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ISOPERIMETRIC INEQUALITIES, SHAPES OF FØLNER SETS AND GROUPS WITH SHALOM’S PROPERTY H_{FD}

by Anna ERSCHLER & Tianyi ZHENG (*)

ABSTRACT. — We prove an isoperimetric inequality for groups. As an application we show that any Grigorchuk group of intermediate growth has at least exponential Følner function. As another application, we obtain lower bounds on Følner functions in various nilpotent-by-cyclic groups. Under a regularity assumption, we obtain a characterization of Følner functions of these groups. As a further application, we evaluate the asymptotics of the Følner function of $\text{Sym}(\mathbb{Z}) \rtimes \mathbb{Z}$. We study examples of groups with Shalom’s property H_{FD} among nilpotent-by-cyclic groups. We show that there exist lacunary hyperbolic groups with property H_{FD} . We find groups with property H_{FD} , which are direct products of lacunary hyperbolic groups and have arbitrarily large Følner functions.

RÉSUMÉ. — Nous prouvons une inégalité isopérimétrique pour les groupes. En application, nous montrons que la fonction de Følner de tout groupe de Grigorchuk à croissance intermédiaire est au moins exponentielle. En tant qu’autre application, nous obtenons des bornes inférieures sur les fonctions de Følner dans divers groupes nilpotents par cycliques. Sous une hypothèse de régularité, nous obtenons une caractérisation des fonctions de Følner de ces groupes. Comme autre application, nous évaluons le comportement asymptotique de la fonction de Følner de $\text{Sym}(\mathbb{Z}) \rtimes \mathbb{Z}$. Nous étudions des exemples de groupes avec la propriété de Shalom H_{FD} parmi les extensions d’un groupe nilpotent par un groupe cyclique. Nous montrons qu’il existe des groupes hyperboliques lacunaires avec la propriété H_{FD} . Nous trouvons des groupes avec la propriété H_{FD} , qui sont des produits directs de groupes hyperboliques lacunaires et ont des fonctions Følner arbitrairement grandes.

1. Introduction

Given a finitely generated group G , equipped with a finite generating set S , we denote by d_S the word metric on G with respect to S and $l_S(g)$ the

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word length $d_S(e, g)$. For a subset $V \subset G$, the boundary $\partial_S V$ of V is the set of elements of V at the word distance one from the complement of V : $v \in V$ such that $d_S(V, G \setminus V) = 1$. We denote by $F_{G,S}(\epsilon)$ the minimum of the cardinality of the sets V such that $\#\partial_S V / \#V \leq \epsilon$, where the sign $\#V$ denotes the cardinality of V . We put $\text{Fol}_{G,S}(n) = F_{G,S}(1/n)$. $\text{Fol}_{G,S}(n)$ is called the Følner function of G . It is well defined whenever G is a finitely generated amenable group.

In this paper we prove an isoperimetric inequality for groups and study its applications. Our result provides information on the structure of Følner sets: we prove that they contain subsets that are “satisfactory” in the sense defined below.

Given a set T in a group G and a subset $V \subset G$, let us say that $v \in V$ is r -good, if there exists at least r distinct elements $u \in T$ such that $vu \in V$. Given a constant $C > 0$ and a finite subset $T \subset G$, we say that V is a C -satisfactory set with respect to T if each $v \in V$ is $C\#T$ -good.

THEOREM 1.1. — *There exist constants $C_1, C_2 \in (0, 1)$ such that the following holds. Let G be a group, S be a finite generating set of G . Let $T \subset G$ be a set such that $l_S(t) \leq r$ for all $t \in T$. Take a subset $V \subset G$ such that $\#\partial_S V / \#V \leq C_1/r$. Then V contains a subset V' which is C_2 -satisfactory with respect to T . In particular, we can take $C_1 = 1/24$ and $C_2 = 1/4$.*

Given radius r , finding an “optimal” set T such that every subset $V \subset G$ such that $\#\partial_S V / \#V \leq C_1/r$ implies V contains a subset which is C_2 -satisfactory with respect to T is related to the *dual isoperimetric problem* discussed in [25, Section 6.3].

Combining Theorem 1.1 with Lemma 2.3, which will be proven in Section 2, we have the following:

COROLLARY 1.2. — *Let G be a group with finite generating set S . Suppose G contains as a subgroup $H = \bigoplus B_i$, $i \in I$, where I is a countable set. Let $v(i, n) = \#\{b \in B_i : b \neq e, l_S(b) \leq n\}$ and*

$$N(n, k) = \#\{i : v(i, n) \geq k\}.$$

Then the Følner function of G satisfies

$$\text{Fol}_{G,S}(n) \geq (k+1)^{CN(n,k)}$$

for any $n, k \in \mathbb{N}$ and some absolute constant $C > 0$.

Corollary 1.2 follows as well from an inequality of Saloff-Coste and the second author [33, Proposition 3.1]; and the case with $k = 1$ from the

inequalities proven in Gromov [25, Section 6.1] where the fibers B_i in the product were assumed to be cyclic assuring that the lower bound is valid not only for the Følner function, but also the linear algebraic Følner function, introduced in [25]. Before explaining examples where the estimate from Theorem 1.1 is better than the one obtained from Corollary 1.2, as in Corollary 1.4 below, we point out some consequences of Corollary 1.2.

Given two functions f_1, f_2 we say that $f_1 \preceq f_2$ if $f_1(x) \leq C f_2(Kx)$ for some positive constants C, K and all x . If $f_1 \preceq f_2$ and $f_2 \preceq f_1$, we say that f_1, f_2 are asymptotically equivalent : $f_1 \simeq f_2$. It is clear that the asymptotic class of the Følner function of G does not depend on the choice of a finite generating set S in G . We recall that the growth function $v_{G,S}(n)$ counts the number of elements of word length $\leq n$. The asymptotic class of $v_{G,S}$ does not depend on the choice of the generating set. We write Fol_G and v_G for the equivalence classes of the corresponding functions.

Recall that the wreath product of A and B , which we denote by $A \wr B$, is a semi-direct product of A and $\sum_A B$, where A acts on $\sum_A B$ by shifts (in many papers, the authors denote it by $B \wr A$). As a particular application of Corollary 1.2, we have that the Følner function of the wreath product $G = A \wr B$, where B is a nontrivial finite group satisfies $\text{Fol}_G(n) \geq \exp(v_A(n))$. This statement has been known previously since it follows from the inequality of Coulhon and Saloff-Coste [6] and isoperimetric inequality for wreath product [11] applied to the case of finite lamp groups. Obtaining this statement as a consequence of Corollary 1.2 provides a unified setting for these inequalities. To see that Corollary 2 the Coulhon–Saloff-Coste inequality $v_A(n) \leq \text{Fol}_A(n)$, for a group A , take a finite group $B = \mathbb{Z}/2\mathbb{Z}$, and fix a finite generating set S of A . Observe that Følner function of $G = A \wr B$ with respect to generating set $\tilde{S} = S \cup \{(\delta_{e_A}, e_A)\}$ is clearly $\leq 2^{\text{Fol}_{A,S}(n)}$. On the other hand, as stated above, we have $\text{Fol}_{G,\tilde{S}}(n) \geq \exp(Cv_{A,S}(n))$. We conclude therefore that $\text{Fol}_{A,S}(n) \geq v_{A,S}(C''n)$, for some positive C'' and all n .

Another application of Corollary 1.2 is the following. See the end of Section 2 for the proof of its statement.

Example 1.3. — The Følner function of the Grigorchuk group G_ω , where $\omega \in \{0, 1, 2\}^{\mathbb{N}}$ is not eventually constant, satisfies

$$\text{Fol}_{G_\omega,S}(n) \geq 2^{Cn}$$

for some absolute constant $C > 0$.

These groups are introduced by Grigorchuk in [19] as first examples of groups of intermediate growth. In particular, the exponential lower bound

on the Følner functions cannot be obtained by the Coulhon–Saloff-Coste inequality mentioned above. It is not known if there exists a finitely generated group with intermediate Følner function, see [21, Problem 12]. The Gap Conjecture of Grigorchuk on Følner functions ([21, Conjecture 12.3(ii)]) states that the Følner function of a finitely generated group is either polynomial or at least exponential. This conjecture is an isoperimetric counterpart of the more well-known Gap Conjecture on volume growth formulated by Grigorchuk in [20], which asks if a finitely generated group G with volume growth function asymptotically strictly smaller than $e^{\sqrt{n}}$ is of polynomial growth. The volume growth lower bound $e^{n^{1/2}}$ for G_ω , where ω is not eventually constant, is due to Grigorchuk (see [19], where it is proven using so called “anti-contraction” property). A weaker form of the Gap Conjecture asks if there exists a $\beta > 0$ such that $v_G(n) = e^{o(n^\beta)}$ implies that G is of polynomial growth.

The lower bound obtained from Theorem 1.1 can be better than the estimate that one gets from the product subgroups as in Corollary 1.2. One of the illustrations is provided in the Corollary 1.4 below.

COROLLARY 1.4. — *Consider a group $\text{Sym}(H)$ of finitary permutations of the elements of a finitely generated group H . Let $G = \text{Sym}(H) \rtimes H$ be the extension of this group by H . Then the Følner function of G is asymptotically greater or equal to $v_H(n)^{v_H(n)}$.*

In particular, the Følner function of $\text{Sym}(\mathbb{Z}) \rtimes \mathbb{Z}$ is asymptotically equivalent to n^n . More generally, the Følner function of $\text{Sym}(\mathbb{Z}^d) \rtimes \mathbb{Z}^d$ (or of $\text{Sym}(N) \rtimes N$, N of growth $\simeq n^d$) is asymptotically equivalent to n^{n^d} .

As another application of the isoperimetric inequalities, we construct nilpotent-by-cyclic groups with prescribed isoperimetry, see Theorem 1.5 below. In particular, for any $a \geq 1$, there exists a nilpotent-by-cyclic group with Følner function asymptotically equivalent to $\exp(n^a)$. Alternatively, we can choose the group to be step-2 nilpotent-by-abelian. In the special case for $a \in [1, 2]$, the group can be chosen to be step-2 nilpotent-by-cyclic.

Gromov states in [25, Section 8.2, Remark (b)] that locally-nilpotent-by-cyclic groups give examples of elementary amenable groups with prescribed Følner functions, provided that the prescribed function has sufficiently fast growing derivatives. Further examples of groups with prescribed Følner functions were provided in the paper of Brieussel and the second named author [3]. In contrast to examples in [3], the groups we consider here are nilpotent-by-cyclic, and the regularity assumption on the prescribed function is milder. In the special case that the prescribed function is comparable to $\exp(n)$ for a union of infinitely many sufficiently long intervals, the group

in Theorem 1.5 mentioned above can be chosen to have some additional properties, namely to have cautious and diffusive random walk along an infinite time subsequence. Recall that for a random walk (W_n) on G , we say it is *diffusive* along a time subsequence if there is a constant $C > 0$ such that

$$(1.1) \quad \liminf_n \mathbb{E}(l_S(W_n))/\sqrt{n} \leq C.$$

Following [16], we say that a μ -random walk on G is *cautious* if for every $c > 0$,

$$(1.2) \quad \limsup_n \mathbb{P}\left(\max_{1 \leq k \leq n} l_S(W_k) \leq c\sqrt{n}\right) > 0.$$

It is not known whether the diffusive condition (1.1) and the cautious condition (1.2) are equivalent.

THEOREM 1.5. — *Let $\mathfrak{c} \in \mathbb{N}$ and $\tau : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing function such that*

$$0 \leq \tau(n) \leq n^{\mathfrak{c}}.$$

Then there exists a nilpotent-by-cyclic group $G = N \rtimes \mathbb{Z}$ and a constant $C > 1$ such that N is nilpotent with class $\leq \mathfrak{c}$ and the Følner function of G satisfies

$$Cn \exp(n + \tau(n)) \geq \text{Fol}_{G,S}(n) \geq \exp\left(\frac{1}{C}(n + \tau(n/C))\right).$$

Alternatively, a group satisfying the estimate above can be chosen as a direct product $G_1 \times G_2$ where each G_i , $i \in \{1, 2\}$, is a nilpotent-by-cyclic group on which simple random walk is cautious.

The rest of the Introduction is devoted to discussions on some properties of the examples we construct in Theorem 1.5 and Section 7 in connection to the phenomenon of oscillating Følner functions and drift functions and Shalom's property H_{FD} .

1.1. Oscillating Følner functions and oscillating drift functions

Theorem 1.5 allows us to construct groups with prescribed Følner functions, in particular, the prescribed Følner function can oscillate. Recall the drift function of a random walk with step distribution μ is defined as

$$L_\mu(n) = \sum_{x \in G} l_S(e, x)\mu^{(n)}(x).$$

The phenomenon of oscillating Følner function and drift function has been observed in various classes of groups. We give a brief review here.

Grigorchuk [19] has constructed groups G_ω of intermediate growth and shown that if ω contains long subsequence of constant $000\dots$ then G_ω in some scale is close to a solvable group which is a subgroup of H^m (m depends on the scale), where H is commensurable with a wreath product on which one can check that simple random walk has drift $L_{H,\nu}(n) \leq C\sqrt{n}$. Thus the results of [19] imply that for any $\alpha(n)$ tending to infinity, there exists a choice of sequence ω such that the resulting group G_ω is of intermediate volume growth and simple random walk μ on it satisfies

$$L_\mu(n_i) \leq \alpha(n_i)\sqrt{n_i}$$

along an infinite subsequence (n_i) . For the same reason we have that along a subsequence (m_i) , $\text{Fol}_{G_\omega}(m_i) \leq \exp(m_i\alpha(m_i))$, and one can find a Følner set V_i with $\#\partial V_i/\#V_i \leq 1/m_i$ within the ball of radius $m_i\alpha(m_i)$. On the other hand, if ω contains long subsequences of $012012012\dots$, then [19] shows that G_ω is close on some scales to a group commensurable with G_{012} . The group G_{012} is defined with sequence $\omega = (012)^\infty$ and is usually referred to as the first Grigorchuk group. The result of [12] shows that the drift function of simple random walk on G_{012} satisfies $L_{G_{012},\mu}(n) \geq n^{0.65}$ for infinitely many n . This implies that by choosing ω appropriately, the drift of simple random walk on G_ω can oscillate between $n^{1/2}\alpha(n)$ and $n^{0.65}/\alpha(n)$ for any $\alpha(n) \rightarrow \infty$ as $n \rightarrow \infty$, see [14, Section 3.1].

The construction of piecewise automatic groups in [13] (see also [14]) shows that in a group of intermediate growth, the drift function $L_\mu(n)$ can oscillate between $n^{1/2}\alpha(n)$ and arbitrarily close to linear; and the Følner function can be bounded from above by $\exp(n\alpha(n))$ along one subsequence, and arbitrarily large along another subsequence.

In all classes mentioned above it is not known how to evaluate the asymptotics of $L_\mu(n)$ and $\text{Fol}_G(n)$.

The evaluation of $L_\mu(n)$ in a family of examples including those with oscillating drift functions was done by Amir and Virág [1], who has shown that a function f satisfies the condition that there is a constant $\gamma \in [3/4, 1)$ such that $a^{3/4}f(x) \leq f(ax) \leq a^\gamma f(x)$ for all $a > 0, x > 0$, then f is equivalent to the drift function of simple random walk on some group. In a paper of Brieussel and the second author [3], via a different construction, it was shown that for any function satisfying $a^{1/2}f(x) \leq f(ax) \leq af(x)$, there is a group such that f is equivalent to the drift function of simple random walk on it. This construction also provides examples of groups with prescribed Følner function.

The examples of G_ω in [19], piecewise automatic groups in [13] and examples of [3] can be chosen to satisfy $L_\mu(n) \leq \sqrt{n}\alpha(n)$ for infinitely many n for any given α tending to infinity, but none of these examples can have $L_\mu(n_i) \leq C\sqrt{n_i}$ along some infinite subsequence (n_i) . Now our examples discussed in Subsection 4.2, Section 7 and those described in [4, Subsection 3.3] provide groups with oscillating drift function satisfying diffusive upper bound along a subsequence (1.1). More precisely, examples in Subsection 4.2 can be chosen to satisfy the following.

COROLLARY 1.6. — *Given any $\beta \in (\frac{1}{2}, 1)$, there exists a nilpotent-by-cyclic group G such that for any symmetric probability measure μ with finite generating support on G , the drift function $L_\mu(n)$ satisfies that $L_\mu(t_i) \leq Ct_i^{1/2}$ along an infinite subsequence (t_i) and $L_\mu(n_i) \geq \frac{1}{C}n_i^\beta$ along another infinite subsequence (n_i) .*

Corollary 1.6 is proved in Subsection 4.3.

1.2. Følner functions and groups admitting controlled Følner pairs along a subsequence

Theorem 1.5 provides groups with Følner function asymptotically equivalent to $e^{n+\tau(n)}$ under some assumptions on τ . Among them some groups have the additional property that they admit controlled Følner pairs along some subsequences. In Subsection 4.4, we explain that if a group G admits a subsequence of controlled Følner pairs, then G has Shalom's property H_{FD} (see Corollary 4.6). Below we recall necessary background and definitions.

Følner pairs were introduced in Coulhon, Grigoryan and Pittet [5] to produce lower bounds on return probability (equivalently upper bounds on ℓ^2 -isoperimetric profiles). We recall the definition: a sequence (F'_n, F_n) of pairs of finite subsets of G with $F'_n \subset F_n$ is called a sequence of *Følner pairs* adapted to an increasing function $\mathcal{V}(n)$ if there is a constant $C < \infty$ and such that for all n , we have

- (i) $\#F_n \leq C\#F'_n$,
- (ii) $d(F'_n, G \setminus F_n) \geq n$,
- (iii) $\#F_n \leq \mathcal{V}(Cn)$.

Note that admitting such a sequence of Følner pairs implies that $\text{Fol}_G(n) \preceq \mathcal{V}(n)$.

Følner pairs provide lower bound for return probability $\mu^{(2n)}(e)$ for a symmetric probability measure μ of finite support on G , see [5]. We mention that the group with prescribed Følner function in Theorem 1.5 admit Følner pairs adapted to a functions of the same order as its Følner function. For example, from Theorem 1.5, we have nilpotent-by-cyclic groups with return probability decay equivalent to e^{-n^α} , for any $\alpha \in [1/3, \infty)$.

The notion of controlled Følner pairs was introduced by Tessera in [37]. We say (F'_n, F_n) with $F'_n \subset F_n$ is a sequence of *controlled Følner pairs* if they satisfy (i) and (ii) as above, and

$$(iii') \quad F_n \subset B(e, Cn).$$

Note that it follows from definition that controlled Følner pairs are adapted to the volume growth function of the group. We say that a group admits a subsequence of controlled Følner pairs if there is an infinite subsequence (n_i) and pairs (F'_{n_i}, F_{n_i}) which satisfy (i), (ii) and (iii').

Recall that as defined by Shalom in [35], a group G have property H_{FD} if every orthogonal G -representation π with non-zero reduced cohomology $\overline{H}^1(G, \pi)$ admits a finite-dimensional sub-representation.

By [35], having property H_{FD} is invariant under quasi-isometries of amenable groups. Thus it seems natural to find sufficient conditions for property H_{FD} which is stable under quasi-isometry as well. The first sufficient criterion for Property H_{FD} which uses random walks is given by Ozawa in [32]. The main application of Ozawa's argument is a new proof of Gromov's polynomial growth theorem. However in [32, Proposition], a sufficient criterion for H_{FD} is formulated for general groups. In [16, Corollary 2.5], it is shown that if G admits a symmetric measure μ with finite support such that for every $c > 0$,

$$(1.3) \quad \limsup_n \mu^n(B(e, c\sqrt{n})) > 0,$$

then G has property H_{FD} . Admitting a subsequence of controlled Følner pairs implies (1.2), see Lemma 4.5, therefore it implies the group has Property H_{FD} by [16, Corollary 2.5].

The conditions in [32, Proposition] are not straightforward to verify even in the simplest examples of exponential growth groups, see [15]. The condition of admitting controlled Følner pairs is more geometric in the sense that it is stable under quasi-isometry (see [37]). It is not known whether the conditions in [32, Proposition] and of [16, Corollary 2.5] are quasi-isometric invariant.

On the other hand, there exist groups with Shalom's property H_{FD} such that simple random walks do not satisfy (1.3), see [4] and Subsection 4.4.

Such examples show that admitting a cautious simple random walk is a strengthening of Property H_{FD} . In particular, we show that groups with property H_{FD} where simple random walks do not satisfy (1.3) can be found among nilpotent-by-cyclic groups considered in Theorem 1.5. Indeed, from the connection of Følner function to the decay of return probability, a lower bound on the Følner function implies an upper bound on the quantity $\mu^n(B(e, r))$, see Remark 4.3. Therefore random walks on groups with Følner functions growing sufficiently fast do not satisfy (1.3).

In Section 7, we construct locally-nilpotent-by-cyclic groups with Shalom's property H_{FD} that are lacunary hyperbolic. By definition, lacunary hyperbolic groups are hyperbolic on some scales, in some sense they are very “large” on these scales. We show that they can be constructed to be at the same time quite “small” on some other scales. Namely, they can have simple random walk which is cautious and diffusive along some infinite subsequence. Among direct products of lacunary hyperbolic groups, there are groups with Property H_{FD} whose Følner function can be arbitrarily large.

There are open questions for some other aspects of lacunary hyperbolic groups, regarding how “small” they can be on the scales that are not hyperbolic. For example, Olshanskii, Osin and Sapir [30] ask whether a lacunary hyperbolic group can have sub-exponential volume growth or non-uniform exponential growth. On the other hand, some questions regarding how “large” an amenable group with Shalom's property H_{FD} can be are also open. We mention that in particular, it is not known whether a finitely generated amenable group admitting a simple random walk with non-trivial Poisson boundary can have property H_{FD} .

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2. Proofs of the isoperimetric inequalities

Let G be a finitely generated group, S be a finite generating set. Recall that l_S denotes the word distance on (G, S) . Take a subset $V \subset G$ such that $\#\partial_S V / \#V \leqslant 1/r$.

LEMMA 2.1 (Generalized Coulhon–Saloff–Coste inequality). — For each $p : 0 < p < 1$, $C_1 > 0$ such that $C_1 < 1 - p$ there exists $C > 0$ such that the following holds. Let $V \subset G$ be a finite subset such that $\#\partial_S V / \#V \leq C_1/r$. Suppose also that T is a subset such that $l_S(t) \leq r$ for any $t \in T$. Then for at least $p\#V$ element $v \in V$ there exists at least $C\#T$ distinct elements $t \in T$ such that $vt \in V$. More precisely, we can take $C : 1 - C = C_1/(1 - p)$.

Proof. — Suppose not. Then for at least $1 - p$ proportion of the elements v of V , there is strictly less than $C\#T$ multiplications by elements in T for which $vt \in V$. The idea of the proof is analogous to one of the known arguments (see e.g. [24, 6.43]) for the proof of the Coulhon–Saloff–Coste inequality [6].

For each $v \in V$ consider the sets $U_v = \{vs, s \in T\}$, and consider the total number N of point of the union U_v intersected with $G \setminus V$, taking into account the multiplicity, that is

$$N = \sum_{v \in V} \#(U_v \cap (G \setminus V)).$$

This cardinality N is at most $\#\partial_S V \#Tr$. Indeed, for each element $t \in T$ fix a geodesics γ_t from e to T . If for some $v \in V$ the element $vt \notin V$, then the shifted geodesic $v, v\gamma_t$ intersects the boundary ∂V at least once. Let $u \in \partial V$ be the first intersection of $v, v\gamma_t$ with the boundary ∂V , and let $d = d(v, u)$. It is clear that $d \leq l_S(t)$ and by the assumption of the lemma we have therefore $d \leq r$. Observe that the point v is uniquely defined by u, t and d . Therefore the upper bound on N follows.

On the other hand, by the assumption of the lemma for at least $(1 - p)\#V$ points v in V there exist at least $(1 - C)\#T$ elements in T such that vt is not in V . Therefore, $N \geq (1 - p)\#V(1 - C)\#T$. We conclude that

$$(1 - p)\#V(1 - C)\#T < N \leq \#\partial V \#Tn \leq \frac{C_1}{r} \#V \#Tr,$$

and hence

$$(1 - p)(1 - C) < C_1,$$

which contradicts the choice of C_1 . \square

The following combinatorial edge removal lemma is along the same line of reasoning as [11, Lemma 1].

LEMMA 2.2. — Let \mathcal{V} be a graph with vertex set V and edge set E . A vertex v is m -good if it has at least m edges connecting to distinct neighbors. Suppose that at least $p\#V$ vertices are m -good, where $p > 2/3$, then \mathcal{V} contains a non-empty subgraph such that every vertex is (cm) -good for where $c < \min\{(3p - 2)/p, 1/2\}$.

Proof. — Consider all the vertices in V that are not (cm) -good. Remove all of them and all the edges adjacent to them. After the removal there can be new vertices that are not (cm) -good. Remove again all of them and their adjacent edges. Repeat the process. We need to show that the process stops before the graph becomes empty.

Orient the removed edges AB as $A \rightarrow B$ if A is removed earlier than B . Suppose v is a m -good vertex, but gets removed after several steps. Consider edges adjacent to v , then at least $(1 - c)m$ edges must be removed in order for v to fail to be (cm) -good. This implies that number of oriented edges ending at v is at least $(1 - c)/c$ as the number of edges starting with v .

Let C_1 be the number edges in \mathcal{V} such that at least one of its end vertices is not m -good. Let C_2 be the number of edges removed that originally both of its end vertices are m -good. Since for every m -good vertex that gets removed at some stage, the number of edges ending at v is at least $(1 - c)/c$ as the number of edges starting with v , we have

$$\text{total number of edges removed} \leq C_1 + C_2 \leq C_1 + \frac{c}{1 - c}C_1 = \frac{1}{1 - c}C_1.$$

Note that $C_1 \leq m(1 - p)\#V$. On the other hand, the total amount of edges in the graph \mathcal{V} satisfies

$$|E| \geq \frac{1}{2}p\#Vm.$$

Since $c < (3p - 2)/p$, we have that the total number of edges removed is strictly less than $|E|$, therefore the remaining graph is non-empty. This finishes the proof of the lemma. \square

Now we return to the proof of Theorem 1.1.

Proof of Theorem 1.1. — Given any $p : 2/3 < p < 1$, $0 < C_1 < 1 - p$, let $V \subset G$ be a finite subset such that $\#\partial_S V/\#V \leq C_1/r$. Then by Lemma 2.1, for at least $p\#V$ elements v in V there exists at least $C\#T$ distinct multiplications by $t \in T$ such that $vt \in V$. Now consider the graph \mathcal{V} for which the vertex set consists of elements in V and $v, u \in V$ are connected by an edge if there is some $t \in T$ such that $u = vt$. Then we have that at least $p\#V$ vertices are $C\#T$ -good. By Lemma 2.2, \mathcal{V} contains a subgraph \mathcal{V}' such that every vertex is $cC\#T$ -good, where $c < \min\{(3p - 2)/p, 1/2\}$. In other words, the vertex set of \mathcal{V}' is cC -satisfactory with respect to T . The choice of constants in the statement is somewhat arbitrary: we take $p = 5/6$, $C_1 = 1/24$ and $c = 1/3$. \square

In order to derive a lower bound for Følner function from Theorem 1.1, we need to estimate the volume of a set V' which is C -satisfactory with respect to T . In the situation that the set T is contained in a product

subgroup $\prod B_i$, one can bound volume of V' from below by the following lemma. It is analogous to [11, Lemma 3]

LEMMA 2.3 (Satisfactory sets for G containing $\bigoplus B_i$). — Let G be a group containing as a subgroup $H = \bigoplus B_i, i \in I$, I is a countable set. Let S_i be a subset of B_i . Given a subset $V \subset G$, consider the cardinality of the following set: for $g \in V$, $k \in \mathbb{N}$,

$$M_k(g) = \#\left\{i \in I : \begin{array}{l} \text{there exists at least } k \text{ distinct } s_i \neq e, s_i \in S_i, \\ \text{such that } gs_i \in V \end{array}\right\}.$$

If for each $g \in V$, the cardinality $\#M_k(g) \geq m$, then the cardinality of the set V is at least $(1 + k)^m$.

Proof. — Induction on m . The statement is obviously true for $m = 1$. Suppose we have proved the statement for $m - 1$. Consider the set of coset classes $G = \bigcup g_\alpha H$, $G_\alpha := g_\alpha H$. $V = \bigcup V_\alpha$. If V satisfies the assumption, all V_α satisfy it also. It is enough therefore to assume that V belongs to one coset class, and without loss of the generality we can assume that $V \subset H = \prod B_i$. Fix one position i where the value B_i takes at least $k + 1$ possible values. For each fixed value apply the induction hypothesis for $m - 1$. The cardinality of V is at least $(k + 1)$ times as much.

Proof of Corollary 1.2. — Let G be a finitely generated group, S is a finite generating set, $N(n, k)$ defined as the statement. For each index $i \in N(n, k)$, choose k distinct elements $\{b_{1,i}, \dots, b_{k,i}\}$ from the set $\{b \in B_i : b \neq e, l_S(b) \leq n\}$. Define T as the union of these elements

$$T = \bigcup_{i \in N(n, k)} \{b_{1,i}, \dots, b_{k,i}\}.$$

Consider a Følner set $V \subset G$ such that $\#\partial V/\#V \leq 1/n$. Then by Theorem 1.1, V contains a subset V' which is C -satisfactory with respect to the set T defined above. It follows from the definition of C -satisfactory and the structure of T that for any $g \in V'$,

$$M_{k'}(g) \geq \frac{C}{2 - C} N(n, k), \quad \text{where } k' = \max\{1, Ck/2\}.$$

By Lemma 2.3 the cardinality of such set V' is at least $(k' + 1)^{\frac{C}{2 - C} N(n, k)}$. Consequently, the cardinality of V admits the same lower bound. We conclude that the Følner function of G satisfies

$$\text{Fol}_G(n) \geq (k' + 1)^{\frac{C}{2 - C} N(n, k)} \geq (k + 1)^{C' N(n, k)},$$

where C, C' are some universal constants. □

Proof of the statement of Example 1.3. — We apply Corollary 1.2 to show that the Følner function of the first Grigorchuk group is at least exponential.

We recall the definition of the first Grigorchuk group $G = G_{012}$, which was defined by Grigorchuk in [18]. The group G_{012} acts on the rooted binary tree T , it is generated by automorphisms a, b, c, d defined as follows. The automorphism a permutes the two subtrees of the root. The automorphim b, c, d are defined recursively:

$$b = (a, c), \quad c = (a, d), \quad d = (1, b).$$

For more details see [18, 19] and also [28, Chapter VIII], [2, Chapter 1].

The rigid stabilizer of a vertex u in G , denoted by $\text{Rist}_G(u)$ is the subgroup of G that consists of these automorphisms that fix all vertices not having u as a prefix. By definition it is clear that automorphisms in rigid stabilizers of different vertices on a given level commute. To obtain a lower bound for the Følner function, given a level k , consider the subgroup $\bigoplus_{u \in T_k} \text{Rist}_G(u)$, where T_k denotes the level k vertices in the tree. We show that for in each summand $\text{Rist}_G(u)$, $u \in T_k$, there is an element $g_u \in \text{Rist}_G(u)$ such that $g_u \neq e$ and $l_S(g) \leqslant 6 \cdot 2^k$.

Consider the substitution σ which was used in the proof of the growth lower bound in [19, Theorem 3.2]:

$$\sigma(a) = aca, \quad \sigma(b) = d, \quad \sigma(c) = b, \quad \sigma(d) = c.$$

First take $g_{1^k} = \sigma^k(abab)$, where $1^k = \overbrace{1 \dots 1}^k$. We verify that g_{1^k} is a non-trivial element in the rigid stabilizer of the vertex 1^k . To show this, it suffices to have

$$g_{1^k} = (1, \sigma^{k-1}(abab)),$$

where 1 is the identity element. By the definition of σ , we have that for any word w in the letters $\{a, b, c, d\}$, $\sigma(w)$ is in the stabilizer of the first level of the tree, hence we can write $\sigma(w) = (w_1, w_2)$. Observe that $w_2 = w$ and w_1 can be obtained from w by sending $a \rightarrow d$, $b \rightarrow \emptyset$, $c \rightarrow a$, $d \rightarrow a$, where \emptyset denote the empty word. Note that the word w_1 is on letters $\{a, d\}$ only. In the first iteration, $\sigma(abab) = acadacad$, by direaction calculation we have that

$$g_1 = \sigma(abab) = (1, abab).$$

For $k \geqslant 1$, $\sigma^k(ab)$ is a word in blocks of $acob$, $acac$ and $acad$. Under the substitution,

$$\sigma(acab) = (dad, acab), \quad \sigma(acac) = (dada, acab), \quad \sigma(acad) = (dada, acad).$$

Let w_k be the word on the left branch of $\sigma^{k+1}(abab)$ under the wreath recursion. Then w_k is a product of blocks dad and $dada$. Since a, d generate the dihedral group $\langle a, d \mid a^2 = 1, d^2 = 1, (ad)^4 = 1 \rangle$, we have that $(dad)^2 = 1$, $(dada)^2 = 1$ and the two elements $dad, dada$ commute. Therefore w_k can only evaluate to $dad, dada$ or their product $daddada = a$ in $\langle a, d \rangle$. In each case we have that $w_k w_k$ evaluates to the identity element. It follows that

$$\sigma^{k+1}(abab) = (w_k w_k, \sigma^k(ab)\sigma^k(ab)) = (1, \sigma^k(abab)).$$

We have proved the claim that $g_{1^{k+1}} = (1, g_{1^k})$. Since σ doubles the word length in each iteration, we have that $l_S(g_{1^k}) \leq 2^{k+2}$.

To get nontrivial elements in Rist_u for other vertices in level k , we take appropriate conjugations of the element g_{1^k} . Since the Schreier graph of the level- k vertices with respect to generators $\{a, b, c, d\}$ is connected (and has diameter 2^k), for any $u \in T_k$, we can fix an element $\rho_u \in G$ such that $u \cdot \rho_u = 1^k$ and $l_S(\rho_u) \leq 2^k$. Take

$$g_u = \rho_u g_{1^k} \rho_u^{-1}.$$

Since g_{1^k} is a nontrivial element in $\text{Rist}_G(1^k)$, it is clear that g_u is a nontrivial element in $\text{Rist}_G(u)$. We have that $l_S(g_u) \leq 2^k + 2^{k+2} + 2^k = 6 \cdot 2^k$. Apply Corollary 1.2 with $H = \bigoplus_{u \in T_k} \text{Rist}_G(u)$ and $n = 6 \cdot 2^k$, we have that for some absolute constant $C > 0$,

$$\text{Fol}_{G,S}(6 \cdot 2^k) \geq 2^{C2^k},$$

which is valid for all k . In other words, we have an exponential lower bound $\text{Fol}_{G,S}(n) \geq 2^{Cn/6}$.

The same argument applies to more general Grigorchuk group G_ω , where ω is not eventually constant. Namely, by taking appropriate substitutions, we can find nontrivial elements in $\text{Rist}_{G_\omega}(u)$ of word length at most $6 \cdot 2^k$ for every $u \in T_k$. It follows from Corollary 1.2 that there is an absolute constant $C > 0$ such that for any $\omega \in \{0, 1, 2\}^{\mathbb{N}}$ which is not eventually constant,

$$\text{Fol}_{G_\omega,S}(n) \geq 2^{Cn}.$$

By [19], these groups G_ω have sub-exponential volume growth. \square

3. Abelian extensions of nilpotent groups: first examples

In this section we consider step-2-nilpotent-by-cyclic groups, which we denote by $G = G_{\mathbb{Z}, \text{Nil}, 2}$, $G_{D, 2}$, and $G_{D, 2, \mathbf{k}}$, see definitions below. Their Følner functions can be estimated by applying the isoperimetric inequalities from the previous section.

Consider 2-step free nilpotent group $N = N_{\mathbb{Z}}$, with generators $b_i, i \in \mathbb{Z}$. Let $N_{\mathbb{Z},2}$ be the quotient of N , over relations $b_i^2 = 1$ for all $i \in \mathbb{Z}$. Let \mathbb{Z} act on the group $N_{\mathbb{Z},2}$ by shifts of the index set, and consider the extension group $G = G_{\mathbb{Z},\text{Nil},2}$.

Consider N and $b_i, i \in \mathbb{Z}$, as above, put $b_{i,j} = [b_i, b_j]$, $i > j$. Note that $b_{i,j}^2 = 1$ and $[b_j, b_i] = [b_i, b_j]^{-1} = b_{i,j}^{-1}$. Given a subset $D \subset \mathbb{Z}$, consider a subgroup B_D of the group generated by $b_{i,j}$, which is generated by $b_{i,j}$ such that $i - j \in D$. Observe that B_D is a normal subgroup of $G_{\mathbb{Z},\text{Nil},2}$. Indeed, subgroup generated by $b_{i,j}$ is central in the subgroup generated by b_i , and the action an element $z \in \mathbb{Z}$ sends $b_{i,j}$ to $b_{i+z,j+z}$, and preserves $i - j$.

Denote by $G_{D,2}$ the quotient group of $G_{\mathbb{Z},\text{Nil},2}$ over B_D .

Moreover, given a sequence $\mathbf{k} = (k_j)$ with $k_j \in \mathbb{N}$, $j \in \mathbb{Z}$, consider the quotient of $G = G_{\mathbb{Z},\text{Nil},2}$ over relations

$$b_{i,i+j} = b_{i+k_j, i+j+k_j}.$$

A subgroup generated by such elements is a normal subgroup, we denote by $G_{\mathbb{Z},\text{Nil},2,\mathbf{k}}$ the quotient of $G = G_{\mathbb{Z},\text{Nil},2}$ by this normal subgroup and we denote by $G_{D,2,\mathbf{k}}$ the quotient of $G_{D,2}$ by the image of this subgroup in $G_{D,2}$.

Example 3.1. — When \mathbf{k} is constant 1, which means $[b_i, b_j] = [b_{i+k}, b_{j+k}]$ for all $i, j, k \in \mathbb{Z}$, the groups $G_{D,2,1}$ were considered in P. Hall [27]. Note that in this special case $G_{D,2,1}$ is a central extension of $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$.

The groups $G_{D,2}$ and $G_{D,2,\mathbf{k}}$ admit the following normal forms for elements. The group $G_{\mathbb{Z},\text{Nil},2}$ is generated by two elements: b_0 and a generator z_0 of \mathbb{Z} . We use this generating set $\{b_0, z_0\}$ for its quotients as well. Any element of $G_{\text{Nil},2}$ can be written a (f, z) , where $z \in \mathbb{Z}$ and $f = \prod_{-\infty}^{\infty} b_i^{\epsilon_i} \prod_{i < j} b_{i,j}^{\epsilon_{i,j}}$, where each $\epsilon_i, \epsilon_{i,j} \in \{0, 1\}$. In the product there are only finitely many terms with ϵ_i or $\epsilon_{i,j}$ nonzero. Note that since the elements $b_{i,j}$ are in the center of $N_{\mathbb{Z},2}$, the ordering of $b_{i,j}$ does not matter. Similarly, any element of $G_{D,2}$ can be written as (f, z) , where $z \in \mathbb{Z}$,

$$f = \prod_{-\infty}^{\infty} b_i^{\epsilon_i} \prod_{i < j, j-i \notin D} b_{i,j}^{\epsilon_{i,j}}.$$

In the further quotient $G_{D,2,\mathbf{k}}$, an element can be written as (f, z) , where $z \in \mathbb{Z}$,

$$f = \prod_{-\infty}^{\infty} b_i^{\epsilon_i} \prod_{\substack{i < j, j-i \notin D, \\ 0 \leq i < k_{j-i}}} b_{i,j}^{\epsilon_{i,j}}.$$

Given a word in $z_0^{\pm 1}, b_0$, the standard commutator collecting procedure (see [26, Chapter 11]) rewrites it into the normal form described above.

As an illustration of the isoperimetric inequality in Corollary 1.2, we first consider the following example.

Example 3.2. — The Følner function of the group $G_{\mathbb{Z}, \text{Nil}, 2}$ is asymptotically equivalent to $\exp(n^2)$.

Proof. — Consider the subgroup $B_r \subset B_{\mathbb{Z}}$ generated by $b_{i,j}$, $-r \leq i < j \leq r$. The length of each $b_{i,j}$ is at most $C'r$, and B_r is isomorphic to the product of $\langle b_{i,j} \rangle$, $-r \leq i < j \leq r$. Applying Corollary 1.2 we conclude that the Følner function of G is at least $\exp(Cn^2)$.

The upper bound that the Følner function is bounded by $\exp(C'n^2)$ is explained in the proof of (1) of the Corollary 3.3 below, applied by taking D to be the empty set. \square

More generally, Corollary 1.2 provides an optimal lower bound for the Følner functions of various quotients of $G_{\text{Nil}, 2}$.

COROLLARY 3.3 (Følner function of $G_{D, 2}$ and $G_{D, 2, \mathbf{k}}$).

- (1) Let $\rho_D(n)$ be the cardinality of $(\mathbb{N} \setminus D) \cap [1, n]$ and $D \subset \mathbb{N}$. Then the Følner function of $G_{D, 2}$ is asymptotically equivalent to

$$\exp(n\rho_D(n) + n).$$

In particular, for any non-decreasing $\rho(n)$ such that $\rho(n+1) - \rho(n) \leq 1$ the function $\exp(n\rho(n) + n)$ is a equivalent to a Følner function of G_D , for some choice of a subset $D \subset \mathbb{N}$.

- (2) Let $\tau_{D, \mathbf{k}}(n) = \sum_{j: 1 \leq j \leq n, j \notin D} \min\{k_j, n\}$. Then the Følner function of $G_{D, \mathbf{k}}$ satisfies

$$\exp(Cn + C\tau_{D, \mathbf{k}}(Cn)) \leq \text{Føl}_{G_{D, \mathbf{k}}}(n) \leq \exp(4n + 2\tau_{D, \mathbf{k}}(2n)).$$

In particular, for any non-decreasing function τ such that $\tau(n+1) - \tau(n) \leq n$ the function $n + \tau(n)$ is asymptotically equivalent to $\log \text{Føl}_G(n)$ for some $G = G_{D, 2, \mathbf{k}}$.

Proof.

(1). — We say that a configuration f is contained in an interval I if in the normal form of f , all the nonzero entries $\epsilon_i, \epsilon_{i,j}$ satisfy $i, j \in I$. For $D \subset \mathbb{Z}$ consider the subset $\Omega_D(n)$ of $G_{D, 2}$,

$$\Omega_D(n) = \{(f, z) : 0 \leq z \leq n, \text{ the configuration } f \text{ is contained in } [0, n]\}.$$

Observe that if $g \in \Omega_D(n)$, then $zb_0 \in \Omega_D(n)$. Therefore, if $g \in \partial\Omega_D(n)$, then $gz_0^{\pm 1} \notin \Omega_D(n)$ and $z = 0$ or $z = n$. For $g \in \Omega_D(n)$, consider its “slice”

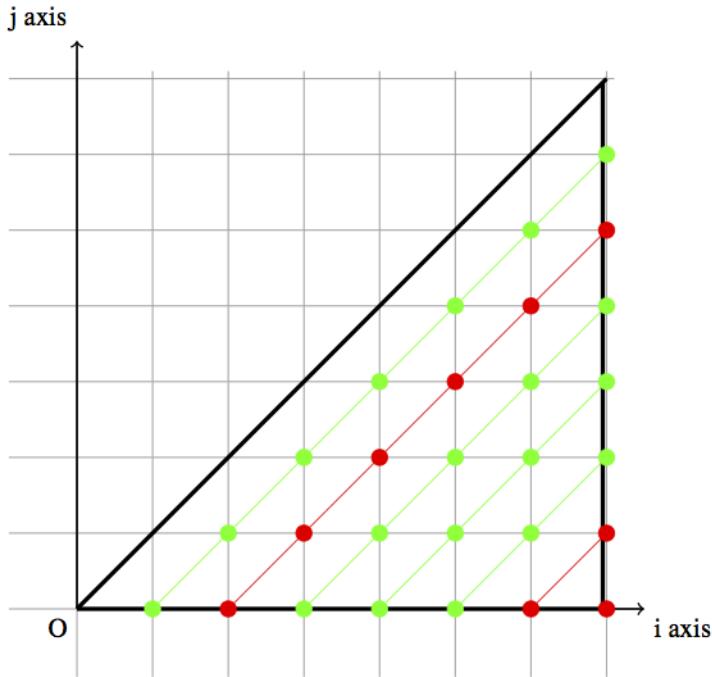


Figure 3.1. Supports of configurations $b_{i,j}$ for g in the Følner set $\Omega(n)$ (of the group $G_{\mathbb{Z}, Nil, 2}$) are nodes inside the black triangle; for $\Omega_D(n)$ (in the group $G_{D, 2}$) these are green nodes inside the black triangle: green diagonals consist of $i, j, i > j$ such that $i - j \notin D$; this picture is for $n = 7$, D contains 2, 6 and 7 and does not contain 1, 3, 4 and 5

in $\Omega_D(n)$: $h \in \Omega_D(n)$ such that $h = fz_0^m$, here z_0 is a generator of \mathbb{Z} and m is some integer. Observe that for each $g \in \Omega_D(n)$, the cardinality of such slice is equal to $n + 1$ and exactly two points of each slice belong to the boundary $\partial\Omega_D(n)$, for all $n \geq 1$.

Therefore for all $n \geq 1$

$$\frac{\#\partial\Omega_D(n)}{\#\Omega_D(n)} = \frac{2}{n+1}.$$

Observe that the cardinality of $\Omega_D(n)$ is equal to

$$(n+1)2^{n+1+\sum_{i \in [0, n] \cap (\mathbb{N} \setminus D)} (n+1-i)}.$$

It implies that for any D , the Følner function of $G_{D, 2}$ is at most $2n \exp(2n + 2n\rho_D(2n))$.

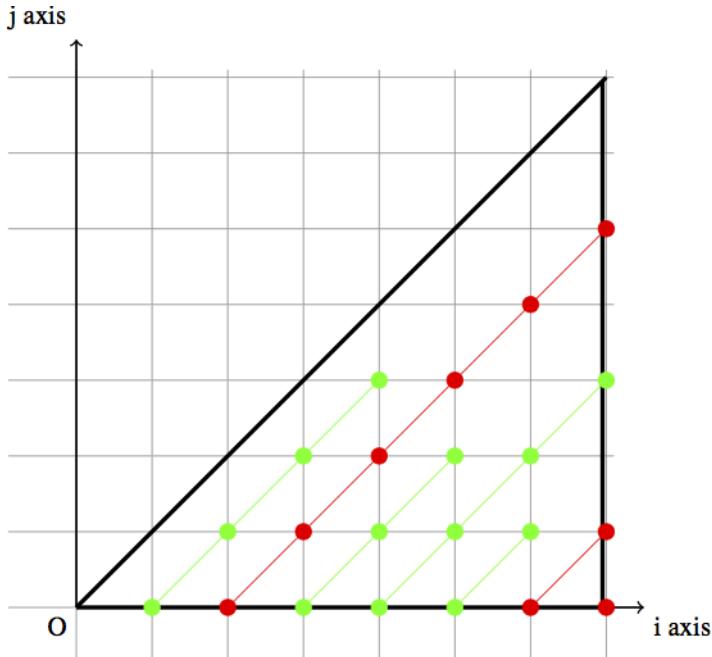


Figure 3.2. Supports of configurations $b_{i,j}$ for g inside the Følner set $\Omega_{D,k_j}(n)$ (of the group $G_{D,2,\mathbf{k}}$) are green nodes of truncated diagonals inside the black triangle. The choice the set of green diagonals (consisting of $i, j : i - j \notin D$) and the choice of the lengths k_i of truncated diagonals is arbitrary in the definition of $G_{D,2,\mathbf{k}}$; this picture is for $n = 7$, D contains 2, 6 and 7 and does not contain 1, 3, 4, and 5, $k_1 = 5$, $k_3 = 4$, $k_4 \geq 5$, $k_5 = 3$

Let us show that Corollary 1.2 implies that there exists $C > 0$ such that the Følner function of $G_{D,2}$ is greater than $\exp(Cn\rho_D(Cn))$. Consider the collection of $b_{i,j}$ with $0 \leq i < j \leq n$, $j - i \notin D$. The length of $b_{i,j}$ is bounded by $l_S(b_{i,j}) \leq 4n$, and the subgroup they generate is isomorphic to a product $\prod \langle b_{i,j} \rangle$ where in the product there are $\sum_{i \in [0,n] \cap (\mathbb{N} \setminus D)} (n+1-i)$ copies of $\mathbb{Z}/2\mathbb{Z}$. By Corollary 1.2, we have that

$$\text{Fol}_{G_{D,2}}(n) \geq \exp \left(C \sum_{i \in [0,n] \cap (\mathbb{N} \setminus D)} (n+1-i) \right) \geq \exp \left(\frac{Cn}{2} \rho_D(n/2) \right).$$

Since $G_{D,2}$ is of exponential growth, we also have $\text{Fol}_{G_{D,2}}(n) \geq \exp(Cn)$. We conclude that

$$\text{Fol}_{G_{D,2}}(n) \sim \exp(n(\rho_D(n) + 1)).$$

Finally, given any non-decreasing function $\rho(n)$ with $\rho(n+1) - \rho(n) \leq 1$, we can select the set $D \subset \mathbb{N}$ such that

$$(\mathbb{N} \setminus D) \cap [0, n] \sim \rho(n).$$

Then the corresponding group $G_{D,2}$ has Følner function as stated.

(2). — The proof is similar. Given a subset $D \subset \mathbb{N}$ and a sequence $\mathbf{k} = (k_j)$, consider the function

$$\tau_{D,\mathbf{k}}(n) = \sum_{i \in [0, n] \cap (\mathbb{N} \setminus D)} \min\{n, k_i - i\}.$$

With the same argument as in the previous part we see that the cardinality of $\Omega_{D,\mathbf{k}}$ is equal to

$$(n+1)2^{n+1+\sum_{i \in [0, n] \cap (\mathbb{N} \setminus D)} \min\{n+1-i, k_i\}},$$

and that for all $n \geq 1$

$$\frac{\#\partial\Omega_{D,\mathbf{k}}(n)}{\#\Omega_{D,\mathbf{k}}(n)} = \frac{2}{n+1}.$$

Observe that

$$\tau_{D,\mathbf{k}}(n/2) \leq \sum_{i \in [0, n] \cap (\mathbb{N} \setminus D)} \min\{n+1-i, k_i\} \leq \tau_{D,\mathbf{k}}(n).$$

The Følner function of $G_{D,2,\mathbf{k}}$ is therefore at most $\exp(n + \tau_{D,\mathbf{k}}(n))$. By the same argument as in the previous part, from Corollary 1.2 we know that the Følner function of $G_{D,2,\mathbf{k}}$ is greater than $\exp(Cn + C\tau_{D,\mathbf{k}}(Cn))$ for some $C > 0$.

Given a prescribed non-decreasing function τ with $\tau(n+1) - \tau(n) \leq n$, we select $D \subset \mathbb{N}$ and k_j , $j \in \mathbb{N}$ by the following rule: if $\lfloor \tau(n) \rfloor = \lfloor \tau(n-1) \rfloor$, then $n \in D$; otherwise $n \notin D$ and $k_n = \lfloor \tau(n) \rfloor - \lfloor \tau(n-1) \rfloor$. Then we have

$$\tau(n/2) \leq \tau_{D,\mathbf{k}}(n) \leq \tau(n).$$

Then the statement follows. \square

Remark 3.4. — Under the assumptions of part (2) of Corollary 3.3, we are not able to say that $\text{Fol}_{G_{D,\mathbf{k}}}(n)$ is asymptotically equivalent to $\exp(n + \tau(n))$ because it is not necessarily true that there exists a constant $A > 1$ such that $\tau(An) \geq 2\tau(n)$ for all n . For example, such constant A does not exist for the following function τ : take a fast growing sequence (n_i) such

that n_{i+1} is much larger than n_i^2 , and take τ to be equal to n_i^2 on the interval $[n_i, n_i^2]$ and linear in between such intervals.

4. Further examples. Groups with strengthened versions of Property H_{FD}

For nilpotent groups of higher nilpotency class, we can take cyclic extensions similar to the previous section. We consider in this section two specific constructions. In these examples, the group G is a quotient of the semi-direct product $N_{\mathbb{Z}, \text{Nil}, c} \rtimes \mathbb{Z}$, where $N = N_{\text{Nil}, \mathbb{Z}, c}$ is the step c free nilpotent group generated by b_i , $i \in \mathbb{Z}$, subject to relation $b_i^2 = 1$ for $i \in \mathbb{Z}$. The upper bound for the Følner function of G follows from taking the standard test sets similar to these in Corollary 3.3. We apply the isoperimetric inequality in Corollary 1.2 to obtain lower bound for the Følner function: we count the rank of the subgroups in the center of the nilpotent group N whose generators are inside the ball of distance n in G .

4.1. A construction similar to $G_{D,2}$

We first recall the notion of basic commutators on letters b_i , $i \in \mathbb{Z}$, as in [26, Ch.11]. The basic commutators, together with their weight and an order, are defined recursively as

- (1) $c_i = b_i$, $i \in \mathbb{Z}$ are the basic commutators of weight 1, $w(b_i) = 1$; ordered with $b_i < b_j$ if $i < j$.
- (2) Having defined the basic commutators of weight less than n , the basic commutators of weight n are $[u, v]$ where u and v are basic with $w(u) + w(v) = n$, $u > v$, and if $u = [u_1, u_2]$ then $u_2 \leqslant v$.
- (3) For a basic commutator u of weight n and a basic commutator v of weight $n - 1$, we have $u > v$. Commutators of the same weight can be ordered arbitrarily, we use the following: for commutators on different strings, order them lexicographically, and for different bracketing of the same string, order them arbitrarily with respect to each other.

The basis theorem (see [26, Theorem 11.2.4]) states that the basic commutators of weight n form a basis for the free abelian group F_n/F_{n+1} , where $F_1 = F$ is the free group on generators $(b_i)_{i \in \mathbb{Z}}$ and $F_n = [F, F_{n-1}]$. Observe that the relations $b_i^2 = 1$ imply that $u^2 = 1$ for any basic commutator u .

Consider the free nilpotent group $N_{\mathbb{Z}, \text{Nilc}}$ as above and the extension $N_{\mathbb{Z}, \text{Nilc}} \rtimes \mathbb{Z}$, where \mathbb{Z} acts by translating generators of $N_{\mathbb{Z}, \text{Nilc}}$, that is $b_i^z = b_{i+z}$. Note that \mathbb{Z} acts also on commutators by translating the indices, and we write u^z for the conjugation: for u a basic commutator on string $(b_{i_1}, \dots, b_{i_k})$, write u^z for the same bracketing on the string $(b_{i_1+z}, \dots, b_{i_k+z})$.

We will take quotients of the group $N_{\mathbb{Z}, \mathfrak{c}} \rtimes \mathbb{Z}$ in what follows. First we set some notations. Let Q be a subset of basic commutators of $N_{\mathbb{Z}, \mathfrak{c}}$ satisfying the condition that

- If $u \in Q$, then $u^z \in Q$ as well for any $z \in \mathbb{Z}$,
- If $[u, v] \in Q$ where u and v are basic commutators, then $u, v \in Q$ as well.

Let $N_{\mathbb{Z}, Q}$ be the quotient of $N_{\mathbb{Z}, \mathfrak{c}}$ with the relations that all basic commutators except those in $Q^{\mathbb{Z}}$ are set to be identity. Note that such an operation is well defined because the basic commutators form a basis in the free nilpotent group $N_{\mathbb{Z}, \mathfrak{c}}$.

The commutators in Q inherit the ordering of basic commutators. The standard word collecting process in $N_{\mathbb{Z}, Q}$ (see [26, Section 11.1]) yields a normal form for elements: every element $f \in N_{\mathbb{Z}, Q}$ can be written uniquely as an ordered product of b_j 's and basic commutators in Q , that is

$$f = \prod_{c \in Q} c^{\delta_c} \prod_{j \in \mathbb{Z}} b_j^{\varepsilon_j},$$

where $b_j, \delta_c \in \{0, 1\}$ and only finitely many b_j and δ_c 's are 1.

Let $E_d(n)$ be the subset of weight d basic commutators on letters from the set $\{b_{-n}, \dots, b_n\}$. Write $E(n) = \bigcup_{d=2}^{\mathfrak{c}} E_d(n)$ and $E_d = \bigcup_{n=1}^{\infty} E_d(n)$. By construction, basic commutators in $Q \cap E_d$ form a basis for the abelian quotients $(N_{\mathbb{Z}, Q})_d / (N_{\mathbb{Z}, Q})_{d+1}$, for each $2 \leq d \leq \mathfrak{c}$.

The extensions by \mathbb{Z} of N_Q is denoted by $G_Q = N_Q \rtimes \mathbb{Z}$.

By a similar reasoning as in the step 2 case, we obtain a lower bound of the Følner function by counting the number of basis element in each abelian quotient.

COROLLARY 4.1. — *Let $\rho_Q(n)$ be the cardinality of the set $Q \cap E(n)$. Then there exists a constant C depending on \mathfrak{c} such that the Følner function of $G_Q = N_{\mathbb{Z}, Q} \rtimes \mathbb{Z}$ satisfies*

$$Cn \exp(\rho_Q(n)) \geq \text{Fol}_{G_Q}(n) \geq \exp\left(\frac{1}{C}\rho_Q(n)\right).$$

Proof. — We show the lower bounds first. Given n , let $d = d(n) \in \{1, \dots, \mathfrak{c}\}$ be an index such that $|E_d(n) \cap Q| \geq \frac{1}{\mathfrak{c}}\rho(n)$. Consider the projection of G_Q to step d quotient, so that now $Q^{\mathbb{Z}} \cap E_d$ is in the center of the nilpotent group. Observe that each basic commutator $Q^{\mathbb{Z}} \cap E_d(n)$ is within distance $2^{d+2}n$ to identity element of G_Q . The number of such non-trivial basis element is $|E_d(n) \cap Q|$. We can now apply Corollary 1.2 to obtain the lower bound on $\text{Fol}_{G_Q}(n)$ in the same way as in Corollary 3.3.

The upper bound on the Følner functions follow from sets Ω_n which is defined as all group elements that can be written as (f, z) where $z \in [-n, n]$ and f is a product of basic commutators only involving generators b_j with $j \in [-n, n]$. \square

4.2. A construction similar to $G_{D,2,k}$ and proof of Theorem 1.5

Now similar to the case when \mathbb{Z} acts on a step-2 nilpotent group, we can consider the case that on a nilpotent group of step \mathfrak{c} , where $\mathfrak{c} \geq 2$, and impose relations such that the \mathbb{Z} -orbits of commutators are periodic. Note that introducing such relations on commutators has consequences on the relations of higher commutators where they appear. Here we will consider a specific way of assigning periodicity, which is sufficient to prove Theorem 1.5 on nilpotent-by-cyclic groups with prescribed Følner function.

Proof of Theorem 1.5. — If τ is bounded, then the lamplighter group $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$ satisfies the inequality in the statement. In what follows we assume that τ is unbounded.

Define a sequence of indices (k_i) recursively as

$$(4.1) \quad k_{i+1} = \min\{n > k_i : \tau(2^n) \geq 2\tau(2^{k_i})\}.$$

By assumption on τ , we have that $\tau(2^{k_i}) \leq (2^{k_i})^{\mathfrak{c}}$. For each i , let \mathfrak{c}_i be the integer such that

$$(2^{k_i})^{\mathfrak{c}_i-1} < \tau(2^{k_i}) \leq (2^{k_i})^{\mathfrak{c}_i}.$$

For each k_i , we consider the group N_i which is a quotient group of $N_{\mathbb{Z}, \mathfrak{c}_i}$. The group N_i is generated by $\{b_j, j \in \mathbb{Z}\}$, with relations

$$b_i^2 = 1, \quad b_i = b_j \text{ if } i \equiv j \pmod{2^{k_i}},$$

and relations imposing that a commutator on a string $(b_{i_1}, b_{i_2}, \dots, b_{i_\ell})$ is trivial whenever

$$\min_{1 \leq j, k \leq \ell} d(\bar{i}_j, \bar{i}_k) < 2^{k_i - \mathfrak{c}_i},$$

where d denotes the distance on the cycle $\mathbb{Z}/2^{k_i}\mathbb{Z}$. Note that the finite cyclic group $\mathbb{Z}/2^{k_i}\mathbb{Z}$ acts on N_i by shift. Let G_i be the extension of N_i by $\mathbb{Z}/2^{k_i}\mathbb{Z}$. By construction, the $\mathbb{Z}/2\mathbb{Z}$ rank of the center of N_i satisfies

$$\frac{1}{C}(2^{k_i})^{\mathfrak{c}_i} \leq \text{Rank}_{\mathbb{Z}/2\mathbb{Z}}(\text{Center}(N_i)) \leq C(2^{k_i})^{\mathfrak{c}_i}.$$

Under the action of \mathbb{Z} , the elements of the center of N_i are divided into $\mathbb{Z}/2^{k_i}\mathbb{Z}$ -orbits and each orbit has size 2^{k_i} .

Further, we choose to keep $\lfloor \tau(2^{k_i})/2^{k_i} \rfloor$ of distinct $\mathbb{Z}/2^{k_i}\mathbb{Z}$ -orbits under the action of $\mathbb{Z}/2^{k_i}\mathbb{Z}$ in the center of N_i . The other orbits in the center of N_i under action of $\mathbb{Z}/2^{k_i}\mathbb{Z}$ are set to be equal to identity element. Since this operation is performed in the center, it does not affect the lower levels. We denote by \bar{N}_i this quotient group of N_i . By construction, the $\mathbb{Z}/2\mathbb{Z}$ -rank of the center of \bar{N}_i is comparable to $\tau(2^{k_i})$.

We denote by \bar{G}_i the extension of \bar{N}_i by $\mathbb{Z}/2^{k_i}\mathbb{Z}$. It is a quotient group of $\Gamma = N_{\mathbb{Z},\mathfrak{c}} \rtimes \mathbb{Z}$. Let π_i be the quotient map

$$\pi_i : \Gamma \rightarrow \bar{G}_i.$$

We take the group G in the claim of the theorem to be

$$G = \Gamma / \bigcap_{i=1}^{\infty} \ker (\pi_i : \Gamma \rightarrow \bar{G}_i).$$

That is, G is the smallest group that projects onto each \bar{G}_i marked with generating set $\{b_0, t\}$.

We now verify that the Følner function of G satisfies the estimate as stated. In order to prove the lower bound for the Følner function, for $n \in [2^{k_i}, 2^{k_{i+1}}]$, consider the quotient group \bar{G}_i , then we have

$$\text{Fol}_{G,S}(n) \geq \text{Fol}_{\bar{G}_i,S}(2^{k_i}).$$

By construction of the group \bar{G}_i , we have $\tau(2^{k_i})$ central nodes within distance $C2^{k_i}$. Therefore, by Corollary 1.2 we have

$$\text{Fol}_{\bar{G}_i,S}(C2^{k_i}) \geq \exp(\tau(2^{k_i})/C).$$

By definition of k_{i+1} we have that $\tau(n/2) < 2\tau(2^{k_i})$, thus we have proved the lower bound on $\text{Fol}_{G,S}(n)$.

In order to show the upper bound of the Følner function, note that for $n \in [2^{k_i}, 2^{k_{i+1}}]$, the ball of radius n around identity in \bar{G}_j with $2^{k_j-\mathfrak{c}} > n$ are the same as the ball of same radius in $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$. Then take the test set to be those elements which have support contained in $[-n, n]$. We have

that the volume of this set is bounded from above by

$$2n2^{2n} \prod_{j: 2^{k_j-\epsilon} < n} \#\bar{G}_j \leq 2n2^{2n} \prod_{j: 2^{k_j-\epsilon} < n} \exp(C\tau(2^{k_j})) \leq \exp(Cn + C\tau(Cn)).$$

In the last inequality, we used the fact that by choice of k_j , $\tau(2^{k_{j-1}}) \leq \frac{1}{2}\tau(2^{k_j})$, thus the summation in the exponential function is bounded by a geometric sum. This completes the proof of the first claim of the theorem.

Next we show the second claim of the theorem. For a sequence of increasing integers (m_i) , and a given prescribed function τ , set

$$\tau_1(n) = \begin{cases} \tau(n) & \text{for } n \in [m_{2j-1}, m_{2j}] \\ \tau(m_{2j}) & \text{for } n \in [m_{2j}, m_{2j+1}]; \end{cases}$$

and

$$\tau_2(n) = \begin{cases} \tau(m_{2j-1}) & \text{for } n \in (m_{2j-1}, m_{2j}) \\ \tau(n) & \text{for } n \in [m_{2j}, m_{2j+1}]. \end{cases}$$

Then both functions satisfy the assumption that $0 \leq \tau_i(n) \leq n^\epsilon$, $i = 1, 2$. Then as in the proof above, there is a nilpotent-by-cyclic group G_i such that $\log F\mathcal{O}l_{G_i}$ is equivalent to $n + \tau_i(n)$. It follows that for the direct product $\Gamma = G_1 \times G_2$, $\log F\mathcal{O}l_\Gamma(n)$ is equivalent to $n + \max\{\tau_1(n), \tau_2(n)\} = n + \tau(n)$.

It remains to verify that for sufficiently fast growing sequence (m_i) , the random walk on each G_i , $i = 1, 2$, is cautious. Since τ_1 is constant on $[m_{2j}, m_{2j+1}]$, it follows from the construction that there is index $i(j)$ such that

$$2^{k_{i(j)}} \leq m_{2j} \text{ and } 2^{k_{i(j)+1}} \geq m_{2j+1}.$$

It implies that in G_1 , the ball of radius cm_{2j+1} is the same as in

$$\tilde{G}_{i(j)} = \Gamma / \left(\bigcap_{1 \leq k \leq i(j)} \ker(\Gamma \rightarrow \bar{G}_k) \cap \ker(\pi_0 : \Gamma \rightarrow \mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})) \right).$$

The group $\tilde{G}_{i(j)}$ fits into the exact sequence

$$1 \rightarrow N_{i(j)} \rightarrow \tilde{G}_{i(j)} \rightarrow \mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z}),$$

where $N_{i(j)}$ is a finite nilpotent group depending only on value of τ_1 on $[0, m_{2j}]$. Take m_{2j+1} to be sufficiently large such that

$$m_{2j+1} \gg \text{Diam}_{\tilde{G}_{i(j)}, S}(N_{i(j)}).$$

Then we have that for simple random walk on G_1 , $k \leq cm_{2j+1}$,

$$|W_k| \leq \text{Diam}_{\tilde{G}_{i(j)}, S}(N_{i(j)}) + |\bar{W}_k|,$$

where \overline{W}_k is the projection of the random walk to $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$. It follows that for $cm_{2j+1} > t \gg \text{Diam}_{\bar{G}_{i(j)}, S}(N_{i(j)})$, for any $c' > 0$

$$\mathbb{P}\left(\max_{1 \leq k \leq t} |W_k|_{G_1} \leq c'\sqrt{t}\right) \geq \mathbb{P}\left(\max_{1 \leq k \leq t} |\overline{W}_k|_{G_1} \leq \frac{c'}{2}\sqrt{t}\right) \geq \delta c',$$

where $\delta(c') > 0$ is a constant only depending on c' . The last inequality used the fact that simple random walk on the lamplighter $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$ is cautious. The argument for the random walk on G_2 is the same, by choosing m_{2j} to be sufficiently larger than m_{2j-1} . \square

Example 4.2. — A few examples of functions that satisfy the assumption of Theorem 1.5:

- $\tau(n) = n^\alpha$, for some $\alpha \in [1, \mathfrak{c}]$.
- The function τ admitting an increasing sequence of integers (n_j) and $\alpha, \beta \in (0, \mathfrak{c}]$ such that

$$\tau(n) = n^\alpha \text{ for } n \in [n_{2j-1}, n_{2j}] \text{ and } \tau(n) = n^\beta \text{ for } n \in [n_{2j}, n_{2j+1}].$$

Note that if $\alpha \leq 1$, then $n + n^\alpha$ is equivalent to n .

- The function τ admitting an increasing sequence of integers (n_j) and $\alpha \in (1, \mathfrak{c}]$ such that

$$\tau(n) = n^\alpha \text{ for } n \in [n_{2j-1}, n_{2j}]$$

and τ is constant on $[n_{2j}, n_{2j+1}]$.

4.3. Corollaries on return probability and drift

By the general relation between isoperimetry and decay of return probability (see e.g. [39, Theorem 14.3]), a lower bound on Følner function implies an upper bound on the return probability $\mu^n(e)$. In particular, if there is some $\delta > 0$ such that the Følner function of G satisfies $\text{Fol}_G(n) \geq \exp(n^{2+\delta})$ over a sequence of sufficiently long sub-intervals, then there exists a time subsequence (n_i) such that

$$\mu^{(2n_i)}(e) \leq \exp\left(-n_i^{\frac{2+\delta}{4+\delta}}\right).$$

Recall the drift function of a random walk with step distribution μ is defined as

$$L_\mu(n) = \sum_{x \in G} d_S(e, x)\mu^{(n)}(x).$$

Remark 4.3. — Although in general it is not sharp, one can use the fact that return probability $\mu^{(2n)}$ attains its maximum at identity e to give the following bounds:

$$\sum_{g \in B(e,r)} \mu^{(2n)}(g) \leq \sum_{g \in B(e,r)} \mu^{(2n)}(e) = v_{G,S}(r) \mu^{(2n)}(e).$$

Since the entropy satisfies $H_\mu(2n) \geq -\log \mu^{(2n)}(e)$, we have

$$L_\mu(2n) \geq c H_\mu(2n) \geq -c \log \mu^{(2n)}(e).$$

It follows that in this case if there is some $\delta > 0$ such that the Følner function of G satisfies $\text{Føl}_G(n) \geq \exp(n^{2+\delta})$ over a sequence of sufficiently long sub-intervals, the μ -random walk is neither cautious nor diffusive.

We now show that groups considered in the previous section with appropriate chosen parameters provide examples with oscillating drift function as stated in Corollary 1.6 in the Introduction.

Proof of Corollary 1.6. — Given β , let α be an exponent such that $\frac{\alpha}{2+\alpha} > \beta$. Take a rapidly growing sequence (m_i) , and set

$$\tau(n) = n^\alpha \text{ for } n \in [m_{2j}, m_{2j+1}] \text{ and } \tau(n) = m_{2j+1}^\alpha \text{ for } n \in [m_{2j+1}, m_{2j+2}).$$

By Theorem 1.5, there is a nilpotent-by-cyclic group G such that

$$\frac{1}{C}(n + \tau(n)) \leq \log \text{Føl}_G(n) \leq Cn + C\tau(n).$$

Recall that in the proof of Theorem 1.5, the group G is defined as the smallest group that projects onto a sequence \bar{G}_i marked with generating set $\{b_0, t\}$, where by construction the Cayley graph of \bar{G}_i coincide with the Cayley graph of the lamplighter $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$ in the ball of radius 2^{k_i} around identity. The sequence (k_i) is determined from the prescribed function τ as in (4.1).

From the definition of the function τ and the sequence (k_i) as in (4.1), we have that there exists a subsequence (p_j) such that

$$k_{p_j+1}/k_{p_j} \geq m_{2j+2}/m_{2j+1}.$$

Indeed, this is because the function τ is constant on the interval (m_{2j+1}, m_{2j+2}) . Mark $\bar{G}_0 = \mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$ with the generating set $\{b_0, t\}$ and write

$$G_j = \Gamma / \bigcap_{i=0}^j \ker(\pi_i : \Gamma \rightarrow \bar{G}_i).$$

Then the Cayley graph of G with respect to the marking $\{b_0, t\}$ and the Cayley graph of G_j coincides in the balls of radius $2^{k_{j+1}}$ around the identities. Note that since each \bar{G}_i , $i \geq 1$ is finite, the group G_j is a finite

extension of the lamplighter group \bar{G}_0 . Note that

$$L_{G,\mu}(n) \leq \text{Diam}_{G_j,S}(\ker(G_j \rightarrow \bar{G}_0)) + L_{\bar{G}_0,\mu}(n) + n\mathbb{P}(|W_n| \geq 2^{k_{j+1}}).$$

When (m_j) grows sufficiently fast such that k_{j+1} is much larger than the diameter of the finite set $\ker(G_j \rightarrow \bar{G}_0)$ in G_j , we have that

$$L_{G,\mu}(t_i) \leq 2L_{\bar{G}_0,\mu}(t_i) \leq C\sqrt{(t_i)}$$

along a time subsequence (t_i) .

Next we show that if (m_j) grows sufficiently fast, then along another subsequence (n_i) , we have

$$\mu^{n_i}(e) \leq \exp\left(-cn_i^{\frac{\alpha}{2+\alpha}}\right).$$

As explained in Remark 4.3, this implies that the μ -random walk is not cautious and $L_\mu(n_i) \geq cn_i^{\frac{\alpha}{2+\alpha}}$.

We conclude that $L_\mu(n)$ oscillates between $n^{\frac{\alpha}{2+\alpha}}$ and $n^{1/2}$, \square

4.4. Examples of groups with property H_{FD}

As we have mentioned in the Introduction, a group G is said to have *Shalom's property H_{FD}* if every orthogonal G -representation π with non-zero reduced cohomology $\overline{H}^1(G, \pi)$ admits a finite-dimensional sub-representation.

By a result of Gournay [17, Theorem 4.7], if the quotient of a group over its *FC*-centre has Shalom's property H_{FD} , then the group also has this property. Recall that the conjugacy class of an element $g \in G$ is the set $\{hgh^{-1} : h \in G\}$. The *FC*-center of the group consists of elements with finite conjugacy classes. We say G is an *FC*-central extension of H if the kernel of the quotient map $G \rightarrow H$ is contained in the *FC*-center of G .

LEMMA 4.4. — *The nilpotent-by-cyclic group G in Theorem 1.5 is an *FC*-central extension of $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$.*

Proof. — A formal commutator u on a string $(b_{i_1}, b_{i_2}, \dots, b_{i_\ell})$ is nontrivial in only finitely many nilpotent groups N_i in the construction. Indeed, to be nontrivial in N_i , we need $\min_{1 \leq j, k \leq \ell} d(\bar{i}_j, \bar{i}_k) \geq 2^{k_i - c_i}$ on the cycle $\mathbb{Z}/2^{k_i}\mathbb{Z}$. Since $G = \Gamma / \bigcap_{i=1}^\infty \ker(\pi_i : \Gamma \rightarrow \bar{G}_i)$, we have that the conjugacy class of image of u in G is contained in a subgroup of the direct product of finitely many N_i 's. Since each N_i is a finite nilpotent group, it follows that the commutator u is in the *FC*-center of G . We conclude that G is an *FC*-central extension of $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$. \square

Therefore by applying Gournay's result and Lemma 4.4, for the groups constructed in Theorem 1.5, we have that they have property H_{FD} . Alternatively, by a generalization of Gournay's result, [3, Proposition 4.7] implies that the group $N_{\text{Nil}, \mathbb{Z}, \mathfrak{c}} \rtimes \mathbb{Z}$ has property H_{FD} . Therefore its quotient groups considered in the previous subsection have property H_{FD} as well.

Recall that by [16, Corollary 2.5], if G admits a cautious random walk μ with finite generating support, then G has Property H_{FD} . A sufficient condition for a symmetric μ -random walk to be cautious is that the ℓ^2 -isoperimetry inside balls satisfies the upper bound in the following lemma. Recall that

$$\begin{aligned} & \lambda_\mu(B(e, r)) \\ &= \inf \left\{ \frac{1}{2} \sum_{x, y \in G} (f(x) - f(xy))^2 \mu(y) : \text{supp}\{f \subseteq B(e, r) : \|f\|_2 = 1\} \right\}. \end{aligned}$$

Note that by Coulhon–Saloff-Coste isoperimetric inequality [6], it always admits a lower bound that for some constant $c = c(\mu) > 0$, for all $r \geq 1$,

$$\lambda_\mu(B(e, r)) \geq cr^{-2}.$$

If the opposite inequality holds along a subsequence of balls $B(e, r_i)$, then the random walk is cautious:

LEMMA 4.5. — *Suppose there exists a constant $C > 0$ and a sequence of balls $B(e, r_i)$ with $r_i \rightarrow \infty$ as $i \rightarrow \infty$, such that*

$$(4.2) \quad \lambda_\mu(B(e, r_i)) \leq Cr_i^{-2},$$

Then the μ -random walk is cautious in the sense of (1.2).

Proof. — Let (W_k) be the μ -random walk trajectory. We first show that

$$\max_{x \in B(e, r)} \mathbb{P}_x(W_k \in B(e, r) \text{ for all } k \leq n) \geq (1 - \lambda_\mu(e, r))^n.$$

This inequality is a known bound which can be obtained from the eigenbasis expansion of the transition probability with Dirichlet boundary on the ball $B(e, r)$. We provide a proof for the convenience of the reader. Let P_n^D be the semigroup with Dirichlet boundary on the ball $B(e, r)$. We recall that by definition

$$P_n^D f(x) = \mathbb{E}_x[f(W_n) \mathbb{1}_{\{n < \tau\}}],$$

where f has zero boundary condition on $B(e, r)$, W_n is the random walk with step distribution μ and τ is the stopping time that the random walk

first exists $B(e, r)$. Let $\lambda_1 < \dots < \lambda_v$ be the eigenvalues of $I - \mu$ with Dirichlet boundary on the ball, and $\varphi_1, \dots, \varphi_v$ be the corresponding eigenfunctions normalized in such a way that $\|\varphi_i\|_2 = 1$. Note that $\lambda_1 = \lambda_\mu(B(e, r))$ and φ_1 is non-negative. Then

$$P_n^D(x, y) = \sum_{i=1}^v (1 - \lambda_i)^n \varphi_i(x) \varphi_i(y).$$

Let x_0 be a point in $B(e, r)$ such that φ_1 achieves its maximum at x_0 . Then

$$\begin{aligned} \sum_{y \in B(e, r)} P_n^D(x_0, y) \varphi_1(x_0) &\geq \sum_{y \in B(e, r)} P_n^D(x_0, y) \varphi_1(y) \\ &= (1 - \lambda_1)^n \varphi_1(x_0) \left(\sum_{y \in B(e, r)} \varphi_1(y)^2 \right) \\ &= (1 - \lambda_1)^n \varphi_1(x_0). \end{aligned}$$

Therefore

$$\mathbb{P}_{x_0}(W_k \in B(e, r) \text{ for all } k \leq n) = \sum_{y \in B(e, r)} P_n^D(x_0, y) \geq (1 - \lambda_\mu(e, r))^n.$$

To see that the assumed bound (4.2) imply cautiousness, note that

$$\mathbb{P}_e \left(\max_{k \leq n} |W_k| \leq 2r \right) \geq \max_{x \in B(e, r)} \mathbb{P}_x(W_k \in B(e, r) \text{ for all } k \leq n).$$

Given any constant $c > 0$, take the time subsequence $n_i = (r_i/c)^2$, we have

$$\mathbb{P}_e \left(\max_{k \leq n_i} |W_k| \leq 2c\sqrt{n_i} \right) \geq (1 - Cr_i^{-2})^{r_i^2/c^2} \geq \delta(c, C) > 0. \quad \square$$

Tessera [37] defined a class of groups (\mathcal{L}) which includes polycyclic groups, solvable Baumslag–Solitar groups and wreath products $\mathbb{Z} \wr F$ with F finite, and showed that a group from this class admits a full sequence of controlled Følner pairs. Moreover, in [38], he proved this property holds for quotient of any solvable algebraic group over a q -adic field where q is a prime. Shalom [35] proved that polycyclic groups have property H_{FD} , using a theorem of Delorme [8] concerning the cohomology of irreducible unitary representation of connected Lie groups. Controlled Følner pairs in polycyclic groups provide another argument to establish property H_{FD} for these groups, which does not use Delorme’s result.

Having a controlled Følner pair $F'_i \subset F_i$ as above implies (see [37, Proposition 4.9])

$$\lambda_\mu(B(e, Cn_i)) \leq Cn_i^{-2}$$

for any finitely supported symmetric probability measure μ .

COROLLARY 4.6. — *Suppose a finitely generated group admits a subsequence of controlled Følner pairs, then it has property H_{FD} .*

Remark 4.7. — By [36, Theorem 1], the asymptotic behavior of the ℓ^2 -isoperimetric profile inside balls $\lambda_\mu(B(e, r))$ is invariant under quasi-isometry. Therefore the property of satisfying the assumption of Lemma 4.5 is stable under quasi-isometry.

In Theorem 1.5 we have shown that group satisfying the claim of the theorem can be chosen to be a direct product of two groups, where each admits a subsequence of controlled Følner pairs. As a consequence of the discussion above, simple random walk on each factor is cautious. Thus an alternative way to prove property H_{FD} of these groups is to use [16, Corollary 2.5] (and the fact that the direct product of groups with Shalom's property also has this property). Corollary 4.6 is used to show Property H_{FD} for certain lacunary hyperbolic groups considered in Section 7 as well.

It was shown in [4] that the construction of [3], as well as a variation of it can be used to provide groups with Shalom's property H_{FD} and prescribed Følner function. Groups we consider in Subsection 3 and 4 provide another simple construction of this kind. These examples show that the property of admitting a cautious simple random walk is strictly stronger than having property H_{FD} .

5. Torsion free examples

We mention here some other extensions of nilpotent groups where the isoperimetric inequality of Corollary 1.2 can be applied to obtain good lower bound for the Følner function. In the construction of the groups in Subsections 3, 4, we can drop the torsion relation that $b_i^2 = 1$ and consider cyclic extensions of torsion free nilpotent groups.

Example 5.1. — Let $N_{\mathbb{Z}, \mathfrak{c}}$ be the free nilpotent group of class \mathfrak{c} on generators $b_i, i \in \mathbb{Z}$. Let $G = G_{\mathbb{Z}, \mathfrak{c}}$ be the extension $N_{\mathbb{Z}, \mathfrak{c}} \times \mathbb{Z}$, where \mathbb{Z} acts by shifting indices. Then $S = \{b_0, t\}$ is a generating set of G . The Følner function of G is asymptotically equivalent to $n^{n^\mathfrak{c}}$.

Proof. — The proof of the lower bound is similar to the torsion case, we look for elements in the center of $N_{\mathbb{Z}, \mathfrak{c}}$ that can be reached within distance n and apply Corollary 1.2.

For a given tuple $i_1 < i_2 < \dots < i_{\mathfrak{c}}$, let $u(i_1, \dots, i_{\mathfrak{c}}) = [[b_{i_1}, b_{i_2}], \dots, b_{i_{\mathfrak{c}}}].$ Note that it is in the center of N and belongs to the collection of basic

commutators. We have that

$$u(i_1, \dots, i_{\mathfrak{c}})^m = [[b_{i_1}, b_{i_2}], \dots, b_{i_{\mathfrak{c}}}^m].$$

Given n , consider the abelian subgroup of the center of N

$$H_n = \prod_{0 < i_1 < \dots < i_{\mathfrak{c}}} \langle u(i_1, \dots, u_{\mathfrak{c}}) \rangle.$$

For each cyclic factor in H_n , we have the length estimate

$$l_S(u(i_1, \dots, u_{\mathfrak{c}})^m) = l_S([[b_{i_1}, b_{i_2}], \dots, b_{i_{\mathfrak{c}}}^m]) \leq C(\mathfrak{c})(n + m),$$

where $C(\mathfrak{c}) > 0$ is a constant only depending on \mathfrak{c} . Now we apply Corollary 1.2 with the choice of length bound $2C(\mathfrak{c})$ and $k = n$. The number $N(2C(\mathfrak{c})n, n)$ of indices such that are least n distinct non-identity elements in the cyclic group $\langle u(i_1, \dots, u_{\mathfrak{c}}) \rangle$ is $\binom{n}{\mathfrak{c}}$. It follows from Corollary 1.2 that

$$\text{Fol}_{G,S}(2C(\mathfrak{c})n) \geq n^{Cn^{\mathfrak{c}}}.$$

The upper bound that $\text{Fol}_{G,S}(n) \leq n^{C'(\mathfrak{c})n^{\mathfrak{c}}}$ follows from choosing the following test set. For an element in G , it can be written uniquely in the normal form (f, z) where $z \in Z$ and f is a ordered product in terms of basic commutators $f = \prod u^{f(u)}$, where $f(u) \in \mathbb{Z}$ is non-zero for only finitely many commutators. Recall that $w(u)$ denotes the weight of u , which is equal to the commutator length, e.g. $w([i_1, i_2]) = 2$. Take the subset

$$\Omega_n = \{(f, z) \in G : \text{supp } f \subset [0, n], |f(u)| \leq n^{w(u)}, 0 < z \leq n\},$$

it is easy to verify that $\#\Omega_n/\Omega_n \leq C/n$. And it is clear that the volume of Ω_n is bounded by

$$\#\Omega_n \leq n \cdot n^{C(\mathfrak{c})n^{\mathfrak{c}}}.$$

□

We can then consider various quotients of $G_{\mathbb{Z}, \mathfrak{c}}$ similar to Subsection 4, namely by adding relations that certain basic commutators vanish or imposing finite orbits under the action of \mathbb{Z} as in the proof of Theorem 1.5. The resulting groups are torsion-free-nilpotent by cyclic. To bound the Følner function from below, in the same way as illustrated in Example 5.1, we look for elements in cyclic factors of the center of the nilpotent group within the ball of radius n of identity in the ambient group and apply Corollary 1.2.

Since torsion free nilpotent groups are left-orderable, their cyclic extensions are left-orderable as well, see e.g. [9, Subsection 2.1.1]. By a result of Gromov [25, Section 3.2], for such left orderable groups, the linear algebraic Følner function (for definition see [25]) is equal to the usual (combinatorial) Følner function. Removing the torsion relations from the construction of

groups in Theorem 1.5, we obtain that for any prescribed increasing function $0 < \tau(n) \leq n^c \log n$, there exists a group $G = N \rtimes \mathbb{Z}$ where N is torsion free nilpotent of step $\leq c$ and $\log F\ddot{o}l_{G,S}(n)$ is asymptotically equivalent to $n \log n + \tau(n)$. Moreover in the lower central series of N , each quotient is torsion-free abelian. Therefore the group G is left-orderable. It follows that the linear algebraic Følner function coincide with the usual combinatorial Følner function, thus satisfying the same estimate.

For the free nilpotent group $N_{\mathbb{Z}^d, c}$ of step c on generators $b_x, x \in \mathbb{Z}^d$, we can consider the extension $N_{\mathbb{Z}^d, c} \rtimes \mathbb{Z}^d$ and its various quotients. The method of looking for elements in cyclic factors in the center of the nilpotent group and applying Corollary 1.2 provides sharp lower bounds for the Følner functions of these groups as well. For example, we have

$$F\ddot{o}l_{N_{\mathbb{Z}^d, c} \rtimes \mathbb{Z}^d}(n) \asymp n^{n^{d^c}}.$$

We mention that none of the torsion free nilpotent-by-abelian groups discussed in this section have Shalom's property H_{FD} , because they all admit $\mathbb{Z} \wr \mathbb{Z}$ as a quotient group. The wreath product $\mathbb{Z} \wr \mathbb{Z}$ does not have property H_{FD} by [35, Theorem 5.4.1].

6. Extensions of a symmetric group

Let H be a finitely generated group equipped with a symmetric finite generating set S , e.g. $H = \mathbb{Z}^d$. Consider the group of permutations of H with finite support. Let Sym_H be the extension of this group by H . It is clear that Sym_H is a finitely generated group, one choice of generators \tilde{S} is transpositions (e, s) , $s \in S$ and the generators S of H . In this section we derive a lower bound on the Følner function on Sym_H in terms of the volume growth of H .

Let $\text{Sym}(X)$ be the symmetric group on a countable set X . For a finite subset $Y \subset X$ denote by $T(Y)$ the set of transpositions between points in Y , $T(Y) = \{(y_1, y_2) : y_1, y_2 \in Y\}$. Here (y_1, y_2) denotes the permutation that transposes y_1 and y_2 . The following lemma provides a lower bound for the cardinality of sets in $\text{Sym}(X)$ which are C -satisfactory with respect to $T(Y)$. By definition, V is C -satisfactory with respect to $T(Y)$ if for each $v \in V$ and at least $C\#T(Y) = \frac{C}{2}\#Y(\#Y - 1)$ multiplications by distinct transpositions $t \in T(Y)$ remain in V , that is, $vt \in V$.

LEMMA 6.1 (Satisfactory sets in finite symmetric groups). — For each $C > 0$, there exists $D = D(C) > 0$ such that the following holds. Let Y be a finite subset of X . Suppose that a finite subset V in $\text{Sym}(X)$ is C -satisfactory with respect to $T(Y)$, then the cardinality of V is at least $(D\#Y)^{D\#Y}$.

Proof. — Write $n = \#Y$. We prove a more general claim. Let $m \in \mathbb{N}$, and we say that $V \subset \text{Sym}(X)$ satisfies the assumption $(*)$ for m if for each $v \in V$, there are at least m distinct elements of $T(Y)$ such that v multiplied with the element (on the right) remains in V . We show that if V verifies assumption $(*)$ for m , then the cardinality of V is at least $(m/(2n))^{m/(2n)}$. The lemma is stated for $m = Cn^2$.

To prove this, observe that for any given $v \in V$, there exists a $y \in Y$, such that $v(y, z) \in V$ for at least m/n distinct z 's, $z \in Y$. We fix one of such v_0 and y and denote by $x = v_0^{-1}(y)$. We subdivide V according to the image of x : consider V_z to be elements v of V such that $v(x) = z$. It is clear that V is a disjoint union of V_z , where the union is taken over $z \in X$. Note that $\#V \geq \frac{m}{n} \min \#V_z$, where the minimum is take over z such $v_0(y, z) \in V$.

Next we show that for each V_z where z is such that $v_0(y, z) \in V$, the satisfactory assumption $(*)$ is verified for $m' = m - n$. Indeed, for $u \in V_z$ there are at least m distinct transpositions in $T(Y)$ such that $ut \in V$. Among them, if $t = (r, s)$ satisfies $ut \in V$ but $ut \notin V_z$, then it implies that either $r = y$ or $s = y$. Changing if necessary the notation, we can assume that $r = y$. It follows that among these m distinct transpositions, there are at most n of them such that $ut \in V$ but $ut \notin V_z$. In other words, for each $u \in V_z$, there are at least $m - n$ distinct transpositions $t \in T(Y)$ such that $ut \in V_z$.

Repeat this process for $m/2n$ steps, we have that the cardinality of V is at least

$$\#V \geq (m/n)((m-n)/n)((m-2n)/n) \cdots \geq (m/(2n))^{m/(2n)}.$$

In particular, for V verifying the assumption $(*)$ for $m \geq Cn^2$, the cardinality of V is at least Dn^{Dn} , for some constant $D > 0$ depending only on C . \square

We now proceed to prove Corollary 1.4. Given $h_1, h_2 \in H$, the transposition (h_1, h_2) has length at most $4l_{H,S_H}(h_1) + 2l_{H,S_H}(h_2)$. Indeed, $(h_1, h_2) = h_1^{-1}(e, h_2h_1^{-1})h_1$. Take a transposition (e, h) where e is the identity element. Write h as a shortest path in the generators: $h = s_{i_1} \dots s_{i_\ell}$ where

$\ell = l_{H,S}(h)$. We have

$$(e, h) = (e, s_{i_1}) \dots (e, s_{i_{\ell-1}})(e, s_{i_\ell})((e, s_{i_1}) \dots (e, s_{i_{\ell-1}}))^{-1}.$$

Therefore, $l_{\text{Sym}_H, \tilde{S}}((e, h)) \leq 2l_{H,S}(h) - 1$. It follows that

$$l_{\text{Sym}_H, \tilde{S}}((h_1, h_2)) \leq 2l_{H,S}(h_1) + 2l_{H,S}(h_2h_1^{-1}) \leq 4l_{H,S}(h_1) + 2l_{H,S}(h_2).$$

Proof of Corollary 1.4. — Given $n \geq 1$ consider the set T_n of transpositions

$$T_n = T(B_H(e, n)) = \{(h, h') : h, h' \in H, \text{ where } l_{H,S}(h), l_{H,S}(h') \leq n\}.$$

Then by the length estimate above, we have that elements of T_n are of length at most $6n$ in Sym_H .

Let V be a Følner set of Sym_H such that $\#\partial_{\tilde{S}}V/\#V \leq 1/200n$. Then by Theorem 1.1, this set is $1/4$ -satisfactory with respect to T_n . Since $T_n \subset \text{Sym}(H)$, we can assume, by taking cosets, that V contains a subset $V' \subset \text{Sym}(H)$ that is $1/4$ -satisfactory with respect to T_n . Apply Lemma 6.1 with $Y = B_{H,S}(e, n)$, it follows that there exists an absolute constant $c > 0$ such that $\#V \geq (cv_{H,S}(n))^{cv_{H,S}(n)}$. \square

Note that the lower bound of $\text{Fol}_{\text{Sym}_H}$ in Corollary 1.4 is better than what we would obtain by applying Corollary 1.2 instead of volume lower bound for satisfactory sets with respect to transpositions. Indeed, for any choice of abelian subgroup of $\text{Sym}(B_{H,S}(e, n))$, by applying Corollary 1.2, we only get the lower bound of order $\exp(n)$.

Example 6.2. — Take $H = \mathbb{Z}^d$, by Corollary 1.4, the Følner function of $G = \text{Sym}(\mathbb{Z}^d) \rtimes \mathbb{Z}^d$ is asymptotically greater to equal than n^{n^d} . The group G admits Følner pairs adapted to the function n^{n^d} . By [5] and the general relation between Følner function and decay of return probability (see e.g. [39, Theorem 14.3]), we deduce that the return probability $\mu^{(2n)}(e)$ of simple random walk on $\text{Sym}(\mathbb{Z}^d) \rtimes \mathbb{Z}^d$ is asymptotically equivalent to $\exp(-n^{d/d+2} \log^{2/d+2} n)$. We mention that random walk on $\text{Sym}(H) \rtimes H$ is called the mixer chain in Yadin [40], where the drift function of the random walk on $\text{Sym}(\mathbb{Z}) \rtimes \mathbb{Z}$ is estimated.

7. Lacunary hyperbolic examples

In this section we show that there exist non-virtually cyclic amenable groups with Shalom's property H_{FD} that are lacunary hyperbolic. Asymptotic cones first appeared in the proof of the polynomial growth theorem by Gromov in [22]. Roughly speaking, an asymptotic cone of a metric space

is what one sees when the space is viewed from infinitely far away. For a definition using ultrafilters see van den Dries and Wilkie [10]. A well-known result of Gromov [23] states that a finitely generated group is hyperbolic if and only if all its asymptotic cones are \mathbb{R} -trees. Recall that a group is lacunary hyperbolic if one of its asymptotic cone is an \mathbb{R} -tree. By Olshanskii, Osin and Sapir [30, Theorem 1.1], lacunary hyperbolic groups can be characterized as certain direct limits of hyperbolic groups. More precisely, a finitely generated group $G = \langle S \rangle$ is lacunary hyperbolic if and only if G is the direct limit of a sequence of hyperbolic groups $G_i = \langle S_i \rangle$, S_i finite, with epimorphisms

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \dots$$

satisfying $\alpha_i(S_i) = S_{i+1}$ and $\delta_i/r_i \rightarrow 0$ as $i \rightarrow \infty$, where δ_i is the hyperbolicity constant of G_i relative to generating set S_i , and r_i is injectivity radius of the map α_i .

Osin, Olshanskii and Sapir in [30] constructed lacunary hyperbolic groups that served as examples/counter examples to various question. In particular, using central extension of lacunary hyperbolic groups they show that there exists a group whose asymptotic cone with countable but non-trivial fundamental group. They prove that what is called “divergence function” (measuring how much the distance between two points outside the ball of given radius increases after removing this ball) can be arbitrary close to linear, but not being linear. They also show that the class of lacunary hyperbolic group is quite large: for example, one can find infinite torsion groups among lacunary hyperbolic groups; some of lacunary hyperbolic groups are amenable. Properties of endomorphisms and automorphisms of lacunary hyperbolic groups were investigated in Coulon–Guirardel [7]. In particular, every lacunary hyperbolic group is Hopfian, see [7, Theorem 4.3].

We now briefly describe the construction of lacunary hyperbolic elementary amenable groups in [30, Section 3.5]. These groups are locally-nilpotent-by- \mathbb{Z} , which was considered previously in [25, Section 8]. Let p be a prime number and $\mathbf{c} = (c_i)$ be an increasing sequence of positive integers. Let $A = A(p, \mathbf{c})$ be the group generated by b_i , $b_i^p = 1$, $i \in \mathbb{Z}$ with the following defining relations:

$$[[b_{i_0}, b_{i_1}], \dots, b_{i_{c_n}}] = 1 \quad \text{if } \max_{j,k} |i_j - i_k| \leq n.$$

The group A is locally nilpotent, and it admits an automorphism $a_i \mapsto a_{i+1}$. Let $G = G(p, \mathbf{c})$ be the extension of A by this automorphism, $G = \langle A, t \rangle$ where $ta_it^{-1} = a_{i+1}$. By [30, Lemma 3.24], if the sequence \mathbf{c} grows fast enough, then G is lacunary hyperbolic.

In what follows, we consider a variation of the construction that introduces a sequence of slow scales where the group G is close to a lamplighter, while on another sequence of scales G can be approximated by hyperbolic groups. In particular, simple random walk on G is cautious along a subsequence of time (t_i) with appropriate choice of parameters.

THEOREM 7.1. — *There exists a finitely generated lacunary hyperbolic non-virtually cyclic amenable group G with Shalom's property H_{FD} . Moreover, the group G can be chosen to be locally-nilpotent-by- \mathbb{Z} and simple random walk on G is cautious and diffusive along an infinite subsequence of time instances.*

Proof. — Let $M = *_{j \in \mathbb{Z}} \langle b_j \rangle$, $b_j^p = 1$ be the free product of copies of $\mathbb{Z}/p\mathbb{Z}$ indexed by \mathbb{Z} , $\Gamma = M \rtimes \mathbb{Z}$ be its cyclic extension where \mathbb{Z} acts by shifting indices. Let π_0 be the quotient map from $\Gamma \rightarrow \mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$. Write $G_0 = \mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$. We define a sequence of quotients of Γ recursively as follows (these quotients are determined by a triple of parameters $(\ell_i, c_i, k_i)_{i \in \mathbb{N}}$).

After we have defined G_i , given parameter ℓ_{i+1} , take the nilpotent subgroup in G_i generated by $b_0, \dots, b_{2^{\ell_{i+1}}-1}$,

$$N_{i+1} = \langle b_0, \dots, b_{2^{\ell_{i+1}}-1} \rangle.$$

Consider the quotient group \widehat{M}_{i+1} of M defined by imposing the relations that for any j , the subgroup generated by $\langle b_j, \dots, b_{j+2^{\ell_{i+1}}-1} \rangle$ is isomorphic to N_{i+1} (isomorphism given by shifting indices by j). Let $\widehat{\Gamma}_{i+1} = \widehat{M}_{i+1} \rtimes \mathbb{Z}$ be the cyclic extension of \widehat{M}_{i+1} . Then $\widehat{\Gamma}_{i+1}$ splits as an HNN-extension of the finite nilpotent group N_{i+1} , therefore $\widehat{\Gamma}_{i+1}$ is virtually free, see [34, Proposition 11]. Therefore $\widehat{\Gamma}_{i+1}$ is hyperbolic, let δ_{i+1} be the hyperbolicity constant of $\widehat{\Gamma}_{i+1}$ with respect to the generating set $S = \{b_0, t\}$. For parameters $c_{i+1}, k_{i+1} \in \mathbb{N}$, consider the quotient group $\bar{\Gamma}_{i+1} = \widehat{\Gamma}_{i+1}(c_{i+1}, k_{i+1})$ of $\widehat{\Gamma}_{i+1}$ subject to additional relations $(*)$

$$\begin{aligned} [[\![b_{j_1}, b_{j_2}]\!], b_{j_3}], \dots, b_{j_m}] &= 0 \text{ for any } m \geq c_{i+1}, \\ b_j &= b_{j+2^{k_{i+1}}} \text{ for all } j \in \mathbb{Z}. \end{aligned}$$

By choosing c_{i+1} and k_{i+1} to be sufficiently large, we can guarantee that the injectivity radius r_i of the quotient map $\widehat{\Gamma}_{i+1} \rightarrow \bar{\Gamma}_{i+1}$ satisfies $r_{i+1} \gg \delta_{i+1}$.

Let $\psi_{i+1} : \Gamma \rightarrow \bar{\Gamma}_{i+1}$ be the projection from Γ to $\bar{\Gamma}_{i+1}$. Recall that π_0 denotes the projection from $\Gamma \rightarrow \mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$. We take the group G_{i+1} to be

$$G_{i+1} = \Gamma / (\ker \psi_{i+1} \cap \ker \pi_0).$$

By construction, if we choose $k_{i+1}, c_{i+1} \gg \ell_{i+1}$, $\ell_{i+1} \gg \max \{\ell_i, k_i, c_i\}$ to be large enough parameters, we have that G_{i+1} and G_i coincide on the

ball of radius $2^{\ell_{i+1}}$ around the identity element, and G_{i+1} coincide with a hyperbolic group $\widehat{\Gamma}_{i+1}$ on the ball of radius r_{i+1} such that $r_{i+1} \gg \delta_{i+1}$, where δ_{i+1} is the hyperbolicity constant of $\widehat{\Gamma}_{i+1}$. Let G be the limit of (G_i) in the Cayley topology, or equivalently,

$$G = \Gamma / \bigcap_{i \in \mathbb{N}} \ker \psi_i.$$

Since along the sequence (r_i) , the ball of radius r_i around identity in G coincide with a ball of same radius in a hyperbolic group with hyperbolicity constant $\delta_i \ll r_i$, it follows that G is lacunary hyperbolic.

We now show that for parameters $\ell_{i+1} \gg \max\{\ell_i, k_i, c_i\}$ large enough, G admits a subsequence of controlled Følner pairs, and simple random walk on G is cautious and diffusive along a subsequence. The argument is along the same line as in the proof of Theorem 1.5.

Since the balls of radius $2^{\ell_{i+1}}$ are the same in G and G_i , up to radius $2^{\ell_{i+1}}$ it is the same to consider the corresponding random walk in G_i . Let H_i be the subgroup of G_i generated by $b_0, \dots, b_{2^{k_i}-1}$. Note that H_i is a finite nilpotent group. Then because of the relations $(*)$ in $\widehat{\Gamma}_i$, G_i fits into the exact sequence

$$1 \rightarrow [H_i, H_i] \rightarrow G_i \rightarrow \mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z}) \rightarrow 1.$$

We use the same letter π_0 to denote the projection $G_i \rightarrow \mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$ as well (π_0 was used for the projection $\Gamma \rightarrow \mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$). For $r \leq 2^{\ell_{i+1}-C}$, let

$$F'_r = \{g \in G_i : \pi_0(g) = (f, z), \text{supp } f \subset [-r, r], |z| \leq r\},$$

$$F_r = \{g \in G_i : \pi_0(g) = (f, z), \text{supp } f \subset [-r, r], |z| \leq r/2\}.$$

Then by definition, (F_r, F'_r) forms a Følner pair. The outer set F'_r has diameter bounded by

$$\text{Diam}_{G_i, S} \leq \text{Diam}_{G_i, S}([H_i, H_i]) + 10r.$$

Since the diameter of the finite subgroup $[H_i, H_i]$ in G_i depends only on the choice of parameters up to index i , we have that for $r \geq r_0(c_i, k_i) \geq \text{Diam}_{G_i, S}([H_i, H_i])$, the set F'_r is contained in the ball $B(e, 20r)$ in G_i . Thus for such r sufficiently large, (F_r, F'_r) is a controlled Følner pair in G_i . Since the balls of radius $2^{\ell_{i+1}}$ are the same in G and G_i , for $\ell_{i+1} \gg c_i, k_i$ sufficiently large such that $r_0(c_i, k_i) \leq 2^{\ell_{i+1}-C}$, (F_r, F'_r) can be identified as a controlled Følner pair in G as well. Therefore if the sequence (ℓ_{i+1}) grows sufficiently fast compared to (c_i, k_i) in the sense described above, we have that G admits a subsequence of controlled Følner pairs. By Lemma 4.5, simple random walk on G is cautious.

Similarly, for simple random walk on G_i , the drift function is bounded by

$$L_{G_i,\mu}(t) \leq \text{Diam}_{G_i,S}([H_i, H_i]) + L_{G_0,\mu}(n),$$

where $G_0 = \mathbb{Z} \wr (\mathbb{Z}/p\mathbb{Z})$. Thus when $r_0(c_i, k_i) \preceq 2^{\ell_{i+1}-C}$, there exists some constant ℓ'_{i+1} depending on c_i, k_i , such that for $t \in [2^{\ell'_{i+1}}, 2^{\ell_{i+1}}]$, the constructed group G_i satisfies

$$L_{G,\mu}(t) = L_{G_i,\mu}(t) \leq C\sqrt{t}.$$

That is, simple random walk on G is diffusive \square

Remark 7.2. — From the construction of the group G , we have that along a sequence of radius (r_i) , the ball of radius r_i around identity of G coincide with the ball of the same radius in a virtually free group. It follows that the drift function can be close to linear along a subsequence of time (t_i) .

Remark 7.3. — The fact that the lamplighter group $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$ can be realized as a direct limit of virtually free group with growing injectivity radius was used by Osin in [31] to show that the Kazhdan constant of a hyperbolic group is not bounded uniformly from below under changing generating sets.

It is known that elementary amenable groups can have arbitrarily fast Følner functions, see [29, Corollary 1.5] (also the remarks in [11, Section3] and [25, Section8]).

Remark 7.4. — By taking the direct product of two lacunary hyperbolic groups as in Theorem 7.1, we obtain locally-nilpotent-by-abelian group with property H_{FD} for which it can be shown (similar to [25, Section8]) that the Følner function is arbitrarily fast. In addition to groups in [3], they can provide another collection of elementary amenable groups with arbitrarily fast Følner functions while simple random walks on them have trivial Poisson boundary. Amenable (but non-elementary amenable)groups of with this property were constructed in [13].

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