.g. in the papers by Grieser [26], Molchanov and Vainberg [48], and in the monograph by Post [59]. Let us recall some basic notions of the theory, mostly following the short presentation given in the paper [55] by Pankrashkin.

Let $d \ge 2$ and $\omega \subset \mathbb{R}^{d-1}$ be a bounded connected Lipschitz domain. We denote by μ the first Dirichlet eigenvalue of ω . By a star waveguide

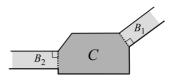


Figure 6.1. Star waveguide Λ with two branches and a dark-shaded center.

we mean a connected Lipschitz domain $\Lambda \subset \mathbb{R}^d$ for which one can find n non-intersecting half-infinite cylinders $B_1, \ldots, B_n \subset \Lambda$, all isometric to $(0, \infty) \times \omega$, such that Λ coincides with the union $B_1 \cup \ldots \cup B_n$ outside a compact set, see Figure 6.1. The cylinders B_j will be called *branches*, the connected bounded domain $C := \Lambda \setminus \overline{B_1 \cup \ldots \cup B_n}$ will be called *center*, which is also assumed Lipschitz. We call such a domain Λ a *star waveguide*. Remark that centers of star waveguides are not defined uniquely: one can attach finite pieces of B_j to a given center to obtain a new center.

For small $\varepsilon > 0$, let $\Omega_{\varepsilon} \subset \mathbb{R}^d$ be a domain composed of finite cylinders $B_{j,\varepsilon}$ isometric to $I_j \times (\varepsilon \omega)$ with $I_j := (0, \ell_j), \ \ell_j > 0, \ j \in \{1, \ldots, J\}$, connected to each other through some bounded Lipschitz domains $C_{k,\varepsilon}$, see Figure 6.2 (a). In the context of the problem, it is natural to refer to $B_{j,\varepsilon}$ as to edges and to $C_{k,\varepsilon}$ as to vertices. We assume that the vertices $C_{k,\varepsilon}$ are isometric to εC_k with some ε -independent domains $C_k, \ k \in \{1, \ldots, K\}$, and that if one considers a vertex $C_{k,\varepsilon}$ and extends the attached cylindrical edges to infinity, then one obtains a domain isometric to $\varepsilon \Lambda_k$ with some ε -independent star waveguide Λ_k having C_k as its center.

In various applications one is interested in the eigenvalues of the Dirichlet laplacian $-\Delta_D^{\Omega_{\varepsilon}}$ in Ω_{ε} as $\varepsilon \to 0^+$. As the domain Ω_{ε} collapses onto its onedimensional skeleton X composed from the segments I_j coupled at the vertices, see Figure 6.2 (b), it is natural to expect that the behavior of the eigenvalues should be determined by an effective operator associated with X. The results of [26, Theorems 2 and 3] can be summarized informally



Figure 6.2. (a) An example of a domain Ω_{ε} with dark shaded vertices. (b) The associated one-dimensional skeleton X.

as follows. Consider the Dirichlet Laplacians $-\Delta_D^{\Lambda_k}$ in the star waveguides Λ_k associated with each vertex as described above: the spectrum consists of the essential part $[\mu, +\infty)$ and of discrete eigenvalues $E_j(-\Delta_D^{\Lambda_k}), j \in \{1, \ldots, N(\Lambda_k)\}, k \in \{1, \ldots, K\}$. Then with some $N \ge N(\Lambda_1) + \cdots + N(\Lambda_K), a_n \in (0, \mu]$ and b > 0 there holds, as $\varepsilon \to 0^+$:

- for $n \in \{1, \ldots, N\}$ there holds $E_n(-\Delta_D^{\Omega_{\varepsilon}}) = a_n/\varepsilon^2 + \mathcal{O}(e^{-b/\varepsilon}),$
- for any fixed $n \in \mathbb{N}$ there holds $E_{N+n}(-\Delta_D^{\Omega_{\varepsilon}}) = \mu/\varepsilon^2 + E_n(L) + \mathcal{O}(\varepsilon)$, where L is a self-adjoint operator in $L^2(X) \simeq \bigoplus_{j=1}^J L^2(0, \ell_j)$ acting as $(f_j) \mapsto (-f''_j)$ with suitable self-adjoint boundary conditions determined by the scattering matrices of $-\Delta_D^{\Lambda_k}$ at the threshold energy μ (see e.g. the paper [28] by Guiloppé for the definition and properties of the scattering matrices).

The operator L, which is the so-called quantum graph laplacian on X (see the monograph [6] by Berkolaiko and Kuchment for an introduction and a review), represents the sought "effective operator" on X, and the associated boundary conditions describe the way how the branches of the network interact through the vertices in the limit $\varepsilon \to 0$. At the same time, finding explicitly the boundary condition in the general case represents a very difficult task.

The above general construction admits an important particular case, which can be formulated in simpler terms. One says that a star waveguide Λ admits a *threshold resonance* if there exists a non-zero function $\Phi \in L^{\infty}(\Lambda)$ satisfying $-\Delta \Phi = \mu \Phi$ in Λ and $\Phi = 0$ at $\partial \Lambda$, then the following result holds [26, Section 8]:

PROPOSITION 6.1. — Assume that none of Λ_k admits a threshold resonance, then for $\varepsilon \to 0^+$ the following asymptotics are valid:

- Denote $N := N(\Lambda_1) + \dots + N(\Lambda_K)$ and let a_1, \dots, a_N be the family of the eigenvalues $E_j(-\Delta_D^{\Lambda_k})$, $j \in \{1, \dots, N(\Lambda_k)\}$, $k \in \{1, \dots, K\}$, enumerated in the non-decreasing order, then for $n \in \{1, \dots, N\}$ one has $E_n(-\Delta_D^{\Omega_{\varepsilon}}) = a_n/\varepsilon^2 + \mathcal{O}(e^{-b/\varepsilon})$, with some b > 0,
- For any fixed $n \ge 1$ there holds

$$E_{N+n}(-\Delta_D^{\Omega_{\varepsilon}}) = \mu/\varepsilon^2 + E_j\left(\bigoplus_{j=1}^J D_j\right) + \mathcal{O}(\varepsilon)$$

with D_i being the Dirichlet Laplacians on $(0, \ell_i)$.

In other word, in the absence of threshold resonances the effective operator L is decoupled and corresponds to the Dirichlet boundary conditions at the vertices. In view of this result, it is important to be able to identify if star waveguides admits no threshold resonance. The following sufficient condition was obtained in [55], which was in turn motivated by the analysis of particular configurations carried out by Bakharev, Nazarov, Matveenko [2], Nazarov [49, 50], Nazarov, Ruotsalainen, Uusitalo [51]. For a star waveguide Λ with a center C we denote by $-\Delta_{DN}^{C}$ the Laplacian in C with the Dirichlet boundary condition of $\partial C \cap \partial \Lambda$ and the Neumann boundary condition at the remaining boundary, then if for some center C one has the strict inequality

(6.2)
$$E_{N(\Lambda)+1}(-\Delta_{DN}^C) > \mu,$$

then Λ has no threshold resonance. In the recent preprint [3] Bakharev and Nazarov prove that the condition (6.2) for some center C is also necessary for the absence of threshold resonance (hence, it is a necessary and sufficient condition).

By comparing Proposition 6.1 with our main Theorem 1.1 one sees that role of the star waveguides attached to the vertices is quite similar to the role of the infinite sectors for the Robin laplacians. In fact our condition of non-resonance (Definition 3.9) is a translation of the condition (6.2)into the framework of Robin sectors. Namely, one may rewrite (6.2) using the center εC of the scaled waveguide $\varepsilon \Lambda$ as $E_{N(\Lambda)+1}(-\Delta_{DN}^{\varepsilon C}) = \mu/\varepsilon^2 +$ c/ε^2 with $c := E_{N(\Lambda)+1}(-\Delta_{DN}^C) - \mu > 0$ and remark that μ/ε^2 is the bottom of the essential spectrum of the Dirichlet laplacian on $\varepsilon \Lambda$. This should be compared with the scaled form of the non-resonance condition $E_{\kappa(\theta)+1}(N^{\delta}_{\theta,\alpha}) \geq -\alpha^2 + c/\delta^2, c > 0$, as $\alpha\delta$ is large, by noting that $-\alpha^2$ is the bottom of the spectrum of the α -Robin laplacian in the infinite sector. We also remark that the result of Proposition 6.1 was obtained earlier by Post [58] under the assumption that each Λ_k admits a center C_k such that $E_1(-\Delta_{DN}^{C_k}) > \mu$, which is exactly the condition (6.2) for $N(\Lambda) = 0$. In fact, the final steps of our proof (especially the construction of the identification map J) are an adaption of those from [58]. In view of the preceding analogies with the waveguides, it would be interesting to find alternative reformulations of our non-resonance condition e.g. in terms of generalized eigenfunctions at the bottom of the essential spectrum, which might help to extend our result to a larger range of angles. It would also be of interest to understand the eigenvalue asymptotics for general angles (i.e. without assuming that the angles are non-resonant), which might involve a development of the scattering theory in infinite sectors similar to the one for waveguides.

Appendix A. Some geometric constructions in curvilinear sectors

Let us introduce a geometric setting which will be used throughout the whole section.

Let Γ_{\pm} be two C^3 curves meeting at a point at an angle $2\theta \in (0, \pi)$. In this section we would like to construct some neighborhoods and cut-off functions near the intersection point. More precisely, let $s_* > 0$ and $\gamma_{\pm} : [-s_*, s_*] \to \mathbb{R}^2$ be the arc length parametrizations of Γ_{\pm} , i.e. both γ_{\pm} are injective C^3 functions with $|\gamma'_{\pm}| = 1$ and $\Gamma_{\pm} = \gamma_{\pm}([-s_*, s_*])$. By applying suitable rotations and translations we assume without loss of generality that

(A.1)
$$\gamma_{\pm}(0) = (0,0), \quad \gamma'_{\pm}(0) = (\cos\theta, \pm \sin\theta), \quad \theta \in \left(0, \frac{\pi}{2}\right).$$

In view of the above assumptions, near the point (0,0) the curves Γ_{\pm} are the graphs of C^3 functions F_{\pm} with $\pm F_+(t) > \pm F_-(t)$ for $\pm t > 0$, and we will be interested in some constructions in the curvilinear sector

 $U := \big\{ (x_1, x_2): \ 0 < x_1 < b, \ F_-(x_1) < x_2 < F_+(x_1) \big\}, \quad b > 0,$

see Figure A.1. For subsequent use we also introduce unit normal vectors $n_{\pm}(s)$ to Γ_{\pm} at $\gamma_{\pm}(s)$ which depend smoothly on s and point to the outside of U for small s. In particular, one has then $n_{\pm}(0) = (-\sin\theta, \pm \cos\theta)$. As n_{\pm} are unit vectors, one has $n'_{\pm}(s) = k_{\pm}(s)\gamma'_{\pm}(s)$, where k_{\pm} are C^1 functions (which coincide up to the sign with the algebraic curvatures on Γ_{\pm}), and $\gamma'_{+}(s) \wedge n_{\pm}(s) \equiv \pm 1$.

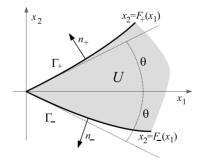


Figure A.1. The curves Γ_{\pm} and the curvilinear sector U. The thin dotted lines correspond to the tangents to Γ_{\pm} at the origin.

LEMMA A.1. — There exist $t_1 > 0$ and a C^2 smooth function $Y : (-t_1, t_1) \to \mathbb{R}^2$ such that for $t \in (0, t_1)$ the point Y(t) is the unique point of U which is at the distance t from both Γ_+ and Γ_- , and the points $A_{\pm}(t) \in \Gamma_{\pm}$ satisfying $|A_{\pm}(t) - Y(t)| = t$ are uniquely defined. Furthermore, $A_{\pm}(t) := \gamma_{\pm}(\lambda_{\pm}(t))$, where λ_{\pm} are C^2 functions defined near 0, and

$$\lambda_{\pm}(0) = 0, \quad \lambda_{\pm}'(0) = \cot a \theta, \quad Y(0) = \begin{pmatrix} 0\\ 0 \end{pmatrix}, \quad Y'(0) = \frac{1}{\sin \theta} \begin{pmatrix} 1\\ 0 \end{pmatrix}.$$

The resulting curve

(A.2)
$$\Sigma := \{ (t, Y(t)) : t \in (-t_1, t_1) \},\$$

can be viewed as the curvilinear angle bisector due to its geometric property: each point of Σ is at equal distances from the curved sides Γ_{\pm} .

Proof. — For $t_0 > 0$ and $s_0 \in (0, s_*)$ consider the maps (see Figure A.2)

$$\Phi_{\pm}: (-s_0, s_0) \times (-t_0, t_0) \to \mathbb{R}^2, \quad \Phi_{\pm}(s, t) = \gamma_{\pm}(s) - tn_{\pm}(s).$$

It is a well known result from the differential geometry that Φ_{\pm} are injective for $t_0 > 0$ small enough, and that dist $(\Phi_{\pm}(s,t),\Gamma_{\pm}) = |t|$ and that they are C^2 -diffeomorphisms from $(-s_0, s_0) \times (-t_0, t_0)$ to its images under Φ_{\pm} . One has

$$\frac{\partial \Phi_{\pm}}{\partial s}(s,t) = \gamma'_{\pm}(s) - tn'_{\pm}(s) = \left(1 - tk_{\pm}(s)\right)\gamma'_{\pm}(s).$$

Define $G: (-s_0, s_0) \times (-s_0, s_0) \times (-t_0, t_0) \rightarrow \mathbb{R}^2$ by

 $G(s_+, s_-, t) := \Phi_+(s_+, t) - \Phi_-(s_-, t),$

then $G(0,0,0) = \gamma_{+}(0) - \gamma_{-}(0) = (0,0)$ and $\partial G/\partial s_{\pm}(s_{+},s_{-},t) = \pm (1-tk_{\pm}(s_{\pm}))\gamma'_{\pm}(s_{\pm})$, and the two vectors $\partial G/\partial s_{\pm}(0,0,0) = \pm \gamma'_{\pm}(0)$ are linearly independent. Hence, it follows by the implicit function theorem that there exist $t_{1} > 0$ and $s_{1} > 0$ and C^{2} functions $\lambda_{\pm} : (-t_{1},t_{1}) \to (-s_{1},s_{1})$

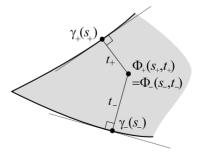


Figure A.2. The maps Φ_{\pm} .

with $\lambda_{\pm}(0) = 0$ such that for $(s_+, s_-, t) \in (-s_1, s_1) \times (-s_1, s_1) \times (-t_1, t_1)$ one has the equivalence: $G(s_+, s_-, t) = 0$ if and only if $s_{\pm} = \lambda_{\pm}(t)$. If one defines a C^2 function $Y : (-t_1, t_1) \to \mathbb{R}^2$ by $Y(t) := \Phi_{\pm}(\lambda_{\pm}(t), t)$, then for any $t \in (0, t_1)$ the point Y(t) is the unique point of U satisfying dist $(Y(t), \Gamma_{\pm}) = t$, and the points $A_{\pm}(t)$ of Γ_{\pm} which are the closest to Y(t) are $A_{\pm}(t) = \gamma_{\pm}(\lambda_{\pm}(t))$. One differentiates $G(\lambda_{+}(t), \lambda_{-}(t), t) = 0$ in tto arrive at

$$\lambda'_{+}(t) \Big[1 - tk_{+}(\lambda_{+}(t)) \Big] \gamma'_{+}(\lambda_{+}(t)) - \lambda'_{-}(t) \Big[1 - tk_{-}(\lambda_{-}(t)) \Big] \gamma'_{-}(\lambda_{-}(t)) \\ - \Big[n_{+}(\lambda_{+}(t)) - n_{-}(\lambda_{-}(t)) \Big] = 0.$$

For t=0 one has $\lambda_+'(0)\gamma_+'(0)-\lambda_-'(0)\gamma_-'(0)=n_+(0)-n_-(0),$ i.e.

$$\begin{pmatrix} \cos\theta & -\cos\theta\\ \sin\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \lambda'_{+}(0)\\ \lambda'_{-}(0) \end{pmatrix} = \begin{pmatrix} 0\\ 2\cos\theta \end{pmatrix},$$

which gives

$$\begin{pmatrix} \lambda'_{+}(0) \\ \lambda'_{-}(0) \end{pmatrix} = \frac{1}{2\sin\theta\cos\theta} \begin{pmatrix} \sin\theta & \cos\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 2\cos\theta \end{pmatrix} = \begin{pmatrix} \cot a \theta \\ \cot a \theta \end{pmatrix}$$

Then

$$Y'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \Phi_{+}(\lambda_{+}(t), t)$$

= $\lambda'_{+}(t) \Big[1 - tk_{+}(\lambda_{+}(t)) \Big] \gamma'_{+}(\lambda_{+}(t)) - n_{+}(\lambda_{+}(t)),$
$$Y'(0) = \lambda'_{+}(0)\gamma'_{+}(0) - n_{+}(0)$$

= $\cot a \theta \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} - \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} = \frac{1}{\sin \theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

Using the objects defined in Lemma A.1 we introduce the following sets V_t :

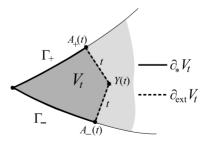


Figure A.3. Construction of the domain V_t

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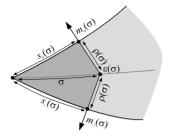


Figure A.4. Parametrization with the arc-length. The unit vectors m_{\pm} are orthogonal to the boundary. The arrows indicate the length of the corresponding arcs. For small σ one has $\rho(\sigma) = \sigma \sin \theta + \mathcal{O}(\sigma^2)$ and $s_{\pm}(\sigma) = \sigma \cos \theta + \mathcal{O}(\sigma^2)$.

DEFINITION A.2. — For $t \in (0, t_1)$ denote by V_t the interior of the curvilinear quadrangle bounded by the pieces of Γ_{\pm} enclosed between the points (0,0) and $A_{\pm}(t)$ and by the straight line segments connecting Y(t) to $A_{\pm}(t)$. We refer to Figure A.3 for an illustration. One will distinguish between two parts of its boundary, i.e. one denotes

$$\partial_* V_t := \partial V_t \cap (\Gamma_+ \cup \Gamma_-), \text{ and } \partial_{\text{ext}} V_t := \partial V_t \setminus \partial_* V_t.$$

Then, we would like to "straighten" V_t in a controlable way in order to obtain a truncated curvilinear sector S_{θ}^r (see Definition 3.1).

LEMMA A.3. — There is a bi-Lipschitz map Φ between two neighborhoods of the origin with $\Phi'(x) = I_2 + \mathcal{O}(|x|)$ for $x \to 0$ and a C^2 smooth function r defined near 0 with r(0) = 0 and $r'(0) = \operatorname{cotan} \theta$ such that $\Phi(\mathcal{S}_{\theta}^{r(t)}) = V_t$, $\Phi(\partial_* \mathcal{S}_{\theta}^{r(t)}) = \partial_* V_t$ and $\Phi(\partial_{\text{ext}} \mathcal{S}_{\theta}^{r(t)}) = \partial_{\text{ext}} V_t$ for all sufficiently small t > 0.

Proof. — Without loss of generality we may assume that $t_1 > 0$ is sufficiently small such that $Y'(t) \neq 0$ for $t \in [-t_1, t_1]$. Let us introduce an arc-length parametrization of the curvilinear angle bisector Σ introduced in (A.2): consider the function σ with $\sigma(0) = 0$ and $\sigma' = |Y'|$, i.e. $\sigma(t)$ is the length of Y([0,t]). One has $\sigma'(0) = |Y'(0)| = 1/\sin\theta$ and $\sigma' = |Y'| > 0$ on $[-t_1,t_1]$. Hence, $\sigma: [-t_1,t_1] \to [-\sigma_-,\sigma_+]$ is a C^2 diffeomorphism for some $\sigma_{\pm} > 0$. Denote by $\rho: [-\sigma_-,\sigma_+] \to [-t_1,t_1]$ its inverse, which is then also C^2 and satisfies $\rho(0) = 0$ and $\rho'(0) = 1/\sigma'(0) = \sin\theta$. Finally, let us pick a small $\delta > 0$ and define $\varepsilon := Y \circ \rho: (-\delta, \delta) \to \mathbb{R}^2$, then one has $|\varepsilon'| = 1$, $\varepsilon'(0) = (1,0)^T$, and $Y([0,t]) = \varepsilon([0,\sigma(t)])$ for small t > 0, i.e. ε is an arclength parametrization of Σ near the origin. By construction, the point $\varepsilon(\sigma)$

is then the unique point of U with dist $(\varepsilon(\sigma), \Gamma_{\pm}) = \rho(\sigma)$, and for small σ one has $\rho(\sigma) = \sigma \sin \theta + \mathcal{O}(\sigma^2)$. Furthermore, if one sets $s_{\pm}(\sigma) := \lambda_{\pm}(\rho(\sigma))$, then $s_{\pm}(\cdot)$ are C^2 functions with $s'_{\pm}(0) = \lambda'_{\pm}(0)\rho'(0) = \cos\theta$, and the points $B_{\pm}(\sigma) := \gamma_{\pm}(s_{\pm}(\sigma))$ of Γ_{\pm} are the closest to $\varepsilon(\sigma)$. We also reparametrize the normal vectors to Γ_{\pm} by setting $m_{\pm}(\sigma) := n_{\pm}(s_{\pm}(\sigma))$, then one has $B_{\pm}(\sigma) = \varepsilon(\sigma) + \rho(\sigma)m_{\pm}(\sigma)$. The above constructions are illustrated in Figure A.4.

For the C^2 maps $\Psi_{\pm}: (\sigma, \tau) \mapsto \varepsilon(\sigma) + \tau m_{\pm}(\sigma)$ one has $\Psi_{\pm}(0, 0) = (0, 0)$ and

$$\Psi'_{\pm}(0,0) = \begin{pmatrix} \frac{\partial \Psi_{\pm}}{\partial \sigma}(0,0) & \frac{\partial \Psi_{\pm}}{\partial \tau}(0,0) \end{pmatrix} = \begin{pmatrix} \varepsilon'(0) & m_{\pm}(0) \end{pmatrix} = \begin{pmatrix} 1 & -\sin\theta\\ 0 & \pm\cos\theta \end{pmatrix},$$

i.e. the Jacobian matrix $\Psi'_{\pm}(0,0)$ is invertible. Therefore, the maps Ψ_{\pm} are diffeomorphisms between suitable neighborhoods of the origin. Furthermore, if for t > 0 one introduces the curvilinear triangles $\Lambda_t := \{(\sigma, \tau) : 0 < \sigma < \sigma(t), 0 < \tau < \rho(\sigma)\}$, then the image $\Psi_{\pm}(\overline{\Lambda_t})$ is exactly the closure of the upper/lower V_t^{\pm} part of V_t , i.e. of the part of V_t lying above/below Σ , and $\Psi_+(\cdot, 0) = \Psi_-(\cdot, 0)$. We now use this observation to construct a map Φ with the sought properties. Namely, in addition to the above curvilinear triangles Λ_t let us consider its "straightened" version $L_t = \{(\sigma, \tau) : 0 < \sigma < \sigma(t), 0 < \tau < \sigma \sin \theta\}$. obtained by replacing ρ through its linear approximation at 0. The map $H : (\sigma, \tau) \mapsto (\sigma, \rho(\sigma)\tau/(\sigma \sin \theta))$ satisfies then $H'(0, 0) = I_2$, hence, it is a diffeomorphism between suitable neighborhoods of the origin, and for sufficiently small t > 0 it is bijective from $\overline{L_t}$ to $\overline{\Lambda_t}$.

Now let us consider the truncated sector $S_t := S_{\theta}^{\sigma(t) \cos \theta}$ and their upper/lower parts $S_t^{\pm} := S_t \cap \{(x_1, x_2) : \pm x_2 > 0\}$. One easily sees that the

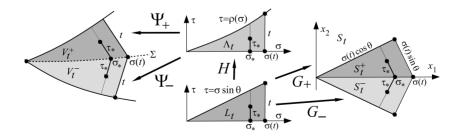


Figure A.5. The maps Ψ_{\pm} , H and G_{\pm} in the proof of Lemma A.3.

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maps

$$G_{\pm} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (\sigma, \tau) \mapsto \sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} -\sin\theta \\ \pm \cos\theta \end{pmatrix}$$

are diffeomorphisms, and $\overline{S}_t^{\pm} = G_{\pm}(\overline{L}_t)$ for small t > 0, and the inverses are given by

$$G_{\pm}^{-1}(x_1, x_2) = \begin{pmatrix} 1 & \pm \tan \theta \\ 0 & \pm \frac{1}{\cos \theta} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We refer to Figure A.5 for a graphical representation of the above maps.

Now let us define Φ by $\Phi(x_1, x_2) = \Psi_{\pm} \circ H \circ G_{\pm}^{-1}(x_1, x_2)$ for $\pm x_2 > 0$, which then extends by continuity to $x_2 = 0$ due to

$$\Psi_{\pm} \circ H \circ G_{\pm}^{-1}(x_1, 0) = \Psi_{\pm} \circ H(x_1, 0) = \Psi_{\pm}(x_1, 0) = \varepsilon(x_1).$$

By construction, the map Φ is C^2 on $\{\pm x_2 \ge 0\}$ and continuous along $x_2 = 0$, hence it is Lipschitz. Furthermore, by construction it defines bijections $S_t^{\pm} \to V_t^{\pm}, S_t \to V_t$ as well as $\partial_* S_t \to \partial_* V_t$ and $\partial_{\text{ext}} S_t \to \partial_{\text{ext}} V_t$. To estimate the Jacobian matrix Φ' we compute

$$(\Psi_{\pm} \circ H \circ G_{\pm}^{-1})'(0,0) = \Psi'_{\pm}(0,0)H'(0,0)(G_{\pm}^{-1})'(0,0)$$

$$= \begin{pmatrix} 1 & -\sin\theta\\ 0 & \pm\cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \pm\tan\theta\\ 0 & \pm\frac{1}{\cos\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix}$$

As Φ_{\pm} are C^1 , it follows that $\Phi'_{\pm} = I_2 + \mathcal{O}(t)$ in V_t , which shows the requested property for Φ' . As Φ_{\pm}^{-1} are C^1 near the origin and Φ^{-1} is continuous by construction, it follows that Φ^{-1} is Lipschitz, therefore, the map Φ is bi-Lipschitz. Hence, we obtain the claim with $r(t) = \sigma(t) \cos \theta$, and $r'(0) = \sigma'(0) \cos \theta = \cot \alpha \theta$.

For later references we mention explicitly the following corollary, which is quite obvious from the geometric point of view:

COROLLARY A.4. — There exist 0 < a < b such that for all sufficiently small t > 0 there holds |x| < bt for $x \in V_t$, and |x| > at for $x \in V_s \setminus \overline{V_t}$ and s > t.

Proof. — Let us use a map Φ and a function r as in Lemma A.3. Remark first that

(A.3)
$$\begin{aligned} |y| &< \frac{r}{\cos \theta} \text{ for } y \in \mathcal{S}_{\theta}^{r} \text{ and } r > 0, \\ |y| &> r \text{ for } y \in \mathcal{S}_{\theta}^{R} \setminus \overline{\mathcal{S}_{\theta}^{r}} \text{ and } R > r > 0. \end{aligned}$$

As $v \in V_t$ iff $v = \Phi(y)$ with $y \in S_{\theta}^{r(t)}$ and $r(t) = \mathcal{O}(t)$, by applying the Taylor expansion of Φ near the origin one obtains $\frac{1}{2}|y| \leq |v| \leq 2|y|$. Using the estimates (A.3) one arrives at the result.

We complete this subsection by a construction of cut-off functions with some special properties:

LEMMA A.5. — Let 0 < a < b, then there exist $\delta_0 > 0$, $\eta > 0$, K > 0and C^2 functions $\varphi_{\delta} : \overline{V_{\eta}} \to \mathbb{R}$ with $\delta \in (0, \delta_0)$ such that:

- (a) $0 \leq \varphi_{\delta} \leq 1$, and for all $\beta \in \mathbb{N}^2$ with $1 \leq |\beta| \leq 2$ there holds $\|\partial^{\beta}\varphi_{\delta}\|_{\infty} \leq K\delta^{-|\beta|}$,
- (b) $\varphi_{\delta} = 1$ in $V_{a\delta}$,
- (c) $\varphi_{\delta} = 0$ in $V_{\eta} \setminus \overline{V_{b\delta}}$,
- (d) the normal derivative of φ_{δ} at Γ_{\pm} is zero.

Proof. — For small $t_0 > 0$ and $s_0 > 0$ consider the maps

$$\Phi_{\pm}: (-s_0, s_0) \times (-t_0, t_0) \to \mathbb{R}^2, \quad \Phi_{\pm}(s, t) = \gamma_{\pm}(s) - tn_{\pm}(s).$$

It is a well known result of differential geometry that Φ_{\pm} are injective for $t_0 > 0$ small enough, with dist $(\Phi_{\pm}(\cdot, t), \Gamma_{\pm}) = |t|$ for $|t| < t_0$, and that they are C^2 -diffeomorphisms from $(-s_0, s_0) \times (-t_0, t_0)$ to its images under Φ_{\pm} . Remark that one has

$$\frac{\partial \Phi_{\pm}}{\partial s}(s,t) = \gamma_{\pm}'(s) - tn_{\pm}'(s) = \left(1 - tk_{\pm}(s)\right)\gamma_{\pm}'(s), \quad \frac{\partial \Phi_{\pm}}{\partial t}(s,t) = -n_{\pm}(s),$$

i.e. if one writes $(\tau_1^{\pm}, \tau_2^{\pm}) := \gamma_{\pm}'$ and $(n_1^{\pm}, n_2^{\pm}) := n_{\pm}$, then

$$\Phi'_{\pm}(s,t) = \begin{pmatrix} (1-tk_{\pm}(s))\tau_1^{\pm}(s) & -n_1^{\pm}(s) \\ (1-tk_{\pm}(s))\tau_2^{\pm}(s) & -n_2^{\pm}(s) \end{pmatrix}.$$

By choosing $\eta > 0$ sufficiently small one can then invert the maps $(s, t) \mapsto \Phi_{\pm}(s, t)$ near the origin in order to obtain C^2 functions s_{\pm} and t_{\pm} on V_{η} . The inverse function theorem gives

(A.4)
$$\nabla s_{\pm}(x) = \pm \frac{1}{1 - t_{\pm}(x)K_{\pm}(x)} \Big(N_2^{\pm}(x), -N_1^{\pm}(x) \Big),$$
$$K_{\pm} := k_{\pm} \circ s_{\pm}, \quad N_j^{\pm} := n_j^{\pm} \circ s_{\pm}.$$

In particular, $s_{\pm}(0,0) = 0$ and $\nabla s_{\pm}(0,0) = (\cos \theta, \pm \sin \theta)$, therefore,

$$s_{\pm}(x_1, x_2) = (\cos \theta, \pm \sin \theta) \cdot (x_1, x_2) + \mathcal{O}(x_1^2 + x_2^2) \text{ for } (x_1, x_2) \to (0, 0).$$

We further remark that for small s one has obviously $s_{\pm}(\gamma_{\pm}(s)) = s$, while

(A.5)
$$s_{\pm}(\gamma_{\mp}(s)) = (\cos\theta, \pm \sin\theta) \cdot \gamma'_{\mp}(0)s + \mathcal{O}(s^2)$$

 $\equiv \cos(2\theta)s + \mathcal{O}(s^2) \text{ for } s \to 0.$

Let us pick some $c \in (a \cot a \theta, b \cot a \theta)$ and then a sufficiently small $\varepsilon > 0$ satisfying

(A.6)
$$[c - \varepsilon, c + \varepsilon] \subset (a \cot a \theta, b \cot a \theta), \quad \cos(2\theta)(c + \varepsilon) < c - \varepsilon.$$

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We remark that the second condition follows from the first one for $\theta \geq \frac{\pi}{4}$. Let $\psi : \mathbb{R} \to [0,1]$ be a C^{∞} function with $\psi(s) = 1$ for $s < c - \varepsilon$ and $\psi(s) = 0$ for $s > c + \varepsilon$. For small $\delta > 0$ we define then $\varphi_{\delta} : \overline{V_{\eta}} \to \mathbb{R}$ by $\varphi_{\delta}(x) = \psi(s_{+}(x)/\delta)\psi(s_{-}(x)/\delta)$. Note that the property (a) is automatically satisfied due to the C^{2} smoothness of the functions s_{\pm} .

In order to see the properties (b) and (c) we first remark that due to Lemma A.1 the definition of the domain V_t for small t can be reformulated as $V_t := \{x \in V_\eta : s_{\pm}(x) < \lambda_{\pm}(t)\}$, and for small δ and a fixed A > 0one has $\lambda_{\pm}(A\delta) = A\delta \operatorname{cotan} \theta + \mathcal{O}(\delta^2)$. In particular, for $x \in V_{a\delta}$ one has $s_{\pm}(x) \leq a\delta \operatorname{cotan} \theta + \mathcal{O}(\delta^2) < (c - \varepsilon)\delta$ as δ is small, which shows that $\varphi_{\delta}(x) = 1$ and proves the claim(b). Furthermore, for $x \notin V_{b\beta}$ one of the following two inequalities holds: $s_{\pm}(x) > \lambda_{\pm}(b\delta)$. As $\lambda_{\pm}(b\delta) = b\delta \operatorname{cotan} \theta + \mathcal{O}(\delta^2) > (c + \varepsilon)\delta$, it follows that at least one of the terms $s_{\pm}(x)/\delta$ is greater than $c + \varepsilon$. As ψ vanishes in $(c + \varepsilon, +\infty)$, it follows that $\varphi_{\delta}(x) = 0$. This proves the claim (c).

Let us finally show the property (d). For a better readability we give the computation of the normal derivative on Γ_+ only, the case of Γ_- is handled in a completely similar way. For $x = \gamma_+(s) \in \Gamma_+$ with s > 0 one has

$$\begin{aligned} \frac{\partial \varphi_{\delta}}{\partial n_{+}}(x) &= n_{+}(s) \cdot (\nabla \varphi_{\delta}) \big(\gamma_{+}(s) \big) \\ &= \frac{1}{\delta} n_{+}(s) \cdot \left[(\nabla s_{+}) \big(\gamma_{+}(s) \big) \psi' \Big(\frac{s_{+} \big(\gamma_{+}(s) \big)}{\delta} \Big) \psi \Big(\frac{s_{-} \big(\gamma_{+}(s) \big)}{\delta} \Big) \\ &+ (\nabla s_{-}) \big(\gamma_{+}(s) \big) \psi' \Big(\frac{s_{-} \big(\gamma_{+}(s) \big)}{\delta} \Big) \psi \Big(\frac{s_{+} \big(\gamma_{+}(s) \big)}{\delta} \Big) \right]. \end{aligned}$$

By (A.4) one has $(\nabla s_+)(\gamma_+(s)) = (n_2^+(s), -n_1^+(s))$, which gives $n_+(s) \cdot (\nabla s_+)(\gamma_+(s)) = 0$, and the preceding expression simplifies to

$$\frac{\partial \varphi_{\delta}}{\partial n_{+}} (\gamma_{+}(s)) = \left[\frac{1}{\delta} n_{+}(s) \cdot (\nabla s_{-}) (\gamma_{+}(s)) \right] \psi' \left(\frac{s_{-} (\gamma_{+}(s))}{\delta} \right) \psi \left(\frac{s}{\delta} \right).$$

Let us show that the product of the last two terms is zero for small δ , i.e. that $\psi'(\xi(s))\psi(s/\delta) = 0$ for $\xi(s) := s_-(\gamma_+(s))/\delta$. First, by construction of ψ the second factor vanishes for $s \ge (c+\varepsilon)\delta$. Therefore, one needs to show that $\psi'(\xi(s)) = 0$ for all $0 < s \le (c+\varepsilon)\delta$ as δ is sufficiently small. Using the Taylor expansion (A.5) for small δ we have $\xi(s) = \cos(2\theta)s/\delta + \mathcal{O}(\delta)$. If $\theta \ge \frac{\pi}{4}$, then $\cos(2\theta) \le 0$, and $\xi(s) \le \mathcal{O}(\delta) < c - \varepsilon$. If $\theta < \frac{\pi}{4}$, then $\cos(2\theta) > 0$, and due to the choice of ε made in (A.6) one obtains $\xi(s) \le \cos(2\theta)(c+\varepsilon) + \mathcal{O}(\delta) < c - \varepsilon$. Therefore, in both cases one has $\xi(s) < c - \varepsilon$ for all $0 < s < (c + \varepsilon)\delta$ as δ is sufficiently small. As ψ was chosen constant on $(-\infty, c - \varepsilon)$, we have $\psi'(\xi(s)) = 0$.

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