ANNALES DE L'INSTITUT FOURIER

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Jet differentials on toroidal compactifications of ball quotients

CENTRE
MERSENNE

JET DIFFERENTIALS ON TOROIDAL COMPACTIFICATIONS OF BALL QUOTIENTS

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ABSTRACT. — We give explicit estimates for the volume of the Green–Griffiths jet differentials of any order on a toroidal compactification of a ball quotient. To this end, we first determine the growth of the logarithmic Green–Griffiths jet differentials on these objects, using a natural deformation of the logarithmic jet space of a given order, to a suitable weighted projective bundle. Then, we estimate the growth of the vanishing conditions that a logarithmic jet differential must satisfy over the boundary to be a standard one.

RÉSUMÉ. — On donne des estimées explicites pour le volume des différentielles de jets de Green–Griffiths à tout ordre sur une compactification toroïdale d'un quotient de la boule. Pour ce faire, on détermine tout d'abord l'ordre de croissance des différentielles de jets de Green–Griffiths logarithmiques sur ces objets, en utilisant une déformation naturelle de l'espace des jets logarithmiques d'un ordre fixé, vers un fibré projectivisé à poids adéquat. Ensuite, on estime la croissance du nombre de conditions d'annulation au bord qu'une différentielle de jets logarithmique doit satisfaire pour être une différentielle de jets standard.

1. Introduction

When dealing with complex hyperbolicity problems, finding global jet differentials on manifolds is an important question, since they permit to give restrictions on the geometry of the entire curves. Let us recall a few basic facts concerning Green–Griffiths jet differentials, which can be found in [9]. Let X be a complex projective manifold. Then, for any k, there exists a (singular) variety $X_k^{GG} \xrightarrow{\pi_k} X$, and an orbifold line bundle $\mathcal{O}_{X_k^{GG}}(1)$ on it, such that for any m, $E_{k,m}^{GG}\Omega_X = (\pi_k)_* \mathcal{O}_k(m)$ is a vector bundle whose

sections are holomorphic differential equations of order k, and degree m, for some suitable notion of weighted degree.

In [8], Demailly proves that if V is a complex projective manifold of general type, then for any k large enough, the Green–Griffiths jet differentials of order k will have maximal growth, or equivalently, $\mathcal{O}_{X_k^{GG}}(1)$ is big. Finding an effective k for which this property holds is an interesting question, whose answer depends on the context: in [8], Demailly uses his metric techniques to give an effective lower bound on k in the case of hypersurfaces of $\mathbb{P}^n$.

We propose here a method to obtain a similar effective result in the case of *toroidal compactifications of ball quotients* (see [14] for the main properties of these manifolds). Specifically, we will find a combinatorial lower bound on the volume of $E_{k,\bullet}^{GG}\Omega_X$, valid for any k :

THEOREM 1.1. — *Let $\overline{X}$ be a toroidal compactification of a ball quotient by a lattice with only unipotent parabolic isometries. Then, for any $k \in \mathbb{N}$, we have the following lower bound on the volume of the k-th order Green–Griffiths jet differentials:*

$$(1.1) \quad \mathrm{vol}(E_{k,\bullet}^{GG}\Omega_{\overline{X}}) \geqslant \frac{1}{(k!)^n} \left[\left(\frac{(K_{\overline{X}} + D)^n}{(n+1)^n} \sum_{\{u_1 \leqslant \cdots \leqslant u_n\} \subset S_{k,n}} \frac{1}{u_1 \ldots u_n} \right) \right.$$

$$\left. + (-D)^n \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant k} \frac{1}{i_1 \ldots i_n} \right],$$

where $S_{k,n}$ is the ordered set

$$S_{k,n} = \{1_1 < \cdots < 1_{n+1} < 2_1 < \cdots < 2_{n+1} < \cdots < k_1 < \cdots < k_{n+1}\}.$$

Here, the fractions $\frac{1}{u_1 \ldots u_n}$ are to be computed by forgetting the indexes on the integers in the set $S_{k,n}$. In [5], the particular case where $k = 1$ (i.e. the case of *symmetric differentials*) was already proved, under the additional assumption that $\Omega_{\overline{X}}$ is nef. Our formula removes this hypothesis, and extends the result to any order k.

Using the results of [3], it is not hard to derive explicit orders k for $E_{k,\bullet}^{GG}\Omega_{\overline{X}}$ to have maximal growth:

COROLLARY 1.2. — *Let $\overline{X} = \overline{\mathbb{B}^n/\Gamma}$ be a toroidal compactification of a ball quotient. Let $k \in \mathbb{N}$. Then, under any of the following hypotheses:*

(1) $n \in [\![4, 5]\!]$ *and* $k > e^{-\gamma} e^{-(-D)^n ((n-2)n!+1)}$;

(2) $n \geqslant 6$ *and* $k > e^{-\gamma} e^{\frac{\frac{\pi^2}{6}(n-2)n!+1}{\frac{n+1}{2\pi}-1}}$,

the line bundle $\mathcal{O}_{\overline{X}_k^{GG}}(1)$ is big. For the first values of $n \geqslant 4$, this yields the lower bounds for $\log k$ displayed in Table 1.1.

Table 1.1. Effective lower bounds on $\log k$ to have $\mathcal{O}_{\overline{X}_k^{GG}}(1)$ big

n	4	5	6	7	8
$\log k$	$-\gamma + 5\,(-D^4)$	$-\gamma + 361\,(D^5)$	$41\,534$	$151\,711$	$920\,325$

The starting point to prove Theorem 1.1 consists in giving a more algebraic interpretation of the central metric construction of [8]. Let us give the main ideas about this construction. For any complex manifold X, the Green–Griffiths jet differential spaces X_k^{GG} can be deformed into a weighted projective bundle, using the standard construction of the Rees algebra. More specifically, there exists a family $\mathcal{X}_k^{GG} \longrightarrow X \times \mathbb{C}$ such that for any $\lambda \in \mathbb{C}^*$, the specialization $(\mathcal{X}_k^{GG})_\lambda \longrightarrow X \times \{\lambda\}$ is isomorphic to X_k^{GG}, and the specialization $(\mathcal{X}_k^{GG})_0 \longrightarrow X \times \{0\}$ is isomorphic to the *weighted projective bundle* $\mathrm{P}(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$. This last bundle is defined to be the quotient of $T_X \oplus \cdots \oplus T_X \longrightarrow X$ by the $\mathbb{C}^*$-action $\lambda \cdot (v_1, \ldots, v_k) = (\lambda v_1, \ldots, \lambda^k v_k)$. Moreover, there is a natural orbifold line bundle $\mathcal{O}_{\mathcal{X}_k^{GG}}(1)$ on the family $\mathcal{X}_k^{GG}$ whose restriction to the fibers $(\mathcal{X}_k^{GG})_\lambda$ gives the tautological bundles of X_k^{GG} and $\mathrm{P}(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$. The metric used in [8] can actually be seen as a singular metric on $\mathcal{O}_{\mathcal{X}_k^{GG}}(1)$; it is constructed in such a way that its specialization to the zero fiber $\mathrm{P}(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$ is induced by some metric on T_X.

One convenient feature about this family $\mathcal{X}_k^{GG}$ is the fact that it permits to interpret the intersection products on the jet spaces X_k^{GG} in terms of the intersection theory on $\mathrm{P}(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$. When dealing with Chow groups computations, these last spaces share many properties with the usual weightless projective spaces. In the first part of our work, we will recall some results about the intersection theory with rational coefficients for a weighted projective spaces $\mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)})$, which were proved first by Al-Amrani [1]. Since we work with rational coefficients instead of integer ones, the study is somewhat simplified; for the reader's convenience, we will explain how we could prove these results by following [11] in a standard way.

The other reason why studying the family $\mathcal{X}_k^{GG}$ is interesting is the fact that the main positivity properties (e.g. nefness, ampleness) of the tautological line bundle $\mathcal{O}(1) \longrightarrow \mathrm{P}(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$ can be extended

from the fiber over 0 to other fibers over $\lambda \in \mathbb{C}^*$, i.e. to the line bundle $\mathcal{O}_{X_k^{GG}}(1)$. Moreover, the positivity properties of $\mathcal{O}(1)$ on $\mathrm{P}(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$ are directly related to the ones of the vector bundle T_X. More generally, we will show in Section 3 that if $E_1^*, \ldots, E_p^*$ are ample (resp. nef), then the orbifold line bundle $\mathcal{O}(1) \longrightarrow \mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)})$ is ample (resp. nef) in the orbifold sense for any choice of weights $a_1, \ldots, a_p$. This will imply in particular the following result:

PROPOSITION 1.3. — *Let X be a complex projective manifold. Assume that Ω_X is ample (resp. nef). Then for any $k \in \mathbb{N}^*$, $\mathcal{O}_{X_k^{GG}}(1)$ is ample (resp. nef) in the orbifold sense.*

Getting back to the case of a ball quotient, we will use the logarithmic version of the previous discussion, and Riemann–Roch theorem in the orbifold case (see [17]) to obtain an estimate for the volume of the Green–Griffiths logarithmic jets differentials $E_{k,\bullet}^{GG}\Omega_{\overline{X}}(\log D)$ in terms of the Segre class of the weighted direct sum $T_{\overline{X}}(-\log D)^{(1)} \oplus \cdots \oplus T_{\overline{X}}(-\log D)^{(k)}$. This last Segre class can be in turn expressed in terms of the standard Segre class $s_\bullet(T_{\overline{X}}(-\log D))$. An application of Hirzebruch proportionality principle in the non-compact case (see [15]) will give our final estimate on $\mathrm{vol}(E_{k,\bullet}^{GG}\Omega_{\overline{X}}(\log D))$, which will be the first member of the estimate (1.1).

Finally, it will remain to relate the growth of the logarithmic jet differentials to the growth of the standard ones. To do this, we will simply bound from above the sections of the coherent sheaves $\mathcal{Q}_{k,m}$, defined for any k and m by the exact sequence

$$0 \longrightarrow E_{k,m}^{GG}\Omega_{\overline{X}} \longrightarrow E_{k,m}^{GG}\Omega_{\overline{X}}(\log D) \longrightarrow \mathcal{Q}_{k,m} \longrightarrow 0.$$

We will find a suitable filtration on the sheaves $\mathcal{Q}_{k,m}$, in such a way that the graded terms are locally free above the boundary D, and can be expressed in terms of the vector bundles Ω_D and $N_{D/\overline{X}}$. Then, using Riemann–Roch computations and the fact that D is a disjoint union of abelian varieties, we will be able to bound $h^0(\mathcal{Q}_{k,m})$ from above, for a fixed k, and m going to $+\infty$. This will give the second term in the estimate (1.1).

Acknowledgments

2. Segre classes of weighted projective bundles

We will now recall some results, first proved by Al-Amrani [1], permitting to construct Chern classes of weighted projective bundles. We will state the results in the simpler setting of Chow rings with rational coefficients.

DEFINITION 2.1. — *Let X be a complex algebraic projective variety. Consider a family $(E_i, a_i)_{1 \leqslant i \leqslant p}$, where the E_i are vector bundles on X, and the a_i are positive integers. The* weighted projectivized bundle *associated with the datum (E_i, a_i) is the projectivized scheme of the graded $\mathcal{O}_X$-algebra $\mathrm{Sym}(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)})^*$, defined as*

$$\mathrm{Sym}(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)})^* = \mathrm{Sym}\ E_1^{*(a_1)} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathrm{Sym}\ E_p^{*(a_p)},$$

where, for any i, $\mathrm{Sym}\ E_i^{*(a_i)}$ is the graded $\mathcal{O}_X$-algebra generated by sections of $E_i^{*(a_i)}$ in degree a_i. We will denote this scheme by $\mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)})$; remark that we use here the geometric convention for projectivized bundles.

We will say, by abuse of language, that $E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}$ is a weighted direct sum, or even a weighted vector bundle.

PROPOSITION 2.2. — *The variety $\mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)})$ has a natural orbifold structure (or a structure of Deligne–Mumford stack), for which the tautological line bundle $\mathcal{O}(1)$ is naturally defined as an orbifold line bundle. Moreover, this orbifold line bundle is* locally ample, *in the sense that the local isotropy groups of the orbifold structure act transitively on the fibres of $\mathcal{O}(1)$ (see for example [16]). Besides, if $\mathrm{lcm}(a_1, \ldots, a_p)|m$, the bundle $\mathcal{O}(m)$ can be identified to a standard line bundle on $\mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_r^{(a_p)})$.*

Proof. — We can naturally endow $\mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_r^{(a_p)})$ with a structure of Artin stack $\mathcal{P}$, since it can be considered as a quotient stack

$$E_1 \oplus \cdots \oplus E_p / \mathbb{C}^*,$$

where $\mathbb{C}^*$ acts by $\lambda \cdot (v_1, \ldots, v_p) = (\lambda^{a_1} v_1, \ldots, \lambda^{a_p} v_p)$.

Locally on X, the weighted projectivized bundle can be trivialized as a product of the base with a weighted projectivized space $\mathbb{P}(a_1, \ldots, a_p)$, where each a_i appears $\mathrm{rk}\ E_i$ times. Consequently, the Artin stack $\mathcal{P}$ has locally an orbifold structure, which makes it an orbifold stack. The claims on $\mathcal{O}(1)$ are local, and they can be proved directly using [10] and [16]. $\square$

Let us start our review of the properties of the Chow groups with rational coefficients of the weighted projectivized bundles.

PROPOSITION 2.3. — *Let $E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}$ be a weighted direct sum. Let us denote the natural projection by $p : \mathrm{P}_X(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}) \longrightarrow X$. For any k, there is an isomorphism*

$$(2.1) \qquad A_k \, \mathrm{P}_X\big(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}\big)_{\mathbb{Q}} \cong \bigoplus_{j=0}^{r} (A_{k-r+j}X)_{\mathbb{Q}},$$

where $r = \sum_{j=1}^{p} \mathrm{rk}\, E_j - 1$.

To prove this result, we can start by checking it in the case where X is an affine scheme. In that case, the weighted projective bundle is a quotient of a standard (trivial) projective bundle by a finite group, and it suffices to use the fact that such a quotient induces an isomorphism on the Chow rings with rational coefficients. We can then use the localization exact sequence to prove the general result.

Using the isomorphism (2.1), we can now define the Segre classes associated with a weighted direct sum $E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}$.

DEFINITION 2.4. — *Let X be a projective algebraic variety of dimension n, and let $E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}$ be a weighted direct sum on X. Let $q : \mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}) \longrightarrow X$ be the natural projection.*

If $k \in [|0, n|]$, the k-th Segre class of $E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}$ is defined as an endomorphism of $(A_* X)_{\mathbb{Q}}$. If $\alpha \in (A_l X)_{\mathbb{Q}}$, let

$$s_k\big(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}\big) \cap \alpha = \frac{1}{m^{k+r}} q_* \big(c_1 \mathcal{O}(m)^{r+k} \cap q^* \alpha\big).$$

where $r = \sum_i \mathrm{rk}\, E_i - 1$, and $m = \mathrm{lcm}(a_1, \ldots, a_p)$.

Remark 2.5. — In Definition 2.4, we could have replaced m by any integer divisible by $\mathrm{lcm}(a_1, \ldots, a_p)$. The important fact used here is that $\mathcal{O}(m)$ is a standard line bundle, which allows us to define its first Chern class in the usual way.

There is a Whitney formula for the weighted projective bundles, which permits to express the Segre classes $s_j(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)})$ in terms of the $s_\bullet(E_j)$ and of the weights (a_j):

PROPOSITION 2.6. — *Let $E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}$ be a weighted projective sum. We have*

$$(2.2) \qquad s_\bullet\big(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}\big) = \frac{\gcd(a_1, \ldots, a_p)}{a_1 \, \ldots \, a_p} \prod_{1 \leqslant j \leqslant p} s_\bullet\big(E_j^{(a_j)}\big),$$

where, for any vector bundle E and any weight $a \in \mathbb{N}$, we have $s_\bullet\big(E^{(a)}\big) = \frac{1}{a^{\mathrm{rk}\, E - 1}} \sum_{j \geqslant 0} \frac{s_j(E)}{a^j}$.

To prove this result, we can use the "splitting principle" to get back to the case where the E_i are all line bundles L_i. Now, denote $P = \mathrm{P}(L_0^{(a_0)} \oplus \cdots \oplus L_r^{(a_r)})$, and $p : P \longrightarrow X$ the canonical projection. Then, for some $m \in \mathbb{N}$, there exists a section of $(p^*L_0)^{\otimes l_0} \otimes \mathcal{O}_P(m)$ cutting out the subvariety $\mathrm{P}(L_1^{(a_1)} \oplus \cdots \oplus L_r^{(a_r)})$ with some computable multiplicity. As in [11], we can use this fact to relate the Segre class $s_\bullet(L_1^{(a_1)} \oplus \cdots \oplus L_r^{(a_r)})$ with the classes $s_\bullet(L_0^{(a_0)} \oplus \cdots \oplus L_r^{(a_r)})$ and $c_\bullet(L_0)$. The formula then follows by induction.

3. Positivity of weighted vector bundles

We now study the extension of the usual positivity properties of vector bundles to the case of weighted vector bundles.

DEFINITION 3.1. — Let $\mathbb{E} = E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}$ be a weighted direct sum. We say that $\mathbb{E}^* = E_1^{*(a_1)} \oplus \cdots \oplus E_p^{*(a_p)}$ is ample (resp. nef) if for any $m \in \mathbb{N}$ divisible enough, the (standard) line bundle $\mathcal{O}(m)$ is ample (resp. nef) on $\mathrm{P}(\mathbb{E})$.

Remark 3.2. — With the terminology of [16], saying that $\mathbb{E}^*$ is ample amounts to saying that $\mathcal{O}(1)$ is orbi-ample on $\mathrm{P}_X(\mathbb{E})$, the tautological orbifold line bundle being locally ample by [10].

We will see that the positivity properties of weighted vector bundles are exactly similar to the ones of the usual vector bundles, and can be proved in the same manner, following [13].

PROPOSITION 3.3. — Assume that $E_1^*,\ldots, E_p^*$ are ample on X. Then,

(1) For any coherent sheaf $\mathcal{F}$ on X, there exists $m_1 \in \mathbb{N}$ such that, for any $m \geqslant m_1$, the sheaf

$$\mathcal{F} \otimes \left(\bigoplus_{a_1 l_1 + \cdots + a_p l_p = m} S^{l_1} E_1^* \otimes \ldots S^{l_p} E_p^* \right)$$

is globally generated.

(2) For any ample divisor H on X, there exists $m_2 \in \mathbb{N}$ such that for any $m \geqslant m_2$, the sheaf

$$\bigoplus_{a_1 l_1 + \cdots + a_p l_p = m} S^{l_1} E_1^* \otimes \cdots \otimes S^{l_p} E_p^*$$

is a quotient of a direct sum of copies of $\mathcal{O}_X(H)$.

(3) If $\mathrm{lcm}(a_1, \ldots, a_p)|m$, then $\mathcal{O}(m)$ is ample on $\mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)})$. In particular, $\mathcal{O}(1)$ is orbi-ample.

Proof. — (1) This is easy to prove by induction on p using the similar characterization of ample vector bundles.

(2) It suffices to apply the point 1. to the sheaf $\mathcal{F} = \mathcal{O}(-H)$.

(3) Because of (2), there exist $m, N \in \mathbb{N}$ and a surjective morphism

$$\mathcal{O}_X(H)^{\oplus N} \longrightarrow \bigoplus_{a_1 l_1 + \ldots a_p l_p = m} S^{l_1} E_1^* \otimes \cdots \otimes S^{l_p} E_p^*.$$

Besides, because of Lemma 3.4, increasing m if necessary, we can suppose that for any $q \in \mathbb{N}$, the following natural morphism of vector bundles on X is surjective:

$$S^q \left[\bigoplus_{a_1 l_1 + \ldots a_p l_p = m} S^{l_1} E_1^* \otimes \cdots \otimes S^{l_p} E_r^* \right]$$

$$\longrightarrow \bigoplus_{a_1 l_1 + \cdots + a_p l_p = mq} S^{l_1} E_1^* \oplus \cdots \oplus S^{l_p} E_p^*.$$

We obtain a surjective morphism of graded $\mathcal{O}_X$-algebras

$$\bigoplus_{q \geqslant 0} S^q \left(\mathcal{O}_X(H)^{\oplus N} \right) \longrightarrow \bigoplus_{q \geqslant 0} \left[\bigoplus_{a_1 l_1 + \cdots + a_p l_p = mq} S^{l_1} E_1^* \otimes \cdots \otimes S^{l_p} E_p^* \right].$$

which determines an embedding

$$\mathrm{P}\big(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)} \big) \hookrightarrow \mathrm{P}\big(\mathcal{O}_X(-H)^{\oplus N} \big),$$

with

$$\mathcal{O}_{\mathrm{P}(\mathcal{O}_X(-H)^{\oplus N})}(1)\big|_{\mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)})} \cong \mathcal{O}(qm).$$

Since the tautological line bundle on $\mathrm{P}(\mathcal{O}_X(-H)^{\oplus N})$ is ample (cf. [13]), this ends the proof. □

LEMMA 3.4. — *Let $E_1, \ldots, E_p$ be $\mathbb{C}$-vector spaces, and let $a_1, \ldots, a_p \in \mathbb{N}^*$. Then, for any $m \in \mathbb{N}$ divisible enough, the natural linear maps*

$$S^q \left[\bigoplus_{a_1 l_1 + \cdots + a_p l_p = m} S^{a_1} E_1 \otimes \cdots \otimes S^{a_p} E_p \right]$$

$$\longrightarrow \bigoplus_{a_1 l_1 + \cdots + a_p l_p = mq} S^{a_1} E_1 \otimes \cdots \otimes S^{a_p} E_p$$

are onto for all $q \geqslant 1$.

Proof. — Because of [10], if m is sufficiently large and divisible by all $a_1, \ldots, a_r$, the (standard) line bundle $\mathcal{O}(m)$ on the weighted projective space $\mathrm{P}_{\mathrm{pt}}(E_1^{*(a_1)} \oplus \cdots \oplus E_p^{*(a_p)})$ is very ample. Consequently, there exists

an integer $q \in \mathbb{N}$ such that $S^p H^0(\mathcal{O}(mq)) \longrightarrow H^0(\mathcal{O}(mqp))$ is onto for all $p \geqslant 1$, which gives the result. $\qquad\square$

We will now study the case of nef line bundles. We will prove the following result.

PROPOSITION 3.5. — Let $E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)}$ be a weighted direct sum. Assume that $E_1^*, \ldots, E_p^*$ are nef. Then, if m is sufficiently divisible, the line bundle $\mathcal{O}(m)$ is nef on $\mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_p^{(a_p)})$.

To this aim, we will use the formalism of vector bundles twisted by rational classes (see [13] for the definition and the positivity properties of these objects). As in the weightless case, we naturally define the notion of ampleness for a weighted sum of twisted vector bundles:

DEFINITION 3.6. — We say that a weighted direct sum of twisted vector bundles of the form

$$E_1\langle a_1\delta\rangle^{(a_1)} \oplus \cdots \oplus E_p\langle a_p\delta\rangle^{(a_p)}$$

is ample, if for any m, divisible by $\mathrm{lcm}(a_1, \ldots, a_p)$ the $\mathbb{Q}$-line bundle $\mathcal{O}(m) \otimes \pi^*\mathcal{O}_X(m\delta)$ is ample on $\mathrm{P}(E_1^{*(a_1)} \oplus \ldots E_p^{*(a_p)})$.

Remark 3.7. — In Definition 3.6, we consider twists of the form $a_1\delta, \ldots, a_r\delta$ with $\delta \in N^1(X)_{\mathbb{Q}}$. This is related to the fact that if $E_1, \ldots, E_r$ are vector bundles, and if L is a line bundle, we have, for any m,

$$\bigoplus_{a_1l_1+\cdots+a_rl_r=m} S^{l_1}(E_1^* \otimes L^{\otimes a_1}) \otimes \cdots \otimes (E_r^* \otimes L^{\otimes a_r})$$

$$= L^{\otimes m} \otimes \bigoplus_{a_1l_1+\cdots+a_rl_r=m} S^{l_1}E_1^* \otimes \cdots \otimes E_r^*,$$

which implies in particular that the weighted projective bundle $P' = \mathrm{P}\left((E_1 \otimes L^{*\otimes a_1})^{(a_1)} \otimes \cdots \otimes (E_p \otimes L^{*\otimes a_p})^{(a_p)}\right)$ is identified to the variety $P = \mathrm{P}(E_1^{(a_1)} \oplus \cdots \oplus E_r^{(a_r)})$, with $\mathcal{O}_{P'}(m) \cong \mathcal{O}_P(m) \otimes p^*L^{\otimes m}$.

LEMMA 3.8. — Let $E_1\langle a_1\delta\rangle, \ldots, E_r\langle a_r\delta\rangle$ be twisted vector bundles on X. Assume that each $E_i^*\langle -a_i\delta\rangle$ is ample. Then the weighted direct sum

$$E_1^*\langle -a_1\delta\rangle^{(a_1)} \oplus \cdots \oplus E_r^*\langle -a_r\delta\rangle^{(a_r)}$$

is ample.

Proof. — We follow directly the proof presented in [13]. Because of Bloch–Gieseker theorem about ramified covers (see [13, Theorem 4.1.10]),

there exist a finite, surjective, flat morphism $f : Y \longrightarrow X$, where Y is a variety, and a divisor A such that $f^*\delta \equiv_{lin} A$. We have a fibered diagram

$$P' = \mathrm{P}_Y(f^*E_1^{*(a_1)} \oplus \cdots \oplus f^*E_r^{*(a_r)}) \xrightarrow{\ g\ } \mathrm{P}_X(E_1^{*(a_1)} \oplus \cdots \oplus E_r^{*(a_r)}) = P$$

$$\begin{array}{ccc} & & \\ Y & \xrightarrow{\quad\quad f \quad\quad} & X. \end{array}$$

Let $Q = \mathrm{P}_Y\big((f^*E_1^* \otimes \mathcal{O}(a_1 A))^{(a_1)} \oplus \cdots \oplus (f^*E_r^* \otimes \mathcal{O}(a_r A))^{(a_r)}\big)$. Then, we have a canonical identification $Q \cong P'$, which leads to identifying the line bundle $\mathcal{O}_Q(m)$ with $\mathcal{O}_{P'}(m) \otimes \pi_Y^* \mathcal{O}_Y(mA)$, as mentioned in Remark 3.7.

Besides, the $\mathbb{Q}$-line bundle

$$(3.1) \qquad\qquad g^* \left(\mathcal{O}_P(m) \otimes \pi^* \mathcal{O}_X (m\delta)\right)$$

is canonically identified to $\mathcal{O}_{P'}(m) \otimes \pi_Y^* \mathcal{O}_Y(mA)$, thus to $\mathcal{O}_Q(m)$. However, since each $E_i^*\langle -a_i\delta\rangle$ is ample, and since f is finite, each vector bundle $f^*E_i^* \otimes \mathcal{O}(-a_i A)$ is ample. Because of Proposition 3.3, the line bundle $\mathcal{O}_Q(m)$ is ample, so the line bundle (3.1) is ample. But g is finite and surjective, so $\mathcal{O}_P(m) \otimes \pi^* \mathcal{O}_X (m\delta)$ is ample on P, which gives the result. $\qquad\square$

Proof of Proposition 3.5. — It suffices to show that for any ample class $h \in N_1(X)_{\mathbb{Q}}$, the class $\mathcal{O}(m) \otimes \pi^*\mathcal{O}(mh)$ is ample. Let h be such a class. Then since each E_i^* is nef, the twisted vector bundles $E_i^*\langle a_i h\rangle$ are ample for any i. Consequently, by Lemma 3.8 and Definition 3.6, if $\mathrm{lcm}(a_1, \ldots, a_r)|m$, the line bundle $\mathcal{O}(m) \otimes \pi^*\mathcal{O}_X(mh)$ is ample on $\mathrm{P}(E_1^{*(a_1)} \oplus \cdots \oplus E_r^{*(a_r)})$. This gives the result. $\qquad\square$

3.1. An example of combinatorial application

We present a simple example of application of the previous discussion, which will turn out to be useful in Section 5, where we deal with jet bundles on a toroidal compactification of a quotient of the ball.

PROPOSITION 3.9. — *Let $k, n \in \mathbb{N}$. We have the following asymptotic upper bound, as $r \longrightarrow +\infty$:*

$$\sum_{j_1+2j_2+\cdots+kj_k=r} \frac{(j_1 + \cdots + j_k)^n}{n!} \leqslant \frac{1}{k!} \left[\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_n\leqslant k} \frac{1}{i_1 \ldots i_n} \right] \frac{r^{n+k-1}}{(n+k-1)!}$$
$$+ O(r^{n+k-2}).$$

Proof. — Let X be an abelian variety of dimension n, endowed with an ample line bundle L. Because of Proposition 3.3, the weighted direct sum $L^{(1)} \oplus \cdots \oplus L^{(k)}$ is ample on X. This means that the orbifold line bundle $\mathcal{O}(1)$ is orbi-ample on $P = \mathrm{P}_X\left(L^{*(1)} \oplus \cdots \oplus L^{*(k)}\right)$. By orbifold asymptotic Riemann–Roch theorem ([17], see also [16]), we have then, for any $m \in \mathbb{N}$,

$$h^0_{orb}(\mathrm{P}, \mathcal{O}(m)) \leqslant \int_P c_1 \mathcal{O}(1)^{n+k-1} \frac{m^{n+k-1}}{(n+k-1)!} + O(m^{n+k-1}).$$

However, because of Definition 2.4 and Proposition 2.6, $\int_X c_1 \mathcal{O}(1)^{n+k-1}$ can be computed as

$$\int_P c_1 \mathcal{O}(1)^{n+k-1} = \int_X s_n \left(L^{*(1)} \oplus \cdots \oplus L^{*(k)}\right)$$
$$= \frac{1}{k!} \int_X \left\{ \left(\sum_i H^i\right) \cdots \left(\sum_i \frac{H^i}{l^i}\right) \cdots \left(\sum_i \frac{H^i}{k^i}\right) \right\}_n,$$

where $H = c_1(L)$. Expending the computation yields

$$\int_P c_1 \mathcal{O}(1)^{n+k-1} = \frac{(L^n)}{k!} \left[\sum_{l_1+\cdots+l_k=n} \frac{1}{1^{l_1} \ldots k^{l_k}} \right]$$
$$= \frac{(L^n)}{k!} \left[\sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant k} \frac{1}{i_1 \ldots i_k} \right].$$

To obtain the result, it suffices to remark that we can identify the vector space $H^0(X, \bigoplus_{j_1+2j_2+\cdots+kj_k=m} L^{\otimes(j_1+\cdots+j_k)})$ to a subspace of the orbifold global sections of $\mathcal{O}(m)$. Thus :

$$h^0\left(X, \bigoplus_{j_1+2j_2+\cdots+kj_k=m} L^{\otimes(j_1+\cdots+j_k)}\right) \leqslant h^0_{orb}(P, \mathcal{O}(m)).$$

Besides, a direct application of Riemann–Roch–Hirzebruch theorem and Kodaira vanishing theorem on X gives

$$h^0(X, L^{\otimes(j_1+\cdots+j_k)}) = \frac{(j_1 + \cdots + j_k)^n}{n!}(L^n)$$

if $j_1 + \ldots j_k \neq 0$. Combining all these equations, we get the inequality. $\quad\square$

We can also get back the following classical result.

PROPOSITION 3.10. — *Let $a_0, \ldots, a_n \in \mathbb{N}^*$. Let $X = \mathbb{P}(a_0, \ldots, a_n)$ be the associated weighted projective space, endowed with its tautological orbifold line bundle $\mathcal{O}_X(1)$. We then have the asymptotic estimate*

$$h^0_{\mathrm{orb}}(X, \mathcal{O}_X(m)) = \frac{\gcd(a_0, \ldots, a_n)}{\prod_j a_j} \frac{m^n}{n!} + O(m^{n-1}).$$

Proof. — It is clear that the weighted direct sum $\mathbb{C}^{(a_0)} \oplus \cdots \oplus \mathbb{C}^{(a_n)} \longrightarrow \operatorname{Spec} \mathbb{C}$ is ample, which means that $\mathcal{O}_X(1)$ is orbi-ample. Then, using [16] and Definition 2.4,

$$h^0_{\mathrm{orb}}(X, \mathcal{O}_X(m)) = \int_X c_1 \mathcal{O}(1)^n \cdot \frac{m^n}{n!} + O(m^{n-1})$$

$$= s_0(\mathbb{C}^{(a_0)} \oplus \cdots \oplus \mathbb{C}^{(a_n)}) \frac{m^n}{n!} + O(m^{n-1}).$$

Besides, because of Proposition 2.6, we have

$$s_0(\mathbb{C}^{(a_0)} \oplus \cdots \oplus \mathbb{C}^{(a_n)}) = \frac{\gcd(a_0, \ldots, a_n)}{\prod_j a_j},$$

which gives the result. $\qquad\square$

4. Green–Griffiths jet bundles

4.1. Deformation of the jet spaces

We first remark that for any projective complex manifold, there is a natural deformation of its Green–Griffiths jets spaces to a weighted projectivized bundles, which will permit us to apply the previous discussion to the study of jet differentials.

Let X be a projective complex manifold. For $k \in \mathbb{N}$, we consider the Green–Griffiths jet differentials algebra $E_{k,\bullet}^{GG}\Omega_X$. Recall (cf. [9]) that $E_{k,\bullet}^{GG}\Omega_X$ is endowed with a natural filtration, which can be described as follows.

For each $(n_1, \ldots, n_k) \in \mathbb{N}^k$, and any coordinate chart $U \subset X$, we define the $(n_1, \ldots, n_k)$-graded term as the following space of local jet differentials:

$$F^{(l_1, \ldots, l_k)} E_{k,m}^{GG}\Omega_X(U)$$

$$= \left\{ \sum_{I=(I_1, \ldots, I_k)} a_I \, (f')^{I_1} \ldots (f^{(k)})^{I_k} \; \middle| \; (|I_1|, \ldots, |I_k|) \leqslant (l_1, \ldots, l_k) \right\}.$$

where for each $I_l = (p_1, \ldots, p_n)$, we write $(f^{(l)})^{I_l} = (f_1^{(l)})^{r_1} \ldots (f_n^{(l)})^{r_n}$. In the above formula, the lexicographic order on $\mathbb{N}^k$ is defined so that $(p_1, \ldots, p_k) < (n_1, \ldots, n_k)$ means that either $p_k < n_k$, or $p_k = n_k$ and $(p_1, \ldots, p_{k-1}) < (n_1, \ldots, n_{k-1})$ in the lexigraphic order for $\mathbb{N}^{k-1}$.

The formula of derivatives of composed maps implies that these local definitions glue together to give a well defined $\mathbb{N}^k$-filtration $F^\bullet E_{k,m}^{GG} \Omega_X$, compatible with the $\mathcal{O}_X$-algebra structure on $E_{k,\bullet}^{GG} \Omega_X$, and which is increasing with respect to the lexicographic order. Moreover, the graded terms occur only for $(l_1, \ldots, l_k)$ such that $l_1 + 2l_2 + \cdots + kl_k = m$, and we have, in this case:

$$\mathrm{Gr}_F^{(l_1, \ldots, l_k)} \left(E_{k,m}^{GG} \right) = \mathrm{Sym}^{l_1} \Omega_X \otimes \cdots \otimes \mathrm{Sym}^{l_k} \Omega_X.$$

By Definition 2.1, this means exactly that, as an $\mathcal{O}_X$-algebra,

$$(4.1) \qquad \mathrm{Gr}_F \left(E_{k,\bullet}^{GG} \right) \cong \mathrm{Sym}\, \Omega_X^{(1)} \otimes \cdots \otimes \mathrm{Sym}\, \Omega_X^{(k)}.$$

We can use the Rees deformation construction (see, e.g., [4]), to construct a $\mathcal{O}_{X \times \mathbb{C}}$-algebra $\mathcal{E}_{k,\bullet}^{GG}$ on $X \times \mathbb{C}$, such that for any $\lambda \in \mathbb{C}^*$, $\mathcal{E}_{k,\bullet}^{GG}|_{X \times \{\lambda\}}$ is identified to $E_{k,\bullet}^{GG} \Omega_X$, and $\mathcal{E}_{k,\bullet}^{GG}|_{X \times \{0\}}$ is identified to $\mathrm{Gr}_F(E_{k,\bullet}^{GG})$.

Let us give a few details about the construction. We first define a graded $\mathcal{O}_{X \times \mathbb{C}^k}$-algebra $\widetilde{\mathcal{E}}_{k,\bullet}^{GG}$, as follows:

$$\widetilde{\mathcal{E}}_{k,m}^{GG}(U \times \mathbb{C}^k) = \left\{ \sum_{l_1, \ldots, l_k} u_{l_1, \ldots, l_k} t_1^{l_1} \ldots t_k^{l_k} \,\middle|\, u_{l_1, \ldots, l_k} \in F^{(l_1, \ldots, l_k)} E_{k,m}^{GG} \Omega_X(U) \right\}$$

where $(t_1, \ldots, t_k)$ are coordinates on $\mathbb{C}^k$. It is easy to check that for any $(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ such that each $\lambda_j \neq 0$, we have $\widetilde{\mathcal{E}}_{k,m}^{GG}|_{X \times \{(\lambda_1, \ldots, \lambda_k)\}} \cong E_{k,m}^{GG} \Omega_X$, and that $\widetilde{\mathcal{E}}_{k,m}^{GG}|_{X \times \{(0, \ldots, 0)\}} \cong \mathrm{Gr}_F(E_{k,m}^{GG} \Omega_X)$. Now, define $\mathcal{E}_{k,\bullet}^{GG}$ to be the pullback of $\widetilde{\mathcal{E}}_{k,\bullet}^{GG}$ by the embedding $(x, t) \in X \times \mathbb{C} \mapsto (x, t, \ldots, t) \in X \times \mathbb{C}^k$.

Remark 4.1. — While $\widetilde{\mathcal{E}}_{k,\bullet}^{GG}$ seems to be the natural object arising in the construction above, it is more tractable to work over $X \times \mathbb{C}$ with the sheaf $\mathcal{E}_{k,\bullet}^{GG}$. To define the latter, we could have used any embedding $t \in \mathbb{C} \longmapsto (t\alpha_1, \ldots, t\alpha_k) \in \mathbb{C}^k$, with $\alpha_i \neq 0$, so our choice $(\alpha_1, \ldots, \alpha_k) = (1, \ldots, 1)$ is rather arbitrary. The same phenomenon occurs in [8], where the metric on $\mathcal{O}_{X_k^{GG}}(m)$ constructed by Demailly depends on some auxiliary parameters $\epsilon_1, \ldots, \epsilon_k \in \mathbb{R}_+^*$.

Applying the Proj functor, we obtain the following result.

PROPOSITION 4.2. — *For any complex projective manifold X, and for any $k \in \mathbb{N}^*$, there exists a morphism of varieties $\mathcal{X}_k^{GG} \longrightarrow X \times \mathbb{C}$, and an orbifold line bundle $\mathcal{O}_{\mathcal{X}_k^{GG}}(1)$ on $\mathcal{X}_k^{GG}$, such that :*

(1) *for any $\lambda \in \mathbb{C}^*$, $\mathcal{X}_k^{GG}\big|_\lambda$ is identified to X_k^{GG}, and $\mathcal{O}_{\mathcal{X}_k^{GG}}(1)\big|_\lambda$ identified to $\mathcal{O}_{X_k^{GG}}(1)$;*

(2) *the fibre $\mathcal{X}_k^{GG}\big|_0$ is identified to the variety $\mathrm{Proj}_X \left(\mathrm{Gr}_F \left(E_{k,\bullet}^{GG} \right) \right) \cong \mathrm{P}_X \left(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)} \right)$, and $\mathcal{O}_{\mathcal{X}_k^{GG}}(1)\big|_0$ is identified to the tautological line bundle of this weighted projective bundle.*

Remark 4.3. — By construction, the identifications mentioned above already occur at the level of sheaves of algebras. To obtain the identifications when taking Proj functors, we just need to check that the gradings on these sheaves of algebras are compatible under these identifications.

Proof. — Let us prove the second point, the first one being similar. By construction, we have a natural identification between

$$\mathcal{E}_{k,\bullet}^{GG}|_{X \times \{0\}} = \bigoplus_{m \geqslant 0} \mathcal{E}_{k,m}^{GG}|_{X \times \{0\}}$$

and

$$\mathrm{Gr}_F(E_{k,\bullet}^{GG} \Omega_X) = \bigoplus_{m \geqslant 0} \mathrm{Gr}_F(E_{k,m}^{GG} \Omega_X)$$

Moreover, this identification is compatible with the grading in m. Besides, by (4.1), the latter sheaf is identified, as a sheaf of *graded* algebras, with

$$\mathrm{Sym}\, \Omega_X^{(1)} \otimes \cdots \otimes \mathrm{Sym}\, \Omega_X^{(k)}$$
$$= \bigoplus_{m \geqslant 0} \left(\bigoplus_{l_1 + 2l_2 + \cdots + kl_k = m} \mathrm{Sym}^{l_1}\, \Omega_X \otimes \cdots \otimes \mathrm{Sym}^{l_k}\, \Omega_X \right).$$

Now, by Definition 2.1, the projectivized bundle associated to the latter sheaf of graded algebras, with respect to the grading in m, is $\mathrm{P}(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$. This implies immediately the identifications of varieties and orbifold line bundles mentioned in the second point. $\square$

We can now show that some usual positivity properties of the cotangent bundle can be transmitted to the higher order jet differentials.

PROPOSITION 4.4. — *If Ω_X is ample (resp. nef), then for any $k \in \mathbb{N}^*$, $E_{k,\bullet}^{GG} \Omega_X$ is ample (resp. nef), meaning that $\mathcal{O}_{X_k^{GG}}(1)$ is ample (resp. nef) as an orbifold line bundle.*

Proof. — Let $\mathcal{X}_k^{GG} \longrightarrow X \times \mathbb{C}$ be the variety given by Proposition 4.2, endowed with its orbifold line bundle $\mathcal{O}_{\mathcal{X}_k^{GG}}(1)$.

Assume first that Ω_X is ample. Then, because of Proposition 3.3, a suitable power of the tautological line bundle $\mathcal{O}(m)$ is ample on $\mathrm{P}(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$ if m is sufficiently divisible. Because of Proposition 4.2, it means that $\mathcal{O}_{\mathcal{X}_k^{GG}}(1)|_0$ is ample. By semi-continuity of the ampleness property, for any $\lambda \in \mathbb{C}^*$ in a Zariski neighborhood of 0, the orbifold line bundle $\mathcal{O}_{\mathcal{X}_k^{GG}}(1)|_\lambda$ is ample. Again because of Proposition 4.2, this means exactly that $E_{k,\bullet}^{GG}\Omega_X$ is ample.

The case where Ω_X is nef is dealt with in the same manner, using Proposition 3.5, and the fact that if $\mathcal{O}_{X_k^{GG}}(1)|_0$ is nef, then $\mathcal{O}_{X_k^{GG}}(1)|_\lambda$ is nef for any very general $\lambda \in \mathbb{C}$ (see [12]). $\qquad\square$

The previous discussion extends naturally to the case of logarithmic jet differentials. We then have the following proposition.

PROPOSITION 4.5. — *Let (X, D) be a smooth log-pair. For any $k \in \mathbb{N}^*$, there exists a morphism $\mathcal{X}_k^{GG,\log} \longrightarrow X \times \mathbb{C}$ and an orbifold line bundle $\mathcal{O}_{\mathcal{X}_k^{GG,\log}}(1)$ on $\mathcal{X}_k^{GG,\log}$ such that*

(1) *for any $\lambda \in \mathbb{C}^*$, $\mathcal{X}_k^{GG,\log}|_\lambda$ is identified to $X_k^{GG,\log}$, and $\mathcal{O}_{\mathcal{X}_k^{GG,\log}}(1)|_\lambda$ identified to $\mathcal{O}_{X_k^{GG}}(1)$;*

(2) *the fibre $\mathcal{X}_k^{GG,\log}|_0$ is identified to*

$$\mathrm{Proj}_X \mathrm{Gr}_F(E_{k,\bullet}^{GG}) \cong \mathbb{P}_X\big(T_X(-\log D)^{(1)} \oplus \cdots \oplus T_X(-\log D)^{(k)}\big),$$

and $\mathcal{O}_{\mathcal{X}_k^{GG,\log}}(1)|_0$ is identified to the tautological orbifold line bundle of this weighted projectivized bundle.

Proof. — As before, it suffices to use the fact that $E_{k,m}^{GG}\Omega_X(\log D)$ admits a filtration whose graded algebra is

$$\mathrm{Sym}\,\Omega_X(\log D)^{(1)} \otimes \cdots \otimes \mathrm{Sym}\,\Omega_X(\log D)^{(k)}. \qquad\square$$

In this setting, Proposition 4.4 extends naturally:

PROPOSITION 4.6. — *Let (X, D) be a smooth log-pair. If $\Omega_X(\log D)$ is nef (resp. ample), then for any $k \in \mathbb{N}^*$, $E_{k,\bullet}^{GG}\Omega_X(\log D)$ is nef (resp. ample), meaning that $\mathcal{O}_{X_k^{GG,\log}}(1)$ is nef (resp. ample) as orbifold line bundle.*

Remark 4.7. — Proposition 4.6 is actually only relevant for the nef property. Indeed, except when X is a curve, the bundle $\Omega_X(\log D)$ cannot be ample in general: if D is smooth and $\dim X \geqslant 2$, we have the residue map

$$\Omega_X(\log D) \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

and the restriction of this map to D shows that $\Omega_X(\log D)|_D$ has a trivial quotient. This implies that $\Omega_X(\log D)$ is not ample.

The following result, combining two theorems of Campana and Păun [7], and Demailly [8], shows that the bigness of the canonical orbifold line bundle $\mathcal{O}(1)$ on $\mathrm{P}(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$ for k large enough suffices to characterize the manifolds of general type.

PROPOSITION 4.8. — *Let X be a projective smooth manifold. The following assertions are equivalent.*

(1) *X is of general type;*
(2) *for large k, $E_{k,\bullet}^{GG} X$ is big, meaning that the usual line bundle $\mathcal{O}(m) \longrightarrow X_k^{GG}$ is big for m sufficiently divisible ;*
(3) *for large k, the orbifold line bundle $\mathcal{O}(1) \longrightarrow \mathrm{P}_X(T_X^{(1)} \otimes \cdots \otimes T_X^{(k)})$ is big, i.e. the line bundle $\mathcal{O}(m)$ is big for m sufficiently divisible.*

Proof.

$(1) \Rightarrow (2)$. — This is the main result of [8].

$(2) \Rightarrow (3)$. — Let $k \in \mathbb{N}^*$ large enough, and consider a sufficiently divisible $m \in \mathbb{N}^*$. Let $\mathcal{X}_k^{GG} \longrightarrow X \times \mathbb{C}$ be the variety given by Proposition 4.2, endowed with its tautological orbifold line bundle $\mathcal{O}_{\mathcal{X}_k^{GG}}(1)$. For any $\lambda \in \mathbb{C}^*$, $\mathcal{O}_{\mathcal{X}_k^{GG}}(m)|_\lambda$ is identified to $\mathcal{O}_{X_k^{GG}}(m)$, which is big. Consequently, there exists a constant $C > 0$, such that for any $\lambda \in \mathbb{C}^*$,

$$h^0(\mathcal{X}_k^{GG}|_\lambda, \mathcal{O}_{\mathcal{X}_k^{GG}}(m)|_\lambda) \geqslant Cm^{n+nk-1}.$$

Since $\mathcal{O}_{\mathcal{X}_k^{GG}}(m)$ is flat on the base $\mathbb{C}$, we deduce by semi-continuity that

$$h^0(\mathcal{X}_k^{GG}|_0, \mathcal{O}_{\mathcal{X}_k^{GG}}(m)|_0) \geqslant Cm^{n+nk-1}.$$

Besides, $X_k^{GG}|_0$ and $\mathcal{O}_{\mathcal{X}_k^{GG}}(1)|_0$ are identified with $\mathrm{P}_X(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$ and to its tautological line bundle, so the previous inequality means exactly that $\mathcal{O}(1)$ is big on $\mathrm{P}(T_X^{(1)} \oplus \cdots \oplus T_X^{(k)})$.

$(3) \Rightarrow (1)$. — This result is proved in [7].   □

5. Application to the toroidal compactifications of ball quotients

Let $\Gamma \in \mathrm{Aut}(\mathbb{B}^n)$ be a lattice with only unipotent parabolic isometries. Then, by [2] and [14], we can compactify the quotient $X = \mathbb{B}^n/\Gamma$ into a *toroidal compactification* $\overline{X} = X \sqcup D$, where D is a disjoint union of abelian

varieties. From now, on, $\overline{X}$ will always denote such a toroidal compactification of a ball quotient.

Let $k \in \mathbb{N}^*$, and let $\mathcal{X}_k^{GG,\log} \longrightarrow X \times \mathbb{C}$ be the family given by Proposition 4.5. Denote by $P_k^{GG,\log} \subset \mathcal{X}_k^{GG,\log}$ the fibre above $0 \subset \mathbb{C}$, which is isomorphic to $P_X(T_X(-\log D)^{(1)} \oplus \cdots \oplus T_X(-\log D)^{(k)})$. Let us also denote by $\mathcal{O}_P(1)$ the orbifold tautological line bundle on this weighted projective bundle.

PROPOSITION 5.1. — *The orbifold line bundle* $\mathcal{O}_{\overline{X}_k^{GG,\log}}(1) \longrightarrow \overline{X}_k^{GG,\log}$ *is nef.*

Proof. — The vector bundle $\Omega_{\overline{X}}(\log D)$ is nef because of [5]. Thus, the result comes from Proposition 4.4. $\qquad\square$

If $m_0 = \mathrm{lcm}(1, \ldots, k)$, the standard line bundle $\mathcal{O}_{\overline{X}_k^{GG,\log}}(m_0)$ is nef. This gives the following asymptotic expansion:

$$(5.1) \quad h^0(\overline{X}, E_{k,lm_0}^{GG} \Omega_{\overline{X}}(\log D))$$
$$= h^0(\overline{X}_k^{GG}, \mathcal{O}_{\overline{X}_k^{GG,\log}}(lm_0))$$
$$= \chi(\overline{X}_k^{GG}, \mathcal{O}_{\overline{X}_k^{GG,\log}}(lm_0)) + O(l^{n+nk-2})$$
$$= \left(\int_{\overline{X}_k^{GG,\log}} c_1 \mathcal{O}(m_0)^{m+nk-1} \right) l^{n+nk-1} + O(l^{n+nk-2}).$$

By Proposition 4.5, $\overline{X}_k^{GG,\log}$ and $P_k^{GG,\log}$ are members of the same flat family $\mathcal{X}_k^{GG,\log} \longrightarrow \mathbb{C}$. Thus, since the first Chern class is a topological invariant, we can compute the leading coefficient of this last expansion, as follows:

$$\int_{\overline{X}_k^{GG,\log}} c_1 \mathcal{O}(m_0)^{m+nk-1} = \int_{P_k^{GG,\log}} c_1 \mathcal{O}_P(m_0)^{m+nk-1}.$$

Then, using Definition 2.4, we find

$$\int_{\overline{X}_k^{GG,\log}} c_1 \mathcal{O}(m_0)^{m+nk-1}$$
$$= m_0^{n+nk-1} \int_{\overline{X}} s_n(T_{\overline{X}}(-\log D)^{(1)} \oplus \cdots \oplus T_{\overline{X}}(-\log D)^{(k)}).$$

If we insert this equation in (5.1), we see that if $m \in \mathbb{N}$ is divisible by m_0, we have

$$
h^0(\overline{X}, E_{k,m}^{GG}\Omega_{\overline{X}}(\log D))
$$
$$
= \int_{\overline{X}} s_n(T_{\overline{X}}(-\log D)^{(1)} \oplus \cdots \oplus T_{\overline{X}}(-\log D)^{(k)}) \frac{m^{n+nk-1}}{(n+nk-1)!}
$$
$$
+ O(m^{n+nk-2}).
$$

This gives the following value for the volume of $E_{k,\bullet}^{GG}\Omega_{\overline{X}}(\log D)$:

$$
(5.2) \quad \mathrm{vol}\left(E_{k,\bullet}^{GG}\Omega_{\overline{X}}(\log D)\right)
$$
$$
= \int_{\overline{X}} s_n(T_{\overline{X}}(-\log D)^{(1)} \oplus \cdots \oplus T_{\overline{X}}(-\log D)^{(k)}).
$$

5.1. Combinatorial expression of the volume. Uniform lower bound in k

The volume (5.2) can be expressed as a certain universal polynomial with rational coefficients in the Chern classes of $T_{\overline{X}}(-\log D)$. The same polynomial, applied to the Chern classes of $T_{\mathbb{P}^n}$ over $\mathbb{P}^n$, permits to compute $\int_{\mathbb{P}^n} s_n(T_{\mathbb{P}^n}^{(1)} \oplus \cdots \oplus T_{\mathbb{P}^n}^{(k)})$, and Hirzebruch proportionality principle in the non-compact case (see [15]) implies

$$
\int_{\overline{X}} s_n(T_{\overline{X}}(-\log D)^{(1)} \oplus \cdots \oplus T_{\overline{X}}(-\log D)^{(k)})
$$
$$
= (-1)^n \frac{(K_{\overline{X}} + D)^n}{(n+1)^n} \int_{\mathbb{P}^n} s_n(T_{\mathbb{P}^n}^{(1)} \oplus \cdots \oplus T_{\mathbb{P}^n}^{(k)})
$$

Using Proposition 2.6, we can give an explicit combinatorial expression of this last quantity. Indeed, if we let $H = c_1 \mathcal{O}_{\mathbb{P}^n}(1)$, since $s_\bullet(T_{\mathbb{P}^n}) = \left(\sum_{i=1}^n (-1)^i H^i\right)^{n+1}$, we find

$$
(-1)^n \int_{\mathbb{P}^n} s_n(T_{\mathbb{P}^n}^{(1)} \oplus \cdots \oplus T_{\mathbb{P}^n}^{(k)})
$$
$$
= \frac{(-1)^n}{(k!)^n} \left\{ \left(\sum_{i=1}^n (-1)^i H^i\right)^{n+1} \left(\sum_{i=1}^n (-1)^i \frac{H^i}{2^i}\right)^{n+1} \cdots \cdots \left(\sum_{i=1}^n (-1)^i \frac{H^i}{k^i}\right)^{n+1} \cdot [\mathbb{P}^n] \right\}_0
$$
$$
= \frac{1}{(k!)^n} \sum_{l_{1,1}+l_{1,2}+\cdots+l_{n+1,k}=n} \frac{1}{1^{l_{1,1}+l_{2,1}+\cdots+l_{n+1,1}} \cdot \cdots \cdot k^{l_{1,k}+\cdots+l_{n+1,k}}} (H^n \cdot [\mathbb{P}^n]).
$$

where each index $l_{i,j}$ ($i \in [[1, n+1]], j \in [[1, k]]$) represents a possible choice of power for H in the i-th factor of the product $\left(\sum_{l=1}^n (-1)^l \frac{H^l}{j^l}\right)^{n+1}$. Thus,

we see that choosing exponents $(l_{i,j})_{1\leqslant i\leqslant n+1,1\leqslant j\leqslant k}$ such that $\sum_{i,j} l_{i,j} = n$ amounts to choosing a non-decreasing sequence $u_1 \leqslant \cdots \leqslant u_n$ of elements of the ordered set

$$S_{k,n} = \{1_1 < \cdots < 1_{n+1} < 2_1 < \cdots < 2_{n+1} < \cdots < k_1 < \cdots < k_{n+1}\},$$

where each integer between 1 and k is repeated $n+1$ times. The bijection between the set of choices of $(l_{i,j})$ and the set of sequences $u_1 \leqslant \cdots \leqslant u_n$ can easily be made explicit : to $(l_{i,j})$, we associate the sequence (u_i), where the element j_m is repeated $l_{m,j}$ times. Thus, we find

$$(-1)^n \int_{\mathbb{P}^n} s_n\left(T_{\overline{X}}^{(1)} \oplus \cdots \oplus T_{\overline{X}}^{(k)}\right) = \frac{1}{(k!)^n} \sum_{\{u_1 \leqslant \cdots \leqslant u_n\} \subset S_{k,n}} \frac{1}{u_1 \ldots u_n},$$

where, in the quotient appearing on the right hand side, we compute the product by treating the elements of S as ordinary integers (we forget their indexes).

We then find an explicit combinatorial formula for the volume of logarithmic jet differentials of order k :

$$(5.3) \quad \mathrm{vol}\left(E_{k,\bullet}^{GG}\Omega_{\overline{X}}(\log D)\right) = \frac{\left(K_{\overline{X}} + D\right)^n}{(n+1)^n (k!)^n} \sum_{\{u_1 \leqslant \cdots \leqslant u_n\} \subset S_{k,n}} \frac{1}{u_1 \ldots u_n}.$$

It is not hard to use this formula to obtain a more tractable lower bound on the volume. Indeed, we have:

$$(5.4) \quad n! \sum_{\{u_1 \leqslant \cdots \leqslant u_n\} \subset S_{k,n}} \frac{1}{u_1 \ldots u_n}$$

$$\geqslant \sum_{(u_1,\ldots,u_n) \in S_{k,n}^n} \frac{1}{u_1 \ldots u_n} = \left(\sum_{u \in S_{k,n}} \frac{1}{u}\right)^n$$

$$= (n+1)^n \left(1 + 1/2 + \cdots + 1/k\right)^n \geqslant (n+1)^n (\log k + \gamma)^n$$

where, in the first inequality, we use the fact that for any ordered set $\{u_1 \leqslant \cdots \leqslant u_n\}$, the number of distinct n-uples $(v_1, \ldots, v_n)$ having the same elements is at least $n!$. The letter γ represents the Euler–Mascheroni constant.

We obtain the following lower bound, *valid for any $k \geqslant 1$*:

$$\mathrm{vol}\left(E_{k,\bullet}^{GG}\Omega_{\overline{X}}(\log D)\right) \geqslant \left(K_{\overline{X}} + D\right)^n \frac{(\log k + \gamma)^n}{(k!)^n \, n!}.$$

This formula can be seen as an effective version of the asymptotic estimates of [8], in the case of logarithmic jet differentials on a toroidal compactification of a ball quotient.

6. Upper bound on the vanishing conditions on the boundary

We will now study the number of vanishing conditions on the boundary that a logarithmic jet differential must satisfy to be a standard one.

For any $k \in \mathbb{N}^*$, and any $m \in \mathbb{N}$, we define a sheaf $\mathcal{Q}_{k,m}$, supported on D, in the following manner:

$$(6.1) \qquad 0 \longrightarrow E_{k,m}^{GG}\Omega_{\overline{X}} \longrightarrow E_{k,m}^{GG}\Omega_{\overline{X}}(\log D) \longrightarrow \mathcal{Q}_{k,m} \longrightarrow 0.$$

Then, we have:

$$(6.2) \qquad h^0(\overline{X}, E_{k,m}^{GG}\Omega_{\overline{X}}) \geqslant h^0(\overline{X}, E_{k,m}^{GG}\Omega_{\overline{X}}(\log D)) - h^0(\overline{X}, \mathcal{Q}_{k,m}).$$

6.1. Filtration on the quotient $\mathcal{Q}_{k,m}$

Our goal is to obtain an upper bound on $h^0(\mathcal{Q}_{k,m})$, as $m \longrightarrow +\infty$, with fixed $k \in \mathbb{N}$. To do this, we will produce a sufficiently sharp filtration on the sheaf $\mathcal{Q}_{k,m}$, so that the graded terms are locally free $\mathcal{O}_D$-modules. We will then the bound from above the number of global sections of these graded terms.

PROPOSITION 6.1. — *The inclusion of* (6.1) *preserves the natural filtrations on* $E_{k,m}^{GG}\Omega_{\overline{X}}$ *and* $E_{k,m}^{GG}\Omega_{\overline{X}}(\log D)$.

Proof. — We only need to check this locally: this inclusion sends an jet differential equation of the form $\prod_{i,l}(f_i^{(l)})^{a_{i,l}}$ on

$$\prod_{i \neq n, l}(f_i^{(l)})^{a_{i,l}} \cdot \prod_l z_n^{a_{i,l}}\left(\frac{f_n^{(l)}}{z_n}\right)^{a_{i,l}}.$$

The exponents of the different $f_i^{(l)}$ are then preserved by the inclusion, so the natural filtrations are also preserved. $\qquad\square$

Consequently, $\mathcal{Q}_{k,m}$ admits a induced filtration F_1, whose graded terms can be written as a quotient of the corresponding graded terms in $E_{k,m}^{GG}\Omega_{\overline{X}}$ and $E_{k,m}^{GG}\Omega_{\overline{X}}(\log D)$:

$$(6.3) \quad \mathrm{Gr}_\bullet^{F_1}\left(\mathcal{Q}_{k,m}\right)$$

$$= \bigoplus_{l_1 + 2l_2 + \cdots + kl_k = m} \frac{\left[S^{l_1}\Omega_{\overline{X}}(\log D) \otimes \cdots \otimes S^{l_k}\Omega_{\overline{X}}(\log D)\right]}{\mathrm{Im}\left[S^{l_1}\Omega_{\overline{X}} \otimes \cdots \otimes S^{l_k}\Omega_{\overline{X}}\right]}.$$

We will now produce successive refinements of the filtration F_1, until we obtain a filtration whose graded terms are all locally free $\mathcal{O}_D$-modules. We

can already simplify the quotient appearing in (6.3), using the following elementary result.

LEMMA 6.2. — Let $\mathcal{E}_1, \ldots \mathcal{E}_l$ be $\mathcal{O}_{\overline{X}}$-modules. For any i, we consider a sub-module $\mathcal{E}'_i \hookrightarrow \mathcal{E}_i$. Then the quotient $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_l \big/ \mathrm{Im}\, (\mathcal{E}'_1 \otimes \cdots \otimes \mathcal{E}'_l)$ admits a filtration whose i-th graded term can be identified with

$$\mathcal{E}'_1 \otimes \cdots \otimes \mathcal{E}'_{i-1} \otimes \left(\mathcal{E}_i \big/ \mathcal{E}'_i \right) \otimes \mathcal{E}_{i+1} \otimes \cdots \otimes \mathcal{E}_l.$$

Proof. — It suffices to consider the filtration induced on the quotient sheaf $\mathcal{E}_1 \otimes \cdots \mathcal{E}_l \big/ \mathrm{Im}\, (\mathcal{E}'_1 \otimes \cdots \otimes \mathcal{E}'_l)$ by the images of any of the sheaves appearing in the sequence of morphisms

$$\mathcal{E}'_1 \otimes \cdots \otimes \mathcal{E}'_l \longrightarrow \ldots \longrightarrow \mathcal{E}'_1 \otimes \cdots \otimes \mathcal{E}'_{i-1} \otimes \mathcal{E}_i \otimes \cdots \otimes \mathcal{E}_l \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_l.$$

$\square$

We deduce from this proposition and the previous one the existence of a filtration F_2 on $\mathcal{Q}_{k,m}$, whose graded module can be written

$$\mathrm{Gr}^{F_2}_\bullet (\mathcal{Q}_{k,m}) = \bigoplus_{l_1 + 2l_2 + \cdots + kl_k = m} \bigoplus_{i=1}^k S^{l_1} \Omega_{\overline{X}} \otimes \cdots \otimes \mathcal{S}_{l_i} \otimes \cdots \otimes S^{l_k} \Omega_{\overline{X}}(\log D),$$

where $\mathcal{S}_l = S^l \Omega_{\overline{X}}(\log D) \big/ S^l \Omega_{\overline{X}}$.

The $\mathcal{O}_{\overline{X}}$-modules $\mathcal{S}_l$ can be in turn filtered in $\mathcal{O}_D$-modules, using a filtration that was first introduced in [6]. For completeness, we will describe this filtration in our special case.

PROPOSITION 6.3. — For any $l \in \mathbb{N}$, $\mathcal{S}_l$ is endowed with a filtration, whose graded terms are $\mathcal{O}_D$-modules, written

$$\mathrm{Gr}_\bullet(\mathcal{S}_l) = \bigoplus_{j=0}^l \bigoplus_{s=0}^j \left(N^*_{D/\overline{X}} \right)^{\otimes s} \otimes S^{l-j} \Omega_D.$$

Proof. — According to [14], each boundary component T_b admits a tubular neighborhood U, quotient of its universal cover $\widehat{U} \subset \mathbb{C}^{n-1} \times \mathbb{C}$ by a lattice $\Lambda \subset \mathbb{C}^{n-1}$. The component T_b can be identified to the quotient of $\mathbb{C}^{n-1} \times \{0\}$ by Λ. Let $D^\circ = \mathbb{C}^{n-1} \times \{0\} \subset \widehat{U}$. The elements $a \in \Lambda$ act on $\Omega_{\widehat{U}}(\log D^0)$ in the following way:

$$\begin{cases} a \cdot \frac{\mathrm{d}\, z_n}{z_n} = \frac{\mathrm{d}\, z_n}{z_n} + \sum_{i=1}^{n-1} \gamma_i(a)\, \mathrm{d}\, z_i; \\ a \cdot \mathrm{d}\, z_i = \mathrm{d}\, z_i \quad \text{if } 1 \leqslant i \leqslant n-1, \end{cases}$$

where the $\gamma_i : \mathbb{C}^{n-1} \longrightarrow \mathbb{C}$ are $\mathbb{R}$-linear maps. The natural filtration by the degree of $\frac{\mathrm{d}\, z_n}{z_n}$ in $S^l \Omega_{\widehat{U}}(\log D^\circ)$ is consequently preserved by Λ, and induces

a filtration G_l on $S^l \Omega_U(\log D))$ whose graded terms are globally trivial and can be written

$$\mathrm{Gr}_j^{G_l}(S^l \Omega_U(\log D)) = \left[\left(\frac{\mathrm{d}\, z_n}{z_n} \right)^j \right] \cdot S^{l-j}\Omega_D.$$

This expression in local coordinates shows that the induced filtration by G_l on $S^l \Omega_U$ admits as general graded term

$$\mathrm{Gr}_j^{G_l \cap S_l \Omega_U}(S^l \Omega_U) = \mathcal{I}_{jD} \otimes_{\mathcal{O}_U} \left[\left(\frac{\mathrm{d}\, z_n}{z_n} \right)^j \right] \cdot S^{l-j}\Omega_D,$$

where $\mathcal{I}_{jD}$ is the sheaf of ideals of the divisor jD. Consequently, G_l induces a new filtration on the quotient $\left. S^l \Omega_U(\log D) \middle/ S^l \Omega_U \right.$, whose graded terms are

$$\mathrm{Gr}_\bullet(\mathcal{S}_l) = \mathcal{O}_{jD} \otimes_{\mathcal{O}_U} S^{l-j}\Omega_D.$$

To obtain the proposition, it suffices to refine this last filtration, remarking that $\mathcal{O}_{jD} = \left. \mathcal{O}_{\overline{X}} \middle/ \mathcal{I}_{jD} \right.$ is itself filtered by

$$0 \subset \left. \mathcal{I}_{(j-1)D} \middle/ \mathcal{I}_{jD} \right. \subset \cdots \subset \left. \mathcal{I}_{lD} \middle/ \mathcal{I}_{jD} \right. \subset \cdots \subset \mathcal{O}_{jD},$$

whose successive quotients can be identified to

$$\left. \mathcal{I}_{lD} \middle/ \mathcal{I}_{(l+1)D} \right. \simeq \left(N_{D/\overline{X}}^* \right)^{\otimes l}. \qquad \square$$

We can consequently refine the filtration F_2, to obtain a new one F_3, whose graded module is

$$\mathrm{Gr}_\bullet^{F_3}(\mathcal{Q}_{k,m})$$

$$= \bigoplus_{i=1}^k \bigoplus_{l_1+2l_2+\cdots+kl_k=m} \bigoplus_{j_i=0}^{l_i} \bigoplus_{s_i=0}^{j_i} \left(S^{l_1}\Omega_{\overline{X}} \otimes \cdots \otimes S^{l_{i-1}}\Omega_{\overline{X}} \right.$$

$$\otimes \left[\left(N_{D/\overline{X}}^* \right)^{\otimes s_i} \otimes S^{l_i - j_i}\Omega_D \right] \otimes S^{l_{i+1}}\Omega_{\overline{X}}(\log D) \otimes \ldots \otimes S^{l_k}\Omega_{\overline{X}}(\log D) \Big).$$

Each one of the terms of this direct sum can be seen as an $\mathcal{O}_D$-module. Besides, we have seen in the proof of Proposition 6.3 that $S^l \Omega_{\overline{X}}(\log D)|_D$ admits a natural filtration whose graded terms are trivial:

$$\mathrm{Gr}_\bullet(S^l \Omega_{\overline{X}}(\log D)) = \bigoplus_{j=0}^l S^j \Omega_D.$$

On the other hand, since each boundary component admit a tubular neighborhood, we have

$$\Omega_{\overline{X}}|_D = N_{D/\overline{X}}^* \oplus \Omega_D,$$

so $S^l \Omega_{\overline{X}}|_D \simeq \bigoplus_{j=0}^l \left(N^*_{D/\overline{X}} \right)^j \otimes S^{l-j} \Omega_D$. We can consequently refine a last time the filtration on $\mathcal{Q}_{k,m}$, to obtain the following proposition.

PROPOSITION 6.4. — For any $k, m \in \mathbb{N}^*$, there exists a filtration $F_\bullet \mathcal{Q}_{k,m}$, whose graded module is an $\mathcal{O}_D$-module written

$$(6.4) \quad \mathrm{Gr}_\bullet^F \left(\mathcal{Q}_{k,m} \right)$$

$$= \bigoplus_{i=1}^k \bigoplus_{l_1 + 2l_2 + \cdots + kl_k = m} \bigoplus_{j_1=0}^{l_i} \cdots \bigoplus_{j_k=0}^{l_k} \bigoplus_{s_i=0}^{j_i} \left(N^*_{D/\overline{X}} \right)^{\otimes (j_1 + \cdots + j_{i-1} + s_i)}$$

$$\otimes S^{l_1 - j_1} \Omega_D \otimes \cdots \otimes S^{l_k - j_k} \Omega_D.$$

where all tensor products are taken over $\mathcal{O}_D$.

6.2. Upper bound on the graded terms of the filtration

We want to obtain an asymptotic upper bound on $h^0 \left(D, \mathrm{Gr}_\bullet^F \left(\mathcal{Q}_{k,m} \right) \right)$ when $m \longrightarrow 0$. We start by changing the indexing of the direct sums, so that we sum over $r = j_1 + 2j_2 + \cdots + kj_k$. If we proceed to the substitution $l_i \leftarrow l_i - j_i$, we find:

$$\mathrm{Gr}_\bullet^F (\mathcal{Q}_{k,m}) = \bigoplus_{r=0}^m \left(\left[\bigoplus_{j_1 + 2j_2 + \cdots + kj_k = r} \bigoplus_{i=1}^k \bigoplus_{s_i=0}^{j_i} \left(N^*_{D/\overline{X}} \right)^{\otimes (j_1 + \cdots + j_{i-1} + s_i)} \right] \right.$$

$$\left. \otimes \left[\bigoplus_{l_1 + 2l_2 + \cdots + kl_k = m - r} S^{l_1} \Omega_D \otimes \cdots \otimes S^{l_k} \Omega_D \right] \right)$$

The term on the right is a trivial vector bundle, because D is made of disjoint abelian varieties. Consequently, we have

$$(6.5) \quad h^0 \left(D, \mathrm{Gr}_\bullet^F \left(\mathcal{Q}_{k,m} \right) \right)$$

$$= \sum_{r=0}^m \left(\left[\sum_{j_1 + 2j_2 + \cdots + kj_k = r} \sum_{i=1}^k \sum_{s_i=0}^{j_i} h^0 \left(D, \left(N^*_{D/\overline{X}} \right)^{\otimes (j_1 + \cdots + j_{i-1} + s_i)} \right) \right] \right.$$

$$\left. \cdot \mathrm{rk} \left[\bigoplus_{l_1 + 2l_2 + \cdots + kl_k = m - r} S^{l_1} \Omega_D \otimes \cdots \otimes S^{l_k} \Omega_D \right] \right).$$

For a fixed $(j_1, \ldots, j_k)$, we now compute

$$\sum_{i=1}^k \sum_{s_i=0}^{j_i} h^0 \left(D, \left(N^*_{D/\overline{X}} \right)^{\otimes (j_1 + \cdots + j_{i-1} + s_i)} \right).$$

Recall that the line bundle $N_{D/X}^*$ is ample (cf. [14]). Consequently, since the boundary is made of abelian varieties, Kodaira vanishing theorem yields

$$\chi(D, (N_{D/\overline{X}}^*)^{\otimes(j_1+j_2+\cdots+s_i)}) = h^0(D, (N_{D/\overline{X}}^*)^{\otimes(j_1+j_2+\cdots+s_i)}),$$

as soon as $j_1 + j_2 + \cdots + s_i \neq 0$.

Besides, still because the boundary is a union of abelian varieties, Hirzebruch–Riemann–Roch theorem gives

$$\chi(D, (N_{D/\overline{X}}^*)^{\otimes(j_1+j_2+\cdots j_{i-1}+s_i)})$$
$$= \frac{1}{(n-1)!}(j_1 + \cdots + j_{i-1} + s_i)^{n-1}\,[-(-D)^n].$$

We can sum this last term on s_i, to find

$$\sum_{s_i=0}^{j_i}(j_1 + \cdots + j_{i-1} + s_i)^{n-1}$$

$$= \sum_{l_1+\cdots+l_i=n-1}\binom{n-1}{l_1,\ldots,\,l_i}j_1^{l_1}\cdots j_{i-1}^{l_{i-1}}\sum_{s_i=0}^{j_i}s_i^{l_i}$$

$$= \sum_{l_1+\cdots+l_i=n-1}\binom{n-1}{l_1,\ldots,\,l_i}j_1^{l_1}\cdots j_{i-1}^{l_{i-1}}\Big[\frac{j_i^{l_i+1}}{l_i+1}+O(j_i^{l_i-1})\Big]$$

$$= \sum_{l_1+\cdots+(l_i+1)=n}\frac{1}{n}\binom{n}{l_1\ldots,\,l_i+1}j_1^{l_1}\cdots j_i^{l_i+1}$$

$$+O\left(\sum_{l_1+\cdots+l_i=n-1}\binom{n-1}{l_1,\ldots,\,l_i}j_1^{l_1}\cdots j_i^{l_i}\right)$$

$$= \frac{1}{n}\big[(j_1+\cdots+j_i)^n-(j_1+\cdots+j_{i-1})^n\big]+O((j_1+\cdots+j_i)^{n-1}),$$

where we use the multinomial formula at the second and fourth lines.

If we sum over i, and using the fact that for a fixed i, there is only one term for which $j_1 + \cdots + j_{i-1} + s_i = 0$, we finally find

$$(6.6)\quad \sum_{i=1}^{k}\sum_{s_i=0}^{j_i}h^0\left(D,\left(N_{D/\overline{X}}^*\right)^{\otimes(j_1+\cdots+j_{i-1}+s_i)}\right)$$

$$= \frac{1}{n!}(j_1+\cdots+j_n)^n\frac{[-(-D)^n]}{n!}+O\left(\sum_i(j_1+\cdots+j_i)^{n-1}\right).$$

6.3. Final asymptotic estimate over $h^0(\mathcal{Q}_{k,m})$ as $m \longrightarrow +\infty$

Applying Proposition 3.9, we can sum (6.6) over the $j_1, \ldots, j_k$ such that $j_1 + 2j_2 + \cdots + kj_k = r$, to find

$$\sum_{j_1+2j_2+\cdots+kj_k=r} \sum_{i=1}^{k} \sum_{s_i=0}^{j_i} h^0\left(D, \left(N_{D/\overline{X}}^*\right)^{\otimes(j_1+\cdots+j_{i-1}+s_i)}\right)$$

$$\leqslant \frac{r^{n+k-1}}{(n+k-1)} \left[\frac{1}{k!} \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_n\leqslant k} \frac{1}{i_1\ldots i_n}\right] [-(-D)^n] + O(r^{n+k-2}).$$

Moreover, according to Proposition 3.10, we have

$$\mathrm{rk}\left(\bigoplus_{l_1+2l_2+\cdots+kl_k=r} S^{l_1}\Omega_D \otimes \cdots \otimes S^{l_k}\Omega_D\right)$$

$$= \frac{1}{(k!)^{n-1}} \frac{m^{(n-1)k-1}}{((n-1)k-1)!} + O(m^{nk-2}).$$

If we put these two asymptotic expressions in (6.5), we obtain the following final estimate on $h^0(\mathcal{Q}_{k,m})$, when $m \longrightarrow +\infty$:

$$(6.7) \quad h^0(\mathcal{Q}_{k,m}) \leqslant h^0(\mathrm{Gr}_\bullet^F(\mathcal{Q}_{k,m}))$$

$$\leqslant \sum_{r=0}^{m} \left[\frac{r^{n+k-1}}{(n+k-1)} \left[\frac{1}{k!} \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_n\leqslant k} \frac{1}{i_1\ldots i_n}\right] [-(-D)^n]\right]$$

$$\cdot \left[\frac{1}{(k!)^{n-1}} \frac{(m-r)^{(n-1)k-1}}{((n-1)k-1)}\right] + O(m^{n+nk-2})$$

$$= \frac{[-(-D)^n]}{(k!)^n} \left[\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_n\leqslant k} \frac{1}{i_1\ldots i_k}\right] \frac{m^{n+nk-1}}{(n+nk-1)!} + O(m^{n+nk-2}).$$

6.4. Uniform lower bound in k on $\mathrm{vol}(E_{k,m}^{GG}\Omega_{\overline{X}})$

Combining (6.2) with (5.3) and (6.7), we finally obtain the lower bound (1.1) on $\mathrm{vol}(E_{k,\bullet}^{GG}\Omega_{\overline{X}})$, which proves Theorem 1.1. The expression (1.1) being valid for any k, we can use the results of [3] to determine an order k after which the algebra $E_{k,\bullet}^{GG}\Omega_{\overline{X}}$ has maximal growth.

For example, it is not hard to obtain an asymptotic expansion of (1.1), with leading coefficient

$$\frac{1}{n!(k!)^n}(\log k)^n \left((K_{\overline{X}} + D)^n + (-D)^n\right) = \frac{1}{n!(k!)^n}(\log k)^k (K_{\overline{X}})^n.$$

When $K_{\overline{X}}$ is nef and big, we get back the asymptotic lower bound of [8].

$$\mathrm{vol}(E_{k,m}^{GG}\Omega_{\overline{X}}) \geqslant \frac{(\log k)^n}{n!(k!)^n}\left(\mathrm{vol}(K_{\overline{X}}) + O((\log k)^{-1})\right).$$

6.5. Explicit orders k to have a big $E_{k,\bullet}^{GG}\Omega_{\overline{X}}$

In this section, we prove Corollary 1.2. We will use (1.1) to determine an effective k after which $E_{k,\bullet}^{GG}\Omega_{\overline{X}}$ is big. Let us begin by determining an upper bound on $\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_n\leqslant k}\frac{1}{i_1\ldots i_n}$. We have

$$\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_n\leqslant k}\frac{1}{i_1\ldots i_k} = \sum_{p=1}^{n}\sum_{\substack{l_1+\cdots+l_p=n\\\forall i,\, l_i>0}}\sum_{1\leqslant j_1<\cdots<j_p\leqslant k}\frac{1}{j_1^{l_1}\ldots j_p^{l_p}},$$

the datum of n integers $1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant k$ in non-decreasing order being equivalent to the one of an integer p giving the number of distinct i_j, of p integers $1 \leqslant j_1 < \cdots < j_p \leqslant k$, and of positive exponents $l_1,\ldots,l_p$ such that $\sum_k l_k = n$. Now, for any $p \geqslant 1$, we have:

$$\sum_{1\leqslant j_1<\cdots<j_p\leqslant n}\frac{1}{j_1\ldots j_p} \leqslant \frac{1}{p!}\sum_{1\leqslant j_1,\ldots,j_p\leqslant k}\frac{1}{j_1\ldots j_p}$$

$$= \frac{1}{p!}\left[\sum_{j=1}^{k}\frac{1}{j}\right]^p$$

$$\leqslant \frac{1}{p!}\left(\log k + \gamma + \frac{1}{2}\right)^p.$$

Let $p \leqslant n-1$, and choose $l_1,\ldots,l_p$ such that $l_1+\cdots+l_p=n$ and $l_i \neq 0$ for any i. Necessarily, at least one l_i is larger than 2, so

$$\sum_{1\leqslant j_1<\cdots<j_p\leqslant n}\frac{1}{j_1^{l_1}\ldots j_p^{l_p}} \leqslant \sum_{1\leqslant j_1<\cdots<j_{p-1}\leqslant n}\sum_{1\leqslant j_p\leqslant n}\frac{1}{j_1\ldots j_{p-1}j_p^2}$$

$$\leqslant \sum_{1\leqslant j_1<\cdots<j_{p-1}\leqslant n}\frac{1}{j_1\ldots j_{p-1}}\cdot\frac{\pi^2}{6}$$

$$\leqslant \frac{1}{(p-1)!}\left(\log k + \gamma + \frac{1}{2}\right)^{p-1}\cdot\frac{\pi^2}{6}.$$

Thus,

$$\sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant k} \frac{1}{i_1 \ldots i_k} \leqslant \frac{1}{n!} \left(\log k + \gamma + \frac{1}{2} \right)^n$$

$$+ \frac{\pi^2}{6} \sum_{p=1}^{n-1} \left(\sum_{\substack{l_1 + \cdots + l_p = n \\ \forall i, \, l_i > 0}} 1 \right) \cdot \frac{1}{(p-1)!} \left(\log k + \gamma + \frac{1}{2} \right)^{p-1}.$$

It is easy to see that $\sum_{\substack{l_1 + \cdots + l_p = n \\ \forall i, \, l_i > 0}} 1 = \binom{n-1}{p-1}$ (choosing the integers l_i amounts to choosing $p-1$ cuts in the set $[|1, n|]$, i.e. among $n-1$ possible cuts). Consequently, we find

$$\sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant k} \frac{1}{i_1 \ldots i_n} \leqslant \frac{\left(\log k + \gamma + \frac{1}{2} \right)^n}{n!}$$

$$+ \frac{\pi^2}{6} \sum_{p=1}^{n-1} \binom{n-1}{p-1} \frac{1}{(p-1)!} \left(\log k + \gamma + \frac{1}{2} \right)^{p-1}.$$

We can use the following upper bound:

$$\sum_{p=1}^{n-1} \binom{n-1}{p-1} \frac{1}{(p-1)!} \left(\log k + \gamma + \frac{1}{2} \right)^{p-1}$$

$$= \sum_{p=0}^{n-2} \binom{n-1}{p} \frac{1}{p!} \left(\log k + \gamma + \frac{1}{2} \right)^p$$

$$\leqslant \sum_{p=0}^{n-2} \binom{n-1}{p} \left(\log k + \gamma + \frac{1}{2} \right)^p$$

$$= \left(\log k + \gamma + \frac{3}{2} \right)^{n-1} - \left(\log k + \gamma + \frac{1}{2} \right)^{n-1}$$

$$\leqslant (n-2) \left(\log k + \gamma + \frac{3}{2} \right)^{n-2},$$

where we used the mean value inequality in the last line. Thus,

$$(6.8) \quad \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_n \leqslant k} \frac{1}{i_1 \ldots i_n}$$

$$\leqslant \frac{\left(\log k + \gamma + \frac{1}{2} \right)^n}{n!} + \frac{\pi^2}{6} (n-2) \left(\log k + \gamma + \frac{3}{2} \right)^{n-2}.$$

Inserting (5.4) and (6.8), in (1.1), we find a lower bound of the form

$$\text{vol}\left(E_{k,\bullet}^{GG}\Omega_{\overline{X}}\right) \geqslant C_k \left[\left(K_{\overline{X}} + D\right)^n + A(k,n)(-D)^n\right]$$

for a certain $C_k \in \mathbb{R}^*$, and

$$A(k,n) = \left[\frac{\log k + \gamma + \frac{1}{2}}{\log k + \gamma}\right]^n + (n-2)n!\frac{\pi^2}{6}\frac{\left(\log k + \gamma + \frac{3}{2}\right)^{n-2}}{\left(\log k + \gamma\right)^n}$$

Let us first deal with the case where $n \geqslant 6$. According to [3], we have $(K_{\overline{X}} + D)^n + \alpha(-D)^n > 0$ for all $\alpha \in \left]0, \left(\frac{n+1}{2\pi}\right)^n\right[$. The only thing left now is to determine an integer k such that $A(k,n) < \left(\frac{n+1}{2\pi}\right)^n$.

Let $j = \log k + \gamma$. We have

$$A(k,n) = \left(1 + \frac{1}{2j}\right)^n + \frac{\pi^2}{6}\frac{(n-2)n!}{j}\left(1 + \frac{3}{2j}\right)^{n-2}$$

$$\leqslant \left(1 + \frac{3}{2j}\right)^{n-2}\left((1 + \frac{1}{2j})^2 + \frac{\pi^2}{6}\frac{(n-2)n!}{j}\right).$$

We see that if $j > \frac{\frac{\pi^2}{6}(n-2)n!+1}{\frac{n+1}{2\pi}-1}$, then $A(k,n) < \left(\frac{n+1}{2\pi}\right)^n$.

Besides, if $n \in [\![4,5]\!]$, then $(K_{\overline{X}})^n = \left(K_{\overline{X}} + D\right)^n + (-D)^n > 0$. Consequently, since $(K_{\overline{X}})^n$ is an integer, $(K_{\overline{X}} + D)^n + (-D)^n \geqslant 1$, and

$$(K_{\overline{X}} + D)^n + \lambda(-D)^n > 0$$

for any $\lambda \in \left]0, 1 + \frac{1}{-(-D)^n}\right[$. Thus, $\text{vol}(E_{k,\bullet}^{GG}\Omega_X) > 0$ as soon as $A(k,n) < 1 + \frac{1}{(-D)^n}$. Performing the same computations as before, we see that it is true if

$$\log k + \gamma > -(-D)^n\left((n-2)n! + 1\right).$$

We have consequently proved Corollary 1.2.

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