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Marc PEIGNÉ, SAMUEL TAPIE & PIERRE VIDOTTO

Orbital counting for some convergent groups

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ORBITAL COUNTING FOR SOME CONVERGENT GROUPS

by Marc PEIGNÉ, Samuel TAPIE & Pierre VIDOTTO (*)

ABSTRACT. — We present examples of geometrically finite manifolds with pinched negative curvature, whose geodesic flow has infinite non-ergodic Bowen–Margulis measure and whose Poincaré series converges at the critical exponent δ_Γ. We obtain an explicit asymptotic for their orbital growth function. Namely, for any $\alpha \in]1, 2[$ and any smooth slowly varying function $L : \mathbb{R} \to (0, +\infty)$, we construct N-dimensional Hadamard manifolds (X, g) of negative and pinched curvature, whose group of oriented isometries possesses convergent geometrically finite subgroups Γ such that, as $R \to +\infty$,

$$N_\Gamma(R) := \sharp\{\gamma \in \Gamma \mid d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\} \sim C_\Gamma(\mathbf{o}) \frac{L(R)}{R^\alpha} \, e^{\delta_\Gamma R},$$

for some $C_\Gamma(\mathbf{o}) > 0$ depending on the base point $\mathbf{o}$.

RÉSUMÉ. — Nous construisons des variétés géométriquement finies à courbure strictement négative pincée, dont le flot géodésique possède une mesure de Bowen-Margulis non ergodique infinie, et dont la série de Poincaré converge à l'exposant δ_Γ, et nous obtenons une estimation précise du comportement asymptotique de la fonction orbitale de ce groupe. Plus précisément, pour tout $\alpha \in]1, 2[$ et toute fonction à variations lentes $L : \mathbb{R} \to (0, +\infty)$, nous construisons des variétés de Hadamard (X, g) de dimension $N \geqslant 2$ dont le groupe des isométries qui préservent l'orientation possède des sous-groupes discrets et géométriquement finis Γ tels que, lorsque $R \to +\infty$,

$$N_\Gamma(R) := \sharp\{\gamma \in \Gamma \mid d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\} \sim C_\Gamma(\mathbf{o}) \frac{L(R)}{R^\alpha} \, e^{\delta_\Gamma R},$$

où $C_\Gamma(\mathbf{o})$ est une constante strictement positive qui dépend du point $\mathbf{o}$.

1. Introduction

We fix $N \geqslant 2$ and consider a N-dimensional Hadamard manifold X of negative, pinched curvature $-B^2 \leqslant K_X \leqslant -A^2 < 0$. Without loss of

generality, we may assume $A \leqslant 1 \leqslant B$. Let Γ be a *Kleinian group* of X, i.e. a discrete, torsion free group of isometries of X, with quotient $X_\Gamma = \Gamma \backslash X$.

In this paper, we study the asymptotic behavior of the *orbital function*:

$$N_\Gamma(\mathbf{x}, \mathbf{y}; R) := \sharp\{\gamma \in \Gamma \,|\, d(\mathbf{x}, \gamma \cdot \mathbf{y}) \leqslant R\}$$

for $\mathbf{x}, \mathbf{y} \in X$. The function N_Γ has been the subject of many investigations since Margulis [11]; see Roblin's book [17] and references therein for an overview of the subject. The first step is to consider its *exponential growth rate*

$$\delta_\Gamma = \limsup_{R \to \infty} \frac{1}{R} \ln(N_\Gamma(\mathbf{x}, \mathbf{y}; R));$$

by the triangular inequality, δ_Γ does not depend on the chosen base points $\mathbf{x}$ and $\mathbf{y}$. The exponent δ_Γ coincides with the *exponent of convergence* of the *Poincaré series* associated with Γ:

$$P_\Gamma(\mathbf{x}, \mathbf{y}; s) := \sum_{\gamma \in \Gamma} e^{-sd(\mathbf{x}, \gamma \cdot \mathbf{y})}, \qquad \mathbf{x}, \mathbf{y} \in X.$$

The real δ_Γ is called the *critical exponent* of Γ. It coincides with the topological entropy of the geodesic flow $(\phi_t)_{t \in \mathbb{R}}$ on the unit tangent bundle $T^1 X_\Gamma$ of X_Γ, restricted to its non-wandering set [12].

Recall that any orbit $\Gamma \cdot \mathbf{x}$ accumulates on a closed subset Λ_Γ of the geometric boundary ∂X of X which does not depend on $\mathbf{x}$. This set is called the *limit set* of Γ, it contains 1, 2 or infinitely many points; in the last case, one says that Γ is *non elementary*. A point $x \in \Lambda_\Gamma$ is said to be *radial* when it is approached by orbit points in some M-neighborhood of any given ray issued from x, for some $M > 0$. The critical exponent also equals the Hausdorff dimension of the *radial limit set* Λ_Γ^{rad} of Γ with respect to some natural metric on ∂X.

The group Γ is said to be *convergent* if $P_\Gamma(\mathbf{x}, \mathbf{y}; \delta_\Gamma) < \infty$, and *divergent* otherwise. Divergence can also be understood in terms of dynamic. Indeed, by Hopf–Tsuji–Sullivan theorem, it is equivalent to ergodicity and total conservativity of the geodesic flow with respect to the Bowen–Margulis measure m_Γ on the non-wandering set of $(\phi_t)_{t \in \mathbb{R}}$ in $T^1 X_\Gamma$ (see again [17] for a complete description of the construction of m_Γ and for a proof of this equivalence).

The following statement is the most general one concerning the asymptotic behavior of the function $N_\Gamma(\mathbf{x}, \mathbf{y}; R)$.

THEOREM 1.1 ([17]). — *Let Γ be a non elementary, discrete subgroup of isometries of X. Then δ_Γ is a true limit. Furthermore, if Γ has a non-arithmetic length spectrum*[(1)], *then it holds, as $R \to +\infty$,*

(1) *if $\|m_\Gamma\| < \infty$, then $N_\Gamma(\mathbf{x}, \mathbf{y}; R) \sim \frac{\|\mu_\mathbf{x}\| \cdot \|\mu_\mathbf{y}\|}{\delta_\Gamma \|m_\Gamma\|} e^{\delta_\Gamma R}$,*

(2) *if $\|m_\Gamma\| = \infty$, then $N_\Gamma(\mathbf{x}, \mathbf{y}; R) = o(e^{\delta_\Gamma R})$,*

where $(\mu_\mathbf{x})_{\mathbf{x} \in X}$ is the family of Patterson δ_Γ-conformal densities of Γ, and m_Γ the Bowen–Margulis measure on $T^1 X_\Gamma$.

By the Poincaré Recurrence Theorem, if $\|m_\Gamma\| < +\infty$ then Γ is divergent. When the Bowen–Margulis measure has infinite mass, Γ may be divergent or convergent.

In this paper, we study the asymptotic behavior of $N_\Gamma(\mathbf{x}, \mathbf{y}; R)$ for a class of *convergent* groups Γ; thus, their Bowen–Margulis measure is infinite. As far as we know, the only known convergent groups Γ with a precise asymptotic for N_Γ are normal subgroups $\Gamma \lhd \Gamma_0$ of a co-compact group Γ_0 such that the quotient Γ_0/Γ is virtually isometric to the lattice $\mathbb{Z}^k$ for some $k \geqslant 3$, see [16].

The finiteness of m_Γ is not easy to establish in general. A precise criterion ensuring this finiteness for *geometrically finite groups* has been obtained in [6] and recently generalized to all non elementary groups in [15].

Recall that a discrete group Γ (or the quotient manifold X/Γ) is said *geometrically finite* if its limit set Λ_Γ decomposes into the radial limit set and the Γ-orbit of finitely many *bounded parabolic points* $x_1, \ldots, x_\ell$, fixed respectively by some parabolic subgroups $P_i, 1 \leqslant i \leqslant \ell$, acting co-compactly on $\partial X \setminus \{x_i\}$. We refer to [3] for a complete description of geometrical finiteness in variable negative curvature. Finite volume manifolds X_Γ (possibly non compact) are particular cases of geometrically finite manifolds.

For geometrically finite groups, the orbital functions $N_{P_i}(\mathbf{x}, \mathbf{y}; \cdot)$ of the parabolic subgroups $P_i, 1 \leqslant i \leqslant \ell$, contain the relevant information about the metric inside the cusps, which may imply m_Γ to be finite or infinite. On the one hand, it is proved in [6] that the divergence of the parabolic subgroups $P \subset \Gamma$ implies $\delta_P < \delta_\Gamma$, which yields that Γ is divergent and $\|m_\Gamma\| < \infty$. On the other hand, there exist geometrically finite groups with parabolic subgroups P satisfying $\delta_P = \delta_\Gamma$: we call such groups *exotic* and say that the parabolic subgroup P (or the corresponding cusps $\mathcal{C}$) is *dominant* when $\delta_P = \delta_\Gamma$. Thus, dominant parabolic subgroups of exotic geometrically finite groups Γ are necessarily convergent. However, the

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[(1)] It means that the set $\{\ell(\gamma) \mid \gamma \in \Gamma\}$ of lengths of closed geodesics of X_Γ is not contained in a discrete subgroup of $\mathbb{R}$.

group Γ itself may be convergent or divergent. Explicit constructions of such groups are given in [6] and [14]; we present a similar construction in Section 2 below.

In this paper, we consider a Schottky product Γ of elementary subgroups $\Gamma_1, \ldots, \Gamma_{p+q}$, of isometries of X (see Section 3 for the definition) with $p+q \geqslant 3$. Such a group is geometrically finite. We assume that Γ is convergent; thus, by [6], it is exotic and possesses factors (say $\Gamma_1, \ldots, \Gamma_p, p \geqslant 1$) which are dominant parabolic subgroups of Γ. We assume that, up to the dominant factor $e^{\delta_\Gamma R}$, the orbital functions $N_{\Gamma_j}(\mathbf{x}, \mathbf{y}; \cdot)$ of these groups satisfy some asymptotic condition of polynomial decay at infinity.

THEOREM 1.2. — *Fix $p, q \in \mathbb{N}$ such that $p \geqslant 1, p + q \geqslant 2$ and let Γ be a Schottky product of elementary subgroups $\Gamma_1, \Gamma_2 \ldots, \Gamma_{p+q}$ of isometries of a pinched negatively curved manifold X, where $\Gamma_1, \ldots, \Gamma_p$ are parabolic. Fix $\mathbf{o} \in X$ and assume that the metric g on X satisfies the following assumptions.*

(H_1) *The group Γ is convergent with Poincaré exponent δ_Γ.*

(H_2) *There exist $\alpha \in \,]1, 2[$, a smooth slowly varying function $L^{(2)}$ and positive constants $c_1, \ldots, c_p$ such that, for any $1 \leqslant j \leqslant p$ and $\Delta > 0$,*

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \sum_{\substack{\gamma \in \Gamma_j \\ R \leqslant d(\mathbf{o}, \gamma \cdot \mathbf{o}) < R+\Delta}} e^{-\delta_\Gamma d(\mathbf{o}, \gamma \cdot \mathbf{o})} = c_j \Delta.$$

(H_3) *For any $p + 1 \leqslant j \leqslant p + q$ and $\Delta > 0$,*

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \sum_{\substack{\gamma \in \Gamma_j \\ R \leqslant d(\mathbf{o}, \gamma \cdot \mathbf{o}) < R+\Delta}} e^{-\delta_\Gamma d(\mathbf{o}, \gamma \cdot \mathbf{o})} = 0.$$

Then, for any $\mathbf{x}, \mathbf{y}$ in X, there exists a constant $C_\Gamma(\mathbf{x}, \mathbf{y}) > 0$ such that, as $R \to +\infty$,

$$\sharp \{\gamma \in \Gamma \, | \, d(\mathbf{x}, \gamma \cdot \mathbf{y}) \leqslant R\} \sim C_\Gamma(\mathbf{x}, \mathbf{y}) \, \frac{L(R)}{R^\alpha} \, e^{\delta_\Gamma R}.$$

Remark 1.3. — Hypothesis (H_1) (resp. (H_2)) deals with the asymptotic behavior of the convergent Poincaré series $P_{\Gamma_j}(\mathbf{o}, \mathbf{o}; \delta_\Gamma)$ for $1 \leqslant j \leqslant p$ (resp. $p+1 \leqslant j \leqslant p+q$). When they are satisfied, the same properties hold for the series $P_{\Gamma_j}(\mathbf{x}, \mathbf{y}; \delta_\Gamma)$, for any $\mathbf{x}, \mathbf{y}$ in X, up to the following modification: for $1 \leqslant j \leqslant p$, the constant c_j has to be replaced by $c_j(\mathbf{x}, \mathbf{y}) = e^{-\delta \mathcal{B}_{x_j}(\mathbf{y}, \mathbf{x})} c_j$ where x_j is the fixed point of the parabolic group Γ_j.

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[2] A function L is said to be "slowly varying" if it is positive, measurable and $L(\lambda t)/L(t) \to 1$ as $t \to +\infty$ for every $\lambda > 0$.

Remark 1.4. — The constants $C_\Gamma(\mathbf{x}, \mathbf{y})$ can be explicited (see Equality (4.21) and Proposition 4.6 for explanations when $\mathbf{x} = \mathbf{y} = \mathbf{o}$). It is (non explicitly but) closely related to similar constant appearing in [18, Theorem C].

Remark 1.5. — In this paper we prove Theorem 1.2 when $p + q \geqslant 3$. In Section 3, we briefly explain how to extend the proof to the case $p + q = 2$.

The importance of the convergence hypothesis (H_1) in the previous theorem is illustrated by the following result, previous work of one of the authors [18], which concerns the case when Γ is divergent with infinite Bowen–Margulis measure.

THEOREM 1.6 ([18, Theorem C]). — *Let Γ be a Schottky product of $p + q \geqslant 2$ elementary subgroups $\Gamma_1, \Gamma_2 \ldots, \Gamma_{p+q}$ of isometries of a pinched negatively curved manifold X with $p \geqslant 1$. Let us fix $\mathbf{o} \in X$ and assume that Γ is divergent with infinite Bowen–Margulis measure and hypotheses (H_2), (H_3) hold.*

Then, for any $\mathbf{x}, \mathbf{y}$ in X, there exists a positive constant $C'_\Gamma(\mathbf{x}, \mathbf{y})$ such that, as $R \to +\infty$,

$$\sharp\{\gamma \in \Gamma \,|\, d(\mathbf{x}, \gamma \cdot \mathbf{y}) \leqslant R\} \sim C'_\Gamma(\mathbf{x}, \mathbf{y}) \, \frac{e^{\delta_\Gamma R}}{R^{2-\alpha} L(R)}.$$

The difference of asymptotic behaviors between Theorems 1.2 and 1.6 may seem surprising, since it is possible to vary smoothly the Riemannian metric $g_{\alpha, L}$ from a divergent to a convergent case, preserving hypotheses (H_2) and (H_3), cf. [14] and Paragraph 2.3 below. Let us explain briefly the reasons of this difference; the details are given in Section 4.

Let $\Gamma = \Gamma_1 * \cdots * \Gamma_{p+q}$ be a Schottky product of elementary subgroups. For any $\gamma \in \Gamma, \gamma \neq \mathrm{Id}$, there exists $k \geqslant 1$ such that γ may be decomposed as $\gamma = a_1 \ldots a_k$, with $a_i \in \Gamma_1 \cup \cdots \cup \Gamma_p$ and $a_i, a_{i+1}, 1 \leqslant i < k$, do not belong to the same subgroup Γ_j; the integer k is called the symbolic length of Γ and we denote $\Gamma(k)$ the set of $\gamma \in \Gamma$ with symbolic length k. When Γ satisfies hypotheses (H_2) and (H_3) (it is either convergent or divergent) then for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that as $R \to +\infty$,

$$\sharp\{\gamma \in \Gamma(k) \,|\, d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\} \sim C_k \frac{L(R)}{R^\alpha} \, e^{\delta_\Gamma R}.$$

On the one hand, when Γ is convergent, the estimates of $\sharp\{\gamma \in \Gamma(k) \,|\, d(\mathbf{o}, \gamma \cdot \mathbf{o})\}$ are summable and the Lebesgue dominated convergence Theorem, yields the asymptotic given in Theorem 1.2.

On the other hand, when Γ is divergent, these estimates are no longer summable and the asymptotic of $\{\gamma \in \Gamma \,|\, d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\}$ as $R \to +\infty$

only depends on the $\gamma = a_1 \ldots a_k$ with $k \gg R$. A more refined estimate is needed and yields Theorem 1.6.

Remark 1.7. — In Section 2.2, we present parabolic groups satisfying Hypothesis (H$_2$). The condition $\alpha > 1$ ensures that the parabolic groups $\Gamma_1, \ldots, \Gamma_p$ are convergent. The additional condition $\alpha < 2$ is used in Lemma 4.3 to obtain a uniform upper bound for the powers of some Markov operator $\widetilde{P}$ introduced in Section 4. The proof of this lemma relies on [18], cf. Proposition 4.5, which is only valid for $\alpha \in (1, 2)$. The statement of Theorem 1.2 remains open when $\alpha \geqslant 2$.

Remark 1.8. — Hypothesis (H$_3$) is satisfied as soon as factors Γ_j, $p+1 \leqslant j \leqslant p + q$, have a critical exponent $\delta_{\Gamma_j} < \delta_\Gamma$: thus, the quantity

$$\sum_{\substack{\gamma \in \Gamma_j \\ R \leqslant d(\mathbf{o}, \gamma \cdot \mathbf{o}) < R + \Delta}} e^{-\delta d(\mathbf{o}, \gamma \cdot \mathbf{o})}$$

decreases exponentially quickly and (H$_3$) holds. For instance, if all the Γ_j are generated by an hyperbolic isometry of X, their critical exponents equal all 0 and (H$_3$) is satisfied.

This article is organized as follows. In Section 2, we recall some background on negatively curved manifolds, and we construct examples of metrics for which the hypotheses of Theorem 1.2 are satisfied.

In Section 3, we introduce the *Ruelle operator*; this is the key analytical tool to establish the asymptotic of the orbital counting function. Eventually, Section 4 is devoted to the proof of Theorem 1.2; firstly, we assume $\mathbf{x} = \mathbf{y} = \mathbf{o}$ and then explain in Subsection 4.5 how to extend it for any points $\mathbf{x}$ and $\mathbf{y}$ in X.

2.   Convergent parabolic groups and Schottky groups

2.1. Geometry of negatively curved manifolds

In the sequel, we fix $N \geqslant 2$ and consider a N-dimensional complete connected Riemannian manifold X with metric g whose sectional curvatures satisfy: $-B^2 \leqslant K_X \leqslant -A^2 < 0$ for fixed constants A and B.

We denote by d the distance on X induced by the metric g.

Let ∂X be the boundary at infinity of X and $\mathbf{z} \in X$. The family of functions $(\mathbf{y} \mapsto d(\mathbf{z}, \mathbf{x}) - d(\mathbf{x}, \mathbf{y}))_{\mathbf{x} \in X}$ converges uniformly on compact sets to the *Busemann function* $\mathcal{B}_x(\mathbf{z}, \cdot)$ when $\mathbf{x} \to x \in \partial X$. The Busemann

function satisfies the fundamental cocycle relation: for any $x \in \partial X$ and any $\mathbf{x}, y, z$ in X,

$$\mathcal{B}_x(\mathbf{x}, \mathbf{z}) = \mathcal{B}_x(\mathbf{x}, \mathbf{y}) + \mathcal{B}_x(\mathbf{y}, \mathbf{z}).$$

The Gromov product between $x, y \in \partial X \times \partial X$, $x \neq y$, is defined by

$$(x|y)_\mathbf{o} = \frac{\mathcal{B}_x(\mathbf{o}, \mathbf{z}) + \mathcal{B}_y(\mathbf{o}, \mathbf{z})}{2}$$

where $\mathbf{z}$ is any point on the geodesic (x, y) joining x to y.

We fix once for all an origin $\mathbf{o} \in X$. For $x \in \partial X$, the *horoballs* $\mathcal{H}_x$ and the *horospheres* $\partial \mathcal{H}_x$ centered at x are respectively the sup-level sets and the level sets of the function $\mathcal{B}_x(\mathbf{o}, \cdot)$. For any $t \in \mathbb{R}$, we set $\mathcal{H}_x(t) :=$ $\{\mathbf{y}/\mathcal{B}_x(\mathbf{o}, \mathbf{y}) \geqslant t\}$ and $\partial \mathcal{H}_x(t) := \{\mathbf{y}/\mathcal{B}_x(\mathbf{o}, \mathbf{y}) = t\}$; the parameter $t = \mathcal{B}_x(\mathbf{o}, \mathbf{y}) - \mathcal{B}_x(\mathbf{o}, \mathbf{x})$ is the *height* of $\mathbf{y}$ with respect to $\mathbf{x}$. When no confusion is possible, we omit the index $x \in \partial X$.

By [2], the expression

$$D(x, y) = e^{-A(x|y)_\mathbf{o}}$$

defines a distance on ∂X satisfying the following property: for any $\gamma \in \Gamma$

$$D(\gamma \cdot x, \gamma \cdot y) = e^{-\frac{A}{2}\mathcal{B}_x(\gamma^{-1} \cdot \mathbf{o}, \mathbf{o})} e^{-\frac{A}{2}\mathcal{B}_y(\gamma^{-1} \cdot \mathbf{o}, \mathbf{o})} D(x, y).$$

In other words, the isometry γ acts on $(\partial X, D)$ as a conformal transformation with coefficient of conformality $|\gamma'(x)|_\mathbf{o} = e^{-A\mathcal{B}_x(\gamma^{-1} \cdot \mathbf{o}, \mathbf{o})}$ at x and satisfies the following equality

$$(2.1) \qquad D(\gamma \cdot x, \gamma \cdot y) = \sqrt{|\gamma'(x)|_\mathbf{o}|\gamma'(y)|_\mathbf{o}} D(x, y).$$

The function $x \mapsto b(\gamma, x) := \mathcal{B}_x(\gamma^{-1} \cdot \mathbf{o}, \mathbf{o})$ plays a central role to describe the action of the isometry γ on the boundary at infinity ∂X. It satisfies the following "cocycle property": for any isometries γ_1, γ_2 of X and any $x \in \partial X$,

$$(2.2) \qquad b(\gamma_1 \gamma_2, x) = b(\gamma_1, \gamma_2 \cdot x) + b(\gamma_2, x).$$

If $\mathcal{P}$ is a parabolic group of isometries of X (not necessarily cyclic), it fixes a unique point $x_\mathcal{P} \in \partial X$. We write $p \to +\infty, p \in \mathcal{P}$ for any sequence $(p_n)_{n \geqslant 1}$ of elements of $\mathcal{P}$ such that $\lim_{n \to +\infty} d(\mathbf{o}, p_n \cdot \mathbf{o}) = +\infty$. Note that

$$\lim_{\substack{p \in \mathcal{P} \\ p \to +\infty}} p \cdot \mathbf{o} = x_\mathcal{P}.$$

The following lemma is an immediate consequence of the definition of the Busemann functions and the Gromov product. It is of great use in the sequel.

LEMMA 2.1.

(1) *For any hyperbolic isometry h with repulsive and attractive fixed point $x_h^- = \lim_{n \to +\infty} h^{-n} \cdot \mathbf{o}$ and $x_h^+ = \lim_{n \to +\infty} h^n \cdot \mathbf{o}$ respectively, we have*

$$b(h^{\pm n}, x) = d(\mathbf{o}, h^{\pm n} \cdot \mathbf{o}) - 2(x_h^{\pm}|x)_{\mathbf{o}} + \epsilon_x(n)$$

with $\lim_{n \to +\infty} \epsilon_x(n) = 0$, the convergence being uniform on the compact sets of $\partial X \setminus \{x_h^{\mp}\}$.

(2) *For any parabolic group $\mathcal{P}$ with fixed point $x_{\mathcal{P}}$, we have*

$$b(p, x) = d(\mathbf{o}, p \cdot \mathbf{o}) - 2(x_{\mathcal{P}}|x)_{\mathbf{o}} + \epsilon_x(p)$$

with $\lim_{\substack{p \in \mathcal{P} \\ p \to +\infty}} \epsilon_x(p) = 0$, the convergence being uniform on the compact sets of $\partial X \setminus \{x_{\mathcal{P}}\}$.

2.2. On the existence of convergent parabolic groups

In this section, we present briefly some known results about the existence of convergent parabolic groups satisfying hypothesis (H$_2$). We refer to [14] and [18] for the details.

We consider on $\mathbb{R}^{N-1} \times \mathbb{R}$ a Riemannian metric of the form

$$g_T = T^2(t)\mathrm{d}x^2 + \mathrm{d}t^2$$

at point $\mathbf{x}_t = (x, t)$ where $\mathrm{d}x^2$ is a fixed euclidean metric on $\mathbb{R}^{N-1}$ and $T : \mathbb{R} \to \mathbb{R}^{*+}$ is a C^∞ decreasing function. The group of isometries of g_T contains all the translations $(x, t) \mapsto (x + \vec{\tau}, t)$ on $\mathbb{R}^{N-1} \times \mathbb{R}$ fixing the last coordinate.

By [4, Chapter 8, Section 3], the sectional curvature at $\mathbf{x} = (x, t)$ equals

- $K_{g_T}(t) = -\frac{T''(t)}{T(t)}$ on any plane $\langle \frac{\partial}{\partial X_i}, \frac{\partial}{\partial t} \rangle, 1 \leqslant i \leqslant N - 1$;
- $-K_{g_T}^2(t)$ on any plane $\langle \frac{\partial}{\partial X_i}, \frac{\partial}{\partial X_j} \rangle, 1 \leqslant i < j \leqslant N - 1$

Note that g_T has negative curvature if and only if T is convex. When $T(t) = e^{-t}$, this provides a model of the hyperbolic space of constant curvature -1.

Let us consider the decreasing function

(2.3) $$u : \begin{cases} \mathbb{R}^{*+} \to \mathbb{R} \\ \quad s \mapsto T^{-1}\left(\frac{1}{s}\right) \end{cases}$$

which satisfies the following implicit equation $T(u(s)) = \frac{1}{s}$. This function u is of interest since it gives precise estimates of the distance between points

lying on the same horosphere $\mathcal{H}_t := \{(x,t) : x \in \mathbb{R}^{N-1}\}, t \in \mathbb{R}$. Namely, the distance between $\mathbf{x}_t = (x,t)$ and $\mathbf{y}_t = (y,t)$ for the metric $T^2(t)\mathrm{d}x^2$ induced by g_T on $\mathcal{H}_t$ is equal to $T(t)\|x-y\|$. Hence, this distance equals 1 when $t = u(\|x-y\|)$ and for this value of t, the union of the 3 segments $[\mathbf{x}_0, \mathbf{x}_t], [\mathbf{x}_t, \mathbf{y}_t]$ and $[\mathbf{y}_t, \mathbf{y}_0]$ lies at a bounded distance of the hyperbolic geodesic joining $\mathbf{x}_0$ and $\mathbf{y}_0$ (see [6, Lemma 4]) : this readily implies that $d(\mathbf{x}_0, \mathbf{y}_0) - 2u(\|x-y\|)$ is bounded.

Now, let $\mathcal{P}$ be a discrete group of isometries of $\mathbb{R}^{N-1}$ spanned by k linearly independent translations $p_{\vec{\tau}_1}, \ldots, p_{\vec{\tau}_k}$ in $\mathbb{R}^{N-1}$, with respective vectors $\vec{\tau}_1, \ldots, \vec{\tau}_k$.

For any $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, we set $\vec{\mathbf{n}} = n_1\vec{\tau}_1 + \cdots + n_k\vec{\tau}_k$. The translations $p_{\vec{\mathbf{n}}}$ are isometries of $(\mathbb{R}^N, g_T)$ and the corresponding Poincaré series of $\mathcal{P}$ is given by, up to finitely many terms,

$$(2.4) \qquad P_{\mathcal{P}}(s) = \sum_{\|\vec{\mathbf{n}}\| > s_\alpha} e^{-sd(\mathbf{o}, p_{\vec{\mathbf{n}}} \cdot \mathbf{o})} = \sum_{\|\vec{\mathbf{n}}\| > s_\alpha} e^{-2su(\|\vec{\mathbf{n}}\|) + sO(1)}.$$

Hence, it is sufficient to choose the function u in a suitable way in order that the series $P_{\mathcal{P}}(s)$ converges at its critical exponent.

We present here two main explicit examples. Example 2.2 comes from [6] where the existence of convergent parabolic groups appeared for the first time. Example 2.3, where we assume $N = 2$ to simplify, is a refinement of Example 2.2.

Example 2.2. — For any $\alpha \geqslant 0$, let us consider the increasing C^2-function $u = u_\alpha$ from $\mathbb{R}^{*+}$ to $\mathbb{R}$ such that

$$u_\alpha(s) = \begin{cases} \ln s & \text{if } 0 < s \leqslant 1 \\ \ln s + \alpha \ln \ln s & \text{if } s \geqslant s_\alpha. \end{cases}$$

We denote T_α the function associated with u_α by relation (2.3); the constant $s_\alpha > 1$ is chosen in such a way the metric

$$g_\alpha = T_\alpha^2(t)\mathrm{d}x^2 + \mathrm{d}t^2$$

on $\mathbb{R}^{N-1} \times \mathbb{R}$ has pinched negative curvature on X, bounded from above by $-A^2$.

In this case, Estimation (2.4) of the Poincaré series of $\mathcal{P}$ becomes, up to finitely many terms,

$$P_{\mathcal{P}}(s) = \sum_{\|\vec{\mathbf{n}}\| > s_\alpha} \frac{e^{sO(1)}}{\|\vec{\mathbf{n}}\|^{2s} \left(\ln \|\vec{\mathbf{n}}\|\right)^{2s\alpha}}.$$

Hence, the Poincaré exponent of $\mathcal{P}$ equals $k/2$ and $\mathcal{P}$ is convergent if and only if $\alpha > \frac{1}{k}$.

Example 2.3. — Without loss of generality, we assume $N = 2$; indeed, the metrics g_T are totally geodesics, hence the geodesic segments $[\mathbf{o}, p \cdot \mathbf{o}], p \in \mathcal{P}$, are included in the vertical plane passing through $\mathbf{o}$ and $p \cdot \mathbf{o}$ and the computation of the distance $d(\mathbf{o}, p^n \cdot \mathbf{o})$ is a 2-dimensional problem.

We fix $\alpha > 1$ and a smooth slowly varying function $L : [0, +\infty[\to (0, +\infty)$. For any real t greater than some $\mathfrak{a} > 0$ to be chosen[3], let us set

$$T(t) = T_{\alpha,L}(t) = e^{-t}\frac{t^\alpha}{L(t)}.$$

We denote by $u_{\alpha,L}$ the function related to $T_{\alpha,L}$ by the implicit equation (2.3) and $g_{\alpha,L}$ the associated metric on $\mathbb{R}^2$. We denote p the isometry $(x, t) \mapsto (x + 1, t)$ on $\mathbb{R}^2$.

The following statement corresponds to [18, Lemma 2.2.3 and Proposition 2.2.4], where explicit examples of functions L are also given.

PROPOSITION 2.4. — *For s large enough,*

$$u(s) = \ln s + \alpha \ln \ln s - \ln L(\ln s) + \epsilon(s)$$

with $\epsilon(s) \to 0$ as $s \to +\infty$.

Consequently, the parabolic group $\mathcal{P} = \langle p \rangle$ on $(\mathbb{R}^2, g_{\alpha,L})$ satisfies the following property: for any $n \in \mathbb{N}$ large enough,

$$d(\mathbf{o}, p^n \cdot \mathbf{o}) = 2\big(\ln n + \alpha \ln \ln n - \ln L(\ln n)\big) + \epsilon(n)$$

with $\lim_{n \to +\infty} \epsilon(n) = 0$.

In particular, for $\alpha > 1$, the group $\mathcal{P}$ is convergent with respect to $g_{\alpha,L}$ and satisfies the assumption (H_2).

The improvement between Example 2.2 and Example 2.3 relies on the observation that the cylinder $\mathbb{R}^2/\mathcal{P}$ endowed with the metric $g_{\alpha,L}$ is a surface of revolution. This allows to use the Clairaut's relations (see for instance [5, Section 4.4, Example 5]) to estimate the distance between $\mathbf{o}$ and $p^n \cdot \mathbf{o}$; computations are detailed in [18, Section 2.2.3].

2.3. On the existence of non elementary exotic groups

Explicit constructions of exotic groups, i.e. non-elementary groups Γ containing a parabolic $\mathcal{P}$ whose Poincaré exponent equals δ_Γ, have been detailed in several papers; first in [6], then in [8], [14], and [18]. Let us describe them in the context of the metrics $g = g_{\alpha,L}$ presented above.

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[3] For technical reasons, we assume in particular $\mathfrak{a} > 4\alpha$, see [18].

For any $a > 0$ and $t \in \mathbb{R}$, we write

$$T_{\alpha,L,a} = \begin{cases} e^{-t} & \text{if } t \leqslant a \\ e^{-a}T_{\alpha,L}(t-a) & \text{if } t \geqslant a, \end{cases}$$

where $T_{\alpha,L}$ is defined in the previous paragraph. As in [14], we consider the metric on $\mathbb{R}^2$ given by $g_{\alpha,L,a} = T_{\alpha,L,a}^2(t)\mathrm{d}x^2 + \mathrm{d}t^2$. It is a complete smooth metric, with pinched negative curvatures, and with constant curvature -1 on $\mathbb{R} \times (-\infty, a)$. Note that $g_{\alpha,L,0} = g_{\alpha,L}$ and $g_{\alpha,L,+\infty}$ is the hyperbolic metric on $\mathbb{H}^2$. From the previous subsection, it follows that for any $a \in (0, +\infty)$ and any $\tau \in \mathbb{R}^*$, a parabolic group of the form $\mathcal{P} = \langle (x,t) \mapsto (x+\tau, t) \rangle$ is convergent for the metric $g_{\alpha,L,a}$. This allows to reproduce the construction of a non-elementary group given in [6] and [14]; let us we present it.

Let h be a hyperbolic isometry of $\mathbb{H}^2$ and p be a parabolic isometry in Schottky position with h (see next section for a precise definition). They generate a free group $\Gamma = \langle h, p \rangle$ which acts discretely without fixed point on $\mathbb{H}^2$. Up to a global conjugacy, we can suppose that p is given by $(x,t) \mapsto (x+\tau, t)$ for some $\tau \in \mathbb{R}^*$. The surface $S = \mathbb{H}^2/\Gamma$ has a cusp, isometric to $\mathbb{R}/\tau\mathbb{Z} \times (a_0, +\infty)$ for some $a_0 > 0$. Therefore, we can replace in the cusp the hyperbolic metric by $g_{\alpha,L,a}$ for any $a \geqslant a_0$; we also denote by $g_{\alpha,L,a}$ the lift of $g_{\alpha,L,a}$ to $\mathbb{R}^2$.

For any $n \in \mathbb{Z}^*$, the group $\Gamma_n = \langle h^n, p \rangle$ acts freely by isometries on $(\mathbb{R}^2, g_{\alpha,L,a})$. It is shown in [6] that, for $n > 0$ large enough, the group Γ_n also converges. This provides a family of examples for Theorem 1.2, since by Remark 1.8 assumption (H$_3$) is automatically satisfied.

By [14], if Γ_n is convergent for some $a_0 > 0$, then there exists $a^* > a_0$ such that for any $a \in [a_0, a^*)$, the group Γ_n acting on $(\mathbb{R}^2, g_{\alpha,L,a})$ is convergent, whereas for $a > a^*$, it has finite Bowen–Margulis measure and hence diverges.

In the case when $a = a^*$, the group Γ also diverges but m_Γ has infinite measure for $\alpha \in]1, 2[$ (see [14]). A precise estimate of the function N_Γ does also exist in this case (cf. Theorem 1.6).

Remark 2.5. — In [8], the authors propose another approach based on a "strong" perturbation of the metric inside the cusp. Starting from a N-dimensional finite volume hyperbolic manifold with cuspidal ends, they modify the metric far inside one end in such a way that the corresponding parabolic group is convergent with Poincaré exponent > 1 and turns the fundamental group of the manifold into a convergent group. In this construction, the sectional curvature of the new metric along certain planes

is < -4 far inside the modified cusp. With this construction, we may extend the validity of t Theorem 1.2 to *finite volume manifolds with infinite Bowen–Margulis measure*. We will not do it here.

2.4. Schottky groups

Let us fix two integers $p \geqslant 1$ and $q \geqslant 0$ such that $\ell := p + q \geqslant 2$ and consider ℓ elementary groups $\Gamma_1, \ldots, \Gamma_\ell$ of isometries of X. For all $j \in \{1, \ldots, \ell\}$, we write $\Gamma_j^* = \Gamma_j \backslash \{\mathrm{Id}\}$; similarly $\Gamma^* = \Gamma \backslash \{\mathrm{Id}\}$.

The elementary groups $\Gamma_1, \ldots, \Gamma_\ell$ are said to be in *Schottky position* if there exist disjoint closed sets F_j in ∂X such that, for any $1 \leqslant j \leqslant \ell$

$$\tag{2.5} \Gamma_j^*(\partial X \setminus F_j) \subset F_j.$$

We set $F = \bigcup_j F_j$.

The group $\Gamma = \langle \Gamma_1, \ldots, \Gamma_\ell \rangle$ spanned by the $\Gamma_j, 1 \leqslant j \leqslant \ell$, is called the *Schottky product* of the Γ_j's.

In this section, we present general properties of such Schottky groups. We emphasize on the coding of the elements of the group by words over a countable alphabet, which is crucial for the proof of Theorem 1.2. We do not require yet that conditions (H_1), (H_2) and (H_3) hold: these hypotheses are only needed in the last section of this paper.

Property (2.5) and Klein's tennis table criterion imply that Γ is the free product of the groups Γ_i:

$$\Gamma = \Gamma_1 \star \Gamma_2 \star \cdots \star \Gamma_\ell.$$

Hence, any $\gamma \in \Gamma^*$ can hence be uniquely written as the product

$$\gamma = a_1 \ldots a_k$$

for some $a_j \in \bigcup \Gamma_j^*$ with the property that no two consecutive elements a_j belong to the same group Γ_j. The set $\mathcal{A} = \bigcup \Gamma_j^*$ is called the *alphabet* of Γ, and $a_1, \ldots, a_k$ the *letters* of γ. The number k of letters of $\gamma = a_1 \ldots a_k$ is the *symbolic length* of γ; we denote by $\Gamma(k)$ the set of elements of Γ with symbolic length k. The last letter of γ plays a special role; the index of the group it belongs to is denoted by l_γ. Applying Property 4.1.3 from [18], one gets

PROPOSITION 2.6. — *There exists a constant $C > 0$ such that for any $\gamma \in \Gamma = \star_i \Gamma_i$ and any $x \in \partial X \setminus F_{l_\gamma}$,*

$$d(\mathbf{o}, \gamma \cdot \mathbf{o}) - C \leqslant \mathcal{B}_x(\gamma^{-1} \cdot \mathbf{o}, \mathbf{o}) \leqslant d(\mathbf{o}, \gamma \cdot \mathbf{o}).$$

This proposition implies in particular the following crucial contraction property [1].

PROPOSITION 2.7. — *There exist a real number $r \in \,]0,1[$ and $C > 0$ such that, for any γ with symbolic length $n \geqslant 1$ and any x belonging to the set $\partial X \setminus F_{l_\gamma}$, it holds*
$$|\gamma'(x)| \leqslant Cr^n.$$

The coding of elements of Γ extends to a coding of a large subset of Λ_Γ; let us restate Proposition 1 of [13].

PROPOSITION 2.8. — *Denote by Σ^+ the set of sequences $(a_n)_{n \geqslant 1}$ for which each letter a_n belongs to the alphabet $\mathcal{A} = \bigcup \Gamma_i^*$ and such that no two consecutive letters belong to the same group (these sequences are called admissible). Fix a point x_0 in $\partial X \setminus F$. Then*

(1) *For any $\mathbf{a} = (a_n)_{n \geqslant 1} \in \Sigma^+$, the sequence $(a_1 \ldots a_n \cdot x_0)_{n \geqslant 1}$ converges to a point $\pi(\mathbf{a})$ in the limit set of Γ, independent on the choice of x_0.*
(2) *The map $\pi : \Sigma^+ \to \Lambda_\Gamma$ is one-to-one and $\pi(\Sigma^+)$ is contained in the radial limit set of Γ.*
(3) *The complement of $\pi(\Sigma^+)$ in the limit set of Γ equals the Γ-orbit of the union of the limits sets Λ_{Γ_i}.*

Hence, up to a countable set of points, the limit set Λ_Γ of Γ coincides with $\pi(\Sigma^+)$. Note that the set Λ_Γ is contained in $F = \bigcup_j F_j$, by (2.5).

For any $1 \leqslant i \leqslant \ell$, let $\Lambda_i = \Lambda_\Gamma \cap F_i$ be the closure of the set of limit points with first letter in Γ_i (not to be confused with the limit set of Γ_i, which is a finite set). Since the F_i are closed disjoint subsets, we get the following useful description of Λ_Γ:

(a) Λ_Γ is the finite union of the sets Λ_i,
(b) the closed sets $\Lambda_i, 1 \leqslant i \leqslant \ell$, are pairwise disjoint,
(c) each of these sets is partitioned into a countable number of closed disjoint subsets:
$$\Lambda_i = \bigcup_{a \in \Gamma_i^*} \cup_{j \neq i} \, a.\Lambda_j \ .$$

Let us fix a point $x_0 \in \partial X \setminus F$. There exists a one-to-one correspondence between $\Gamma \cdot x_0$ and Γ. Furthermore, the point $\gamma \cdot x_0 \in F_j$ for any $\gamma \in \Gamma^*$ with first letter in Γ_j. We set $\widetilde{\Sigma}_+ = \Sigma^+ \cup \Gamma$ and notice that, by the previous Proposition, the natural map $\pi : \widetilde{\Sigma}_+ \to \Lambda_\Gamma \cup \Gamma \cdot x_0$ is one-to-one with image $\pi(\Sigma^+) \cup \Gamma \cdot x_0$. Thus we introduce the following notations:

(a) $\widetilde{\Lambda} = \Lambda_\Gamma \cup \Gamma \cdot x_0$;

(b) $\widetilde{\Lambda}_i = \widetilde{\Lambda}_\Gamma \cap F_i$ for any $1 \leqslant i \leqslant \ell$.

The set $\widetilde{\Lambda}$ is the disjoint union of $\{x_0\}$ and the sets $\widetilde{\Lambda}_i, 1 \leqslant i \leqslant \ell$. Furthermore, each $\widetilde{\Lambda}_i$ is partitioned into a countable number of subsets with disjoint closures:

$$\widetilde{\Lambda}_i = \bigcup_{a \in \Gamma_i^*} a \left(\bigcup_{j \neq i} \widetilde{\Lambda}_j \cup \{x_0\} \right).$$

The cocycle b defined in (2.2) plays a central role in the sequel. Now, we introduce an extension of this cocycle (denoted also b) which allows to control the distance (and not the value of the Buseman function) between two points of the orbit $\Gamma \cdot \mathbf{o}$. It is defined on $\widetilde{\Lambda}$ as follows: for any $\gamma \in \Gamma$ and $x \in \widetilde{\Lambda}$,

$$b(\gamma, x) := \begin{cases} \mathcal{B}_x(\gamma^{-1}\mathbf{o}, \mathbf{o}) & \text{if } x \in \Lambda_\Gamma; \\ d(\gamma^{-1} \cdot \mathbf{o}, g \cdot \mathbf{o}) - d(\mathbf{o}, g \cdot \mathbf{o}) & \text{if } x = g \cdot x_0 \text{ for some } g \in \Gamma. \end{cases}$$

The cocycle equality (2.2) is still valid for this extended function b; furthermore,

$$b(\gamma, x_0) = d(\mathbf{o}, \gamma \cdot \mathbf{o}).$$

3. Ruelle operators for Schottky groups

Let X be a Hadamard manifold with origin $\mathbf{o} \in X$, and $\Gamma = \Gamma_1 \star \cdots \star \Gamma_\ell$ a Schottky product of elementary groups of isometries of X, as defined in the previous section.

3.1. Orbital counting function and Ruelle operators

Let us decompose the orbital counting function according to the symbolic length of the elements of Γ, then introduce in a natural way the main analytical tool of the proof: the Ruelle operators.

We write $\delta = \delta_\Gamma$ and $N_\Gamma(\mathbf{o}; R) = N_\Gamma(\mathbf{o}, \mathbf{o}; R)$. For all $R > 0$,

$$N_\Gamma(\mathbf{o}; R) = \sum_{\gamma \in \Gamma} \sum_{n \in \mathbb{N}} \mathbb{1}_{[n,n+1[}(R - d(\mathbf{o}, \gamma \cdot \mathbf{o}))$$

$$= e^{\delta R} \sum_{\gamma \in \Gamma} e^{-\delta d(\mathbf{o}, \gamma \cdot \mathbf{o})} \sum_{n \in \mathbb{N}} e_n(R - d(\mathbf{o}, \gamma \cdot \mathbf{o}))$$

$$= e^{\delta R} \sum_{n \in \mathbb{N}} \sum_{\gamma \in \Gamma} e^{-\delta b(\gamma, x_0)} e_n(R - b(\gamma, x_0))$$

where for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$,

(3.1) $$e_n(t) = e^{-\delta t} \mathbb{1}_{[n,n+1[}.$$

Thus, the quantity $N_\Gamma(\mathbf{o}; R)$ decomposes as

(3.2) $$N_\Gamma(\mathbf{o}; R) = e^{\delta R} \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \sum_{\gamma \in \Gamma(k)} e^{-\delta b(\gamma, x_0)} e_n(R - b(\gamma, x_0))$$

where $\Gamma(k)$ is the set of elements of Γ with symbolic length k.

For all $\phi \in L^\infty(\widetilde{\Lambda})$, all $u : \mathbb{R} \to R$ with compact support, all $s > 0$ and all $(x, t) \in \partial X \times \mathbb{R}$, let us set

$$\widetilde{\mathcal{L}}_s(\phi \otimes u)(x, t) = \sum_{\gamma \in \Gamma(1)} \mathbb{1}_{x \notin \widetilde{\Lambda}_{l_\gamma}} e^{-sb(\gamma, x)} \phi(\gamma \cdot x) u(t - b(\gamma, x)).$$

This is a finite sum as soon as u has compact support.

By the cocycle property of b, the iterates $\widetilde{\mathcal{L}}_s^k$ of the operators $\widetilde{\mathcal{L}}_s$ may be written as follows

$$\widetilde{\mathcal{L}}_s^k(\phi \otimes u)(x, t) = \sum_{\gamma \in \Gamma(k)} \mathbb{1}_{x \notin \widetilde{\Lambda}_{l_\gamma}} e^{-sb(\gamma, x)} \phi(\gamma \cdot x) u(t - b(\gamma, x))$$

(more detailed explanations are given in the next section for the iterates of the classical Ruelle operators).

Thus, equation (3.2) may be rewritten as

(3.3) $$N_\Gamma(\mathbf{o}; R) = e^{\delta R} \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \widetilde{\mathcal{L}}_\delta^k (\mathbb{1}_{\widetilde{\Lambda}} \otimes e_n)(x_0, R).$$

These operators $\widetilde{\mathcal{L}}_s$ can be seen as an $\mathbb{R}$-extension of the well known class of *Ruelle operators* $\mathcal{L}_s$ associated with the Schottky group Γ, defined formally by: for any function $\phi \in L^\infty(\widetilde{\Lambda})$ and any $x \in \widetilde{\Lambda}$,

(3.4) $$\mathcal{L}_s \phi(x) = \sum_{\gamma \in \Gamma(1)} \mathbb{1}_{x \notin \widetilde{\Lambda}_{l_\gamma}} e^{-sb(\gamma, x)} \phi(\gamma \cdot x)$$

These operators $\mathcal{L}_s$ are classical tools in hyperbolic dynamic, in particular to study the geodesic flow on $T^1 X_\Gamma$, cf. for instance [1], [7] or [18]. In the present paper, we do not develop further considerations about the geodesic flow since we will not use it to estimate the orbital function.

3.2. Poincaré series versus Ruelle operators

From now on, we assume that all the Poincaré series of the $\Gamma_j, j = 1, \ldots, \ell$, do converge at $\delta = \delta_\Gamma$. For instance, this condition holds when hypotheses (H_1), (H_2) and (H_3) are satisfied.

Definition (3.4) may be rewritten as

$$\mathcal{L}_s\phi(x) = \sum_{j=1}^{\ell} \sum_{\gamma\in\Gamma_j^*} \mathbb{1}_{x\notin\tilde{\Lambda}_j} e^{-sb(\gamma,x)}\phi(\gamma\cdot x)$$

For any $1 \leqslant j \leqslant \ell$, the sequence $(\gamma\cdot\mathbf{o})_{\gamma\in\Gamma_j}$ accumulates on the fixed point(s) of Γ_j; hence, the sequence $(b(\gamma,x) - d(\mathbf{o},\gamma\cdot\mathbf{o}))_{\gamma\in\Gamma_j}$ is bounded, uniformly in $x \notin \tilde{\Lambda}_j$. Therefore, since the Poincaré series of the Γ_j does converge at $\delta = \delta_\Gamma$, the quantity $\mathcal{L}_s\phi(x)$ is well defined as soon as ϕ is bounded and $s \geqslant \delta := \max\{\delta_{\Gamma_j} \mid 1 \leqslant j \leqslant \ell\}$.

For any $s \geqslant 0$ and γ in Γ^*, let $w_s(\gamma, \cdot)$ be the *weight function* defined on $\tilde{\Lambda}$ by: for any $s \geqslant \delta$ and $\gamma \in \Gamma$,

$$w_s(\gamma, x) := \begin{cases} 1 & \text{if } \gamma = \mathrm{Id}, \\ e^{-sb(\gamma,x)} & \text{if } x \in \tilde{\Lambda}_j, \, j \neq l_\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that these functions are continuous on $\tilde{\Lambda}$ and

$$\mathcal{L}_s\phi(x) = \sum_{\gamma\in\Gamma(1)} w_s(\gamma, x)\phi(\gamma \cdot x).$$

By a straightforward computation, one may thus check that $\mathcal{L}_s$ is a bounded linear operator on $(C(\tilde{\Lambda}), |\cdot|_\infty)$ when $s \geqslant \delta$; we denote by $\rho_s(\infty)$ its spectral radius on this space.

Let us now compute the iterates $\mathcal{L}_s^k, k \geqslant 1$, of the operators $\mathcal{L}_s$. The functions $w_s(\gamma, \cdot)$ also satisfy the following cocycle relation: if $\gamma_1, \gamma_2 \in \mathcal{A}$ do not belong to the same group Γ_j, then

$$w_s(\gamma_1\gamma_2, x) = w_s(\gamma_1, \gamma_2 \cdot x)w_s(\gamma_2, x).$$

Due to this cocycle property, we may write, for any $k \geqslant 1$, any bounded function $\phi : \tilde{\Lambda} \to \mathbb{R}$ and any $x \in \tilde{\Lambda}$,

$$\begin{aligned} \mathcal{L}_s^k\phi(x) &= \sum_{\gamma\in\Gamma(k)} w_s(\gamma, x)\phi(\gamma \cdot x) \\ &= \sum_{\gamma\in\Gamma(k)} \mathbb{1}_{x\notin\tilde{\Lambda}_{l_\gamma}} e^{-sb(\gamma,x)}\phi(\gamma \cdot x). \end{aligned}$$

By the "ping-pong dynamic" between the subgroups $\Gamma_j, 1 \leqslant j \leqslant \ell$, and Proposition 2.6, the difference $b(\gamma, x) - d(\mathbf{o}, \gamma \cdot \mathbf{o})$ is bounded uniformly in $k \geqslant 0, \gamma \in \Gamma(k)$ and $x \notin \tilde{\Lambda}_{l_\gamma}$. Consequently, there exists a constant $C > 0$

such that, for any $x \in \widetilde{\Lambda}$, any $k \geqslant 1$ and any $s \geqslant \delta$,

$$\mathcal{L}_s^k 1(x) \overset{c}{\asymp} \sum_{\gamma \in \Gamma(k)} e^{-sd(\mathbf{o}, \gamma \cdot \mathbf{o})}$$

where c is a positive constant and $A \overset{c}{\asymp} B$ means $\frac{A}{c} \leqslant B \leqslant cA$. Hence,

$$(3.5) \qquad P_\Gamma(s) := \sum_{\gamma \in \Gamma} e^{-sd(\mathbf{o}, \gamma \cdot \mathbf{o})} = +\infty \quad \Longleftrightarrow \quad \sum_{k \geqslant 0} \mathcal{L}_s^k 1(x) = +\infty.$$

In particular

$$(3.6) \qquad \delta_\Gamma = \sup\{s \geqslant \delta \,|\, \rho_s(\infty) \geqslant 1\} = \inf\{s \geqslant \delta \,|\, \rho_s(\infty) \leqslant 1\}.$$

We prove in the next paragraph that Γ is convergent if and only if $\rho_\delta(\infty) < 1$.

3.3. On the spectrum of the operators $\mathcal{L}_s, s \geqslant \delta$

Following the classical approach to study the spectrum of the transfer operators $\mathcal{L}_s$ (see [1], [7] and comments therein), we consider their restriction to the space $\mathrm{Lip}(\widetilde{\Lambda})$ of Lipschitz functions from $\widetilde{\Lambda}$ to $\mathbb{C}$ defined by

$$\mathrm{Lip}(\widetilde{\Lambda} = \{\phi \in C(\widetilde{\Lambda}) \,|\, \|\phi\| = |\phi|_\infty + [\phi] < +\infty\}$$

$$\text{where} [\phi] = \sup_{0 \leqslant i \leqslant l} \sup_{\substack{x, y \in \tilde{\Lambda}_j \\ x \neq y}} \frac{|\phi(x) - \phi(y)|}{D(x, y)}$$

is the Lipschitz coefficient of ϕ on $(\partial X, D)$.

The space $(\mathrm{Lip}(\widetilde{\Lambda}), \|\cdot\|)$ is a Banach space and by Arzela–Ascoli Theorem, the identity map from $(\mathrm{Lip}(\widetilde{\Lambda}), \|\cdot\|)$ into $(C(\widetilde{\Lambda}), |\cdot|_\infty)$ is compact. It is proved in [1] that the operators $\mathcal{L}_s, s \geqslant \delta$, act both on $(C(\Lambda_\Gamma), |\cdot|_\infty)$ and $(\mathrm{Lip}(\Lambda_\Gamma), \|\cdot\|)$. P. Vidotto has extended in [18] this property to the Banach spaces $(C(\widetilde{\Lambda}), |\cdot|_\infty)$ and $(\mathrm{Lip}(\widetilde{\Lambda}), \|\cdot\|)$. We denote by ρ_s the spectral radius of $\mathcal{L}_s$ on $\mathrm{Lip}(\widetilde{\Lambda})$. The following proposition gathers the spectral properties of the $\mathcal{L}_s$ which we need in the present paper.

PROPOSITION 3.1. — Assume that $\ell = p + q \geqslant 3$. For any $s \geqslant \delta$,

(1) $\rho_s = \rho_s(\infty)$;

(2) ρ_s is a simple eigenvalue of $\mathcal{L}_s$ acting on $\mathrm{Lip}(\widetilde{\Lambda})$ and the associated eigenfunction h_s is positive on $\widetilde{\Lambda}$;

(3) there exists $0 \leqslant r < 1$ such that the rest of the spectrum of $\mathcal{L}_s$ on $\mathrm{Lip}(\widetilde{\Lambda})$ is included in a disc of radius $\leqslant r\rho_s$.

Sketch of the proof. — We refer to [1] and [18] for more details. In [1], it is proved that the restriction of the functions $w_s(\gamma, \cdot), \gamma \in \Gamma$, to the set Λ_Γ belong to $\mathrm{Lip}(\Lambda_\Gamma)$ and that for any $s \geqslant \delta$ there exists $C = C(s) > 0$ such that, for any γ in Γ^*

$$\|w_s(\gamma, \cdot)\| \leqslant Ce^{-sd(\mathbf{o}, \gamma \cdot \mathbf{o})}.$$

In [18, Proposition 8.3.1], P. Vidotto has proved that the same inequality holds for the functions $w_s(\gamma, \cdot)$ on $\widetilde{\Lambda}$. Thus, the operator $\mathcal{L}_s$ is bounded on $\mathrm{Lip}(\widetilde{\Lambda})$ when $s \geqslant \delta$.

In order to describe its spectrum on $\mathrm{Lip}(\widetilde{\Lambda})$, we first write a "contraction property" for the iterated operators $\mathcal{L}_s^k$: for all $x, y \in \widetilde{\Lambda}$, all $k \in \mathbb{N}$ and all $s \geqslant \delta$, it holds

$$|\mathcal{L}_s^k\phi(x) - \mathcal{L}_s^k\phi(y)|$$
$$\leqslant \sum_{\gamma \in \Gamma(k)} |w_s(\gamma, x)| \, |\phi(\gamma \cdot x) - \phi(\gamma \cdot y)| + \sum_{\gamma \in \Gamma(k)} [w_s(\gamma, \cdot)] \, |\phi|_\infty D(x, y).$$

By Proposition 2.7 and the mean value relation (2.1), there exist $C > 0$ and $0 \leqslant r < 1$ such that $D(\gamma \cdot x, \gamma \cdot y) \leqslant Cr^k D(x, y)$ whenever $x, y \in \widetilde{\Lambda}_j$, $j \neq l_\gamma$ and $\gamma \in \Gamma(k)$. Hence, we get

(3.7) $$[\mathcal{L}_s^k\phi] \leqslant r_k[\phi] + R_k|\phi|_\infty$$

where $r_k = (Cr^k)|\mathcal{L}_s^k 1|_\infty$ and $R_k = \sum_{\gamma \in \Gamma(k)}[w_s(\gamma, \cdot)]$. Observe that

$$\limsup_k r_k^{1/k} = r \limsup_k |\mathcal{L}_s^k 1|_\infty^{1/k} = r\rho_s(\infty)$$

where $\rho_s(\infty)$ is the spectral radius of the positive operator $\mathcal{L}_s$ on $C(\widetilde{\Lambda})$. By Hennion's work [10], Inequality (3.7) implies that $\mathcal{L}_s$ is quasi-compact and its essential spectral radius on $\mathrm{Lip}(\widetilde{\Lambda})$ is less than $r\rho_s(\infty)$. In other words, any spectral value of $\mathcal{L}_s$ with modulus strictly larger than $r\rho_s(\infty)$ is an eigenvalue with finite multiplicity and is isolated in the spectrum of $\mathcal{L}_s$.

This implies in particular $\rho_s = \rho_s(\infty)$. Indeed, the inequality $\rho_s \geqslant \rho_s(\infty)$ is obvious since the function 1 belongs to $\mathrm{Lip}(\widetilde{\Lambda})$. Conversely, the strict inequality would imply the existence of a function $\phi \in \mathrm{Lip}(\widetilde{\Lambda})$ such that $\mathcal{L}_s\phi = \lambda\phi$ for some $\lambda \in \mathbb{C}$ of modulus $> \rho_s(\infty)$; this yields $|\lambda||\phi| \leqslant \mathcal{L}_s|\phi|$ so that $|\lambda| \leqslant \rho_s(\infty)$: a contradiction.

It remains to control the nature of the value ρ_s in the spectrum of $\mathcal{L}_s$. By the previous argument, we know that ρ_s is an eigenvalue of $\mathcal{L}_s$ with (at least) one associated eigenfunction $h_s \geqslant 0$. This function is positive on $\widetilde{\Lambda}$: otherwise, there exist $1 \leqslant j \leqslant p + q$ and a point $y_0 \in \widetilde{\Lambda}_j$ such that $h_s(y_0) = 0$. The equality $\mathcal{L}_s h_s(y_0) = \rho_s h_s(y_0)$ implies $h_s(\gamma \cdot y_0) = 0$ for any

$\gamma \in \Gamma$ with last letter $\neq j$. The minimality of the action of Γ on Λ_Γ and the fact that $\Gamma \cdot x_0$ accumulates on Λ implies $h_s = 0$ on $\widetilde{\Lambda}$: a contradiction.

In order to prove that ρ_s is a simple eigenvalue on $\mathrm{Lip}(\widetilde{\Lambda})$, let us introduce the so-called "Doob transform" of $\mathcal{L}_s$. For any $s \geqslant \delta$, we denote by P_s the operator defined formally by: for any bounded Borel function $\phi : \widetilde{\Lambda} \to \mathbb{C}$ and $x \in \widetilde{\Lambda}$,

$$P_s\phi(x) = \frac{1}{\rho_s h_s(x)} \mathcal{L}(h_s\phi)(x) = \frac{1}{\rho h_s(x)} \sum_{\gamma \in \Gamma(1)} \mathbb{1}_{x \notin \widetilde{\Lambda}_{l_\gamma}} e^{-sb(\gamma \cdot x)} h(\gamma \cdot x)\phi(\gamma \cdot x).$$

The iterates of P_s are given by: $P_s^0 = \mathrm{Id}$ and for $k \geqslant 1$

$$(3.8) \qquad P_s^k\phi(x) = \frac{1}{\rho_s^k h_s(x)} \sum_{\gamma \in \Gamma(k)} \mathbb{1}_{x \notin \widetilde{\Lambda}_{l_\gamma}} e^{-sb(\gamma,x)} h(\gamma \cdot x)\phi(\gamma \cdot x).$$

The operator P_s acts on $\mathrm{Lip}(\widetilde{\Lambda})$ as a *Markov operator*, i.e. $P_s\phi \geqslant 0$ if $\phi \geqslant 0$ and $P_s \mathbb{1}_{\widetilde{\Lambda}} = \mathbb{1}_{\widetilde{\Lambda}}$. It inherits the spectral properties of $\mathcal{L}_s$ and is in particular quasi-compact with essential spectral radius < 1. The spectral value 1 is an eigenvalue and it remains to prove that the associated eigenspace is $\mathbb{C} \cdot 1$. Let $f \in \mathrm{Lip}(\widetilde{\Lambda})$ such that $P_s f = f$ and $1 \leqslant j \leqslant p + q$ and $y_0 \in \widetilde{\Lambda}_j$ such that $|f(y_0)| = |f|_\infty$. An argument of convexity applied to the inequality $P_s|f| \leqslant |f|$ readily implies $|f(y_0)| = |f(\gamma \cdot y_0)|$ for any $\gamma \in \Gamma$ with last letter $\neq j$. By minimality of the action of Γ on $\widetilde{\Lambda}$, it follows that the modulus of f is constant on $\widetilde{\Lambda}$. Applying again an argument of convexity, the minimality of the action of Γ on $\widetilde{\Lambda}$ and the fact that $\Gamma \cdot x_0$ accumulates on Λ, one proves that f is in fact constant on $\widetilde{\Lambda}$. Therefore, the eigenspace of $\mathcal{L}_s$ associated with ρ_s equals $\mathbb{C} \cdot h_s$.

A similar proof, using the fact that $\ell \geqslant 3$, shows that ρ_s is the unique eigenvalue with modulus ρ_s; the argument is detailed in [1, Proposition III.4] and [18, Proposition 8.3.2]. □

When $p + q = 2$, an analogous property holds (see [7, Lemma VII.6] for the details):

PROPOSITION 3.2. — *Assume that $\ell = p + q = 2$. For any $s \geqslant \delta$,*

(1) $\rho_s = \rho_s(\infty)$;

(2) ρ_s *is a simple eigenvalue of $\mathcal{L}_s$ acting on $\mathrm{Lip}(\widetilde{\Lambda})$ with an associated eigenfunction h_s^+ positive on $\widetilde{\Lambda} = \widetilde{\Lambda}_1 \cup \widetilde{\Lambda}_2$;*

(3) $-\rho_s$ *is a simple eigenvalue of $\mathcal{L}_s$ acting on $\mathrm{Lip}(\widetilde{\Lambda})$, with an associated eigenfunction h_s^- which is positive on $\widetilde{\Lambda}_1$ and negative on $\widetilde{\Lambda}_2$;*

(4) *there exists $0 \leqslant r < 1$ such that the rest of the spectrum of $\mathcal{L}_s$ on $\mathrm{Lip}(\widetilde{\Lambda})$ is included in a disc of radius $\leqslant r\rho_s$.*

In the sequel, to lighten the proof of Theorem 1.2, we only consider the case $p + q \geqslant 3$. Adapting the proof when $p + q = 2$ is a little bit technical but straightforward, using Proposition 3.2 instead of Proposition 3.1.

Expression (3.8) yields the following.

Notation 3.3. — *For any $s \geqslant \delta$, $x \in \widetilde{\Lambda}$, $k \geqslant 0$ and any $\gamma \in \Gamma(k)$, we set*

$$(3.9) \qquad p_s(\gamma, x) := \frac{1}{\rho_s^k} \frac{h_s(\gamma \cdot x)}{h_s(x)} w_s(\gamma, x).$$

As for the $w_s(\gamma, \cdot)$, these "weight functions" are positive and satisfy the cocycle property: for all $s \geqslant \delta, x \in \widetilde{\Lambda}$ and $\gamma_1, \gamma_2 \in \Gamma$,

$$p_s(\gamma_1 \gamma_2, x) = p_s(\gamma_1, \gamma_2 \cdot x) \cdot p_s(\gamma_2, x).$$

Let us emphasize that the operator P_s is Markovian, i.e.

$$\sum_{\gamma \in \Gamma(k)} p_s(\gamma, x) = 1.$$

COROLLARY 3.4. — *The group Γ is convergent if and only if $\rho_\delta < 1$.*

Proof. — If $\rho_\delta = \rho_\delta(\infty) < 1$ then $\rho_s < 1$ for any $s \geqslant \delta$, since $s \mapsto \rho_s(\infty) = \rho_s$ is decreasing on $[\delta, +\infty[$. Equality (3.6) implies $\delta_\Gamma \leqslant \delta$ and so $\delta_\Gamma = \delta$; by (3.5), it follows that Γ is convergent.

Assume now $\rho_\delta \geqslant 1$. When Γ is non exotic, it is divergent by [6]. Otherwise, $\delta_\Gamma = \delta$ and, since the eigenfunction h_δ is positive on $\widetilde{\Lambda}$, we have, for any $k \geqslant 1$ and $x \in \widetilde{\Lambda}$

$$\mathcal{L}_\delta^k 1(x) \asymp \mathcal{L}_\delta^k h_\delta(x) = \rho_\delta^k h_\delta(x) \asymp \rho_\delta^k.$$

Consequently $\sum_{k \geqslant 0} \mathcal{L}_\delta^k 1(x) = +\infty$ and the group Γ is divergent, by (3.5).   □

4. Counting for convergent groups

This last section is devoted to the proof of Theorem 1.2. Let us fix a N-dimensional Hadamard manifold X with pinched negative curvatures, and a base point $\mathbf{o} \in X$. Fix $\alpha \in \,]1, 2[$ and let Γ be a Schottky group of isometries of X satisfying hypotheses (H_1), (H_2) and (H_3) of Theorem 1.2. Recall that it implies that all the elementary groups Γ_j, $j = 1, \dots, \ell$, as well as the full group Γ, have a Poincaré series which converges at $\delta = \delta_\Gamma$. This implies that

- the Ruelle operators $\mathcal{L}_s$ and $\widetilde{\mathcal{L}}_s$ described in the previous section are well defined provided $s \geqslant \delta$;
- the spectral radius of $\mathcal{L}_\delta$ on $\mathrm{Lip}(\widetilde{\Lambda})$ is $\rho_\delta < 1$, by Corollary 3.4.

4.1. Proof of Theorem 1.2

From now on, definitions and notations of Section 3.2 are used freely. By (3.3), for all $R > 0$, it holds

$$N_\Gamma(\mathbf{o}; R) = e^{\delta R} \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \widetilde{\mathcal{L}}_\delta^k (\mathbb{1}_{\widetilde{\Lambda}} \otimes e_n)(x_0, R),$$

where e_n is defined by (3.1). Let us remark that, for all $R > 0$ and $x \in \widetilde{\Lambda}$, the map

$$M_{x,R} : u \mapsto M_{x,R}(u) = \sum_{k \in \mathbb{N}} \widetilde{\mathcal{L}}_\delta^k (\mathbb{1} \otimes u)(x, R)$$

is a positive bounded linear form on the space of continuous function with compact support, i.e. a locally finite Radon measure on $\mathbb{R}$. The orbital counting function can hence be rewritten as

$$N_\Gamma(\mathbf{o}; R) = e^{\delta R} \sum_{n \in \mathbb{N}} M_{x_0, R}(e_n).$$

With this notation, Theorem 1.2 is an immediate consequence of the following statement.

PROPOSITION 4.1. — *For any $x \in \widetilde{\Lambda}$, there exists $C(x) > 0$ such that, for any continuous map $u : \mathbb{R} \to \mathbb{R}$ with compact support,*

$$(4.1) \qquad \lim_{R \to +\infty} \frac{R^\alpha}{L(R)} M_{x,R}(u) = C(x) \int_{\mathbb{R}} u(t) \mathrm{d}t.$$

Firstly, let us prove that Theorem 1.2 is a direct consequence of Proposition 4.1.

Proof of Theorem 1.2. — From Beppo–Levi monotone convergence Theorem, (4.1) also holds for all positive functions. Therefore,

$$\lim_{R \to +\infty} \frac{e^{-\delta R} R^\alpha}{L(R)} N_\Gamma(\mathbf{o}; R) = \lim_{R \to +\infty} \sum_{n \in \mathbb{N}} \frac{R^\alpha}{L(R)} M_{x_0, R}(e_n)$$

$$= \sum_{n \in \mathbb{N}} \lim_{R \to +\infty} \frac{R^\alpha}{L(R)} M_{x_0, R}(e_n)$$

$$= C(x_0) \sum_{n \in \mathbb{N}} \int_{\mathbb{R}} e_n(t) \mathrm{d}t$$

$$(4.2) \qquad = C(x_0) \int_{\mathbb{R}} e^{-\delta t} = \frac{C(x_0)}{\delta}.$$

The above change of order between series and limit is valid since all terms are positive. This is exactly Theorem 1.2, with $C_\Gamma = \frac{C(x_0)}{\delta}$. The value of $C(x_0)$ is explicited in Subsection 4.4. $\qquad \square$

The rest of this paragraph is devoted to the proof of Proposition 4.1.

Proof of Proposition 4.1. — Let us fix a continuous function $u : \mathbb{R} \to \mathbb{R}^+$ with compact support. By definition, for all $x \in \widetilde{\Lambda}$ and all $R > 0$,

$$M_{x,R}(u) = \sum_{k \geqslant 0} M_{x,R}^{(k)}(u)$$

where

$$M_{x,R}^{(k)}(u) = \widetilde{\mathcal{L}}_\delta^k(\mathbb{1} \otimes u)(x, R)$$

$$= \sum_{\gamma \in \Gamma(k)} \mathbb{1}_{x \notin \tilde{\Lambda}_{l_\gamma}} e^{-\delta b(\gamma, x)} \mathbb{1}_{\widetilde{\mathbf{\Lambda}}}(\gamma \cdot x) u(R - b(\gamma, x)).$$

In the same way as in Paragraph 3.3 we have associated the Markovian operator P_δ to $\mathcal{L}_\delta$, we consider the Markovian operator $\widetilde{P}_\delta$ on $\widetilde{\Lambda} \times \mathbb{R}$, defined by: for any $\phi \in \mathrm{Lip}(\widetilde{\Lambda})$, any continuous function with compact support $u : \mathbb{R} \to \mathbb{R}$ and any $(x, t) \in \widetilde{\Lambda} \times \mathbb{R}$,

$$(4.3) \qquad \widetilde{P}_\delta(\phi \otimes u)(x, t) = \frac{1}{\rho_\delta h_\delta(x)} \widetilde{\mathcal{L}}_\delta(h_\delta \phi \otimes u)(x, t)$$

$$(4.4) \qquad\qquad = \sum_{\gamma \in \Gamma(1)} p_\delta(\gamma, x) \phi(\gamma \cdot x) u(t - b(\gamma, x)).$$

This operator $\widetilde{P}_\delta$ commutes with the action of the translations on $\mathbb{R}$ and its iterates are given by: for any $k \geqslant 1$,

$$\widetilde{P}_\delta^k(\phi \otimes u)(x, t) = \sum_{\gamma \in \Gamma(k)} p_\delta(\gamma, x) \phi(\gamma \cdot x) u(t - b(\gamma, x)).$$

From now on, to lighten notations we write $P = P_\delta$, $\widetilde{P} = \widetilde{P}_\delta$, $h = h_\delta$, $\rho = \rho_\delta$ and $p = p_\delta$. Hence, we can rewrite the quantity $M_{x,R}^{(k)}(u)$ as

$$M_{x,R}^{(k)}(u) = \rho^k h(x) \widetilde{P}^k \left(\frac{1}{h} \otimes u \right)(x, R),$$

so that,

$$(4.5) \qquad\qquad M_{x,R}(u)) = h(x) \sum_{k \geqslant 0} \rho^k \widetilde{P}^k \left(\frac{1}{h} \otimes u \right)(x, R).$$

The end of the proof of Proposition 4.1 follows from the two following lemmas, whose proofs are postponed to Sections 4.2 and 4.3.

We first control the behavior as $R \to +\infty$ of the quantity $M_{x,R}^{(1)}(u)$.

LEMMA 4.2. — *For any continuous function $u : \mathbb{R} \to \mathbb{R}$ with compact support, there exists a constant $C_u > 0$ such that, for any $\phi \in \mathrm{Lip}(\widetilde{\Lambda})$, any $x \in \widetilde{\Lambda}$ and $R \geqslant 1$,*

$$(4.6) \qquad \left| \widetilde{P}(\phi \otimes u)(x, R) \right| \leqslant C_u \|\phi\|_\infty \times \frac{L(R)}{R^\alpha}.$$

Furthermore,

$$(4.7) \qquad \lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \widetilde{P}(\phi \otimes u)(x, R) = \sum_{j=1}^{p} C_j(x)\phi(x_j) \int_{\mathbb{R}} u(t)\mathrm{d}t,$$

where C_j is defined by: for $1 \leqslant j \leqslant p$,

$$(4.8) \qquad C_j(x) := c_j \frac{h(x_j)}{\rho h(x)} \times \begin{cases} e^{2\delta(x_j \,|\, x)_\mathbf{o}} & \text{if } x \in \Lambda_\Gamma \backslash \widetilde{\Lambda}_j; \\ e^{\delta(\mathcal{B}_{x_j}(\mathbf{o}, g \cdot \mathbf{o}) + d(\mathbf{o}, g \cdot \mathbf{o}))} & \text{if } x = g \cdot x_0 \notin \widetilde{\Lambda}_j; \\ 0 & \text{if } x \in \widetilde{\Lambda}_j, \end{cases}$$

where, for any $1 \leqslant j \leqslant p$, the constant c_j comes from (H_2) and x_j is the unique fixed point of Γ_j.

A similar statement holds for all the $M_{x,R}^{(k)}, k \geqslant 1$.

LEMMA 4.3. — *For any continuous function $u : \mathbb{R} \to \mathbb{R}^+$ with compact support:*

- *there exists a constant $C_u > 0$ such that, for any $\phi \in \mathrm{Lip}(\widetilde{\Lambda})$, any $x \in \widetilde{\Lambda}$, any $k \geqslant 1$ and any $R \geqslant 1$,*

$$(4.9) \qquad \left| \widetilde{P}^k (\phi \otimes u) (x, R) \right| \leqslant C_u \, k^2 \, \|\phi\|_\infty \times \frac{L(R)}{R^\alpha};$$

- *for any $\phi \in \mathrm{Lip}(\widetilde{\Lambda})$, any $x \in \widetilde{\Lambda}$ and any $k \geqslant 1$,*

$$(4.10) \qquad \lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \widetilde{P}^k (\phi \otimes u) (x, R)$$
$$= \sum_{j=1}^{p} \left(\sum_{l=0}^{k-1} P^l C_j(x) P^{k-1-l}\phi(x_j) \right) \int_{\mathbb{R}} u(t)\mathrm{d}t$$

where the Lipschitz functions C_j, $1 \leqslant j \leqslant p$, are given by (4.8).

Proposition 4.1 follows immediately from these statements. Indeed, Equation (4.5), Lemma 4.3 and the dominated convergence theorem yield

$$(4.11) \quad \lim_{R\to+\infty} \frac{R^\alpha}{L(R)} M^{(k)}_{x,R}$$

$$= \left(h(x) \sum_{k\geqslant 1} \rho^k \sum_{j=1}^{p} \left(\sum_{l=0}^{k-1} P^l C_j(x) P^{k-1-l} \left(\frac{1}{h}\right)(x_j) \right) \right) \times \int_{\mathbb{R}} u(t)\mathrm{d}t. \quad \square$$

4.2. Proof of Lemma 4.2

Let us fix $x \in \widetilde{\Lambda}$ and u a continuous function on $\mathbb{R}$ whose support is included in the interval $[a,b]$.

Proof of the upperbound (4.6). — For any $R \geqslant b$, it holds

$$\widetilde{P}(\phi\otimes u)(x,R) = \frac{1}{\rho h(x)} \sum_{j=1}^{p+q} \sum_{\gamma\in\Gamma_j} e^{-\delta b(\gamma x)} \mathbb{1}_{x\notin\tilde{\Lambda}_j} h(\gamma\cdot x)\phi(\gamma\cdot x)u(R-b(\gamma,x)).$$

By hypotheses (H$_2$) and (H$_3$) and Lemma 2.1, for any $j = 1,\ldots,p+q$ there exists a constant $K_j > 0$ such that, for any $R \geqslant 1$,

$$\sum_{\substack{\gamma\in\Gamma_j \\ R-b\leqslant b(\gamma,x)\leqslant R-a}} e^{-\delta b(\gamma,x)} \leqslant K_j(b-a)\frac{L(R-b)}{(R-b)^\alpha}.$$

Together with the fact that L is slowly varying, this implies (4.6). $\square$

Proof of the asymptotic (4.7). — In order to establish (4.7), it is sufficient to prove that for any $j = 1,\ldots,p+q$,

$$(4.12) \quad \lim_{R\to+\infty} \frac{R^\alpha}{L(R)} \sum_{\gamma\in\Gamma_j} p(\gamma,x)\phi(\gamma\cdot x)u(R-b(\gamma,x)) = C_j(x)\phi(x_j)\int_{\mathbb{R}} u(t)\mathrm{d}t,$$

where $C_j(x)$ is given by (4.8) for $1 \leqslant j \leqslant p$ and $C_j(x) = 0$ for $j = p+1,\ldots,q$. By a classical approximation argument, it is sufficient to check that (4.12) holds when u is the characteristic function of some interval $[a,b]$, in which case

$$\sum_{\gamma\in\Gamma_j} p(\gamma,x)\phi(\gamma\cdot x)u(R-b(\gamma,x))$$

$$= \frac{1}{h(x)} \sum_{\substack{\gamma\in\Gamma_j \\ R-b\leqslant b(\gamma,x)\leqslant R-a}} e^{-\delta b(\gamma x)} \mathbb{1}_{x\notin\tilde{\Lambda}_j} h(\gamma\cdot x)\phi(\gamma\cdot x).$$

First, assume that $x = g \cdot x_0$ belongs to $\Gamma \cdot x_0$. For any $j = 1, \ldots, p$ and γ in Γ_j, $\gamma \neq \mathrm{Id}$, the sequence $(\gamma^n \cdot o)_{n \geqslant 1}$ tends to x_j as $n \to \pm\infty$; it yields

$$b(\gamma^n, x) - d(\mathbf{o}, \gamma^n \cdot \mathbf{o}) = d(\gamma^{-n} \cdot \mathbf{o}, g \cdot \mathbf{o}) - d(\gamma^{-n} \cdot \mathbf{o}, \mathbf{o}) - d(\mathbf{o}, g \cdot \mathbf{o})$$
$$\overset{n \to \pm\infty}{\longrightarrow} -\mathcal{B}_{x_j}(\mathbf{o}, g \cdot \mathbf{o}) - d(\mathbf{o}, g \cdot \mathbf{o}).$$

When $x \in \Lambda$, Lemma 2.1 yields

$$\lim_{n \to \pm\infty} b(\gamma^n, x) - d(\mathbf{o}, \gamma^n \cdot \mathbf{o}) = -2(x_j \,|\, x)_{\mathbf{o}}.$$

Eventually, by hypotheses (H_2) and (H_3), for any $1 \leqslant j \leqslant p + q$,

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \sum_{\substack{\gamma \in \Gamma_j \\ R - b \leqslant d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R - a}} p(\gamma, x) = C_j(x)|b - a|.$$

Hence,

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \sum_{\gamma \in \Gamma_j} p(\gamma, x)\phi(\gamma \cdot x) u(R - b(\gamma, x)) = C_j(x)\phi(x_j)|b - a|$$

which is exactly (4.12) when u equals the characteristic function of the interval $[a, b]$. □

4.3. Proof of Lemma 4.3

For the convenience of the reader, we assume that all subgroups $\Gamma_j, 1 \leqslant j \leqslant p + q$, are parabolic. Hence, they have a unique fixed point at infinity x_j and for any $x \in \widetilde{\Lambda}$,

$$\lim_{\substack{\gamma \in \Gamma_j \\ \gamma \to +\infty}} \gamma \cdot x = x_j.$$

When some $\Gamma_j, p + 1 \leqslant j \leqslant p + q$, is generated by an hyperbolic isometry h_j, we have to distinguish between positive and negative power of h_j; this overcharges the notations and present no more interest.

The proof of inequality (4.9) is based on upper estimates given in [18], whose proofs follow the approach developed in [9]. Let us present them briefly.

Firstly, we introduce the notion of "Schottky cocycle" which is of interest in the sequel.

DEFINITION 4.4. — *A Schottky cocycle is a map* $B : \Gamma \times \widetilde{\Lambda} \to \mathbb{R}$ *such that for all* $\gamma_1, \gamma_2 \in \Gamma$ *such that the last letter of* γ_1 *and the first letter of* γ_2 *do not belong to the same elementary factor, we have*

$$B(\gamma_1 \gamma_2, x) = B(\gamma_1, \gamma_2 \cdot x) + B(\gamma_2, x).$$

By (2.2), the map b introduced at the end of Section 2 is a cocycle, hence a Schottky cocycle. However, the map $B : \Gamma \times \widetilde{\Lambda} \to \mathbb{R}$ defined for all $\gamma \in \Gamma(k)$ with symbolic length k by

$$B(\gamma, x) = b(\gamma, x) + \sigma k$$

where $\sigma > 0$ is fixed, is not a cocycle but a Schottky cocycle.

For all such Schottky cocycles B and all $\delta > 0$, we consider the Ruelle operators $\mathcal{L}_{\delta,B}$ and their $\mathbb{R}$-extensions $\widetilde{\mathcal{L}}_{\delta,B}$ defined by: for all $\phi \in L^\infty(\widetilde{\Lambda})$, $u \in \mathcal{C}_c(\mathbb{R})$ and all $x \in \widetilde{\Lambda}$ and $R \in \mathbb{R}$,

$$\mathcal{L}_{\delta,B}(\phi)(x) = \sum_{\gamma \in \Gamma(1)} \mathbb{1}_{x \notin \tilde{\Lambda}_{l_\gamma}} e^{-\delta B(\gamma,x)} \phi(\gamma \cdot x)$$

and

$$\widetilde{\mathcal{L}}_{\delta,B}(\phi \otimes u)(x,R) = \sum_{\gamma \in \Gamma(1)} \mathbb{1}_{x \notin \tilde{\Lambda}_{l_\gamma}} e^{-\delta B(\gamma,x)} \phi(\gamma \cdot x) u(R - B(\gamma,x)).$$

Since B is a Schottky cocycle, for all $k \in \mathbb{N}$, it holds

$$\mathcal{L}_{\delta,B}^k(\phi)(x) = \sum_{\gamma \in \Gamma(k)} \mathbb{1}_{x \notin \tilde{\Lambda}_{l_\gamma}} e^{-\delta B(\gamma,x)} \phi(\gamma \cdot x)$$

and an analogous equality for $\widetilde{\mathcal{L}}_{\delta,B}$. The reader may check that in Sections 3 and 4 we have only used the Schottky cocycle property of b, and not its full cocycle property. We gather the results from [18] which we need in the following proposition.

PROPOSITION 4.5 ([18]). — *With the previous notations, let $B : \Gamma \times \widetilde{\Lambda} \to \mathbb{R}$ be a Schottky cocycle such that there exists $\beta \in (0,1)$ and a smooth slowly varying function $L : \mathbb{R} \to (0, +\infty)$ satisfying the following hypotheses:*

- *for all $j \in \{1, \ldots, p\}$, there exists $C_j > 0$ such that, as $T \to +\infty$,*

$$\sum_{\alpha \in \Gamma_j, d(\mathbf{o}, \alpha \cdot \mathbf{o}) \geqslant T} e^{-\delta B(\alpha, x_0)} \sim C_j \frac{L(T)}{T^\beta};$$

- *for all $j \in \{p+1, \ldots, p+q\}$, as $T \to +\infty$,*

$$\sum_{\alpha \in \Gamma_j, d(\mathbf{o}, \alpha \cdot \mathbf{o}) \geqslant T} e^{-\delta B(\alpha, x_0)} = o\left(\frac{L(T)}{T^\beta}\right);$$

- *for all $\Delta > 0$, there exists $C_\Delta > 0$ such that*

$$\sum_{\alpha \in \Gamma_j, T-\Delta \leqslant d(\mathbf{o}, \alpha \cdot \mathbf{o}) \leqslant T+\Delta} e^{-\delta B(\alpha, x_0)} \leqslant C_\Delta \frac{L(T)}{T^{1+\beta}};$$

- for all $\gamma \in \Gamma$, the map $x \mapsto B(\gamma, x)$ is Lipschitz continuous on $\widetilde{\Lambda}$;
- $\mathcal{L}_{\delta,B}$ is a bounded and positive operator on $\mathrm{Lip}(\widetilde{\Lambda})$ with spectral radius 1, this spectral value 1 is a simple eigenvalue, whose corresponding eigenfunction h is positive, and the rest of the spectrum is contained in a disc of radius $r < 1$.

Let $\widetilde{P}_B$ be the operator defined by: for all $\phi \in \mathrm{Lip}(\widetilde{\Lambda})$ and all $u \in \mathcal{C}_c(\mathbb{R})$,

$$\widetilde{P}_B(\phi \otimes u) = \frac{1}{h}\widetilde{L}_{\delta,B}(h\phi \otimes u).$$

At last, let $(a_k)_{k \in \mathbb{N}}$ be the sequence defined implicitly by: for all $k \in \mathbb{N}$,

$$\frac{a_k^\beta}{L(a_k)} = k.$$

Then, there exists $C > 0$ such that, for all $R \geqslant 0$ and all $k \in \mathbb{N}$:

(4.13) $$\left|\widetilde{P}_B^k(\phi \otimes u)(x, R)\right| \leqslant C\frac{\|\phi\|_\infty \|u\|_{L^1}}{a_k} \quad \text{if } R \leqslant 2a_k;$$

and

(4.14) $$\left|\widetilde{P}_B^k(\phi \otimes u)(x, R)\right| \leqslant Ck\frac{L(R)}{R^{1+\beta}}\|\phi \otimes u\|_\infty \quad \text{if } R \geqslant 2a_k.$$

Sketch of proof. — Let us first suppose that Γ is a divergent group with infinite Bowen–Margulis measure and $B = b$, i.e. for all $\gamma \in \Gamma$, $B(\gamma, x_0) = d(\mathbf{o}, \gamma \cdot \mathbf{o})$. If B satisfies the three first hypotheses of Proposition 4.5, we are exactly in the setting of [18]. As already mentioned in Section 3, it is proved in [18, Chapter 4] that $\mathcal{L}_{\delta,B}$ has spectral radius 1 on $\mathrm{Lip}(\widetilde{\Lambda})$, that 1 is a simple eigenvalue associated to a positive eigenfunction h, and the rest of the spectrum is contained in disc of radius $r < 1$.

Thus, the upperbound (4.13) is a byproduct of Proposition A.1 and the upperbound (4.14) by [18, Proposition A.2]. The proof detailed in [18] for the cocycle b works verbatim for any other Schottky cocycle B satisfying the hypotheses of Proposition 4.5 above. It is long and technical and we do not write it out here. $\square$

Proof of the upperbound (4.9). — Recall that the group Γ satisfies the hypotheses of Theorem 1.2. Let us write $\beta = \alpha - 1 \in (0, 1)$, and for all $k \in \mathbb{N}$, all $\gamma \in \Gamma$ and all $x \in \widetilde{\Lambda}$, let us set

$$B(\gamma, x) := \widetilde{b}(\gamma, x) + \frac{k \ln \rho}{\delta},$$

where $\rho \in (0, 1)$ is the spectral radius of $\mathcal{L}_\delta$ introduced in Section 3. Note that by definition, $\widetilde{P}_B = \widetilde{P}_\delta = \widetilde{P}$, where $\widetilde{P}_\delta$ is defined in (4.3).

By (H_2) and (H_3) and Subsection 3.3, the function B is a Schottky cocycle satisfying hypotheses of Proposition 4.5. It yields:

and

$C_k(x, r, R)$

$$:= \sum_{\substack{\gamma \in \Gamma(k) \\ d(\mathbf{o}, \gamma \cdot \mathbf{o}) > r}} \sum_{\substack{\beta \in \Gamma(1) \\ d(\mathbf{o}, \beta \cdot \mathbf{o}) > r}} p(\gamma, \beta \cdot x) p(\beta, x) \phi(\gamma \beta \cdot x) u\big(R - b(\gamma, \beta \cdot x) - b(\beta, x)\big).$$

Step 1. — Let us first prove that

$$(4.15) \quad \lim_{R \to +\infty} \frac{R^\alpha}{L(R)} A_k(x, r, R)$$

$$= \sum_{\substack{\beta \in \Gamma(1) \\ d(\mathbf{o}, \beta \cdot \mathbf{o}) \leqslant r}} p(\beta, x) \times \lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \widetilde{P}^k \left(\phi \otimes u\right) (\beta \cdot x, R).$$

Indeed, the set of $\beta \in \Gamma(1)$ such that $d(\mathbf{o}, \beta \cdot \mathbf{o}) \leqslant r$ is finite and $b(\beta, x) \leqslant r$ for such an isometry β; furthermore, if $p(\beta, x) \neq 0$ then $\frac{R}{2} \leqslant R - b(\beta, x) \leqslant R + C$ where $C > 0$ is the constant which appears in Proposition 2.6. Using the induction hypothesis, it yields, for any $\beta \in \Gamma(1)$ such that $d(\mathbf{o}, \beta \cdot \mathbf{o}) \leqslant r$,

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} p(\beta, x) \sum_{\gamma \in \Gamma(k)} p(\gamma, \beta \cdot x) \phi(\gamma \beta \cdot x) u\big(R - b(\beta, x) - b(\gamma, \beta \cdot x)\big)$$

$$= p(\beta, x) \times \lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \widetilde{P}^k \left(\phi \otimes u\right) (\beta \cdot x, R - b(\beta, x)).$$

Convergence (4.15) follows, summing over β. It yields, by Convergence (4.7) in Lemma 4.2,

$$(4.16) \quad \lim_{r \to +\infty} \lim_{R \to +\infty} \frac{R^\alpha}{L(R)} A_k(x, r, R)$$

$$= \sum_{j=1}^{p} \left(\sum_{l=1}^{k} P^l C_j(x) P^{k-l} \phi(x_j) \right) \times \int_{\mathbb{R}} u(t) \mathrm{d}t.$$

Step 2. — Now, let us prove that there exists $\epsilon(r) > 0$, with $\lim_{r \to +\infty} \epsilon(r) = 0$, such that, for any $k \geqslant 1$,

$$\liminf_{R \to +\infty} \frac{R^\alpha}{L(R)} B_k(x, r, R) \overset{\epsilon(r)}{\simeq} \limsup_{R \to +\infty} \frac{R^\alpha}{L(R)} B_k(x, r, R)$$

$$(4.17) \qquad\qquad \overset{\epsilon(r)}{\simeq} \sum_{j=1}^{p} \sum_{\substack{\gamma \in \Gamma(k) \\ d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant r}} p(\gamma, x_j) \phi(\gamma \cdot x_j) C_j(x) \int_{\mathbb{R}} u(t) \mathrm{d}t,$$

where we write $a \overset{\epsilon}{\simeq} b$ if $1 - \epsilon \leqslant \frac{a}{b} \leqslant 1 + \epsilon$.

Since each Γ_j has a unique fixed point x_j, there exists a map $\epsilon \colon (0, +\infty) \to (0, +\infty)$ which tends to 0 as $r \to +\infty$, such that

$$\frac{p(\gamma, \beta \cdot x)}{p(\gamma, x_j)} \overset{\epsilon(r)}{\simeq} 1$$

for any $j = 1, \ldots, p+q$, any $\beta \in \Gamma_j$ with $d(\mathbf{o}, \beta \cdot \mathbf{o}) \geqslant r$, any $x \in \widetilde{\Lambda}$ and any $\gamma \in \Gamma$ with $l_\gamma \neq j$.

The set of $\gamma \in \Gamma(k)$ such that $d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant r$ is finite; furthermore, for such γ and any $\beta \in \Gamma(1)$, it holds $\frac{R}{2} \leqslant R - b(\gamma, \beta \cdot x) \leqslant R + C$, as above. Therefore,

$$\sum_{\substack{\gamma \in \Gamma(k) \\ d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant r}} \sum_{\substack{\beta \in \Gamma(1) \\ d(\mathbf{o}, \beta \cdot \mathbf{o}) > r}} p(\gamma, \beta \cdot x) p(\beta, x) \phi(\gamma\beta \cdot x) u\big(R - b(\gamma, \beta \cdot x) - b(\beta, x)\big)$$

$$\overset{\epsilon(r)}{\simeq} \sum_{j=1}^{p+q} \sum_{\substack{\gamma \in \Gamma(k) \\ d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant r}} p(\gamma, x_j) \phi(\gamma \cdot x_j) \sum_{\substack{\beta \in \Gamma_j \\ d(\mathbf{o}, \beta \cdot \mathbf{o}) > r}} p(\beta, x) u\big(R - b(\gamma, \beta \cdot x) - b(\beta, x)\big)$$

Convergence (4.17) follows, using (4.12). In particular, letting $r \to +\infty$ and using again statement Convergence (4.7) in Lemma 4.2, it holds

$$\lim_{r \to +\infty} \liminf_{R \to +\infty} \frac{R^\alpha}{L(R)} B_k(x, r, R) = \lim_{r \to +\infty} \limsup_{R \to +\infty} \frac{R^\alpha}{L(R)} B_k(x, r, R)$$

$$\text{(4.18)} \qquad\qquad\qquad = \sum_{j=1}^{p} P^k \phi(x_j) C_j(x) \int_{\mathbb{R}} u(t) \mathrm{d}t.$$

Step 3. — Eventually, we prove that there exists a constant $C > 0$ such that, for any $R \geqslant 2r \geqslant 1$,

$$\text{(4.19)} \qquad\qquad C_k(x, r, R) \leqslant C k^2 \|\phi\|_\infty \frac{L(R)}{R^\alpha} \sum_{n=[r]}^{+\infty} \frac{L(n)}{n^\alpha}.$$

By Proposition 2.6, the condition $u(R - b(\gamma\beta \cdot x) - b(\beta, x)) \neq 0$ implies

$$d(\mathbf{o}, \gamma \cdot \mathbf{o}) + d(\mathbf{o}, \beta \cdot \mathbf{o}) = R \pm c \quad \text{and} \quad b(\gamma\beta \cdot x) + b(\beta, x) = R \pm c \quad \text{[4]}$$

for some constant $c > 0$ which depends on the support of u. We decompose $C_k(x, r, R)$ into $C_k(x, r, R) = C_{k,1}(x, r, R) + C_{k,2}(x, r, R)$ with

$C_{k,1}(x, r, R)$

$$:= \sum_{\substack{\gamma \in \Gamma(k) \\ r < d(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R/2}} \sum_{\substack{\beta \in \Gamma(1) \\ d(\mathbf{o}, \beta\mathbf{o}) > r}} p(\gamma, \beta \cdot x) p(\beta, x) \phi(\gamma\beta \cdot x) u\big(R - b(\gamma, \beta \cdot x) - b(\beta, x)\big).$$

---

[4] the notation $A = B \pm c$ means $|A - B| \leqslant c$.

and

$$C_{k,2}(x,r,R)$$
$$:= \sum_{\substack{\gamma \in \Gamma(k) \\ d(\mathbf{o},\gamma \cdot \mathbf{o}) \geqslant R/2}} \sum_{\substack{\beta \in \Gamma(k) \\ d(\mathbf{o},\beta \cdot \mathbf{o}) > r}} p(\gamma, \beta \cdot x) p(\beta, x) \phi(\gamma \beta \cdot x) u \big(R - b(\gamma, \beta \cdot x) - b(\beta, x) \big).$$

Firstly, we control the term $C_{k,1}(x,r,R)$. Assuming $c \geqslant 1$, one may write

$$C_{k,1}(x,r,R)$$
$$\leqslant \|\phi\|_\infty \|u\|_\infty \sum_{n=[r]}^{[R/2]} \sum_{\substack{\gamma \in \Gamma(k) \\ d(\mathbf{o},\gamma \cdot \mathbf{o}) = n \pm c}} \sum_{\substack{\beta \in \Gamma(k) \\ d(\mathbf{o},\beta \cdot \mathbf{o}) = R - n \pm c}} p(\gamma, \beta \cdot x) p(\beta, x)$$
$$\leqslant \|\phi\|_\infty \|u\|_\infty \sum_{n=[r]}^{[R/2]} \sum_{\substack{\beta \in \Gamma(1) \\ d(\mathbf{o},\beta \cdot \mathbf{o}) = R - n \pm c}} p(\beta, x) \left(\sum_{\substack{\gamma \in \Gamma(k) \\ d(\mathbf{o},\gamma \cdot \mathbf{o}) = n \pm c}} p(\gamma, \beta \cdot x) \right).$$

By (4.9), this yields, for some constant $c' > 0$,

$$C_{k,1}(x,r,R) \leqslant c' k^2 \|\phi\|_\infty \|u\|_\infty \sum_{n=[r]}^{[R/2]} \frac{L(R-n)}{(R-n)^\alpha} \frac{L(n)}{n^\alpha}$$
$$\leqslant c' k^2 \|\phi\|_\infty \|u\|_\infty \frac{L(R)}{R^\alpha} \sum_{n=[r]}^{+\infty} \frac{L(n)}{n^\alpha},$$

where the last inequality is based on the facts that $R - n \geqslant R/2 - 1$ and L is slowly varying. The same inequality holds for $C_{k,2}(x,r,R)$, reversing the roles of γ and β in the argument. Hence,

(4.20) $$\lim_{r \to +\infty} \limsup_{R \to +\infty} \frac{R^\alpha}{L(R)} C_k(x,r,R) = 0.$$

The asymptotic formula (4.10) follows, combining (4.16), (4.18) and (4.20). □

4.4. Specification of the constant $C_\Gamma(\mathbf{o},\mathbf{o})$

In this paragraph, we prove the following statement.

PROPOSITION 4.6. — *The constant $C_\Gamma(\mathbf{o}, \mathbf{o})$ appearing in the statement of Theorem 1.2 is given by*

$$C_\Gamma(\mathbf{o}, \mathbf{o}) = \frac{1}{\delta} \sum_{j=1}^{p} c_j \left(\sum_{\substack{\gamma \in \Gamma \\ l_\gamma \neq j}} e^{-\delta \mathcal{B}_{x_j}(\gamma^{-1} \cdot \mathbf{o}, \mathbf{o})} \right)^2 .$$

Proof. — By Formula (4.2) in Subsection 4.1, it holds $C_\Gamma(\mathbf{o}, \mathbf{o}) = \frac{C(x_0)}{\delta_\Gamma}$, where $C(x_0)$ is given by (4.11):

$$C(x_0) = h(x_0) \sum_{k \geqslant 1} \rho^k \sum_{j=1}^{p} \left(\sum_{l=0}^{k-1} P^l C_j(x_0) P^{k-1-l} \left(\frac{1}{h} \right) (x_j) \right)$$

$$= h(x_0) \rho \sum_{j=1}^{p} \left(\sum_{k \geqslant 0} \rho^k P^k \left(\frac{1}{h} \right) (x_j) \right) \left(\sum_{l \geqslant 0} \rho^l P^l C_j(x_0) \right)$$

(the last equality holds since P has spectral radius 1).

On the one hand, for any $k \geqslant 0$,

$$\rho^k P^k \left(\frac{1}{h} \right) (x_j) = \frac{1}{h(x_j)} \sum_{\gamma \in \Gamma(k)} \mathbb{1}_{x_j \notin \tilde{\Lambda}_{l_\gamma}} e^{-\delta b(\gamma, x_j)} = \frac{1}{h(x_j)} \mathcal{L}_\delta^k(\mathbb{1})(x_j).$$

On the other hand, for any $l \geqslant 0$, by the definition of $C_j(\,\cdot\,)$ in (4.8),

$$\rho^l P^l C_j(x_0) = \frac{c_j h(x_j)}{\rho h(x_0)} \sum_{\gamma \in \Gamma(l)} \mathbb{1}_{\gamma \cdot x_0 \notin \tilde{\Lambda}_j} e^{-\delta b(\gamma, x_0)} e^{\delta(\mathcal{B}_{x_j}(\mathbf{o}, \gamma \cdot \mathbf{o}) + d(\mathbf{o}, \gamma \cdot \mathbf{o}))}$$

$$= \frac{c_j h(x_j)}{\rho h(x_0)} \sum_{\gamma \in \Gamma(l)} \mathbb{1}_{\gamma \cdot x_0 \notin \tilde{\Lambda}_j} e^{-\delta b(\gamma^{-1}, x_j)}$$

$$= \frac{c_j h(x_j)}{\rho h(x_0)} \sum_{g \in \Gamma(l)} \mathbb{1}_{x_j \notin \tilde{\Lambda}_{l_g}} e^{-\delta b(g, x_j)} \text{ setting } g = \gamma^{-1}$$

$$= \frac{c_j h(x_j)}{\rho h(x_0)} \mathcal{L}_\delta^l(\mathbb{1})(x_j).$$

The result follows, summing over k and l and using the definition of $\mathcal{L}_\delta$ and the expression of its powers. □

4.5. General orbital couting estimate

Let $\mathbf{x}, \mathbf{y} \in X$ be any two points and let us mention briefly how the proof of Theorem 1.2 can be adapted to get the general statement of Theorem 1.2.

We fix $x_0 \in \partial X \backslash \Lambda_\Gamma$, set $\widetilde{\Lambda} = \Gamma \cdot x_0 \cup \Lambda_\Gamma$ and, instead of b, we consider a new function $\bar{b}$, defined as follows: for all $\gamma \in \Gamma$ and all $\xi \in \widetilde{\Lambda}$,

$$\bar{b}(\gamma, \xi) = \begin{cases} b(\gamma, \xi) = \mathcal{B}_\xi(\gamma^{-1} \cdot \mathbf{x}, \mathbf{x}) & \text{if } \xi \in \Lambda_\Gamma; \\ d(\gamma^{-1} \cdot \mathbf{x}, g \cdot \mathbf{y}) - d(\mathbf{x}, g \cdot \mathbf{y}) & \text{if } \xi = g \cdot x_0 \text{ with } g \in \Gamma^*; \\ d(\gamma^{-1} \cdot \mathbf{x}, \mathbf{y}) & \text{if } \xi = x_0. \end{cases}$$

An immediate computation shows that $\bar{b}$ is a cocycle. As was done in [18, Chapter 8] for the cocycle b, one can show that for all $\gamma \in \Gamma$, the map $\bar{b}(\gamma, \cdot)$ is Lipschitz continuous on $\widetilde{\Lambda}$. This new cocycle is suitable for the estimation of $N_\Gamma(\mathbf{x}, \mathbf{y}; R)$ since for all $\gamma \in \Gamma$,

$$\bar{b}(\gamma, x_0) = d(\mathbf{x}, \gamma \cdot \mathbf{y}).$$

One can then reproduce the study done in Section 3 for the Ruelle operator $\mathcal{L}_{\delta, \bar{b}}$ associated with $\bar{b}$. Under the hypotheses of Theorem 1.2, this operator is quasi-compact on $\mathrm{Lip}(\widetilde{\Lambda})$ and hs a simple dominant eigenvalue $\rho \in (0, 1)$. The rest of the proof can be carried verbatim, replacing b by $\bar{b}$; the constant $C_\Gamma(\mathbf{x}, \mathbf{y})$ is given by

$$(4.21) \qquad C_\Gamma(\mathbf{x}, \mathbf{y}) = \frac{1}{\delta} \sum_{j=1}^{p} c_j \left(\sum_{\substack{\gamma \in \Gamma \\ l_\gamma \neq j}} e^{-\delta \mathcal{B}_{x_j}(\gamma^{-1} \cdot \mathbf{x}, \mathbf{y})} \right)^2.$$

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Marc PEIGNÉ
Institut Denis Poisson, UMR 7013
Université de Tours
Université d'Orléans
CNRS 37200 Tours (France)
peigne@univ-tours.fr

Samuel TAPIE
Laboratoire Jean Leray
2 rue de la Houssinière
BP92208 44322 Nantes Cedex 3 (France)
samuel.tapie@univ-nantes.fr

Pierre VIDOTTO
Laboratoire Jean Leray
2 rue de la Houssinière
BP92208 44322 Nantes Cedex 3 (France)
pierre.vidotto@yahoo.fr