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COUNTING PROBLEMS FOR SPECIAL-ORTHOGONAL ANOSOV REPRESENTATIONS

by León CARVAJALES (*)

ABSTRACT. — For positive integers p and q let $G := \mathrm{PSO}(p, q)$ be the projective indefinite special-orthogonal group of signature (p, q). We study counting problems in the Riemannian symmetric space X_G of G and in the pseudo-Riemannian hyperbolic space $\mathbb{H}^{p,q-1}$. Let $S \subset X_G$ be a totally geodesic copy of $X_{\mathrm{PSO}(p,q-1)}$. We look at the orbit of S under the action of a projective Anosov subgroup of G. For certain choices of such a geodesic copy we show that the number of points in this orbit which are at distance at most t from S is finite and asymptotic to a purely exponential function as t goes to infinity. We provide an interpretation of this result in $\mathbb{H}^{p,q-1}$, as the asymptotics of the amount of space-like geodesic segments of maximum length t in the orbit of a point.

RÉSUMÉ. — Pour des entiers positifs p et q soit $G := \mathrm{PSO}(p, q)$ le groupe projectif spécial-orthogonal indéfini de signature (p, q). Nous étudions des problèmes de comptage dans l'espace symétrique Riemannien X_G de G et dans l'espace hyperbolique pseudo-Riemannien $\mathbb{H}^{p,q-1}$. Soit $S \subset X_G$ une copie totalement géodésique de $X_{\mathrm{PSO}(p,q-1)}$. Nous examinons l'orbite de S sous l'action d'un sous-groupe de G de type projectivement Anosov. Pour certains choix d'une telle copie géodésique, nous montrons que le nombre de points dans cette orbite qui se trouvent à une distance maximale t de S est fini et asymptotiquement purement exponentiel lorsque t tend vers l'infini. Nous fournissons une interprétation de ce résultat dans $\mathbb{H}^{p,q-1}$, comme l'asymptotique de la quantité de segments géodésiques de type espace de longueur maximale t dans l'orbite d'un point.

1. Introduction

Let X be a proper non compact metric space and o be a point in X. Given a discrete group Δ of isometries of X, consider the *orbital counting function*

$$N_\Delta(o, t) := \#\{g \in \Delta : \ d_X(o, g \cdot o) \leqslant t\},$$

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where $t \geqslant 0$. The *orbital counting problem* consists on the study of the asymptotic behaviour of $N_\Delta(o,t)$ as $t \longrightarrow \infty$.

When $X = \mathbb{R}^2$ and $\Delta = \mathbb{Z}^2$ this is known as the *Gauss circle problem* (see Phillips–Rudnick [45]). For a negatively curved Hadamard manifold X and Δ co-compact, this problem was studied by Margulis in his PhD Thesis [34]: the author shows a purely exponential asymptotic for $N_\Delta(o,t)$, the exponent being the topological entropy of the geodesic flow of the quotient space $\Delta\backslash X$. Many authors have generalized the work of Margulis to different contexts, see Roblin [49] and references therein for a fairly complete picture in the negatively curved setting.

When X is a (not necessarily Riemannian) symmetric space associated to a semisimple Lie group G and $\Delta < G$ is a lattice, these kind of problems were studied notably by Eskin–McMullen [16] and Duke–Rudnick–Sarnak [15]. In the non-lattice case but restricted to Riemannian symmetric spaces, one also finds the work of Quint [48] and Sambarino [51]. Quint deals with the case in which Δ is a Schottky group (in the sense of Benoist [3]). Sambarino treats more generally the case of Anosov subgroups (in the full flag variety of G) introduced by Labourie [30].

Before stating precise results we discuss in an informal way the problems adressed by this paper. Fix $d := p + q$ where $p \geqslant 1$ and $q \geqslant 2$, and let $\langle \cdot, \cdot \rangle_{p,q}$ be the bilinear symmetric form on $\mathbb{R}^d$ defined by

$$\langle (x_1, \dots, x_d), (y_1, \dots, y_d) \rangle_{p,q} := \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{d} x_i y_i.$$

We denote by $G := \mathrm{PSO}(p,q)$ the group of projectivized matrices in $\mathrm{SL}(d,\mathbb{R})$ preserving $\langle \cdot, \cdot \rangle_{p,q}$ and by X_G the *Riemannian symmetric space* of G, that is, the space of q-dimensional subspaces of $\mathbb{R}^d$ on which the form $\langle \cdot, \cdot \rangle_{p,q}$ is negative definite. Let d_{X_G} be the distance in X_G induced by a G-invariant Riemannian metric. For closed subsets A and B of X_G, set

$$d_{X_G}(A,B) := \inf\{d_{X_G}(a,b) : \quad a \in A, b \in B\}.$$

On the other hand, the *pseudo-Riemannian hyperbolic space of signature* $(p, q-1)$ is the set

$$\mathbb{H}^{p,q-1} := \left\{ o = [\hat{o}] \in \mathbb{P}(\mathbb{R}^d) : \quad \langle \hat{o}, \hat{o} \rangle_{p,q} < 0 \right\},$$

endowed with a G-invariant pseudo-Riemannian metric coming from restriction of the form $\langle \cdot, \cdot \rangle_{p,q}$ to tangent spaces.

Let Δ be a discrete subgroup of G and fix a point o in $\mathbb{H}^{p,q-1}$. In this paper we study counting problems in X_G and in $\mathbb{H}^{p,q-1}$.

- *Counting in X_G:* Denote by
$$S^o := \{\tau \in X_G : \ o \subset \tau\}.$$
It is a totally geodesic sub-manifold of X_G isometric to the Riemannian symmetric space of $\mathrm{PSO}(p, q-1)$. We define two counting functions in this setting. The first one is
$$N_\Delta(S^o, t) := \#\{g \in \Delta : \ d_{X_G}(S^o, g \cdot S^o) \leqslant t\}.$$
For the second one we pick a point $\tau \in S^o$ and define
$$N_\Delta(S^o, \tau, t) := \#\{g \in \Delta : \ d_{X_G}(\tau, g \cdot S^o) \leqslant t\}.$$
- *Counting in $\mathbb{H}^{p,q-1}$:* We provide a geometric interpretation of the function $N_\Delta(S^o, t)$ in $\mathbb{H}^{p,q-1}$. It is the amount of *space-like* geodesic segments[(1)] of length at most t, that connect o with points of $\Delta \cdot o$. The function $N_\Delta(S^o, \tau, t)$ has a geometric interpretation in this setting as well, which is more involved, and that we postpone until Subsection 1.2.

Remark 1.1. — If $q = 1$ one has $\mathbb{H}^p = X_G = \mathbb{H}^{p,q-1}$ and $o = S^o = \tau$. We have as well the equalities
$$N_\Delta(o, t) = N_\Delta(S^o, t) = N_\Delta(S^o, \tau, t)$$
and our results correspond to the classical and well-known counting theorems already quoted.

In contrast with the counting function $N_\Delta(o, t)$ described at the beginning, the functions $N_\Delta(S^o, t)$ and $N_\Delta(S^o, \tau, t)$ could in general be equal to infinity for large values of t. Part of the results that we present here concern the study of conditions for the choice of o (and τ) that guarantee that the new counting functions are real-valued for every $t \geqslant 0$. Once this is established, one may ask if the *exponential growth rate*
$$\limsup_{t \longrightarrow \infty} \frac{\log N_\Delta(S^o, t)}{t}$$
is positive, finite and independent on the choice of o (and the analogue questions for $N_\Delta(S^o, \tau, t)$). A more subtle problem is to find an asymptotic for the functions $N_\Delta(S^o, t)$ and $N_\Delta(S^o, \tau, t)$ as $t \longrightarrow \infty$. The main goal of this paper is to give an answer to this more subtle problem for an interesting class of subgroups Δ: images of word hyperbolic groups under *projective Anosov representations*.

(1) That is, geodesic segments which are tangent to positive vectors.

1.1. Main results in X_G

In order to formally state our results we need to recall some basic facts concerning (projective) Anosov representations. Anosov representations are (a stable class of) faithful and discrete representations from word hyperbolic groups into semisimple Lie groups that share many geometrical and dynamical features with holonomies of convex co-compact hyperbolic manifolds. They were introduced by Labourie [30] in his study of the Hitchin component and further extended to arbitrary word hyperbolic groups by Guichard–Wienhard [21]. After that, Anosov representations had been object of intensive research in the field of geometric structures on manifolds and their deformation spaces (see for instance the surveys of Kassel [26] or Wienhard [53] and references therein).

Let $P_1^{p,q}$ be the stabilizer of an isotropic line in $\mathbb{R}^d$, i.e. a line on which the form $\langle \cdot , \cdot \rangle_{p,q}$ equals zero. Then $P_1^{p,q}$ is a parabolic subgroup of G and the quotient space $\partial\mathbb{H}^{p,q-1} := G/P_1^{p,q}$, called the *boundary* of $\mathbb{H}^{p,q-1}$, identifies with the set of isotropic lines in $\mathbb{R}^d$.

Fix a non elementary word hyperbolic group Γ and let $\partial_\infty\Gamma$ be its Gromov boundary. Let $\rho : \Gamma \longrightarrow G$ be a $P_1^{p,q}$-Anosov representation. By definition (see Section 5) this means that there exists a continuous equivariant map

$$\xi : \partial_\infty\Gamma \longrightarrow \partial\mathbb{H}^{p,q-1}$$

with the following properties:

- *Transversality:* Let $\cdot^{\perp_{p,q}}$ denote the orthogonal complement with respect to the form $\langle \cdot , \cdot \rangle_{p,q}$. Then the map $\eta := \xi^{\perp_{p,q}}$ satisfies $\xi(x) \oplus \eta(y) = \mathbb{R}^d$ for every $x \neq y$ in $\partial_\infty\Gamma$.
- *Uniform hyperbolicity:* Some flow associated to ρ satisfies a uniform contraction/dilation property (see [21, 30]).

When ρ is $P_1^{p,q}$-Anosov all infinite order elements in $\rho(\Gamma)$ are *proximal*. This means that they act on $\mathbb{P}(\mathbb{R}^d)$ with a unique attractive fixed line and a unique repelling hyperplane. The *limit set* of ρ is, by definition, the closure of the set of attractive fixed lines of proximal elements in $\rho(\Gamma)$. It is denoted by $\Lambda_{\rho(\Gamma)}$ and coincides with the image of ξ.

Define

$$\mathbf{\Omega}_\rho := \{o = [\hat{o}] \in \mathbb{H}^{p,q-1} : \langle \hat{o}, \hat{\xi} \rangle_{p,q} \neq 0 \text{ for all } \xi = [\hat{\xi}] \in \Lambda_{\rho(\Gamma)}\}.$$

In the study of discrete groups of projective transformations, it is standard to consider sets similar to $\mathbf{\Omega}_\rho$ (see for instance Danciger–Guéritaud–Kassel [13, 14] and references therein). Without any further assumption the set $\mathbf{\Omega}_\rho$ could be empty. An important class of Anosov representations for which $\mathbf{\Omega}_\rho$ is non empty is given by $\mathbb{H}^{p,q-1}$*-convex co-compact subgroups* introduced in [14]. However in our results we do not assume that ρ is $\mathbb{H}^{p,q-1}$-convex co-compact, we only need that $\mathbf{\Omega}_\rho \neq \emptyset$ (see Example 6.1).

Proposition 1.2 (Propositions 6.7 and 6.8). — *Let $\rho : \Gamma \longrightarrow G$ be a $P_1^{p,q}$-Anosov representation, a point $o \in \mathbf{\Omega}_\rho$ and $\tau \in S^o$. Then for every $t \geqslant 0$ one has*[(2)]

$$N_{\rho(\Gamma)}(S^o, \tau, t) < \infty \quad \textit{and} \quad N_{\rho(\Gamma)}(S^o, t) < \infty.$$

The main results of this paper in the Riemannian context are Theorems 1.3 and 1.4. The notation $f(t) \sim g(t)$ stands for

$$\lim_{t \longrightarrow \infty} \frac{f(t)}{g(t)} = 1.$$

Theorem 1.3. — *A Let $\rho : \Gamma \longrightarrow G$ be a $P_1^{p,q}$-Anosov representation and $o \in \mathbf{\Omega}_\rho$. There exist positive constants $h = h_\rho$ and $M = M_{\rho,o}$ such that*

$$N_{\rho(\Gamma)}(S^o, t) \sim \frac{e^{ht}}{M}.$$

Theorem 1.4. — *B Let $\rho : \Gamma \longrightarrow G$ be a $P_1^{p,q}$-Anosov representation, a point $o \in \mathbf{\Omega}_\rho$ and $\tau \in S^o$. There exist positive constants $h = h_\rho$ and $M' = M'_{\rho,\tau}$ such that*

$$N_{\rho(\Gamma)}(S^o, \tau, t) \sim \frac{e^{ht}}{M'}.$$

The constant h is the same in both Theorems 1.3 and 1.4 and it is independent on the choice of o in $\mathbf{\Omega}_\rho$ (and τ in S^o). It coincides with the topological entropy of the *geodesic flow* ϕ^ρ of ρ, introduced by Bridgeman–Canary–Labourie–Sambarino [10], and can be computed as

$$h = \limsup_{t \longrightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \ \lambda_1(\rho(\gamma)) \leqslant t\}}{t}.$$

Here $[\gamma]$ denotes the conjugacy class of γ and $\lambda_1(\rho(\gamma))$ denotes the logarithm of the spectral radius of $\rho(\gamma)$. The constants M and M' are related to the total mass of specific measures in the Bowen–Margulis measure class of ϕ^ρ (recall that the Bowen–Margulis measure class is the homothety class of measures maximizing entropy of ϕ^ρ).

Since the work of Margulis [34], in order to obtain a counting result one usually studies the ergodic properties of a well chosen dynamical system. In

(2) Even though finiteness of $N_{\rho(\Gamma)}(S^o, \tau, t)$ follows directly from finiteness of $N_{\rho(\Gamma)}(S^o, t)$, in our proof we first show $N_{\rho(\Gamma)}(S^o, \tau, t) < \infty$ and use it to prove $N_{\rho(\Gamma)}(S^o, t) < \infty$.

order to find a dynamical system adapted to Theorem 1.3 we introduce a decomposition of a specific subset of G, analogue to the Cartan Decomposition, but replacing *the* maximal compact subgroup of G by $\mathrm{PSO}(p, q-1)$ and *the* Cartan subspace by a smaller abelian subalgebra (Subsection 3.4). For Theorem 1.4 we use the more studied *polar decomposition* of G (Subsection 3.3).

Relation with the work of Oh–Shah

Motivated by the study of *Apollonian circle packings* on the Riemann sphere, Oh–Shah [41] studied counting problems similar to ours. Indeed, let $p = 1$ and $q = 3$. Then $\mathbb{H}^{1,2}$ identifies with the space of circles of the Riemann sphere or, equivalently, the space of totally geodesic isometric copies of $\mathbb{H}^2$ inside $\mathbb{H}^3$. In [41, Theorem 1.5] the cited authors prove that for a well-chosen $S^o \cong \mathbb{H}^2 \subset \mathbb{H}^3$ and any point $\tau \in \mathbb{H}^3$ one has

$$\#\{g \in \Delta : \quad d_{\mathbb{H}^3}(\tau, g \cdot S^o) \leqslant t\} \sim M^{-1} e^{ht}.$$

Hence Theorem 1.4 can be interpreted as a higher rank generalization of this result. We note however that, for $p = 1$ and $q = 3$, our results only concern convex co-compact groups, while Oh–Shah's Theorem applies to a wider class of geometrically finite Kleinian subgroups. A slightly different counting theorem in $\mathbb{H}^{1,2}$ was obtained by the cited authors in [39]. Effective versions of Oh–Shah's results (i.e. with an error term) have been obtained by Lee–Oh [32] and Mohammadi–Oh [38]. Our Theorem 1.3 seems to be new even in this setting.

The approach by Oh–Shah is similar to the one of Eskin–McMullen [16]: they study the equidistribution, with respect to certain measures, of the orthogonal translates of S^o under the geodesic flow of $\Delta\backslash\mathbb{H}^3$ (see Oh–Shah [40] for precisions). Here we use different techniques. We follow the approach by Sambarino [50] and construct a dynamical system on a compact space that contains the required geometric information.

1.2. Interpretation in $\mathbb{H}^{p,q-1}$

Another part of our contributions concern geometric interpretations of Theorems 1.3 and 1.4 in $\mathbb{H}^{p,q-1}$. We now state these interpretations.

Geodesics in $\mathbb{H}^{p,q-1}$ are intersections of projectivized 2-dimensional subspaces of $\mathbb{R}^d$ with $\mathbb{H}^{p,q-1}$ and they are classified in three types, depending on the sign of the form $\langle \cdot, \cdot \rangle_{p,q}$ on its tangent vectors (see Subsection 2.2.2).

We are mainly interested in *space-like geodesics*, i.e. geodesics associated to planes on which the form $\langle\cdot,\cdot\rangle_{p,q}$ has signature $(1,1)$. Let $o,o'\in\mathbb{H}^{p,q-1}$ be two points joined by a space-like geodesic and let $\ell_{o,o'}$ be the length of this geodesic segment (see Subsection 2.2.4). We denote by $\mathscr{C}_o^{>}$ the set of points of $\mathbb{H}^{p,q-1}$ that can be joined to o by a space-like geodesic and we set

$$\mathscr{C}_{o,G}^{>}:=\{g\in G:\quad g\cdot o\in\mathscr{C}_o^{>}\}.$$

Proposition 1.5 (Proposition 3.8). — *Let $o\in\mathbb{H}^{p,q-1}$ and $g\in\mathscr{C}_{o,G}^{>}$. Then*

$$\ell_{o,g\cdot o}=d_{X_G}(S^o,g\cdot S^o).$$

In Corollary 6.3 we prove that given a $P_1^{p,q}$-Anosov representation $\rho:\Gamma\longrightarrow G$ and o in $\mathbf{\Omega}_\rho$, then apart from possibly finitely many exceptions γ in Γ one has $\rho(\gamma)\in\mathscr{C}_{o,G}^{>}$. By Proposition 6.8 we have

$$\#\{\gamma\in\Gamma:\quad\rho(\gamma)\in\mathscr{C}_{o,G}^{>}\text{ and }\ell_{o,\rho(\gamma)\cdot o}\leqslant t\}<\infty$$

for every positive t. Moreover, Theorem 1.3 implies that this function is asymptotic to $M^{-1}e^{ht}$ as $t\longrightarrow\infty$.

In order to state the corresponding geometric interpretation of Theorem 1.4 we follow Kassel–Kobayashi [27, p. 151]. Let $o\in\mathbb{H}^{p,q-1}$ and $\tau\in S^o$. Then

$$\mathbb{H}^p_\tau:=(o\oplus\tau^{\perp_{p,q}})\cap\mathbb{H}^{p,q-1}$$

is a space-like totally geodesic copy of $\mathbb{H}^p$ passing through o. Let K^τ be the (maximal compact) subgroup of G stabilizing τ. As we shall see, for every g in G the point $g\cdot o$ lies in the K^τ-orbit of a point o_g in $\mathbb{H}^p_\tau$. The counterpart of Theorem 1.4 in $\mathbb{H}^{p,q-1}$ is provided by the following proposition.

Proposition 1.6 (Proposition 3.5). — *For every g in G one has*

$$\ell_{o,o_g}=d_{X_G}(\tau,g\cdot S^o).$$

Relation with the work of Glorieux–Monclair and Kassel–Kobayashi

Glorieux–Monclair [18] introduced an orbital counting function for $\mathbb{H}^{p,q-1}$-convex co-compact representations that differs from

$$t\mapsto\#\{\gamma\in\Gamma:\quad\rho(\gamma)\in\mathscr{C}_{o,G}^{>}\text{ and }\ell_{o,\rho(\gamma)\cdot o}\leqslant t\}$$

by a constant. Indeed, they define an $\mathbb{H}^{p,q-1}$-*distance*

$$d_{\mathbb{H}^{p,q-1}}(o,o'):=\begin{cases}\ell_{o,o'} & \text{if } o'\in\mathscr{C}_o^{>}\\ 0 & \text{otherwise,}\end{cases}$$

and show that it satisfies a version of the triangle inequality in the convex hull of the limit set of ρ. This is used to prove that the exponential growth rate of the counting function

$$t \mapsto \#\{\gamma \in \Gamma : \quad d_{\mathbb{H}^{p,q-1}}(o, \rho(\gamma) \cdot o) \leqslant t\}$$

is independent on the choice of the basepoint o. The authors interpret this exponential rate as a *pseudo-Riemannian Hausdorff dimension* of the limit set of ρ, with the purpose of finding upper bounds for this number ([18, Theorem 1.2]). A consequence of Theorem 1.3 and Proposition 3.8 (see Remarks 6.9 and 7.15) is that this rate coincides with the topological entropy h of ϕ^ρ.

On the other hand, as we shall see in Section 3 the number ℓ_{o,o_g} is related to the *polar projection* of g and therefore Theorem 1.4 addresses the problems treated by Kassel–Kobayashi in [27, Section 4]. In [27] the authors study the orbital counting function of Theorem 1.4 for *sharp* subgroups of a real reductive symmetric space (see [27, Section 4]). Kassel–Kobayashi obtain some estimates on the growth of this function, but no precise asymptotic is established.

The method of [18] is based on pseudo-Riemannian geometry: they construct analogues of Busemann functions, Gromov products and Patterson–Sullivan densities in $\mathbb{H}^{p,q-1}$ using this viewpoint. Our approach is inspired by [27] and has Lie-theoretic flavor: we study linear algebraic interpretations of the geometric quantities involved in the definition of the counting functions. This allows us to establish finiteness of these functions, to make a link between the different symmetric spaces and to apply Ledrappier's [31] framework to our setting.

1.3. Outline of the proof

There are three major steps in the proof of Theorems 1.3 and 1.4.

First step

As we said, we interpret the geometric quantities involved in Theorems 1.3 and 1.4 as linear algebraic quantities.

Let us be more precise. Fix $o \in \mathbb{H}^{p,q-1}$ and denote by H^o the stabilizer in G of this point. If we consider the symmetry of $\mathbb{R}^d$ given by $J^o := \mathrm{id}_o \oplus (-\mathrm{id}_{o^{\perp_{p,q}}})$, we have that H^o equals the fixed point set of the involution

$$\sigma^o : g \mapsto J^o g J^o$$

of G (see Subsection 2.2.1). This identifies the tangent space at o of $\mathbb{H}^{p,q-1}$ with the subspace of $\mathfrak{so}(p,q)$ defined by $\mathfrak{q}^o := \{\mathrm{d}\sigma^o = -1\}$. In Propositions 3.8 and 3.10 we prove that for every $g \in \mathscr{C}^{>}_{o,G}$ one has

$$d_{X_G}(S^o, g \cdot S^o) = \frac{1}{2}\lambda_1(J^o g J^o g^{-1}). \tag{1.1}$$

The main ingredient in the proof of equality (1.1) is the following version of the classical Cartan Decomposition of G.

Proposition 1.7 (Proposition 3.7). — *Let $o \in \mathbb{H}^{p,q-1}$ and $\mathfrak{b}^+ \subset \mathfrak{q}^o$ be a ray such that $\exp(\mathfrak{b}^+) \cdot o$ is space-like. Given $g \in \mathscr{C}^{>}_{o,G}$ there exists $h, h' \in H^o$ and a unique $X \in \mathfrak{b}^+$ such that*

$$g = h \exp(X) h'.$$

On the other hand, the linear algebraic interpretation of the quantity $d_{X_G}(\tau, g \cdot S^o)$ is the following: the choice of τ induces a norm $\|\cdot\|_\tau$ on $\mathbb{R}^d$ invariant under the action of K^τ. We show in Propositions 3.5 and 3.6 that for every $g \in G$ the following equality holds

$$d_{X_G}(\tau, g \cdot S^o) = \frac{1}{2}\log \|J^o g J^o g^{-1}\|_\tau. \tag{1.2}$$

Once again the proof of this equality relies on a generalization of Cartan Decomposition (see Schlichtkrull [52, Chapter 7]): every $g \in G$ can be written as

$$g = k \exp(X) h$$

for some $k \in K^\tau$, $h \in H^o$ and a unique $X \in \mathfrak{b}^+$.

Second step

In order to simplify the exposition we assume that Γ is torsion free. In this case every $\gamma \neq 1$ in Γ has a unique attractive (resp. repelling) fixed point in $\partial_\infty\Gamma$, denoted by γ_+ (resp. γ_-). Consider $\rho : \Gamma \longrightarrow G$ a $P_1^{p,q}$-Anosov representation. The key feature of choosing o in $\mathbf{\Omega}_\rho$ is that it guarantees some *transversality condition* for the proximal matrices $J^o\rho(\gamma)J^o$ and $\rho(\gamma^{-1})$ and this allows to estimate the quantities (1.1) and (1.2) in terms of the spectral radius of $\rho(\gamma)$.

More precisely, we will see in Proposition 2.6 that

$$\mathbf{\Omega}_\rho = \{o \in \mathbb{H}^{p,q-1} : \quad J^o \cdot \xi(x) \notin \eta(x) \text{ for all } x \in \partial_\infty\Gamma\}. \tag{1.3}$$

Fix $o \in \mathbf{\Omega}_\rho$ and a distance d in $\mathbb{P}(\mathbb{R}^d)$ induced by the choice of an inner product in $\mathbb{R}^d$. By compactness of $\partial_\infty\Gamma$ there exists a positive constant r such that

$$d(J^o \cdot \xi(x), \eta(x)) \geqslant r$$

holds for every $x \in \partial_\infty\Gamma$ (here $d(J^o \cdot \xi(x), \eta(x))$ is the minimal distance between $J^o \cdot \xi(x)$ and the lines included in $\eta(x)$). Further, if γ_+ is uniformly far from γ_-, with respect to some visual distance in $\partial_\infty\Gamma$, then $\xi(\gamma_+)$ (resp. $\xi(\gamma_-)$) is uniformly far from $\eta(\gamma_-)$ (resp. $\eta(\gamma_+)$). In Lemma 6.6 we combine all these facts with Benoist's work [4] to conclude that the product $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$ is proximal. Moreover, we obtain a comparison between the quantity (1.1) (resp. (1.2)) and

$$\lambda_1(\rho(\gamma))$$

with very precise control on the error made in this comparison.

Third step

We apply Sambarino's outline [50] to our particular context(3). To a Hölder cocycle c on $\partial_\infty\Gamma$ the author associates a Hölder reparametrization ψ_t^c of the geodesic flow of Γ. Recall that a *Hölder cocycle* is a map $c : \Gamma \times \partial_\infty\Gamma \longrightarrow \mathbb{R}$ satisfying

$$c(\gamma_0\gamma_1, x) = c(\gamma_0, \gamma_1 \cdot x) + c(\gamma_1, x)$$

for every γ_0, γ_1 in Γ and $x \in \partial_\infty\Gamma$ and such that the map $c(\gamma_0, \cdot\,)$ is Hölder (with the same exponent for every γ_0). The cocycle c' is said to be *cohomologous* to c if there exists a Hölder continuous function $U : \partial_\infty\Gamma \longrightarrow \mathbb{R}$ such that for every γ in Γ and x in $\partial_\infty\Gamma$ one has

$$c(\gamma, x) - c'(\gamma, x) = U(\gamma \cdot x) - U(x).$$

In that case ψ_t^c is conjugate to $\psi_t^{c'}$ (see [50, Section 3]). By considering a Markov coding and applying Parry–Pollicott's Prime Orbit Theorem [42], Sambarino obtains an asymptotic for the number of periodic orbits of ψ_t^c of period less than or equal to t (see [50, Corollary 4.1]). Obviously this is a purely dynamical result, i.e. changing ψ_t^c in its conjugacy class does not affect the asymptotics.

However our problem is more subtle: one must find a particular cocycle, with some geometric meaning, and not just *any* cocycle in the given cohomology class. Indeed, the cocycles that we consider to prove Theorems 1.3 and 1.4 are cohomologous, but only the specific choices in such a cohomology class yield the respective results.

(3) The results in [50] are proved for fundamental groups of closed negatively curved manifolds. However, all the results obtained there remain valid when Γ is an arbitrary word hyperbolic group admitting an Anosov representation. This is explained in detail in Appendix A.

Let us briefly sketch the proof of Theorem 1.3 (Theorem 1.4 is proved in a similar way). Fix $o \in \mathbf{\Omega}_\rho$ and consider

$$c_o : \Gamma \times \partial_\infty\Gamma \longrightarrow \mathbb{R} : \quad c_o(\gamma, x) := \frac{1}{2}\log\left|\frac{\langle \rho(\gamma)\cdot v_x, J^o\rho(\gamma)\cdot v_x\rangle_{p,q}}{\langle v_x, J^o\cdot v_x\rangle_{p,q}}\right|$$

where $v_x \neq 0$ is any vector in $\xi(x)$[(4)]. This is a well-defined function thanks to (1.3) and it is a Hölder cocycle.

Let $\partial^2_\infty\Gamma$ be the set of pairs of distinct points in $\partial_\infty\Gamma$ and consider the action of Γ on $\partial^2_\infty\Gamma \times \mathbb{R}$ given by

$$\gamma\cdot(x,y,s) := (\gamma\cdot x, \gamma\cdot y, s - c_o(\gamma, y)).$$

We denote by $\mathrm{U}_o\Gamma$ the quotient space. The *translation flow* on $\partial^2_\infty\Gamma\times\mathbb{R}$ given by

$$\psi_t(x,y,s) := (x,y,s-t)$$

descends to a flow $\psi_t = \psi^o_t$ on $\mathrm{U}_o\Gamma$. As Sambarino shows in [50, Theorem 3.2 (1)] (see also Lemma A.7) the flow ψ_t is conjugate to a Hölder reparametrization of the geodesic flow of Γ introduced by Gromov [19]. We will show (see Lemma A.7) that periodic orbits of ψ_t are parametrized by conjugacy classes of *primitive* elements in Γ, i.e. elements which cannot be written as a power of another element. If γ is primitive, the corresponding period is given by

$$\ell_{c_o}(\gamma) := \lambda_1(\rho(\gamma)).$$

We show the following property concerning spectral radii in a projective Anosov representation.

Proposition 1.8 (Proposition A.2). — *Let ρ be a projective Anosov representation of Γ. Then the set $\{\lambda_1(\rho(\gamma))\}_{\gamma\in\Gamma}$ spans a non discrete subgroup of $\mathbb{R}$.*

Denote by h the topological entropy of ψ_t. The probability of maximal entropy of ψ_t can be constructed as follows: define the *Gromov product*

$$[\,\cdot\,,\cdot\,]_o : \partial^2_\infty\Gamma \longrightarrow \mathbb{R} : \quad [x,y]_o := -\frac{1}{2}\log\left|\frac{\langle v_x, J^o\cdot v_x\rangle_{p,q}\langle v_y, J^o\cdot v_y\rangle_{p,q}}{\langle v_x, v_y\rangle_{p,q}\langle v_y, v_x\rangle_{p,q}}\right|.$$

This function is well-defined thanks to (1.3) and transversality of ξ and η. One can prove that

$$[\gamma\cdot x, \gamma\cdot y]_o - [x,y]_o = -(c_o(\gamma,x) + c_o(\gamma,y))$$

(4) When $q = 1$ this coincides with the *Busemann cocycle* of $\mathbb{H}^p$, i.e. $c_o(\gamma, x) = \beta_{\xi(x)}(\rho(\gamma^{-1})\cdot o, o)$ where $\beta_\cdot(\,\cdot\,,\cdot\,) : \partial\mathbb{H}^p\times\mathbb{H}^p\times\mathbb{H}^p \longrightarrow \mathbb{R}$ is the Busemann function.

holds for every γ in Γ and $(x,y)\in\partial^2_\infty\Gamma$. Let μ_o be a *Patterson–Sullivan probability* associated to c_o, that is, μ_o is a probability on $\partial_\infty\Gamma$ that satisfies

$$\frac{\mathrm{d}\gamma_*\mu_o}{\mathrm{d}\mu_o}(x)=e^{-hc_o(\gamma^{-1},x)}$$

for every $\gamma\in\Gamma^{(5)}$. For the existence of such a probability see Subsection A.2.2. The measure

$$e^{-h[\,\cdot\,,\cdot\,]_o}\mu_o\otimes\mu_o\otimes\mathrm{d}t$$

on $\partial^2_\infty\Gamma\times\mathbb{R}$ is Γ-invariant. It induces on the quotient $\mathrm{U}_o\Gamma$ the measure of maximal entropy of ψ_t, which is unique up to scaling (see [50, Theorem 3.2 (2)] or Proposition A.12).

Denote by $C^*_c(\partial^2_\infty\Gamma)$ the dual of the space of compactly supported real continuous functions on $\partial^2_\infty\Gamma$ equipped with the weak-star topology. For x in $\partial_\infty\Gamma$ let δ_x be the Dirac mass at x. Inspired by the work of Roblin [49], Sambarino [50, Proposition 4.3] shows

$$Me^{-ht}\sum_{\gamma\in\Gamma,\ell_{c_o}(\gamma)\leqslant t}\delta_{\gamma_-}\otimes\delta_{\gamma_+}\longrightarrow e^{-h[\,\cdot\,,\cdot\,]_o}\mu_o\otimes\mu_o$$

on $C^*_c(\partial^2_\infty\Gamma)$ as $t\longrightarrow\infty$ (for a proof in our context see Proposition A.13). The constant $M=M_{\rho,o}>0$ equals the product of h with the total mass of $e^{-h[\,\cdot\,,\cdot\,]_o}\mu_o\otimes\mu_o\otimes\mathrm{d}t$ on the quotient space $\mathrm{U}_o\Gamma$.

As we show in Lemma 7.9, the number $[\gamma_-,\gamma_+]_o$ is the precise error term in the comparison between $\ell_{c_o}(\gamma)$ and

$$\frac{1}{2}\lambda_1(J^o\rho(\gamma)J^o\rho(\gamma^{-1}))=d_{X_G}(S^o,\rho(\gamma)\cdot S^o)$$

provided by Benoist's Theorem 4.6. This is the geometric step: we replace the period $\ell_{c_o}(\gamma)$ by the number $d_{X_G}(S^o,\rho(\gamma)\cdot S^o)$ in the previous sum, using the Gromov product.

Proposition 1.9 (Proposition 7.11). — *Let Γ be a torsion free word hyperbolic group, $\rho:\Gamma\longrightarrow G$ be a $P^{p,q}_1$-Anosov representation and $o\in\mathbf{\Omega}_\rho$. Then*

$$Me^{-ht}\sum_{\gamma\in\Gamma,d_{X_G}(S^o,\rho(\gamma)\cdot S^o)\leqslant t}\delta_{\gamma_-}\otimes\delta_{\gamma_+}\longrightarrow\mu_o\otimes\mu_o$$

on $C^*(\partial_\infty\Gamma\times\partial_\infty\Gamma)$ as $t\longrightarrow\infty$.

---

(5) Recall that if $f:X\longrightarrow Y$ is a map and m is a measure on X then $f_*(m)$ denotes the measure on Y defined by $A\mapsto m(f^{-1}(A))$.

The proof of Proposition 7.11 follows line by line the proof of [50, Theorem 6.5], which is again inspired by Roblin's work [49].

It turns out that the previous proposition can be used to deduce Theorem 1.3 in the general case, that is, if we admit torsion elements in Γ.

Proposition 1.10 (Proposition 7.13). — *Let $\rho : \Gamma \longrightarrow G$ be a $P_1^{p,q}$-Anosov representation and $o \in \mathbf{\Omega}_\rho$. Then*

$$Me^{-ht} \sum_{\gamma\in\Gamma, d_{X_G}(S^o,\rho(\gamma)\cdot S^o)\leqslant t} \delta_{\rho(\gamma^{-1})\cdot o^{\perp_{p,q}}} \otimes \delta_{\rho(\gamma)\cdot o} \longrightarrow \eta_*(\mu_o) \otimes \xi_*(\mu_o)$$

on $C^*(\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d))$ as $t \longrightarrow \infty$.

1.4. Organization of the paper

In Section 2 we recall basic facts on the symmetric spaces X_G and $\mathbb{H}^{p,q-1}$. Of particular importance is Subsection 2.2.6, which is devoted to the study of end points of space-like geodesics passing through our preferred point $o \in \mathbb{H}^{p,q-1}$. We give several characterizations of this set that will allow us to understand $\mathbf{\Omega}_\rho$ in different ways, all of them used indistinctly in Sections 6, 7 and 8. In Section 3 we study the geometric quantities involved in Theorems 1.3 and 1.4. Equalities (1.1) and (1.2) are proven respectively in Subsections 3.4 and 3.3. In Section 4 we recall Benoist's results on products of proximal matrices and Section 5 is devoted to reminders on Anosov representations. In Section 6 we define the set $\mathbf{\Omega}_\rho$ and study the action of Γ on this set. We show in particular that the orbital counting functions involved in Theorems 1.3 and 1.4 are well-defined (Proposition 6.8 and Proposition 6.7). We also obtain some estimates for the spectral radius and operator norm of elements $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$ which are of major importance (cf. Lemma 6.6). In Section 7 (resp. Section 8) we prove Theorem 1.3 (resp. Theorem 1.4). Finally, in Appendix A we explain how to adapt the results of [50] to the context of arbitrary word hyperbolic groups admitting an Anosov representation.

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2. Two symmetric spaces associated to $\mathrm{PSO}(p,q)$

Fix two integers $p, q \geqslant 1$ and let $d := p+q$. We assume $d > 2$. Denote by $\mathbb{R}^{p,q}$ the vector space $\mathbb{R}^d$ endowed with the quadratic form

$$\langle (x_1,\dots,x_d),(y_1,\dots,y_d)\rangle_{p,q} := \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{d} x_i y_i.$$

From now on we denote by $G := \mathrm{PSO}(p,q)$ the subgroup of $\mathrm{PSL}(d,\mathbb{R})$ consisting on elements whose lifts to $\mathrm{SL}(d,\mathbb{R})$ preserve the form $\langle \cdot,\cdot\rangle_{p,q}$.

For a subspace π of $\mathbb{R}^d$ we denote by $\pi^{\perp_{p,q}}$ its orthogonal complement with respect to $\langle \cdot,\cdot\rangle_{p,q}$, i.e.

$$\pi^{\perp_{p,q}} := \{x \in \mathbb{R}^d : \ \langle x,y\rangle_{p,q} = 0 \text{ for all } y \in \pi\}.$$

Let $\mathfrak{g} := \mathfrak{so}(p,q)$ be the Lie algebra of G. If $\cdot^t$ denotes the *usual* transpose operator one has that $\mathfrak{g}$ equals the set of matrices of the form

$$\begin{pmatrix} X_1 & X_2 \\ X_2^t & X_3 \end{pmatrix}$$

where X_1 is of size $p\times p$, X_3 is of size $q \times q$ and both are skew-symmetric with respect to $\cdot^t$. The *Killing form* of G is the symmetric bilinear form κ on $\mathfrak{g}$ defined by

$$\kappa(X,Y) := \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y),$$

where $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is the adjoint representation. It can be seen that the following equality holds:

$$\kappa(X,Y) = (d-2)\mathrm{tr}(XY)$$

(see Helgason [22, p. 180 & p. 189]).

2.1. The Riemannian symmetric space X_G

A *Cartan involution* of G is an involutive automorphism $\tau : G \longrightarrow G$ such that the bilinear form

$$(X,Y) \mapsto -\kappa(X, \mathrm{d}\tau(Y))$$

is positive definite. The fixed point set K^τ of such an involution is a maximal compact subgroup of G (see Knapp [29, Theorem 6.31]). The *Riemannian symmetric space* of G is the set consisting on Cartan involutions of G. It is denoted by X_G and it is equipped with a natural action of G which is transitive (cf. [29, Corollary 6.19]). The stabilizer of τ is K^τ, thus

$$G/K^\tau \cong X_G.$$

Remark 2.1. — The space X_G can be identified with the space of q-dimensional subspaces of $\mathbb{R}^d$ on which the form $\langle \cdot , \cdot \rangle_{p,q}$ is negative definite. Explicitly, to a q-dimensional negative definite subspace π one associates the Cartan involution of G determined by the inner product of $\mathbb{R}^d$ which equals $-\langle \cdot , \cdot \rangle_{p,q}$ (resp. $\langle \cdot , \cdot \rangle_{p,q}$) on π (resp. $\pi^{\perp_{p,q}}$) and for which π and $\pi^{\perp_{p,q}}$ are orthogonal.

The choice of a point τ in X_G determines a *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{p}^\tau \oplus \mathfrak{k}^\tau$$

where $\mathfrak{p}^\tau := \{\mathrm{d}\tau = -1\}$ and $\mathfrak{k}^\tau := \{\mathrm{d}\tau = 1\}$. The group K^τ is tangent to $\mathfrak{k}^\tau$ and one has a G-equivariant identification

$$\mathfrak{p}^\tau \cong T_\tau X_G \tag{2.1}$$

given by $X \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \exp(tX) \cdot \tau$ (see [22, Theorem 3.3 of Ch. IV]).

Example 2.2. — Consider the involution of G defined by $\tau(g) := (g^{-1})^t$. One sees that $\tau \in X_G$ and $\mathfrak{p}^\tau$ (resp. $\mathfrak{k}^\tau$) is the set of symmetric matrices (resp. skew-symmetric matrices) in $\mathfrak{so}(p, q)$. Moreover K^τ is the subgroup $\mathrm{PS}(\mathrm{O}(p) \times \mathrm{O}(q))$.

The Killing form κ is positive definite (resp. negative definite) on $\mathfrak{p}^\tau$ (resp. $\mathfrak{k}^\tau$). Thanks to (2.1) any positive multiple of κ induces a G-invariant Riemannian metric on X_G. It is well-known (see [22, Theorem 4.2 of Chapter IV]) that X_G equipped with any of these metrics is a symmetric space which is non-positively curved.

We already mentioned that in this paper we study counting problems not only in X_G but also in $\mathbb{H}^{p,q-1}$. In the next section we construct $\mathbb{H}^{p,q-1}$, whose metric is induced by the form $\langle \cdot , \cdot \rangle_{p,q}$. However, we will see that the Killing form induces as well a G-invariant metric on $\mathbb{H}^{p,q-1}$. These two metrics differ by the scaling factor $(2(d-2))^{-1}$ (see Remark 2.3 for further precisions). Since we want a simultaneous treatment of the geometry of the spaces X_G and $\mathbb{H}^{p,q-1}$, we fix the following normalization for the metric on X_G:

$$d_{X_G}(\tau, \exp(X)\cdot\tau) := \left(\frac{1}{2(d-2)}\kappa(X,X)\right)^{\frac{1}{2}} \tag{2.2}$$

for all $\tau \in X_G$ and all $X \in \mathfrak{p}^\tau$.

2.2. The pseudo-Riemannian hyperbolic space $\mathbb{H}^{p,q-1}$

Let

$$\hat{\mathbb{H}}^{p,q-1} := \{\hat{o} \in \mathbb{R}^{p,q} : \ \langle \hat{o}, \hat{o}\rangle_{p,q} = -1\}$$

endowed with the restriction of the form $\langle \cdot, \cdot \rangle_{p,q}$ to tangent spaces. This metric induces on

$$\mathbb{H}^{p,q-1} := \{o = [\hat{o}] \in \mathbb{P}(\mathbb{R}^{p,q}) : \ \langle \hat{o}, \hat{o}\rangle_{p,q} < 0\}$$

a pseudo-Riemannian structure invariant under the projective action of G. This space is called the *pseudo-Riemannian hyperbolic space of signature* $(p, q-1)$. The *boundary* of $\mathbb{H}^{p,q-1}$ is the space of *isotropic lines* defined by

$$\partial\mathbb{H}^{p,q-1} := \{\xi = [\hat{\xi}] \in \mathbb{P}(\mathbb{R}^{p,q}) : \ \langle \hat{\xi}, \hat{\xi}\rangle_{p,q} = 0\}.$$

It is also equipped with the natural (transitive) action of G. If we denote by $P_1^{p,q}$ the (parabolic) subgroup of G stabilizing an isotropic line, then

$$\partial\mathbb{H}^{p,q-1} \cong G/P_1^{p,q}.$$

2.2.1. Structure of symmetric space

The action of G on $\mathbb{H}^{p,q-1}$ is transitive, hence $\mathbb{H}^{p,q-1} \cong G/H^o$ where H^o is the stabilizer in G of the point $o \in \mathbb{H}^{p,q-1}$. For instance, when $o = [0, \ldots, 0, 1] \in \mathbb{H}^{p,q-1}$ one has

$$H^o = \left\{ \begin{bmatrix} \hat{g} & 0 \\ 0 & 1 \end{bmatrix} \in G : \ \hat{g} \in \mathrm{O}(p, q-1) \right\}.$$

Fix any $o \in \mathbb{H}^{p,q-1}$. Since o and $o^{\perp_{p,q}}$ are transverse we can consider the matrix

$$J^o := \mathrm{id}_o \oplus (-\mathrm{id}_{o^{\perp_{p,q}}}).$$

It follows that $H^o = \mathrm{Fix}(\sigma^o)$ where σ^o is the involution of G defined by

$$\sigma^o(g) := J^o g J^o. \tag{2.3}$$

Thus $\mathbb{H}^{p,q-1} \cong G/H^o$ is a symmetric space of G.

Remark 2.3. — Let $o \in \mathbb{H}^{p,q-1}$ and $\mathfrak{q}^o := \{\mathrm{d}\sigma^o = -1\}$. There exists a G-equivariant identification

$$\mathfrak{q}^o \cong T_o\mathbb{H}^{p,q-1}$$

given by $X \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \exp(tX) \cdot o$. We denote by $\langle \cdot, \cdot \rangle$ the pull-back of the $(p, q-1)$-form on $T_o\mathbb{H}^{p,q-1}$ under this map and, for $X \in \mathfrak{q}^o$, we set $|X| := \langle X, X \rangle$[6].

Recall that κ is the Killing form of $\mathfrak{so}(p, q)$. From explicit computations (that we omit) one can conclude that the equality

$$|X| = \frac{1}{2(d-2)}\kappa(X, X) \tag{2.4}$$

holds for every $X \in \mathfrak{q}^o$. This justifies the choice of normalization made in Subsection 2.1.

Remark 2.4. — Let $o \in \mathbb{H}^{p,q-1}$. Then the action of the connected component of H^o containing the identity is conjugate to the action of $\mathrm{SO}(p, q-1)$ on $\mathbb{R}^{p,q-1}$.

2.2.2. Geodesics of $\mathbb{H}^{p,q-1}$

Geodesics of $\mathbb{H}^{p,q-1}$ are the intersections of straight lines of $\mathbb{P}(\mathbb{R}^{p,q})$ with $\mathbb{H}^{p,q-1}$. They are divided in three types:

- *Space-like geodesics:* associated to 2-dimensional subspaces of $\mathbb{R}^d$ on which $\langle \cdot, \cdot \rangle_{p,q}$ has signature $(1, 1)$. They have positive speed and meet the boundary $\partial\mathbb{H}^{p,q-1}$ in two distinct points.
- *Time-like geodesics:* associated to 2-dimensional subspaces of $\mathbb{R}^d$ on which $\langle \cdot, \cdot \rangle_{p,q}$ has signature $(0, 2)$. They have negative speed and do not meet the boundary (they are closed).
- *Light-like geodesics:* associated to 2-dimensional subspaces of $\mathbb{R}^d$ on which $\langle \cdot, \cdot \rangle_{p,q}$ has signature $(0, 1)$, that is, is degenerate but has a negative eigenvalue. They have zero speed and meet the boundary in a single point.

For a point $o \in \mathbb{H}^{p,q-1}$ we denote by $\mathscr{C}_o^0$ (resp. $\mathscr{C}_o^>$) the set of points of $\mathbb{H}^{p,q-1}$ that can be joined with o by a light-like (resp. space-like) geodesic. Its closure in $\mathbb{P}(\mathbb{R}^{p,q})$ is denoted by $\overline{\mathscr{C}_o^0}$ (resp. $\overline{\mathscr{C}_o^>}$).

(6) This number can be positive, negative or zero for $X \neq 0$ in $\mathfrak{q}^o$.

2.2.3. Light-cones

The following lemma is proved by Glorieux–Monclair in [18, Lemma 2.2].

Lemma 2.5. — *Let* $o \in \mathbb{H}^{p,q-1}$*. Then* $\overline{\mathscr{C}_o^0} \cap \partial\mathbb{H}^{p,q-1} = o^{\perp_{p,q}} \cap \partial\mathbb{H}^{p,q-1}$.

2.2.4. Lenghts of space-like geodesics

For a point o' in $\mathscr{C}_o^>$ we denote by $\ell_{o,o'}$ the length of the geodesic segment connecting o with o'. For instance the geodesic

$$s \mapsto [\sinh(s), 0 \ldots, 0, \cosh(s)] \in \mathbb{H}^{p,q-1} \tag{2.5}$$

is parametrized by arc-length.

2.2.5. Space-like copies of $\mathbb{H}^p$

Let π be a $(p+1)$-dimensional subspace of $\mathbb{R}^d$ of signature $(p,1)$. Then $\mathbb{P}(\pi) \cap \mathbb{H}^{p,q-1}$ identifies with

$$\{o = [\hat{o}] \in \mathbb{P}(\mathbb{R}^{p,1}) \;\; \langle \hat{o}, \hat{o} \rangle_{p,1} < 0\}.$$

It follows that $\mathbb{P}(\pi) \cap \mathbb{H}^{p,q-1}$ is a totally geodesic isometric copy of $\mathbb{H}^p$ inside $\mathbb{H}^{p,q-1}$. Moreover this sub-manifold is space-like, in the sense that any of its tangent vectors has positive norm.

2.2.6. End points of space-like geodesics

Let o be a point in $\mathbb{H}^{p,q-1}$. Note that J^o preserves the form $\langle \cdot, \cdot \rangle_{p,q}$ and thus acts on $\partial\mathbb{H}^{p,q-1}$. Set

$$\mathscr{O}^o := \{\xi \in \partial\mathbb{H}^{p,q-1} : \;\; J^o \cdot \xi \neq \xi\}.$$

Proposition 2.6. — *Let* $o \in \mathbb{H}^{p,q-1}$*. Then the following equalities hold:*

$$\begin{aligned} \mathscr{O}^o &= \{\xi \in \partial\mathbb{H}^{p,q-1} : \;\; J^o \cdot \xi \notin \xi^{\perp_{p,q}}\} \\ &= \partial\mathbb{H}^{p,q-1} \setminus o^{\perp_{p,q}} \\ &= \partial\mathbb{H}^{p,q-1} \setminus \overline{\mathscr{C}_o^0}. \end{aligned}$$

We conclude that, unless $q = 1$, the set $\mathscr{O}^o$ is not the whole boundary of $\mathbb{H}^{p,q-1}$.

Proof of Proposition 2.6. — The equality $\partial\mathbb{H}^{p,q-1} \setminus o^{\perp_{p,q}} = \partial\mathbb{H}^{p,q-1} \setminus \overline{\mathscr{C}_o^0}$ is a consequence of Lemma 2.5. The other equalities follow from the definitions. □

3. Generalized Cartan decompositions

The goal of this section is to define two generalized Cartan projections and to provide a link between them and Theorems 1.3 and 1.4. The first one (Subsection 3.3) is called the *polar projection* of G and it is well-known. The second one (Subsection 3.4) is new and can only be defined for elements in G that satisfy some special property with respect to the choice of the basepoint o.

3.1. Notations

Through this section we fix a point $o \in \mathbb{H}^{p,q-1}$ and let $H^o = \mathrm{Fix}(\sigma^o)$ be its stabilizer in G (cf. Subsection 2.2.1). Let $\mathfrak{h}^o$ be the Lie algebra of fixed points of $\mathrm{d}\sigma^o$ and $\mathfrak{q}^o := \{\mathrm{d}\sigma^o = -1\}$. One has the following decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \mathfrak{h}^o \oplus \mathfrak{q}^o.$$

Moreover, this decomposition is orthogonal with respect to the Killing form of $\mathfrak{g}$.

Let τ be a Cartan involution commuting with σ^o: such involutions always exist and two of them differ by conjugation by an element in H^o (see Matsuki [36, Lemma 4]). Let $K^\tau := \mathrm{Fix}(\tau)$, which is a maximal compact subgroup of G. Let $\mathfrak{p}^\tau$ and $\mathfrak{k}^\tau$ be the subspaces defined in Subsection 2.1. As σ^o and τ commute, the following holds:

$$\mathfrak{g} = (\mathfrak{p}^\tau \cap \mathfrak{q}^o) \oplus (\mathfrak{p}^\tau \cap \mathfrak{h}^o) \oplus (\mathfrak{k}^\tau \cap \mathfrak{q}^o) \oplus (\mathfrak{k}^\tau \cap \mathfrak{h}^o).$$

Let $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ be a (necessarily abelian) maximal subalgebra: two of them differ by conjugation by an element in $K^\tau \cap H^o$. We will consider closed Weyl chambers in $\mathfrak{b}$ corresponding to positive systems of restricted roots of $\mathfrak{b}$ in $\mathfrak{g}^{\sigma^o\tau} := (\mathfrak{p}^\tau \cap \mathfrak{q}^o) \oplus (\mathfrak{k}^\tau \cap \mathfrak{h}^o)$. These closed Weyl chambers will be denoted by $\mathfrak{b}^+$.

Example 3.1. — Let $o = [0, \dots, 0, 1]$. Then H^o is the upper left corner embedding of $\mathrm{O}(p, q-1)$ in G and the involution σ^o is obtained by conjugation by $J^o = \mathrm{diag}(-1, \dots, -1, 1)$. One sees that $\mathfrak{h}^o$ equals the upper left corner embedding of $\mathfrak{so}(p, q-1)$ in $\mathfrak{so}(p, q)$ and that

$$\mathfrak{q}^o = \left\{ \begin{pmatrix} 0 & 0 & Y_1 \\ 0 & 0 & Y_2 \\ Y_1^t & -Y_2^t & 0 \end{pmatrix} : \; Y_1 \in \mathrm{M}(p \times 1, \mathbb{R}), \;\; Y_2 \in \mathrm{M}((q-1) \times 1, \mathbb{R}) \right\}.$$

Let τ be the Cartan involution of Example 2.2. One observes that τ commutes with σ^o and

$$\mathfrak{p}^\tau \cap \mathfrak{q}^o = \{X \in \mathfrak{q}^o : \quad Y_2 = 0\} \quad \mathfrak{k}^\tau \cap \mathfrak{q}^o = \{X \in \mathfrak{q}^o : \quad Y_1 = 0\}\,.$$

Pick $\mathfrak{b}$ to be the subset of $\mathfrak{p}^\tau \cap \mathfrak{q}^o$ of matrices with Y_1 of the form

$$\begin{pmatrix} s \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for some $s \in \mathbb{R}$: this is a maximal subalgebra of $\mathfrak{p}^\tau \cap \mathfrak{q}^o$. A closed Weyl chamber $\mathfrak{b}^+$ is defined by the inequality $s \geqslant 0$.

The following remark will be used repeatedly in the sequel.

Remark 3.2. — Even though G does not act on $\mathbb{R}^d$, it makes sense to ask if an element g of G preserves a norm on $\mathbb{R}^d$ (this notion does not depend on the choice of a lift of g to $\mathrm{SL}(d,\mathbb{R})$). Given a Cartan involution τ commuting with σ^o, let $\|\cdot\|_\tau$ be a norm on $\mathbb{R}^d$ preserved by K^τ. We claim that this norm is preserved by J^o. Indeed, this is obvious for the choices of Example 3.1 and follows in general by conjugating by an element g in G that takes $[0,\dots,0,1]$ to the point o.

3.2. The sub-manifold S^o

Define

$$S^o := \{\tau \in X_G : \quad \tau\sigma^o = \sigma^o\tau\}.$$

Remark 3.3. — Recall from Remark 2.1 that X_G can be identified with the space of q-dimensional negative definite subspaces of $\mathbb{R}^d$. Under this identification S^o corresponds to the set of subspaces that contain the line o. By considering the $\langle\cdot,\cdot\rangle_{p,q}$-orthogonal complement we see that S^o parametrizes the space of totally geodesic space-like copies of $\mathbb{H}^p$ inside $\mathbb{H}^{p,q-1}$ passing through o (cf. Subsection 2.2.5).

Using the fact that two elements of S^o differ by conjugation by an element in H^o one observes that for any $\tau \in S^o$ the following holds

$$S^o = H^o \cdot \tau.$$

Further, the group H^o has several connected components but one can see that the connected component containing the identity acts transitively on S^o. Hence S^o is connected and one can show that

$$S^o = \exp(\mathfrak{p}^\tau \cap \mathfrak{h}^o) \cdot \tau.$$

It follows that S^o is a totally geodesic sub-manifold of X_G and $T_\tau S^o \cong \mathfrak{p}^\tau \cap \mathfrak{h}^o$ (see [22, Theorem 7.2 of Ch. IV]).

3.3. $K\exp(\mathfrak{b}^+)H$-decomposition

For the rest of this section we fix a Cartan involution $\tau \in S^o$, a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ and a closed Weyl chamber $\mathfrak{b}^+ \subset \mathfrak{b}$. By Schlichtkrull [52, Proposition 7.1.3] the following decomposition of G holds:

$$G = K^\tau \exp(\mathfrak{b}^+) H^o \tag{3.1}$$

where the $\exp(\mathfrak{b}^+)$-component is uniquely determined and one can define

$$b^\tau : G \longrightarrow \mathfrak{b}^+ \tag{3.2}$$

by taking the log of this component. This is a continuous map called the *polar projection* of G associated to the choice of τ and $\mathfrak{b}^+$. It generalizes the usual Cartan projection of G.

Remark 3.4. — Note that b^τ is not proper (unless $q = 1$). However it descends to a map $\mathbb{H}^{p,q-1} \cong G/H^o \longrightarrow \mathfrak{b}^+$ which, by definition, is proper.

We now discuss geometric interpretations of the polar projection b^τ. The geometric interpretation in $\mathbb{H}^{p,q-1}$ follows Kassel–Kobayashi [27, p. 151], while the geometric interpretation in X_G is inspired by the work of Oh–Shah [41] for the case $p = 1$ and $q = 3$.

Let us begin with the interpretation in the pseudo-Riemannian setting. By Remark 3.3, the choice of $\tau \in S^o$ determines a totally geodesic space-like copy of the p-dimensional hyperbolic space, inside $\mathbb{H}^{p,q-1}$ and passing through o. We denote this copy by $\mathbb{H}^p_\tau$. From explicit computations one can show that

$$\mathbb{H}^p_\tau = \exp(\mathfrak{p}^\tau \cap \mathfrak{q}^o) \cdot o.$$

In particular $\mathbb{H}^p_\tau$ contains the geodesic ray $\exp(\mathfrak{b}^+) \cdot o$ starting from o. Equality (3.1) tell us that for every g in G the point $g \cdot o$ lies in the K^τ-orbit of $o_g := \exp(b^\tau(g)) \cdot o$ (see Figure 3.1). The geometric interpretation

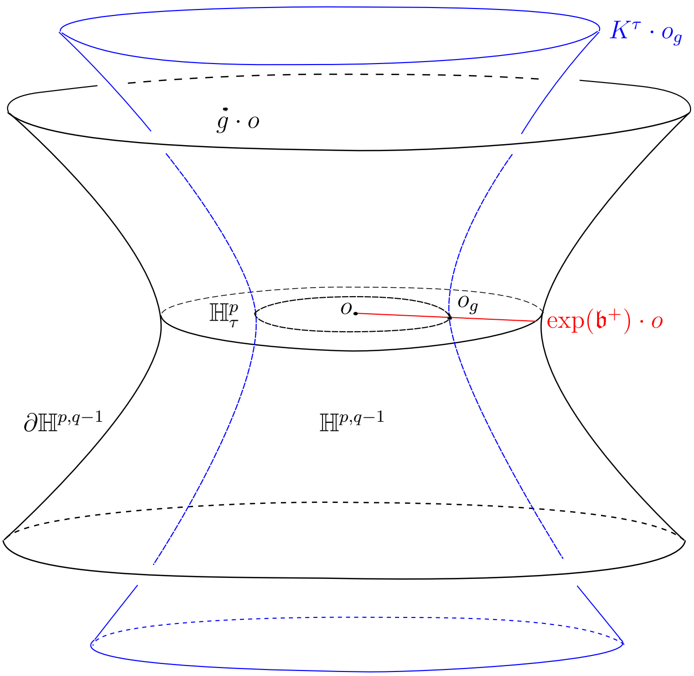

Figure 3.1. Geometric interpretation of polar projection in $\mathbb{H}^{p,q-1}$.

of the polar projection is now clear: the number $|b^\tau(g)|^{\frac{1}{2}}$ equals the length of the geodesic segment connecting o with o_g[(7)].

We now turn our attention to the Riemannian symmetric space X_G.

Proposition 3.5. — *For every g in G one has*

$$|b^\tau(g)|^{\frac{1}{2}} = d_{X_G}(g^{-1} \cdot \tau, S^o).$$

Proof. — The function $g \mapsto d_{X_G}(g^{-1} \cdot \tau, S^o)$ is K^τ-invariant on the left and H^o-invariant on the right, hence it suffices to check that the equality of the statement holds when $g = \exp(X)$ for some $X \in \mathfrak{b}^+$.

Since X_G is non-positively curved, there exists a unique geodesic through $\exp(-X) \cdot \tau$ which is orthogonal to $S^o = \exp(\mathfrak{p}^\tau \cap \mathfrak{h}^o) \cdot \tau$. This geodesic is $\exp(\mathfrak{b}) \cdot \tau$ and intersects S^o in τ, hence

$$d_{X_G}(\exp(-X) \cdot \tau, S^o) = d_{X_G}(\exp(-X) \cdot \tau, \tau).$$

Thanks to Remark 2.3 and (2.2) the proof is complete. □

(7) Recall that $|\cdot|$ is the form on $\mathfrak{q}^o$ defined in Remark 2.3.

We finish this subsection with a linear algebraic interpretation of the polar projection. Let $\|\cdot\|_\tau$ be a norm on $\mathbb{R}^d$ invariant under the action of K^τ.

Proposition 3.6. — *For every g in G one has*

$$|b^\tau(g)|^{\frac{1}{2}} = \frac{1}{2}\log \|J^o g J^o g^{-1}\|_\tau.$$

Proof. — We prove the proposition for the particular choices of Example 3.1, the general case follows from this one by conjugating by appropriate elements of G.

By Remark 3.2 the matrix J^o preserves $\|\cdot\|_\tau$ thus

$$\frac{1}{2}\log \|J^o g J^o g^{-1}\|_\tau = \frac{1}{2}\log \|g J^o g^{-1}\|_\tau.$$

The map $g \mapsto \frac{1}{2}\log\|gJ^og^{-1}\|_\tau$ is K^τ-invariant on the left and H^o-invariant on the right, hence it remains to check that the equality of the statement holds on $\exp(\mathfrak{b}^+)$. Let $X \in \mathfrak{b}^+$, that is,

$$X = \begin{pmatrix} & & & s \\ & & 0 & \\ & \iddots & & \\ 0 & & & \\ s & & & \end{pmatrix}$$

for some $s \geqslant 0$. Since $X \in \mathfrak{q}^o$, one has $J^o \exp(-X) = \exp(X) J^o$ and thus

$$|X|^{\frac{1}{2}} = s = \frac{1}{2}\log \|\exp(X)J^o\exp(-X)\|_\tau. \qquad \square$$

3.4. $H\exp(\mathfrak{b}^+)H$-decomposition

Recall from Subsection 2.2.2 the definition of the set $\mathscr{C}_o^>$ and define

$$\mathscr{C}_{o,G}^> := \{g \in G : \quad g\cdot o \in \mathscr{C}_o^>\}.$$

Proposition 3.7. — *For every g in $\mathscr{C}_{o,G}^>$ one can write*

$$g = h\exp(X)h'$$

for some $h, h' \in H^o$ and a unique $X \in \mathfrak{b}^+$.

It is clear that this decomposition of g can only hold when $g \in \mathscr{C}_{o,G}^>$.

Proof of Proposition 3.7. — Take h in H^o such that $h^{-1}g\cdot o \in \exp(\mathfrak{b}^+)\cdot o$. There exists then $X \in \mathfrak{b}^+$ and $h' \in H^o$ such that $h^{-1}g = \exp(X)h'$. Note that X is unique since it is determined by the length of the geodesic segment connecting o with $g \cdot o$. □

We define the map

$$b^o : \mathscr{C}^{>}_{o,G} \longrightarrow \mathfrak{b}^+ : \quad g \mapsto b^o(g) \tag{3.3}$$

where $g = h\exp(b^o(g))h'$ for some $h, h' \in H^o$. Note that b^o descends to the quotient $\mathscr{C}^{>}_{o}$ but this map is not proper (compare with Remark 3.4).

Proposition 3.8. — *For every g in $\mathscr{C}^{>}_{o,G}$ one has*

$$\ell_{o,g\cdot o} = |b^o(g)|^{\frac{1}{2}} = d_{X_G}(S^o, g\cdot S^o).$$

Proof. — The first equality was already discussed in the proof of Proposition 3.7. For the second one write $g = h\exp(b^o(g))h'$. Since $S^o = H^o\cdot\tau$ we have

$$d_{X_G}(S^o, h\exp(b^o(g))h'\cdot S^o) = d_{X_G}(H^o\cdot\tau, \exp(b^o(g))H^o\cdot\tau).$$

Set $X := b^o(g)$. If $X = 0$ there is nothing to prove, so assume $X \neq 0$. In that case $H^o\cdot\tau$ is disjoint from $\exp(X)H^o\cdot\tau$: since the action of $\mathfrak{b}$ on the geodesic $\exp(\mathfrak{b})\cdot\tau$ is free, this follows from the fact that X_G is non-positively curved and the fact that $\exp(\mathfrak{b})\cdot\tau$ intersects orthogonally $H^o\cdot\tau$ (resp. $\exp(X)H^o\cdot\tau$) in τ (resp. $\exp(X)\cdot\tau$).

To finish the proof we need the following lemma.

Lemma 3.9. — *Take $\tau' \in H^o\cdot\tau$ and $\tau'' \in \exp(X)H^o\cdot\tau$. Then the following holds:*

$$d_{X_G}(\tau',\tau'') \geqslant d_{X_G}(\tau, \exp(X)\cdot\tau).$$

Proof of Lemma 3.9. — Let $\beta_1 \subset H^o\cdot\tau$ (resp. $\beta_2 \subset \exp(X)H^o\cdot\tau$) be the unit-speed geodesic connecting $\beta_1(0) = \tau$ (resp. $\beta_2(0) = \exp(X)\cdot\tau$) with τ' (resp. τ''). Then β_1 and β_2 are disjoint and from the fact that X_G is non-positively curved follows that the map

$$(t,s) \mapsto d_{X_G}(\beta_1(t), \beta_2(s))$$

is smooth (see Petersen [44, p. 129]). Moreover, since $\exp(\mathfrak{b})\cdot\tau$ is orthogonal both to $H^o\cdot\tau$ and $\exp(X)H^o\cdot\tau$ we conclude that the differential at $(0,0)$ of this map is zero.

Take $t_0 > 0$ such that $\beta_1(t_0) = \tau'$ and a positive a such that the geodesic $t \mapsto \beta_2(at)$ equals τ'' in t_0. By Busemann [11, Theorem 3.6] the map

$$t \mapsto d_{X_G}(\beta_1(t), \beta_2(at))$$

is convex. Since it has a critical point at $t=0$ the proof of the lemma is finished. □

Thanks to Remark 2.3 and (2.2) the proof of Proposition 3.8 is now complete. □

Recall that $\lambda_1(g)$ denotes the logarithm of the spectral radius of $g\in G$.

PROPOSITION 3.10. — *For every g in $\mathscr{C}^{>}_{o,G}$ one has*

$$|b^o(g)|^{\frac{1}{2}}=\frac{1}{2}\lambda_1(J^o g J^o g^{-1}).$$

Proof. — It suffices to prove the proposition for the choices of o and $\mathfrak{b}^+$ of Example 3.1. Write $g=h\exp(b^o(g))h'$ with

$$b^o(g)=\begin{pmatrix} & & & s\\ & & 0 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \\ s & & & \end{pmatrix}$$

for some $s\geqslant 0$. We have $|b^o(g)|^{\frac{1}{2}}=s$. On the other hand, J^o commutes with elements of H^o and thus the number $\frac{1}{2}\lambda_1(J^o g J^o g^{-1})$ equals to

$$\frac{1}{2}\lambda_1(J^o h\exp(b^o(g))J^o\exp(b^o(g))^{-1}h^{-1}).$$

Further, this number coincides with

$$\frac{1}{2}\lambda_1(J^o\exp(b^o(g))J^o\exp(b^o(g))^{-1}).$$

Since $b^o(g)\in\mathfrak{q}^o$ we have $J^o\exp(b^o(g))^{-1}=\exp(b^o(g))J^o$ and the proof is complete. □

4. Proximality

In this section we recall basic facts on product of proximal matrices, the main one being Benoist's Theorem 4.6. This results are well-known but we provide proofs for those which are not explicitly stated in the literature (the reader familiarized with these concepts may skip this section). Standard references are the works of Benoist [2, 3, 4].

4.1. Notations and basic definitions

A norm $\|\cdot\|$ on $\mathbb{R}^d$ will be fixed in the whole section. For $\xi_1,\xi_2\in\mathbb{P}(\mathbb{R}^d)$ define the distance

$$d(\xi_1,\xi_2):=\inf\{\|v_{\xi_1}-v_{\xi_2}\|:\ \ v_{\xi_i}\in\xi_i\ \text{ and }\ \|v_{\xi_i}\|=1 \text{ for all } i=1,2\}.$$

Let $\mathsf{Gr}_{d-1}(\mathbb{R}^d)$ be the Grassmannian of $(d-1)$-dimensional subspaces of $\mathbb{R}^d$. There exists a G-equivariant identification $\mathbb{P}((\mathbb{R}^d)^*)\longrightarrow\mathsf{Gr}_{d-1}(\mathbb{R}^d)$ given by

$$\theta\mapsto\ker\theta$$

where the action of G on the left side is given by $g\cdot\theta:=\theta\circ g^{-1}$. This identification will be used from now on whenever convenient.

For $\eta_1,\eta_2\in\mathsf{Gr}_{d-1}(\mathbb{R}^d)$ we let

$$d(\xi_1,\eta_1):=\min\{d(\xi_1,\xi):\ \ \xi\in\mathbb{P}(\eta_1)\}$$

and we denote by $d^*(\eta_1,\eta_2)$ the distance on $\mathbb{P}((\mathbb{R}^d)^*)$ induced by the operator norm on $(\mathbb{R}^d)^*$. Given a positive ε we set

$$b_\varepsilon(\xi_1):=\{\xi\in\mathbb{P}(\mathbb{R}^d):\ \ d(\xi_1,\xi)<\varepsilon\}$$

and

$$B_\varepsilon(\eta_1):=\{\xi\in\mathbb{P}(\mathbb{R}^d):\ \ d(\xi,\eta_1)\geqslant\varepsilon\}.$$

On the other hand, let

$$\mathbb{P}^{(2)}:=\{(\theta,v)\in\mathbb{P}((\mathbb{R}^d)^*)\times\mathbb{P}(\mathbb{R}^d):\ \ v\notin\ker\theta\}$$

and

$$\mathbb{P}^{(4)}:=\{(\theta,v,\phi,u)\in\mathbb{P}^{(2)}\times\mathbb{P}^{(2)}:\ \ v\notin\ker\phi \text{ and } u\notin\ker\theta\}.$$

Observe that

$$\mathscr{G}_{\|\cdot\|}=\mathscr{G}:\mathbb{P}^{(2)}\longrightarrow\mathbb{R}:\ \ \mathscr{G}(\theta,v):=\log\frac{|\theta(v)|}{\|\theta\|\|v\|}\tag{4.1}$$

is well-defined. Similarly the following map is well-defined

$$\mathbb{B}:\mathbb{P}^{(4)}\longrightarrow\mathbb{R}:\ \ \mathbb{B}(\theta,v,\phi,u):=\log\left|\frac{\theta(u)}{\theta(v)}\frac{\phi(v)}{\phi(u)}\right|\tag{4.2}$$

and is called de *cross-ratio* of (θ,v,ϕ,u)[8]. Both $\mathscr{G}$ and $\mathbb{B}$ are continuous.

[8] Sometimes $e^{\mathbb{B}}$ is called the cross-ratio.

4.2. Product of proximal matrices

Given g in $\mathrm{End}(\mathbb{R}^d) \setminus \{0\}$ we denote by

$$\lambda_1(g) \geqslant \cdots \geqslant \lambda_d(g)$$

the logarithms of the moduli of the eigenvalues of g, repeated with multiplicity (we use the convention $\log 0 = -\infty$). The matrix g is said to be *proximal* in $\mathbb{P}(\mathbb{R}^d)$ if $\lambda_1(g)$ is simple. In that case we let g_+ (resp. g_-) to be the attractive fixed line (resp. repelling fixed hyperplane) of g in $\mathbb{P}(\mathbb{R}^d)$. Note that if g is non invertible then g_- contains the kernel of g.

We now define a quantified version of proximality. The definition that we propose is (slightly) weaker than the one given by Benoist in [2, 3, 4]. We provide proofs of the basic facts established in those works when necessary.

Definition 4.1. — *Let $0 < \varepsilon \leqslant r$ and $g \in \mathrm{End}(\mathbb{R}^d) \setminus \{0\}$ be a proximal matrix. The matrix g is called* (r, ε)-proximal *if $d(g_+, g_-) \geqslant 2r$ and one has $g \cdot B_\varepsilon(g_-) \subset b_\varepsilon(g_+)$.*

Lemma 4.2 (Benoist [2, Corollaire 6.3]). — *Let $0 < \varepsilon \leqslant r$. There exists a constant $c_{r,\varepsilon} > 0$ such that for every (r, ε)-proximal matrix g one has*

$$\log \|g\| - c_{r,\varepsilon} \leqslant \lambda_1(g) \leqslant \log \|g\|.$$

The following criterion of (r, ε)-proximality will be very useful in the sequel.

Lemma 4.3 (Benoist [2, Lemme 6.2]). — *Let g be an element in $\mathrm{End}(\mathbb{R}^d) \setminus \{0\}$, $\eta \in \mathsf{Gr}_{d-1}(\mathbb{R}^d)$, $\xi \in \mathbb{P}(\mathbb{R}^d)$ and $0 < \varepsilon \leqslant r$. If $d(\xi, \eta) \geqslant 6r$ and $g \cdot B_\varepsilon(\eta) \subset b_\varepsilon(\xi)$ then g is $(2r, 2\varepsilon)$-proximal with $d(g_+, \xi) \leqslant \varepsilon$ and $d^*(g_-, \eta) \leqslant \varepsilon$.*

Proof. — Consider the Hilbert distance on the convex set $B_\varepsilon(\eta)$ (see [5]). The condition $g \cdot B_\varepsilon(\eta) \subset b_\varepsilon(\xi)$ implies that g is contracting for this metric and thus has a unique fixed point in $B_\varepsilon(\eta)$, which belongs in fact to $b_\varepsilon(\xi)$. The proof now finishes as in [2, Lemme 6.2]. □

Corollary 4.4 (Benoist [4, Lemme 1.4]). — *Let $0 < \varepsilon \leqslant r$. If g_1 and g_2 are (r, ε)-proximal and satisfy*

$$d(g_{1_+}, g_{2_-}) \geqslant 6r \quad \text{and} \quad d(g_{2_+}, g_{1_-}) \geqslant 6r$$

then $g_1 g_2$ is $(2r, 2\varepsilon)$-proximal.

Let g_1 and g_2 be two matrices as in Corollary 4.4. The goal now is to state a theorem (Theorem 4.6) which provides a comparison between the spectral radius and operator norm of $g_1 g_2$ in terms of the spectral radii of g_1 and g_2 and the maps $\mathscr{G}$ and $\mathbb{B}$.

Lemma 4.5. — *Fix $r > 0$ and $\delta > 0$. For every ε small enough, the following property is satisfied: for every pair of (r, ε)-proximal elements g_1 and g_2 such that*

$$d(g_{1_+}, g_{2_-}) \geqslant 6r \quad \text{and} \quad d(g_{2_+}, g_{1_-}) \geqslant 6r$$

one has

$$|\mathscr{G}(g_{2_-}, g_{1_+}) - \mathscr{G}((g_1g_2)_-, (g_1g_2)_+)| < \delta.$$

Proof. — For every $0 < \varepsilon \leqslant r$, consider the compact set $C_{r,\varepsilon}$ of pairs (g_1, g_2) of norm-one (r, ε)-proximal matrices in $\mathrm{End}(\mathbb{R}^d) \setminus \{0\}$ satisfying

$$d(g_{1_+}, g_{2_-}) \geqslant 6r \quad \text{and} \quad d(g_{2_+}, g_{1_-}) \geqslant 6r.$$

The function

$$(g_1, g_2) \mapsto |\mathscr{G}(g_{2_-}, g_{1_+}) - \mathscr{G}((g_1g_2)_-, (g_1g_2)_+)|$$

is continuous and equals zero on $C_r := \cap_{\varepsilon>0} C_{r,\varepsilon} \subset \mathrm{End}(\mathbb{R}^d) \setminus \{0\}$. □

Theorem 4.6 (Benoist [4, Lemme 1.4]). — *Fix $r > 0$ and $\delta > 0$. Then for every ε small enough, the following properties are satisfied: for every pair of (r, ε)-proximal elements g_1 and g_2 such that*

$$d(g_{1_+}, g_{2_-}) \geqslant 6r \quad \text{and} \quad d(g_{2_+}, g_{1_-}) \geqslant 6r$$

one has:

(1) *The number*

$$\left|\lambda_1(g_1g_2) - (\lambda_1(g_1) + \lambda_1(g_2)) - \mathbb{B}(g_{1_-}, g_{1_+}, g_{2_-}, g_{2_+})\right|$$

is less than δ.

(2) *The number*

$$\left|\log \|g_1g_2\| - (\lambda_1(g_1) + \lambda_1(g_2)) - \mathbb{B}(g_{1_-}, g_{1_+}, g_{2_-}, g_{2_+}) + \mathscr{G}(g_{2_-}, g_{1_+})\right|$$

is less than δ.

Proof.

(1). — See [4, Lemme 1.4].

(2). — Let ε be as in (1). For every g_1 and g_2 as in the statement, Corollary 4.4 implies that g_1g_2 is $(2r, 2\varepsilon)$-proximal. By [50, Lemma 5.6] (and taking ε smaller if necessary) we have

$$|\log \|g_1g_2\| - \lambda_1(g_1g_2) + \mathscr{G}((g_1g_2)_-, (g_1g_2)_+)| < \delta.$$

Lemma 4.5 finishes the proof. □

5. Projective Anosov representations

Anosov representations were introduced by Labourie [30] for surface groups and extended by Guichard–Wienhard [21] to word hyperbolic groups. In this section we recall the definition of (projective) Anosov representations and some well-known facts concerning (r,ε)-proximality of matrices in the image of such a representation.

5.1. Singular values

The most useful characterization of Anosov representations for our purposes is the one given in terms of *singular values*. We begin by recalling this notion and we fix also some notations that we will use in the rest of the paper.

Let τ be a q-dimensional subspace of $\mathbb{R}^d$ which is negative definite for $\langle\cdot,\cdot\rangle_{p,q}$. Consider $\langle\cdot,\cdot\rangle_\tau$ to be the inner product of $\mathbb{R}^d$ that coincides with $-\langle\cdot,\cdot\rangle_{p,q}$ (resp. $\langle\cdot,\cdot\rangle_{p,q}$) on τ (resp. $\tau^{\perp_{p,q}}$) and for which τ and $\tau^{\perp_{p,q}}$ are orthogonal. Given g in $\mathrm{PSL}(d,\mathbb{R})$, we let g^{*_τ} to be the adjoint operator with respect to $\langle\cdot,\cdot\rangle_\tau$. Set

$$a_1^\tau(g) \geqslant \cdots \geqslant a_d^\tau(g)$$

to be the logarithms of the eigenvalues of $\sqrt{g^{*_\tau}g}$ repeated with multiplicity. These are called the *τ-singular values* of g. Geometrically, they represent the (logarithms of the) lengths of the semi axes of the ellipsoid which is the image by g of the unit sphere

$$\mathbb{S}_\tau^{d-1} := \{x \in \mathbb{R}^d : \ \langle x,x\rangle_\tau = 1\}.$$

Let $i = 1,\ldots,d-1$. Given an element g in $\mathrm{PSL}(d,\mathbb{R})$ such that $a_i^\tau(g) > a_{i+1}^\tau(g)$ we denote by $U_i(g)$ the i-dimensional subspace of $\mathbb{R}^d$ spanned by the i biggest axes of $g\cdot\mathbb{S}_\tau^{d-1}$. We also set

$$S_{d-i}(g) := U_{d-i}(g^{-1}).$$

Remark 5.1. — Let $\varepsilon > 0$. It follows from Singular Value Decomposition (see Horn–Johnson [23, Section 7.3 of Chapter 7]), that there exists $L > 0$ such that for every g in $\mathrm{PSL}(d,\mathbb{R})$ satisfying $a_1^\tau(g) - a_2^\tau(g) > L$ one has

$$g\cdot B_\varepsilon(S_{d-1}(g)) \subset b_\varepsilon(U_1(g)),$$

where $B_\varepsilon(S_{d-1}(g))$ and $b_\varepsilon(U_1(g))$ are defined as in Subsection 4.1.

5.2. The definition of projective Anosov representations

A lot of work has been done in order to simplify the original definition of Anosov representations, here we follow mainly the work of Bochi–Potrie–Sambarino [6] (see also Guichard–Guéritaud–Kassel–Wienhard [20] or Kapovich–Leeb–Porti [24]).

Fix τ as in the previous subsection and let Γ be a finitely generated group. Consider a finite symmetric generating set S of Γ and take $|\cdot|_\Gamma$ to be the associated word length: for γ in Γ, it is the minimum number required to write γ as a product of elements of S[9]. Let $\rho : \Gamma \longrightarrow \mathrm{PSL}\,(d, \mathbb{R})$ be a representation. We say that ρ is *projective Anosov* if there exist positive constants C and α such that for all $\gamma \in \Gamma$ one has

$$a_1^\tau(\rho(\gamma)) - a_2^\tau(\rho(\gamma)) \geqslant \alpha|\gamma|_\Gamma - C. \tag{5.1}$$

By Kapovich–Leeb–Porti [25, Theorem 1.4] (see also [6, Section 3]), condition (5.1) implies that Γ is word hyperbolic[10]. We assume in this paper that Γ is non elementary. Let $\partial_\infty\Gamma$ be the Gromov boundary of Γ and Γ_{H} be the set of infinite order elements in Γ. Every γ in Γ_{H} has exactly two fixed points in $\partial_\infty\Gamma$: the attractive one denoted by γ_+ and the repelling one denoted by γ_-. The dynamics of γ on $\partial_\infty\Gamma$ is of type *north-south*.

Fix $\rho : \Gamma \longrightarrow \mathrm{PSL}(d, \mathbb{R})$ a projective Anosov representation. By [6, 20, 24] we know that there exist continuous equivariant maps

$$\xi : \partial_\infty\Gamma \longrightarrow \mathbb{P}(\mathbb{R}^d) \quad \text{and} \quad \eta : \partial_\infty\Gamma \longrightarrow \mathsf{Gr}_{d-1}(\mathbb{R}^d)$$

which are *transverse*, i.e. for every $x \neq y$ in $\partial_\infty\Gamma$ one has

$$\xi(x) \oplus \eta(y) = \mathbb{R}^d. \tag{5.2}$$

One can see that condition (5.1) implies that for every γ in Γ_{H} the matrix $\rho(\gamma)$ is proximal. Equivariance of ξ and η implies that

$$\xi(\gamma_+) = \rho(\gamma)_+ \quad \text{and} \quad \eta(\gamma_+) = \rho(\gamma^{-1})_-.$$

It follows that both ξ and η are homeomorphisms onto their images. In fact, these homeomorphisms are Hölder (see Bridgeman–Canary–Labourie–Sambarino [10, Lemma 2.5]).

We denote by $\Lambda_{\rho(\Gamma)} \subset \mathbb{P}(\mathbb{R}^d)$ the image of ξ, which is called the *limit set* of $\rho(\Gamma)$: it is the closure of the set of attractive fixed points in $\mathbb{P}(\mathbb{R}^d)$

(9) This number depends on the choice of S. However, the set S will be fixed from now on hence we do not emphasize the dependence on this choice in the notation.
(10) We refer the reader to the book of Ghys–de la Harpe [17] for definitions and standard facts on word hyperbolic groups.

of proximal elements in $\rho(\Gamma)$. The image of η is called the *dual limit set* of $\rho(\Gamma)$.

Here is another characterization of the limit sets which is very useful. An explicit reference is [20, Theorem 5.3] (it can also be deduced from [6, Subsection 3.4]). Let $d = d_\tau$ (resp. $d^* = d^*_\tau$) be the distance on $\mathbb{P}(\mathbb{R}^d)$ (resp. $\mathbb{P}((\mathbb{R}^d)^*)$) associated to $\langle \cdot, \cdot \rangle_\tau$.

Proposition 5.2. — *Let $\rho : \Gamma \longrightarrow \mathrm{PSL}(d, \mathbb{R})$ be a projective Anosov representation. Then $\xi(\partial_\infty \Gamma)$ (resp. $\eta(\partial_\infty \Gamma)$) equals the set of accumulation points of sequences $\{U_1(\rho(\gamma_n))\}_n$ (resp. $\{S_{d-1}(\rho(\gamma_n))\}_n$) where $\gamma_n \longrightarrow \infty$. Moreover, given a positive ε there exists $L > 0$ such that for every γ in Γ_{H} with $|\gamma|_\Gamma > L$ one has*

$$d(U_1(\rho(\gamma)), \rho(\gamma)_+) < \varepsilon \quad \text{and} \quad d^*(S_{d-1}(\rho(\gamma)), \rho(\gamma)_-) < \varepsilon.$$

We are interested in projective Anosov representations whose image is contained in $G = \mathrm{PSO}(p, q)$. The following remark is then important for our purposes.

Remark 5.3. — Let $\rho : \Gamma \longrightarrow \mathrm{PSL}(d, \mathbb{R})$ be a projective Anosov representation. If $\rho(\Gamma)$ is contained in G we say that ρ is $P_1^{p,q}$-*Anosov* (recall that $P_1^{p,q}$ denotes the (parabolic) subgroup of G stabilizing an isotropic line). In this case, the image of ξ is contained in $\partial \mathbb{H}^{p,q-1}$ and the dual map η equals $\xi^{\perp_{p,q}}$.

5.3. Proximality properties

The following lemma will be useful in the next section.

Lemma 5.4 (cf. [50, Lemma 5.7]). — *Let $\rho : \Gamma \longrightarrow \mathrm{PSL}(d, \mathbb{R})$ be a projective Anosov representation and $0 < \varepsilon \leqslant r$. Then*

$$\#\{\gamma \in \Gamma_{\mathrm{H}} : \ d(\rho(\gamma)_+, \rho(\gamma)_-) \geqslant 2r \ \text{ and } \ \rho(\gamma) \text{ is not } (r, \varepsilon)\text{-proximal}\} < \infty.$$

Proof. — Consider a sequence $\gamma_n \longrightarrow \infty$ in Γ_{H} such that $d(\rho(\gamma_n)_+, \rho(\gamma_n)_-) \geqslant 2r$ for all n. By Proposition 5.2 for every n big enough the following holds

$$b_{\frac{\varepsilon}{2}}(U_1(\rho(\gamma_n))) \subset b_\varepsilon(\rho(\gamma_n)_+)$$

and

$$B_\varepsilon(\rho(\gamma_n)_-) \subset B_{\frac{\varepsilon}{2}}(S_{d-1}(\rho(\gamma_n))).$$

By Remark 5.1 and (5.1) the condition $\rho(\gamma_n) \cdot B_\varepsilon(\rho(\gamma_n)_-) \subset b_\varepsilon(\rho(\gamma_n)_+)$ is satisfied for sufficiently large n. □

6. The set $\mathbf{\Omega}_\rho$

Let $\rho : \Gamma \longrightarrow G$ be a $P_1^{p,q}$-Anosov representation and define

$$\mathbf{\Omega}_\rho := \{o \in \mathbb{H}^{p,q-1} : \quad J^o \cdot \xi(x) \notin \eta(x) \text{ for all } x \in \partial_\infty \Gamma\}.$$

This section is structured as follows. In Subsection 6.1 we prove that the action of Γ on $\mathbf{\Omega}_\rho$ is properly discontinuous. Moreover, we show that if o is a point in $\mathbf{\Omega}_\rho$ then the geodesic connecting o with $\rho(\gamma) \cdot o$ is space-like (apart from possibly finitely many exceptions $\gamma \in \Gamma$). In Subsection 6.2 we study the matrices $J^o \rho(\gamma) J^o \rho(\gamma^{-1})$ for a point o in $\mathbf{\Omega}_\rho$: we apply to them Benoist's work on proximality. Finiteness of our counting functions is proved in Subsection 6.3. Finally, in Subsection 6.4 we prove a proposition that will be needed in the proof of Proposition 7.11.

Before we start, let us discuss some examples for which $\mathbf{\Omega}_\rho$ is non empty. From Proposition 2.6 we know that the following alternative description of $\mathbf{\Omega}_\rho$ holds

$$\mathbf{\Omega}_\rho = \{o = [\hat{o}] \in \mathbb{H}^{p,q-1} : \quad \langle \hat{o}, \hat{\xi} \rangle_{p,q} \neq 0 \text{ for all } \xi = [\hat{\xi}] \in \Lambda_{\rho(\Gamma)}\}.$$

We have the following important example.

Example 6.1.

- Let Γ be the fundamental group of a convex co-compact hyperbolic manifold of dimension $m \geqslant 2$ and $\iota_0 : \Gamma \longrightarrow \mathrm{SO}(m,1)$ be the holonomy representation. Fix $p \geqslant m$ and $q \geqslant 2$. Consider the embedding $\mathbb{R}^{m,1} \hookrightarrow \mathbb{R}^{p,q}$ given by

  $$\mathbb{R}^{m,1} \cong \operatorname{span}\{e_{p-m+1}, \dots, e_{p+1}\},$$

  where e_i is the vector of $\mathbb{R}^d$ with all entries equal to zero except for the i-th entry which is equal to one. This induces a projection $j : \mathrm{SO}(m,1) \longrightarrow G$ and a representation $\rho_0 : \Gamma \longrightarrow G$ defined by

  $$\rho_0 := j \circ \iota_0.$$

  Thus ρ_0 is $P_1^{p,q}$-Anosov, because ι_0 is $P_1^{m,1}$-Anosov. The set $\mathbf{\Omega}_{\rho_0}$ is non empty: every point $o \in \mathbb{H}^{p,q-1}$ for which the subspace

  $$\operatorname{span}\{o, e_{p+2}, \dots, e_d\}$$

  has signature $(0,q)$ belongs to $\mathbf{\Omega}_{\rho_0}$. Since the condition of being Anosov is open in the space of representations of Γ into G and the limit map ξ varies continuously with the representation (see Guichard–Wienhard [21, Theorem 5.13]), we obtain that if ρ is a small deformation of ρ_0 then $\mathbf{\Omega}_\rho$ is non empty.

- The previous example generalizes to a large class of representations introduced by Danciger–Guéritaud–Kassel in [14, 13] called $\mathbb{H}^{p,q-1}$-*convex co-compact*(11). Let $\Gamma < G$ be a $\mathbb{H}^{p,q-1}$-convex co-compact group and $\rho : \Gamma \longrightarrow G$ be the inclusion representation, which is $P_1^{p,q}$-Anosov as proved in [13, Theorem 1.25]. Let Ω be a non empty Γ-invariant properly convex open subset of $\mathbb{H}^{p,q-1}$. By [13, Proposition 4.5], Ω is contained in $\mathbf{\Omega}_\rho$.
- There exist examples of $P_1^{p,q}$-Anosov representations ρ whose image is not $\mathbb{H}^{p,q-1}$-convex co-compact but satisfy $\mathbf{\Omega}_\rho \neq \emptyset$ (see [14, Examples 5.2 & 5.3]).

6.1. Dynamics on $\mathbf{\Omega}_\rho$

Observe that $\mathbf{\Omega}_\rho$ is Γ-invariant. The following proposition is well-known, we include a proof for completeness.

Proposition 6.2. — *Let $\rho : \Gamma \longrightarrow G$ be a $P_1^{p,q}$-Anosov representation. Then the action of Γ on $\mathbf{\Omega}_\rho$ is properly discontinuous, that is, for every compact set $C \subset \mathbf{\Omega}_\rho$ one has*

$$\#\{\gamma \in \Gamma : \quad \rho(\gamma) \cdot C \cap C \neq \emptyset\} < \infty.$$

Moreover, for any point o in $\mathbf{\Omega}_\rho$ the set of accumulation points of $\rho(\Gamma) \cdot o$ in $\mathbb{H}^{p,q-1} \cup \partial\mathbb{H}^{p,q-1}$ coincides with the limit set $\Lambda_{\rho(\Gamma)}$.

Proof. — Let $C \subset \mathbf{\Omega}_\rho$ be a compact set and fix a norm on $\mathbb{R}^d$. By definition of $\mathbf{\Omega}_\rho$ we can take a positive ε such that

$$C \cap \bigcup_{x \in \partial_\infty \Gamma} b_\varepsilon(\xi(x)) = \emptyset \quad \text{and} \quad C \subset \bigcap_{x \in \partial_\infty \Gamma} B_\varepsilon(\eta(x)).$$

By Proposition 5.2, Remark 5.1 and (5.1) we know that, apart from possibly finitely many exceptions γ in Γ, the following holds:

$$b_{\frac{\varepsilon}{2}}(U_1(\rho(\gamma))) \subset \bigcup_{x \in \partial_\infty \Gamma} b_\varepsilon(\xi(x)),$$

$$\bigcap_{x \in \partial_\infty \Gamma} B_\varepsilon(\eta(x)) \subset B_{\frac{\varepsilon}{2}}(S_{d-1}(\rho(\gamma)))$$

(11) These are inclusion representations induced by taking an infinite discrete subgroup $\Gamma < G$ which preserves some properly convex non empty open set $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ whose boundary is strictly convex and of class C^1. One requires that Γ preserves some *distinguished* non empty convex subset of Ω on which the action is co-compact (see [13, 14] for precisions).

and

$$\rho(\gamma) \cdot B_{\frac{\varepsilon}{2}}(S_{d-1}(\rho(\gamma))) \subset b_{\frac{\varepsilon}{2}}(U_1(\rho(\gamma))).$$

For these γ we have then that $\rho(\gamma) \cdot C$ is contained in the ε-neighbourhood of $\Lambda_{\rho(\Gamma)}$ and thus is disjoint from C.

We have shown that the action of Γ on $\mathbf{\Omega}_\rho$ is properly discontinuous and that for any point o in $\mathbf{\Omega}_\rho$ the accumulation points of $\rho(\Gamma) \cdot o$ belong to $\Lambda_{\rho(\Gamma)}$. Conversely, the Γ-orbit of any point in $\Lambda_{\rho(\Gamma)}$ is dense in the limit set and now the proof is complete. □

Let $o \in \mathbf{\Omega}_\rho$ and recall the notations introduced in Subsection 2.2.2. Given an open set $W \subset \partial\mathbb{H}^{p,q-1}$ disjoint from $\overline{\mathscr{C}_o^0} \cap \partial\mathbb{H}^{p,q-1}$ we denote by $\mathscr{C}_o^{>W}$ the subset of $\mathscr{C}_o^{>}$ consisting of points o' such that the (space-like) geodesic ray connecting o with o' has its end point in W.

The following corollary has been proved by Glorieux–Monclair [18] for $\mathbb{H}^{p,q-1}$-convex co-compact groups.

Corollary 6.3. — *Let $\rho : \Gamma \longrightarrow G$ be a $P_1^{p,q}$-Anosov representation, a point $o \in \mathbf{\Omega}_\rho$ and $W \subset \partial\mathbb{H}^{p,q-1}$ an open set containing $\Lambda_{\rho(\Gamma)}$ with closure disjoint from $\overline{\mathscr{C}_o^0} \cap \partial\mathbb{H}^{p,q-1}$. Then apart from possibly finitely many exceptions γ in Γ one has $\rho(\gamma) \cdot o \in \mathscr{C}_o^{>W}$. In particular the geodesic joining o with $\rho(\gamma) \cdot o$ is space-like.*

Proof. — Let C be the closure of $\mathbb{H}^{p,q-1} \setminus \mathscr{C}_o^{>W}$ in $\mathbb{H}^{p,q-1} \cup \partial\mathbb{H}^{p,q-1}$. Note that C is compact and by Proposition 6.2 does not contain accumulation points of $\rho(\Gamma) \cdot o$, hence $\rho(\Gamma) \cdot o \cap C$ is finite. Since $\gamma \mapsto \rho(\gamma) \cdot o$ is proper the proof is complete. □

6.2. Proximality of $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$

For the rest of the section we fix a $P_1^{p,q}$-Anosov representation $\rho : \Gamma \longrightarrow G$, a point $o \in \mathbf{\Omega}_\rho$ and a Cartan involution $\tau \in S^o$.

The next lemma is a direct consequence of Proposition 5.2, transversality condition (5.2) and the definition of $\mathbf{\Omega}_\rho$.

Lemma 6.4. — *Let d_τ be the distance on $\mathbb{P}(\mathbb{R}^d)$ induced by the norm $\|\cdot\|_\tau$. There exists a positive constant D such that*

$$\#\{\gamma \in \Gamma : \ d_\tau(J^o \cdot U_1(\rho(\gamma)), S_{d-1}(\rho(\gamma^{-1}))) < D\} < \infty.$$

Lemma 6.5. — *There exist $0 < \varepsilon \leqslant r$ such that, apart from possibly finitely many exceptions $\gamma \in \Gamma$, the matrix $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$ is (r, ε)-proximal.*

Proof. — We apply a *ping-pong* argument together with Lemma 4.3. By Lemma 6.4 we can take a positive constant r and a finite subset $F \subset \Gamma$ such that for every $\gamma \in \Gamma \setminus F$ one has

$$d_\tau(J^o \cdot U_1(\rho(\gamma)), S_{d-1}(\rho(\gamma^{-1}))) \geqslant 6r. \tag{6.1}$$

Take $0 < \varepsilon \leqslant r$ such that for every $\gamma \in \Gamma \setminus F$ one has

$$b_\varepsilon(J^o \cdot U_1(\rho(\gamma))) \subset B_\varepsilon(S_{d-1}(\rho(\gamma^{-1}))).$$

By Remark 3.2 the matrix J^o preserves d_τ thus

$$J^o \cdot b_\varepsilon(U_1(\rho(\gamma))) \subset B_\varepsilon(S_{d-1}(\rho(\gamma^{-1}))).$$

By taking F larger if necessary we have that

$$\rho(\gamma^{-1}) \cdot B_\varepsilon(S_{d-1}(\rho(\gamma^{-1}))) \subset b_\varepsilon(U_1(\rho(\gamma^{-1})))$$

holds for every γ in $\Gamma \setminus F$. It follows that

$$J^o\rho(\gamma^{-1}) \cdot B_\varepsilon(S_{d-1}(\rho(\gamma^{-1}))) \subset B_\varepsilon(S_{d-1}(\rho(\gamma)))$$

and applying $\rho(\gamma)$ we obtain

$$\rho(\gamma)J^o\rho(\gamma^{-1}) \cdot B_\varepsilon(S_{d-1}(\rho(\gamma^{-1}))) \subset b_\varepsilon(U_1(\rho(\gamma))).$$

Then

$$J^o\rho(\gamma)J^o\rho(\gamma^{-1}) \cdot B_\varepsilon(S_{d-1}(\rho(\gamma^{-1}))) \subset b_\varepsilon(J^o \cdot U_1(\rho(\gamma))).$$

By (6.1) and Lemma 4.3 the proof is finished. □

The following is a strengthening of Lemma 6.5. It provides a link between the generalized Cartan projections b^o and b^τ and the spectral radii of proximal elements in $\rho(\Gamma)$. For the remainder of the section we fix a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ and a closed Weyl chamber $\mathfrak{b}^+$.

Lemma 6.6. — *Fix any $\delta > 0$ and A and B two compact disjoint sets in $\partial_\infty\Gamma$. Then there exist $0 < \varepsilon \leqslant r$ such that, apart from possibly finitely many exceptions $\gamma \in \Gamma_{\mathrm{H}}$ with $\gamma_- \in A$ and $\gamma_+ \in B$, the following holds:*

1. *The matrices $J^o\rho(\gamma)J^o$ and $\rho(\gamma^{-1})$ are (r,ε)-proximal.*
2. *$d_\tau(J^o \cdot \rho(\gamma)_+, \rho(\gamma^{-1})_-) \geqslant 6r$ and $d_\tau(\rho(\gamma^{-1})_+, J^o \cdot \rho(\gamma)_-) \geqslant 6r$.*
3. *$d_\tau((J^o\rho(\gamma)J^o)_+, \rho(\gamma^{-1})_-) \geqslant 6r$ and $d_\tau(\rho(\gamma^{-1})_+, (J^o\rho(\gamma)J^o)_-) \geqslant 6r$.*
4. *The matrix $\rho(\gamma)$ belongs to $\mathscr{C}^{>}_{o,G}$ and the number*

$$2\left(|b^o(\rho(\gamma))|^{\frac{1}{2}} - \lambda_1(\rho(\gamma))\right)$$

is at distance at most 2δ from

$$\mathbb{B}(J^o \cdot \rho(\gamma)_-, J^o \cdot \rho(\gamma)_+, \rho(\gamma^{-1})_-, \rho(\gamma^{-1})_+).$$

(5) *The number*

$$2\left(|b^\tau(\rho(\gamma))|^{\frac{1}{2}} - \lambda_1(\rho(\gamma))\right)$$

is at distance at most 2δ *from*

$$\mathbb{B}(J^o\cdot\rho(\gamma)_-, J^o\cdot\rho(\gamma)_+, \rho(\gamma^{-1})_-, \rho(\gamma^{-1})_+) - \mathscr{G}_\tau(\rho(\gamma^{-1})_-, J^o\cdot\rho(\gamma)_+).$$

Proof. — By transversality condition (5.2) there exists $r>0$ such that

$$d_\tau(\xi(x),\eta(y)) \geqslant 2r \quad\text{and}\quad d_\tau(\xi(y),\eta(x)) \geqslant 2r \tag{6.2}$$

for all $(x,y)\in A\times B$. Further, since $o\in\mathbf{\Omega}_\rho$ we may assume

$$d_\tau(J^o\cdot\xi(x),\eta(x)) \geqslant 6r \tag{6.3}$$

for all $x\in\partial_\infty\Gamma$. Given these $r>0$ and $2\delta>0$, we consider $\varepsilon>0$ as in Benoist's Theorem 4.6.

By Lemma 5.4 there exists a finite subset F of Γ_{H} outside of which elements satisfying $d_\tau(\rho(\gamma)_+,\rho(\gamma)_-)\geqslant 2r$ are (r,ε)-proximal. Thanks to (6.2), for all $\gamma\in\Gamma_{\mathrm{H}}\setminus F$ with $\gamma_-\in A$ and $\gamma_+\in B$ one has that $\rho(\gamma^{\pm1})$ is (r,ε)-proximal. Moreover, since $J^o=(J^o)^{-1}$ preserves $\|\cdot\|_\tau$ we have that $J^o\rho(\gamma)J^o$ is (r,ε)-proximal with $(J^o\rho(\gamma)J^o)_\pm = J^o\cdot\rho(\gamma)_\pm$. In fact, by (6.3) we have

$$d_\tau(J^o\cdot\rho(\gamma)_+,\rho(\gamma^{-1})_-)\geqslant 6r \quad\text{and}\quad d_\tau(\rho(\gamma^{-1})_+, J^o\cdot\rho(\gamma)_-)\geqslant 6r.$$

Thanks to Proposition 3.10 (and Corollary 6.3), Proposition 3.6, Theorem 4.6 and the fact that $\lambda_1(\rho(\gamma^{-1}))$ equals $\lambda_1(\rho(\gamma))$ for all γ, the proof is finished. □

6.3. The orbital counting functions of Theorems 1.3 and 1.4

Proposition 6.7. — *For every* $t\geqslant 0$ *one has*

$$\#\left\{\gamma\in\Gamma:\ |b^\tau(\rho(\gamma))|^{\frac{1}{2}}\leqslant t\right\}<\infty.$$

Proof. — By Remark 3.4 the map b^τ descends to a proper map in $\mathbb{H}^{p,q-1}\cong G/H^o$, that we still denote by b^τ. Hence

$$C:=\{o'\in\mathbb{H}^{p,q-1}:\ |b^\tau(o')|\leqslant t^2\}$$

is compact. By Proposition 6.2, apart from possibly finitely many exceptions γ in Γ, we have that $\rho(\gamma)\cdot o$ does not belong to C. □

The next proposition follows from a combination of Propositions 3.10 and 3.6, Lemmas 6.5 and 4.2, and the previous proposition.

PROPOSITION 6.8. — *For every $t \geqslant 0$ one has*

$$\#\left\{\gamma \in \Gamma : \quad \rho(\gamma) \in \mathscr{C}^{>}_{o,G} \text{ and } |b^o(\rho(\gamma))|^{\frac{1}{2}} \leqslant t\right\} < \infty.$$

Remark 6.9. — Assume that ρ is $\mathbb{H}^{p,q-1}$-convex co-compact and the basepoint o belongs to the convex hull of the limit set of ρ. By Corollary 6.3 and Proposition 3.8 we have that

$$\limsup_{t \longrightarrow \infty} \frac{\log \#\{\gamma \in \Gamma : \quad \rho(\gamma) \in \mathscr{C}^{>}_{o,G} \text{ and } |b^o(\rho(\gamma))|^{\frac{1}{2}} \leqslant t\}}{t}$$

coincides with

$$\limsup_{t \longrightarrow \infty} \frac{\log \#\{\gamma \in \Gamma : \quad d_{\mathbb{H}^{p,q-1}}(o, \rho(\gamma)\cdot o) \leqslant t\}}{t},$$

where $d_{\mathbb{H}^{p,q-1}}$ is the $\mathbb{H}^{p,q-1}$-distance introduced in [18].

6.4. Weak triangle inequality

The following proposition is inspired by [18, Theorem 3.5].

PROPOSITION 6.10. — *There exists a constant $L > 0$ such that for every $f \in \Gamma$ there exists $D_f > 0$ with the following property: for every $\gamma \in \Gamma$ with $|\gamma|_\Gamma > L$ one has*

$$\frac{1}{2}\lambda_1(J^o\rho(f)\rho(\gamma)J^o\rho(\gamma^{-1})\rho(f^{-1})) \leqslant D_f + \frac{1}{2}\lambda_1(J^o\rho(\gamma)J^o\rho(\gamma^{-1})).$$

We can think about the content of Proposition 6.10 as follows. Fix $f \in \Gamma$ such that $\rho(f) \in \mathscr{C}^{>}_{o,G}$. By Corollary 6.3 for every γ with $|\gamma|_\Gamma$ large enough one has $\rho(\gamma) \in \mathscr{C}^{>}_{o,G}$ and $\rho(f)\rho(\gamma) \in \mathscr{C}^{>}_{o,G}$. Thanks to Proposition 3.8 and Proposition 3.10, the inequality established in Proposition 6.10 can be stated as

$$\ell_{o,\rho(f)\rho(\gamma)\cdot o} \leqslant D_f + \ell_{\rho(f)\cdot o,\rho(f)\rho(\gamma)\cdot o},$$

where the constant D_f depends on the choice of o and f (and ρ) but not on the choice of γ. Even though the function $\ell_{\cdot,\cdot}$ is not a distance, we can heuristically think about D_f as the term that replaces $\ell_{o,\rho(f)\cdot o}$ in the usual triangle inequality for distances.

Proof of Proposition 6.10. — Take $0 < \varepsilon \leqslant r$ as in Lemma 6.5. Let $L > 0$ such that for every γ in Γ with $|\gamma|_\Gamma > L$ the matrix $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$ is (r, ε)-proximal. Fix $f \in \Gamma$ and let γ be a element in Γ with $|\gamma|_\Gamma > L$. We have

$$\frac{1}{2}\lambda_1(J^o\rho(f)\rho(\gamma)J^o\rho(\gamma^{-1})\rho(f^{-1})) \leqslant \frac{1}{2}\log\|J^o\rho(f)\rho(\gamma)J^o\rho(\gamma^{-1})\rho(f^{-1})\|_\tau.$$

By Remark 3.2 the right side number equals

$$\frac{1}{2}\log\|\rho(f)\rho(\gamma)J^o\rho(\gamma^{-1})\rho(f^{-1})\|_\tau$$

which is less than or equal to

$$D'_f+\frac{1}{2}\log\|J^o\rho(\gamma)J^o\rho(\gamma^{-1})\|_\tau$$

where $D'_f := \frac{1}{2}\log\|\rho(f)\|_\tau + \frac{1}{2}\log\|\rho(f^{-1})\|_\tau$. Since $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$ is (r,ε)-proximal, we conclude by applying Lemma 4.2. □

7. Distribution of the orbit of o with respect to b^o

In this section we prove Theorem 1.3. The section is structured as follows: in Subsection 7.1 we define a Hölder cocycle on $\partial_\infty\Gamma$ and the corresponding flow. In Subsection 7.2 we study the associated Gromov product. Theorem 1.3 in the torsion free case (resp. general case) is proved in Subsection 7.3 (resp. Subsection 7.4).

For the rest of the section we fix $\rho:\Gamma\longrightarrow G$ a $P_1^{p,q}$-Anosov representation and a point o in $\mathbf{\Omega}_\rho$.

7.1. The cocycle c_o

Observe that by definition of $\mathbf{\Omega}_\rho$ and equivariance of the curves ξ and η the following map is well-defined.

Definition 7.1. — *Let*

$$c_o:\Gamma\times\partial_\infty\Gamma\longrightarrow\mathbb{R}:\quad c_o(\gamma,x):=\frac{1}{2}\log\left|\frac{\theta_x\left(\rho(\gamma^{-1})J^o\rho(\gamma)\cdot v_x\right)}{\theta_x\left(J^o\cdot v_x\right)}\right|,$$

where $\theta_x:\mathbb{R}^d\longrightarrow\mathbb{R}$ is a non-zero linear functional whose kernel equals $\eta(x)$ and $v_x\neq 0$ belongs to $\xi(x)$.

A geometric interpretation of the map c_o is provided by the following remark. This characterization will not be used in the sequel.

Remark 7.2. — One can prove that for every $\gamma\in\Gamma$ and $x\in\partial_\infty\Gamma$ one has

$$c_o(\gamma,x)=\beta_{\xi(x)}(\rho(\gamma^{-1})\cdot o,o)$$

where $\beta_\cdot(\,\cdot\,,\cdot\,)$ is the pseudo-Riemannian Busemann function defined by Glorieux–Monclair [18, Definition 3.8].

Recall that a *Hölder cocycle* is a function $c:\Gamma\times\partial_\infty\Gamma\longrightarrow\mathbb{R}$ satisfying that for every γ_0,γ_1 in Γ and $x\in\partial_\infty\Gamma$ one has

$$c(\gamma_0\gamma_1,x)=c(\gamma_0,\gamma_1\cdot x)+c(\gamma_1,x)$$

and such that the map $c(\gamma_0,\cdot\,)$ is Hölder (with the same exponent for every γ_0). The *period* of (an infinite order element) $\gamma\in\Gamma_{\mathrm{H}}$ is defined by $\ell_c(\gamma):=c(\gamma,\gamma_+)$.

Lemma 7.3. — *The map c_o is a Hölder cocycle. The period of $\gamma\in\Gamma_{\mathrm{H}}$ is given by*

$$\ell_{c_o}(\gamma)=\lambda_1(\rho(\gamma))>0.$$

Proof. — A direct computation shows that c_o is a Hölder cocycle. On the other hand let $\gamma\in\Gamma_{\mathrm{H}}$ and fix a particular choice of a linear functional θ_{γ_+}. Since $\lambda_1(\rho(\gamma))=\lambda_1(\rho(\gamma^{-1}))$ one sees that $\theta_{\gamma_+}\circ(\pm\rho(\gamma^{-1}))$ coincides with $e^{\lambda_1(\rho(\gamma))}\theta_{\gamma_+}$ up to a sign (here $\pm\rho(\gamma^{-1})$ denotes some lift of $\rho(\gamma^{-1})$ to $\mathrm{SO}(p,q)$). The proof is now complete. □

Set $\partial^2_\infty\Gamma:=\{(x,y)\in\partial_\infty\Gamma\times\partial_\infty\Gamma:\ \ x\neq y\}$ and consider the *translation flow* on $\partial^2_\infty\Gamma\times\mathbb{R}$ defined by

$$\psi_t(x,y,s):=(x,y,s-t). \tag{7.1}$$

The group Γ acts on $\partial^2_\infty\Gamma\times\mathbb{R}$ by

$$\gamma\cdot(x,y,s):=(\gamma\cdot x,\gamma\cdot y,s-c_o(\gamma,y)). \tag{7.2}$$

This action is proper and co-compact and we denote the quotient space by $\mathrm{U}_o\Gamma$. The flow ψ_t descends to a flow on $\mathrm{U}_o\Gamma$, still denoted ψ_t, which is a Hölder reparametrization of the Gromov geodesic flow of Γ [19]. This is the analogue of Sambarino's Theorem [50, Theorem 3.2 (1)] (see also Lemma A.7).

We say that an element γ in Γ is *primitive* if cannot be written as a positive power of another element in Γ. Periodic orbits of ψ_t are in one-to-one correspondence with conjugacy classes of primitive elements in Γ. If $[\gamma]$ is such a conjugacy class, the period of the corresponding periodic orbit is

$$\ell_{c_o}(\gamma)=\lambda_1(\rho(\gamma))$$

(see Fact A.1 and Lemma A.7). The topological entropy of ψ_t coincides with the *entropy* of ρ defined by Bridgeman–Canary–Labourie–Sambarino [10]:

$$h_\rho:=\limsup_{t\longrightarrow\infty}\frac{\log\#\{[\gamma]\in[\Gamma]:\ \ \gamma\text{ is primitive and }\lambda_1(\rho(\gamma))\leqslant t\}}{t}.$$

It is positive and finite (cf. Fact A.3) and will be denoted by h from now on.

Remark 7.4. — One can prove that if we *push* all this construction by the limit map $\xi : \partial_\infty \Gamma \longrightarrow \Lambda_{\rho(\Gamma)}$, we recover the geodesic flow defined in [18, Subsection 6.1] for $\mathbb{H}^{p,q-1}$-convex co-compact groups. This remark will not be used in the sequel.

7.2. Dual cocycle and Gromov product

Thanks to transversality condition (5.2) and the fact that o belongs to $\boldsymbol{\Omega}_\rho$ the following map is well-defined.

Definition 7.5. — *Let*

$$[\,\cdot\,,\cdot\,]_o : \partial^2_\infty \Gamma \longrightarrow \mathbb{R} : \quad [x,y]_o := -\frac{1}{2}\log\left|\frac{\theta_x\left(J^o\cdot v_x\right)\theta_y\left(J^o\cdot v_y\right)}{\theta_x\left(v_y\right)\theta_y\left(v_x\right)}\right|,$$

where θ_x (resp. θ_y) is a non-zero linear functional whose kernel is $\eta(x)$ (resp. $\eta(y)$) and v_x (resp. v_y) is a non-zero vector in $\xi(x)$ (resp. $\xi(y)$).

Remark 7.6. — The map $[\,\cdot\,,\cdot\,]_o$ coincides, up to a sign, with the Gromov product introduced in [18, Subsection 3.5]. The authors give geometric interpretations of this function using pseudo-Riemannian geometry.

Remark 7.7. — The cocycle c_o is dual to itself, i.e. $\ell_{c_o}(\gamma) = \ell_{c_o}(\gamma^{-1})$ for every $\gamma \in \Gamma_{\mathrm{H}}$. Indeed, this follows from Lemma 7.3 and the fact that $\lambda_1(g) = \lambda_1(g^{-1})$ for all g in G.

The proof of the following lemma is a direct computation.

Lemma 7.8. — *The map $[\,\cdot\,,\cdot\,]_o$ is a Gromov product for the pair $\{c_o, c_o\}$, that is, for every $\gamma \in \Gamma$ and every $(x,y) \in \partial^2_\infty \Gamma$ one has*

$$[\gamma\cdot x, \gamma\cdot y]_o - [x,y]_o = -(c_o(\gamma,x) + c_o(\gamma,y)).$$

The following lemma will be very important in the proof of Theorem 1.3. It provides a geometric interpretation of the Gromov product different from the one given in Remark 7.6.

Lemma 7.9. — *Let γ be an element of Γ_{H}. Then*

$$[\gamma_-,\gamma_+]_o = -\frac{1}{2}\mathbb{B}(J^o\cdot\rho(\gamma)_-, J^o\cdot\rho(\gamma)_+, \rho(\gamma^{-1})_-, \rho(\gamma^{-1})_+).$$

Proof. — From Section 5 we know that $\rho(\gamma^{\pm 1})$ is proximal and that the following holds:

$$\rho(\gamma)_+ = \xi(\gamma_+), \quad \rho(\gamma^{-1})_+ = \xi(\gamma_-), \quad \rho(\gamma)_- = \eta(\gamma_-), \quad \rho(\gamma^{-1})_- = \eta(\gamma_+).$$

Since $J^o = (J^o)^{-1}$, the matrix $J^o\rho(\gamma)J^o$ is proximal and one has the equalities

$$(J^o\rho(\gamma)J^o)_+ = J^o \cdot \xi(\gamma_+) \quad \text{and} \quad (J^o\rho(\gamma)J^o)_- = J^o \cdot \eta(\gamma_-).$$

The proof finishes by a direct computation. □

7.3. Distribution of attractors and repellors with respect to b^o

Recall that $h = h_{\mathrm{top}}(\psi_t)$ and let μ_o be a *Patterson–Sullivan probability* on $\partial_\infty\Gamma$ associated to c_o, i.e. μ_o satisfies

$$\frac{\mathrm{d}\gamma_*\mu_o}{\mathrm{d}\mu_o}(x) = e^{-hc_o(\gamma^{-1},x)}$$

for every $\gamma \in \Gamma$. Such a probability exists (see Subsection A.2.2). By Lemma 7.8 the measure

$$e^{-h[\,\cdot\,,\cdot\,]_o}\mu_o \otimes \mu_o \otimes \mathrm{d}t \tag{7.3}$$

on $\partial^2_\infty\Gamma \times \mathbb{R}$ is Γ-invariant and induces on the quotient $\mathrm{U}_o\Gamma$ a ψ_t-invariant measure. By Sambarino [50, Theorem 3.2 (2)] this measure is, up to scaling, the probability of maximal entropy of ψ_t (see Proposition A.12).

For a metric space X we denote by $C^*_c(X)$ the dual of the space of compactly supported continuous real functions on X equipped with the weak-star topology. If x is a point in X, let $\delta_x \in C^*_c(X)$ be the Dirac mass at x.

Proposition 7.10 (Sambarino [50, Proposition 4.3][12]). — *There exists a constant $M = M_{\rho,o} > 0$ such that*

$$Me^{-ht} \sum_{\gamma\in\Gamma_{\mathrm{H}},\ell_{c_o}(\gamma)\leqslant t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \longrightarrow e^{-h[\,\cdot\,,\cdot\,]_o}\mu_o \otimes \mu_o$$

as $t \longrightarrow \infty$ on $C^*_c(\partial^2_\infty\Gamma)$.

From Proposition 7.10 we deduce Proposition 7.11 which directly implies Theorem 1.3 in the torsion free case.

Fix a point $\tau \in S^o$, a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ and a closed Weyl chamber $\mathfrak{b}^+$ contained in $\mathfrak{b}$.

(12) For a proof in our setting see Proposition A.13.

Proposition 7.11. — *There exists a constant* $M = M_{\rho,o} > 0$ *such that*

$$Me^{-ht} \sum_{\gamma\in\Gamma_{\mathrm{H}}, |b^o(\rho(\gamma))|^{\frac{1}{2}} \leqslant t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \longrightarrow \mu_o \otimes \mu_o$$

as $t \longrightarrow \infty$ *on* $C^*(\partial_\infty\Gamma \times \partial_\infty\Gamma)$.

Recall that the generalized Cartan projection b^o is defined in the set $\mathscr{C}^{>}_{o,G}$. The sum in Proposition 7.11 is taken then over all elements $\gamma \in \Gamma_{\mathrm{H}}$ for which $\rho(\gamma) \in \mathscr{C}^{>}_{o,G}$ and $|b^o(\rho(\gamma))|^{\frac{1}{2}} \leqslant t$. To make the formula more readable we do not emphasize the fact that $\rho(\gamma)$ must belong to $\mathscr{C}^{>}_{o,G}$. On the other hand, by Corollary 6.3 this condition holds apart from finitely many exceptions $\gamma \in \Gamma$.

Proof of Proposition 7.11. — Set

$$\theta_t := Me^{-ht} \sum_{\gamma\in\Gamma_{\mathrm{H}}, |b^o(\rho(\gamma))|^{\frac{1}{2}} \leqslant t} \delta_{\gamma_-} \otimes \delta_{\gamma_+}.$$

We first prove the statement outside the diagonal, that is, on subsets of $\partial^2_\infty\Gamma$. Let $\delta > 0$ and $A, B \subset \partial_\infty\Gamma$ disjoint open sets. Consider an element $\gamma \in \Gamma_{\mathrm{H}}$ such that $\gamma_- \in A$ and $\gamma_+ \in B$ and let $s := [\gamma_-, \gamma_+]_o$. By taking A and B smaller we may assume

$$|[x, y]_o - s| < \delta \tag{7.4}$$

for all $(x, y) \in A \times B$.

By Lemma 6.6, apart from possibly finitely many exceptions $\gamma \in \Gamma_{\mathrm{H}}$ with $(\gamma_-, \gamma_+) \in A \times B$, the number

$$\left| |b^o(\rho(\gamma))|^{\frac{1}{2}} - \lambda_1(\rho(\gamma)) - \frac{1}{2}\mathbb{B}(J^o \cdot \rho(\gamma)_-, J^o \cdot \rho(\gamma)_+, \rho(\gamma^{-1})_-, \rho(\gamma^{-1})_+) \right|$$

is less than δ. Applying Lemma 7.3 and Lemma 7.9 we conclude that

$$\left| |b^o(\rho(\gamma))|^{\frac{1}{2}} - \ell_{c_o}(\gamma) + [\gamma_-, \gamma_+]_o \right| < \delta.$$

By (7.4) it follows that

$$\ell_{c_o}(\gamma) - s - 2\delta < |b^o(\rho(\gamma))|^{\frac{1}{2}} < \ell_{c_o}(\gamma) - s + 2\delta$$

holds apart from finitely many exceptions $\gamma \in \Gamma_{\mathrm{H}}$ such that $\gamma_- \in A$ and $\gamma_+ \in B$. From now on, the proof of the convergence

$$\theta_t(A \times B) \longrightarrow \mu_o(A)\mu_o(B)$$

follows line by line the proof of [50, Theorem 6.5].

It remains to prove the convergence in the diagonal $\{(x, x) : \ x \in \partial_\infty\Gamma\}$, but once again, the proof is the same as the one given in [50, Theorem 6.5]. For completeness we briefly sketch it.

Since μ_o has no atoms (see Lemma A.10), for every γ in Γ the diagonal has $(\mu_o \otimes \gamma_* \mu_o)$-measure equal to zero. We fix two elements $\gamma_0, \gamma_1 \in \Gamma_{\mathrm{H}}$ with no common fixed point in $\partial_\infty \Gamma$ and let $\varepsilon_0 > 0$. There exists a finite open covering $\mathscr{U}$ of $\partial_\infty \Gamma$ such that for $i = 0, 1$ one has

$$\sum_{U \in \mathscr{U}} \mu_o(U) \mu_o(\gamma_i^{-1} \cdot U) < \varepsilon_0.$$

We can assume that for every $U \in \mathscr{U}$ there exists $i \in \{0, 1\}$ such that $\gamma_i^{-1} \cdot \overline{U}$ is disjoint from $\overline{U}$. There exists an open covering $\mathscr{V}$ of $\partial_\infty \Gamma$ with the following properties:

(1) $\sum_{V \in \mathscr{V}} \mu_o(V) \mu_o(\gamma_i^{-1} \cdot V) < \varepsilon_0$ for $i = 0, 1$.

(2) The closure of every element in $\mathscr{U}$ is contained in a unique element of $\mathscr{V}$ and if $\gamma_i^{-1} \cdot \overline{U}$ is disjoint from $\overline{U}$ the same holds for this element in $\mathscr{V}$.

(3) Suppose that $\gamma_i^{-1} \cdot \overline{U} \cap \overline{U} = \emptyset$ and let $V \in \mathscr{V}$ be the unique element such that $\overline{U} \subset V$. Then apart from finitely many exceptions γ such that $\gamma_\pm \in U$ one has $(\gamma_i^{-1} \gamma)_- \in V$ and $(\gamma_i^{-1} \gamma)_+ \in \gamma_i^{-1} \cdot V$.

Set $D := \max_{i=0,1} \{D_{\gamma_i^{-1}}\}$ where $D_{\gamma_i^{-1}}$ is the constant given by Proposition 6.10 and take $U \in \mathscr{U}$ as in (3). By Proposition 6.10 we have

$$\theta_t(U \times U) \leqslant M e^{-ht} \sum_{\gamma \in \Gamma_{\mathrm{H}}, |b^o(\rho(\gamma))|^{\frac{1}{2}} \leqslant t + D} \delta_{\gamma_-}(V) \delta_{\gamma_+}(\gamma_i^{-1} \cdot V) + M e^{-ht} \# F$$

where F is a finite set independent of t. Since $V \times \gamma_i^{-1} \cdot V$ is far from the diagonal the right side converges to

$$e^D \mu_o(V) \mu_o(\gamma_i^{-1} \cdot V)$$

as $t \longrightarrow \infty$. Adding up in $U \in \mathscr{U}$ we conclude

$$\limsup_{t \longrightarrow \infty} \sum_{U \in \mathscr{U}} \theta_t(U \times U) \leqslant 2 e^D \varepsilon_0.$$

Hence $\theta_t(\{(x, x) : \ x \in \partial_\infty \Gamma\})$ converges to zero and since the diagonal has measure zero for $\mu_o \otimes \mu_o$ the proof is finished. □

7.4. Proof of Theorem 1.3

The following is a corollary of Proposition 7.11.

Corollary 7.12. — *There exists a constant $M = M_{\rho,o} > 0$ such that*

$$M e^{-ht} \sum_{\gamma \in \Gamma_{\mathrm{H}}, |b^o(\rho(\gamma))|^{\frac{1}{2}} \leqslant t} \delta_{\rho(\gamma^{-1}) \cdot o^{\perp_{p,q}}} \otimes \delta_{\rho(\gamma) \cdot o} \longrightarrow \eta_*(\mu_o) \otimes \xi_*(\mu_o)$$

on $C^*(\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d))$ *as* $t \longrightarrow \infty$*.*

Proof. — Set

$$\nu_t^{\mathrm{H}} := Me^{-ht} \sum_{\gamma \in \Gamma_{\mathrm{H}}, |b^o(\rho(\gamma))|^{\frac{1}{2}} \leqslant t} \delta_{\rho(\gamma^{-1}) \cdot o^{\perp_{p,q}}} \otimes \delta_{\rho(\gamma) \cdot o}$$

and take θ_t the measure defined in the proof of Proposition 7.11. We know that

$$(\eta, \xi)_*(\theta_t) \longrightarrow \eta_*(\mu_o) \otimes \xi_*(\mu_o).$$

Hence we only have to show the following convergence

$$\nu_t^{\mathrm{H}} - (\eta, \xi)_*(\theta_t) \longrightarrow 0. \tag{7.5}$$

Take a small positive δ. By Proposition 5.2 and the proof of Proposition 6.2 we know that, apart from finitely many exceptions γ in Γ_{H}, one has

$$d(\rho(\gamma) \cdot o, \rho(\gamma)_+) < \delta \quad \text{and} \quad d(\rho(\gamma^{-1}) \cdot o, \rho(\gamma^{-1})_+) < \delta.$$

By taking $\cdot^{\perp_{p,q}}$ we can assume further that $d^*(\rho(\gamma^{-1}) \cdot o^{\perp_{p,q}}, \rho(\gamma)_-) < \delta$. Now the proof of (7.5) follows from evaluation on continuous functions of $\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d)$. □

We now include torsion elements to the previous statement and finish the proof of Theorem 1.3.

Proposition 7.13. — *There exists a constant* $M = M_{\rho,o} > 0$ *such that*

$$Me^{-ht} \sum_{\gamma \in \Gamma, |b^o(\rho(\gamma))|^{\frac{1}{2}} \leqslant t} \delta_{\rho(\gamma^{-1}) \cdot o^{\perp_{p,q}}} \otimes \delta_{\rho(\gamma) \cdot o} \longrightarrow \eta_*(\mu_o) \otimes \xi_*(\mu_o)$$

on $C^*(\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d))$ *as* $t \longrightarrow \infty$*.*

Proof. — The structure of the proof is the same as that of Proposition 7.11, that is, we first prove the statement outside the diagonal and deduce from that the statement on the diagonal. Here by *diagonal* we mean the set

$$\Delta := \{(\theta, v) \in \mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d) : \ \theta(v) = 0\}.$$

Let

$$\nu_t := Me^{-ht} \sum_{\gamma \in \Gamma, |b^o(\rho(\gamma))|^{\frac{1}{2}} \leqslant t} \delta_{\rho(\gamma^{-1}) \cdot o^{\perp_{p,q}}} \otimes \delta_{\rho(\gamma) \cdot o}$$

and take ν_t^{H} as in the proof of Corollary 7.12.

Consider first a continuous function f on $\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d)$ whose support $\mathrm{supp}(f)$ is disjoint from Δ. We prove the following.

Lemma 7.14. — *One has*

$$\#\{\gamma\in\Gamma:\ (\rho(\gamma^{-1})\cdot o^{\perp_{p,q}},\rho(\gamma)\cdot o)\in\operatorname{supp}(f)\text{ and }\gamma\notin\Gamma_{\mathrm{H}}\}<\infty.$$

Proof of Lemma 7.14. — Fix a positive D such that for every $(\theta,v)\in\operatorname{supp}(f)$ one has $d(\theta,v)>D$. As we saw in the proof of Proposition 6.2, the distances

$$d(\rho(\gamma)\cdot o,U_1(\rho(\gamma)))\quad\text{and}\quad d^*(\rho(\gamma^{-1})\cdot o^{\perp_{p,q}},S_{d-1}(\rho(\gamma)))$$

converge to zero as $\gamma\longrightarrow\infty$. We conclude that, apart from possibly finitely many exceptions γ in Γ with $(\rho(\gamma^{-1})\cdot o^{\perp_{p,q}},\rho(\gamma)\cdot o)\in\operatorname{supp}(f)$, one has

$$d(U_1(\rho(\gamma)),S_{d-1}(\rho(\gamma)))>D.$$

Now apply (5.1), Remark 5.1 and Benoist's Lemma 4.3 to conclude that for $|\gamma|_\Gamma$ large enough the matrix $\rho(\gamma)$ is proximal. □

From Lemma 7.14 we conclude that

$$\lim_{t\longrightarrow\infty}\nu_t(f)=\lim_{t\longrightarrow\infty}\nu_t^{\mathrm{H}}(f)$$

which by Corollary 7.12 equals $(\eta_*(\mu_o)\otimes\xi_*(\mu_o))(f)$.

It remains to prove the convergence on the diagonal. It suffices to prove that for every positive ε_0 there exists an open covering $\{U^*\times U\}$ of Δ such that

$$\limsup_{t\longrightarrow\infty}\nu_t\left(\bigcup(U^*\times U)\right)\leqslant\varepsilon_0.$$

The proof is the same as in Proposition 7.11. Namely, take two elements γ_0,γ_1 in Γ_{H} with no common fixed point in $\partial_\infty\Gamma$ and a coverings $\mathscr{U}=\{U^*\times U\}$ and $\mathscr{V}=\{V^*\times V\}$ of Δ by open sets with the following properties:

(1) For every $U^*\times U$ in $\mathscr{U}$ there exists $i=0,1$ such that $\rho(\gamma_i^{-1})\cdot\overline{U}$ is transverse to $\overline{U^*}$.
(2) $\sum_{V^*\times V\in\mathscr{V}}(\eta_*(\mu_o)\otimes\xi_*(\mu_o))(V^*\times\rho(\gamma_i^{-1})\cdot V)<\varepsilon_0$ for $i=0,1$.
(3) The closure of every element in $\mathscr{U}$ is contained in a unique element of $\mathscr{V}$ and if $\rho(\gamma_i^{-1})\cdot\overline{U}$ is transverse to $\overline{U^*}$ the same holds for this element in $\mathscr{V}$.
(4) Suppose that $\rho(\gamma_i^{-1})\cdot\overline{U}$ is transverse to $\overline{U^*}$ and let $V^*\times V\in\mathscr{V}$ be the unique element such that $\overline{U}\subset V$ and $\overline{U^*}\subset V^*$. Then, apart from possibly finitely many exceptions γ such that

$$(\rho(\gamma^{-1})\cdot o^{\perp_{p,q}},\rho(\gamma)\cdot o)\in U^*\times U,$$

one has

$$(\rho((\gamma_i^{-1}\gamma)^{-1})\cdot o^{\perp_{p,q}},\rho(\gamma_i^{-1}\gamma)\cdot o)\in V^*\times\rho(\gamma_i^{-1})\cdot V.$$

Provided with this construction, the proof finishes in the same way as that of Proposition 7.11. □

Remark 7.15. — From Proposition 7.13 we deduce that

$$\lim_{t\longrightarrow\infty}\frac{\log\#\{\gamma\in\Gamma:\ \ \rho(\gamma)\in\mathscr{C}^{>}_{o,G}\text{ and }|b^o(\rho(\gamma))|^{\frac{1}{2}}\leqslant t\}}{t}$$

coincides with the entropy $h=h_\rho$ of ρ.

8. Distribution of the orbit of o with respect to b^τ

The proof of Theorem 1.4 follows the same lines of the proof of Theorem 1.3, we just have to pick a (slightly) different flow ψ_t.

Fix a $P_1^{p,q}$-Anosov representation $\rho:\Gamma\longrightarrow G$, a point o in $\mathbf{\Omega}_\rho$ and $\tau\in S^o$.

8.1. The cocycle c_τ

Let $\|\cdot\|_\tau$ be the norm introduced in Subsection 5.1.

Definition 8.1. — *Let*

$$c_\tau:\Gamma\times\partial_\infty\Gamma\longrightarrow\mathbb{R}:\ \ c_\tau(\gamma,x):=\frac{1}{2}\log\left(\frac{\|\rho(\gamma)\cdot\theta_x\|_\tau\|\rho(\gamma)\cdot v_x\|_\tau}{\|\theta_x\|_\tau\|v_x\|_\tau}\right)$$

where $\theta_x:\mathbb{R}^d\longrightarrow\mathbb{R}$ is a non-zero linear functional whose kernel equals $\eta(x)$ and $v_x\neq 0$ belongs to $\xi(x)$.

Remark 8.2. — One can prove that for every $\gamma\in\Gamma$ and $x\in\partial_\infty\Gamma$ one has

$$c_\tau(\gamma,x)=\log\frac{\|\rho(\gamma)\cdot v_x\|_\tau}{\|v_x\|_\tau},$$

that is, c_τ coincides with the map $\beta_1(\,\cdot\,,\cdot\,)$ of [50, Section 5]. This remark will not be used in the sequel.

The following lemma holds by straightforward computations.

Lemma 8.3. — *The function c_τ is a Hölder cocycle. The period of γ in Γ_{H} is given by*

$$\ell_{c_\tau}(\gamma)=\lambda_1(\rho(\gamma))>0.$$

The quotient space of $\partial^2_\infty\Gamma\times\mathbb{R}$ by the action of Γ induced by c_τ will be denoted by $\mathrm{U}_\tau\Gamma$. It is equipped with a flow that lifts to the translation flow (7.1) on $\partial^2_\infty\Gamma\times\mathbb{R}$.

8.2. Dual cocycle and Gromov product

Definition 8.4. — *Let*

$$[\,\cdot\,,\cdot\,]_\tau : \partial^2_\infty \Gamma \longrightarrow \mathbb{R} : \quad [x,y]_\tau := \frac{1}{2} \log \left| \frac{\theta_y\,(v_x)\,\theta_x\,(v_y)}{\theta_x\,(J^o \cdot v_x)\,\|\theta_y\|_\tau \|v_y\|_\tau} \right|.$$

Remark 8.5. — Recall that c_o is the cocycle defined in Section 7. The cocycle c_τ is dual to c_o, i.e. $\ell_{c_o}(\gamma) = \ell_{c_\tau}(\gamma^{-1})$ for every $\gamma \in \Gamma_{\mathrm{H}}$.

The proof of the following lemma is a direct computation.

Lemma 8.6. — *For every $\gamma \in \Gamma$ and every $(x,y) \in \partial^2_\infty \Gamma$ one has*

$$[\gamma \cdot x, \gamma \cdot y]_\tau - [x,y]_\tau = -(c_o(\gamma,x) + c_\tau(\gamma,y)).$$

Lemma 8.7. — *Let γ be an element of Γ_{H}. Then the number $[\gamma_-,\gamma_+]_\tau$ equals*

$$-\frac{1}{2}\mathbb{B}(J^o \cdot \rho(\gamma)_-, J^o \cdot \rho(\gamma)_+, \rho(\gamma^{-1})_-, \rho(\gamma^{-1})_+) + \frac{1}{2}\mathscr{G}_\tau(\rho(\gamma^{-1})_-, J^o \cdot \rho(\gamma)_+).$$

Proof. — Recall the definition of $[\,\cdot\,,\cdot\,]_o$ from Subsection 7.2. One has

$$[\gamma_-,\gamma_+]_\tau = [\gamma_-,\gamma_+]_o + \frac{1}{2} \log \frac{\left|\theta_{\gamma_+}(J^o \cdot v_{\gamma_+})\right|}{\|\theta_{\gamma_+}\|_\tau \|v_{\gamma_+}\|_\tau}.$$

The proof then follows from Lemma 7.9 and Remark 3.2. □

8.3. Distribution of attractors and repellors with respect to b^τ

Let μ_τ be a Patterson–Sullivan probability on $\partial_\infty\Gamma$ associated to c_τ and recall that μ_o is the one associated to c_o. The analogue of Proposition 7.10 is available for the flow on $\mathrm{U}_\tau\Gamma$. The limit measure can be written in this case as(13)

$$e^{-h[\,\cdot\,,\cdot\,]_\tau} \mu_o \otimes \mu_\tau.$$

Let $\mathfrak{b}^+$ be a closed Weyl chamber of a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$.

Proposition 8.8. — *There exists a constant $M' = M'_{\rho,\tau} > 0$ such that*

$$M' e^{-ht} \sum_{\gamma \in \Gamma_{\mathrm{H}}, |b^\tau(\rho(\gamma))|^{\frac{1}{2}} \leqslant t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \longrightarrow \mu_o \otimes \mu_\tau$$

as $t \longrightarrow \infty$ on $C^*(\partial_\infty\Gamma \times \partial_\infty\Gamma)$.

Proof. — The proof is the same that the one given in Proposition 7.11 adapted to the pair $\{c_o, c_\tau\}$ and the Gromov product $[\,\cdot\,,\cdot\,]_\tau$: apply Lemma 6.6 (5) and Lemma 8.7. □

(13) For a proof, see Remark A.14.

8.4. Proof of Theorem 1.4

The following proposition, which implies Theorem 1.4, can be proved in the same way as Proposition 7.13.

Proposition 8.9. — *There exists a constant $M' = M'_{\rho,\tau} > 0$ such that*

$$M' e^{-ht} \sum_{\gamma\in\Gamma, |b^{\tau}(\rho(\gamma))|^{\frac{1}{2}} \leqslant t} \delta_{\rho(\gamma^{-1})\cdot o^{\perp_{p,q}}} \otimes \delta_{\rho(\gamma)\cdot o} \longrightarrow \eta_*(\mu_o) \otimes \xi_*(\mu_\tau)$$

on $C^*(\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d))$ as $t \longrightarrow \infty$.

Appendix A. Distribution of periodic orbits in $\mathrm{U}_o\Gamma$ and $\mathrm{U}_\tau\Gamma$

The goal of this appendix is to describe the distribution of periodic orbits of the flows defined in Sections 7 and 8 (Proposition A.13 and Remark A.14). For the case on which Γ is the fundamental group of a closed negatively curved manifold, this result is covered by [50, Proposition 4.3]. Here we treat the case of word hyperbolic groups admitting an Anosov representation.

In [50, Proposition 4.3] the author applies the thermodynamic formalism to reparametrizations of the geodesic flow of the manifold. Here we benefit from the fact that a projective Anosov representation ρ is given and use the *geodesic flow* of ρ, introduced by Bridgeman–Canary–Labourie–Sambarino in [10], as a reference flow. This is a canonical flow associated to a projective Anosov representation and we show that it is Hölder conjugate to the flows on the spaces $\mathrm{U}_o\Gamma$ and $\mathrm{U}_\tau\Gamma$. Since the techniques of the thermodynamic formalism are available for the geodesic flow of the representation (see [10, 12]), the adaptations needed in our context are straightforward.

The appendix is structured as follows. In Subsection A.1 we recall the definition of the geodesic flow of a representation and its main properties. We are interested in two descriptions of its probability of maximal entropy (Facts A.3 and A.6). In Subsection A.2 we translate these results to the flows on $\mathrm{U}_o\Gamma$ and $\mathrm{U}_\tau\Gamma$.

A.1. The geodesic flow $\mathrm{U}_\rho\Gamma$

We fix from now on a projective Anosov representation $\rho : \Gamma \longrightarrow G$.

A.1.1. Definition and the metric Anosov property

The standard reference for this subsection is [10]. Given $(x,y) \in \partial^2_\infty \Gamma$ let

$$\mathrm{M}(x,y) := \{(\theta, v) \in \eta(x) \times \xi(y) : \quad \theta(v) = 1\}/\sim$$

where $(\theta, v) \sim (-\theta, -v)$. Consider the line bundle over $\partial^2_\infty \Gamma$ defined by

$$F_\rho := \{(x,y,\theta,v) : \ \ (x,y) \in \partial^2_\infty \Gamma \text{ and } (\theta,v) \in \mathrm{M}(x,y)\}.$$

FACT A.1 (Bridgeman–Canary–Labourie–Sambarino [10, Sections 4–5]). *The following holds:*

- *The group Γ acts naturally on F_ρ and this action is proper and co-compact. The quotient space is denoted by $\mathrm{U}_\rho \Gamma$.*
- *The flow ϕ_t on F_ρ defined by*

$$\phi_t(x,y,\theta,v) := (x,y,e^{-t}\theta, e^t v)$$

descends to a flow on $\mathrm{U}_\rho\Gamma$, still denoted by ϕ_t, and called the geodesic flow *of ρ. The geodesic flow of ρ is conjugate, by a Hölder homeomorphism, to a Hölder reparametrization of the Gromov geodesic flow of Γ (see Mineyev [37]).*
- *Periodic orbits of ϕ_t are in one-to-one correspondence with conjugacy classes of primitive elements γ in Γ. The corresponding period is $\lambda_1(\rho(\gamma))$.*
- *The geodesic flow ϕ_t is a transitive* metric Anosov flow. *Very informally, this means that there exists laminations W^{ss}, W^{uu}, W^{cs} and W^{cu} of $\mathrm{U}_\rho\Gamma$, called respectively* strong stable lamination, strong unstable lamination, central stable lamination *and* central unstable lamination, *defining* a local product structure *and with the property that W^{ss} (resp. W^{uu}) is exponentially contracted by the flow (resp. the inverse flow). For precise definitions see* [10, Subsection 3.2].

  Explicitly, for a point $Z_0 = (x_0, y_0, \theta_0, v_0)$ in $\mathrm{U}_\rho\Gamma$ the strong stable and strong unstable leaves through Z_0 are given by:

$$W^{ss}(Z_0) = \{(x, y_0, \theta, v_0) \in \mathrm{U}_\rho\Gamma : \ \ \theta \in \eta(x) \text{ and } \theta(v_0) = 1\}$$

  and

$$W^{uu}(Z_0) = \{(x_0, y, \theta_0, v) \in \mathrm{U}_\rho\Gamma : \ \ v \in \xi(y) \text{ and } \theta_0(v) = 1\}.$$

  The central stable and central unstable leaves are given by:

$$W^{cs}(Z_0) = \{(x, y_0, \theta, v) \in \mathrm{U}_\rho\Gamma : \ \theta \in \eta(x), \ v \in \xi(y_0) \text{ and } \theta(v) = 1\}$$

  and

$$W^{cu}(Z_0) = \{(x_0, y, \theta, v) \in \mathrm{U}_\rho\Gamma : \ \theta \in \eta(x_0), \ v \in \xi(y) \text{ and } \theta(v) = 1\}.$$

A.1.2. Entropy and distribution of periodic orbits

A flow is said to be *topologically weakly-mixing* if all the periods of its periodic orbits are not multiple of a common constant.

Proposition A.2. — *The geodesic flow of ρ is topologically weakly-mixing.*

Before proving Proposition A.2 let us state the main result of this subsection. Indeed, the following fact is a consequence of the existence of a strong Markov coding for ϕ_t (see [10, 12]) together with the weak-mixing property. For Axiom A flows it was originally proved by Bowen [8] (the counting result is due to Parry–Pollicott [42]). In order to obtain it in our more general context, we need to apply Pollicott's work [46, Subsection 3.5].

Fact A.3. — *The following holds:*

- *The topological entropy of ϕ_t is positive and finite. It is given by*
$$\limsup_{t\longrightarrow\infty} \frac{\log \#\{[\gamma]\in[\Gamma] :\ \gamma \text{ is primitive and } \lambda_1(\rho(\gamma)) \leqslant t\}}{t}$$
and it is denoted by $h = h_\rho$.
- *As $t \longrightarrow \infty$, one has*
$$hte^{-ht}\#\{[\gamma]\in[\Gamma] :\ \gamma \text{ is primitive and } \lambda_1(\rho(\gamma)) \leqslant t\} \longrightarrow 1.$$
- *There exists a unique probability $m = m_\rho$ of maximal entropy for ϕ_t, called the emphBowen–Margulis probability.*
- *Periodic orbits become equidistributed with respect to m: if $\mathrm{Leb}_{[\gamma]}$ denotes the Lebesgue measure of length $\lambda_1(\rho(\gamma))$ supported on the periodic orbit $[\gamma]$, then*
$$hte^{-ht}\sum \frac{1}{\lambda_1(\rho(\gamma))}\mathrm{Leb}_{[\gamma]} \longrightarrow m$$
in the weak-star topology as $t \longrightarrow \infty$. Here the sum is taken over all conjugacy classes of primitive elements γ such that $\lambda_1(\rho(\gamma)) \leqslant t$.

We finish this subsection with an elementary proof of Proposition A.2 inspired by the work of Benoist [4].

Proof of Proposition A.2. — Suppose by contradiction that ϕ_t is not topologically weakly-mixing. By Fact A.1 this implies that there exists a constant $a > 0$ such that the group spanned by the set $\{\lambda_1(\rho(\gamma))\}_{\gamma\in\Gamma}$ is contained in $a\mathbb{Z}$.

Set

$$\partial^4_\infty\Gamma := \{(x_1, x_2, x_3, x_4) \in (\partial_\infty\Gamma)^4 :\ (x_i, x_j) \in \partial^2_\infty\Gamma \text{ for all } i \neq j\}.$$

Since $\{(\gamma_-,\gamma_+)\}_{\gamma\in\Gamma_{\mathrm{H}}}$ is dense in $\partial^2_\infty\Gamma$ (see Gromov [19, Corollary 8.2.G]), Benoist's Theorem 4.6 implies that

$$\{\mathbb{B}(\eta(x'),\xi(y'),\eta(x),\xi(y)) : \ (x',y',x,y)\in\partial^4_\infty\Gamma\}\subset a\mathbb{Z}. \tag{A.1}$$

Fix three different points x',y' and y in $\partial_\infty\Gamma$. Transversality condition (5.2) and the definition of the cross-ratio implies the following: for every $x\in\partial_\infty\Gamma$ such that $(x',y',x,y)\in\partial^4_\infty\Gamma$ there exists a neighbourhood V of x and a point $\xi_{x,y,y'}$ in the projective line $\xi(y)\oplus\xi(y')$ such that

$$\eta(\tilde{x})\cap(\xi(y)\oplus\xi(y'))=\{\xi_{x,y,y'}\} \tag{A.2}$$

holds for every $\tilde{x}\in V$. We have the following.

Lemma A.4. — *Assume that* (A.2) *holds. Then the limit set $\Lambda_{\rho(\Gamma)}$ is not contained in $\xi(y)\oplus\xi(y')$.*

Proof of Lemma A.4. — Suppose by contradiction that $\Lambda_{\rho(\Gamma)}\subset\xi(y)\oplus\xi(y')$. Transversality condition (5.2) implies that for every $x\in\partial_\infty\Gamma$ different from y' and y one has

$$\eta(x)\cap(\xi(y)\oplus\xi(y'))=\{\xi(x)\}.$$

Then by (A.2) the map ξ is not injective and this is a contradiction. □

Because of Lemma A.4 we can take y'' in $\partial_\infty\Gamma$ such that $\xi(y'')$ does not belong to $\xi(y)\oplus\xi(y')$. We can assume further that $y''\neq x'$.

By (A.1) we have again the following: for every $x\notin\{x',y,y',y''\}$ there exists a neighbourhood V of x and a point $\xi_{x,y,y''}$ in the projective line $\xi(y)\oplus\xi(y'')$ such that

$$\eta(\tilde{x})\cap(\xi(y)\oplus\xi(y''))=\{\xi_{x,y,y''}\}$$

holds for every $\tilde{x}\in V$.

As in Lemma A.4 we conclude that $\Lambda_{\rho(\Gamma)}$ cannot be contained in $\xi(y)\oplus\xi(y')\oplus\xi(y'')$ and now an inductive argument yields the desired contradiction. □

A.1.3. The invariant measure of the strong stable lamination

As shown by Margulis [35], for Anosov flows there exists an invariant measure of the strong stable lamination which is exponentially contracted by the flow. In our context this measure is also available: this follows from the thermodynamic formalism as explained by Bowen–Marcus [9, Section 4]. As we shall see in Fact A.6, the importance for us of this measure relies on the fact that describes the probability of maximal entropy of ϕ_t in a different way that the one provided by Fact A.3.

The statement that we need is the following (for precisions see [9]).

Fact A.5. — *Given any $Z_0 \in \mathrm{U}_\rho\Gamma$ and any small relative neighbourhood $W^{cu}_{\mathrm{loc}}(Z_0)$ of Z_0 in $W^{cu}(Z_0)$, there exists a positive and finite Borel measure $\nu^{cu}_{\mathrm{loc}}(Z_0)$ on $W^{cu}_{\mathrm{loc}}(Z_0)$ such that:*

- *The family $\{\nu^{cu}_{\mathrm{loc}}(Z_0)\}_{Z_0\in\mathrm{U}_\rho\Gamma}$ is W^{ss}-invariant*(14)*.*
- *There exists a real number $h^u \geqslant 0$ such that for every t and every $Z_0 \in \mathrm{U}_\rho\Gamma$ one has*

$$(\phi_t)_*(\nu^{cu}_{\mathrm{loc}}(Z_0)) = e^{-h^u t}\nu^{cu}_{\mathrm{loc}}(\phi_t(Z_0)).$$

A.1.4. The Bowen–Margulis probability

By reversing time and disintegrating along flow lines, Fact A.5 yields a family of measures $\{\nu^{ss}_{\mathrm{loc}}(Z_0)\}$ on local strong stable leaves which is expanded by the flow. In the case of Anosov flows, Margulis [35] first showed how the families $\{\nu^{cu}_{\mathrm{loc}}(Z_0)\}$ and $\{\nu^{ss}_{\mathrm{loc}}(Z_0)\}$ with the above properties combine to produce a ϕ_t-invariant finite Borel measure ν in the whole space. This measure coincides, up to scaling, with the Bowen–Margulis probability of the flow.

The statement that we need in our context is the following. Once again, this is a standard fact and the reader is referred for instance to Katok–Hasselblatt's book [28, Section 5 of Chapter 20] for a proof in the case of Anosov flows. With obvious adaptations the same proof works in our setting.

Fact A.6. — *Suppose that $\{\nu^{ss}_{\mathrm{loc}}(Z_0)\}_{Z_0\in\mathrm{U}_\rho\Gamma}$ is a family of measures on the local strong stable leaves with the following properties:*

- *There exists a real number $h^s \geqslant 0$ such that for every t and every $Z_0 \in \mathrm{U}_\rho\Gamma$ one has*

$$(\phi_t)_*(\nu^{ss}_{\mathrm{loc}}(Z_0)) = e^{h^s t}\nu^{ss}_{\mathrm{loc}}(\phi_t(Z_0)).$$

- *For every $Z_0 \in \mathrm{U}_\rho\Gamma$ and every open set A contained in a neighbourhood of Z_0 with local product structure, the map*

$$W^{cu}_{\mathrm{loc}}(Z_0) \longrightarrow \mathbb{R} : \quad Z \mapsto \nu^{ss}_{\mathrm{loc}}(Z)(A \cap W^{ss}_{\mathrm{loc}}(Z))$$

is upper semi continuous.

(14) The precise definition of a W^{ss}-emphinvariant measure can be found in [9, p. 43]. Very informally, this means that if we have a map between two local leaves $W^{cu}_{\mathrm{loc}}(Z_0)$ and $W^{cu}_{\mathrm{loc}}(Z_1)$ which is defined emphfollowing the leaves of W^{ss}, then this map sends the measure $\nu^{cu}_{\mathrm{loc}}(Z_0)$ to the measure $\nu^{cu}_{\mathrm{loc}}(Z_1)$.

Consider the family $\{\nu^{cu}_{\mathrm{loc}}(Z_0)\}_{Z_0\in \mathrm{U}_\rho\Gamma}$ provided by Fact A.5. Then the following holds:

- *If A is an open set contained in a neighbourhood of $Z_0 \in \mathrm{U}_\rho\Gamma$ with local product structure, set*
$$\nu(A) := \int_{Z\in W^{cu}_{\mathrm{loc}}(Z_0)} \nu^{ss}_{\mathrm{loc}}(Z)(A\cap W^{ss}_{\mathrm{loc}}(Z))\mathrm{d}\nu^{cu}_{\mathrm{loc}}(Z_0)(Z).$$
Then this measure extends to a finite Borel measure ν on $\mathrm{U}_\rho\Gamma$ such that for every $t\in\mathbb{R}$ the following holds:
$$(\phi_t)_*\nu = e^{(h^s-h^u)t}\nu.$$
Evaluating the previous equality on $\mathrm{U}_\rho\Gamma$, we obtain that $h^s = h^u$ and that ν is ϕ_t-invariant.
- *The number h^u equals the topological entropy h of the flow and the probability proportional to ν is the Bowen–Margulis probability of ϕ_t.*

A.2. The flows on $\mathrm{U}_o\Gamma$ and $\mathrm{U}_\tau\Gamma$

A.2.1. Explicit conjugations between $\mathrm{U}_\rho\Gamma$, $\mathrm{U}_o\Gamma$ and $\mathrm{U}_\tau\Gamma$

Recall that $\psi_t = \psi^o_t$ is the flow on $\mathrm{U}_o\Gamma$ induced by the translation flow (7.1).

The following lemma implies in particular that the action of Γ on $\partial^2_\infty\Gamma\times\mathbb{R}$ via c_o is proper and co-compact.

Lemma A.7. — *There exists a Hölder homeomorphism $\mathrm{U}_\rho\Gamma \longrightarrow \mathrm{U}_o\Gamma$ that conjugates the flows ϕ_t and ψ_t. Further, for every point $(x_0,y_0,t_0)\in \mathrm{U}_o\Gamma$ the central unstable and strong stable leaves through (x_0,y_0,t_0) are given by*
$$W^{cu}(x_0,y_0,t_0) = \{(x_0,y,t)\in \mathrm{U}_o\Gamma : \ \ y\in\partial_\infty\Gamma\setminus\Gamma\cdot x_0 \text{ and } t\in\mathbb{R}\}$$
and
$$W^{ss}(x_0,y_0,t_0) = \{(x,y_0,t_0)\in \mathrm{U}_o\Gamma : \ \ x\in\partial_\infty\Gamma\setminus\Gamma\cdot y_0\}.$$

Proof. — Consider the map $F_\rho \longrightarrow \partial^2_\infty\Gamma\times\mathbb{R}$ defined by
$$(x,y,\theta,v)\mapsto \left(x,y,-\frac{1}{2}\log|\langle v, J^o\cdot v\rangle_{p,q}|\right),$$
which is easily seen to be Hölder continuous, injective and equivariant. Moreover one can prove that it is proper and surjective, hence a homeomorphism.

The statement involving the flows and the laminations is straightforward. □

We now turn our attention to the translation flow on $\mathrm{U}_\tau\Gamma$. An analogue of Lemma A.7 is also available. In fact, the analogue holds because of the following remark.

Remark A.8. — The cocycles c_o and c_τ are cohomologous. Indeed, this follows from the fact that c_o and c_τ have the same periods and a theorem due to Livšic [33]. Explicitly, let

$$U:\partial_\infty\Gamma\longrightarrow\mathbb{R}:\quad U(x):=\frac{1}{2}\log\frac{\|v_x\|_\tau\|\theta_x\|_\tau}{|\theta_x(J^o\cdot v_x)|}.$$

Then for every γ in Γ and x in $\partial_\infty\Gamma$ one has

$$c_\tau(\gamma,x)-c_o(\gamma,x)=U(\gamma\cdot x)-U(x).$$

A.2.2. Patterson–Sullivan probabilities for c_o and c_τ

The goal of this subsection is to show the existence of a *Patterson–Sullivan probability of dimension* h^u for the cocycle c_o, that is, a probability measure μ_o on $\partial_\infty\Gamma$ such that

$$\frac{\mathrm{d}\gamma_*\mu_o}{\mathrm{d}\mu_o}(x)=e^{-h^u c_o(\gamma^{-1},x)}\tag{A.3}$$

holds for every $\gamma\in\Gamma$[15]. We will see in the next subsection that in fact one has $h^u=h$. The existence of a Patterson–Sullivan probability μ_τ for c_τ follows directly from this one by Remark A.8.

When Γ is the fundamental group of a closed negatively curved manifold, the existence (and uniqueness) of such a probability is proved by Ledrappier [31]. When $\rho(\Gamma)$ is Zariski dense one can apply the work of Quint [47] and for the case of $\mathbb{H}^{p,q-1}$-convex co-compact groups we find also the construction presented by Glorieux–Monclair [18].

Even though Patterson's method [43] works correctly in our setting and produces directly a Patterson-Sullivan probability of dimension h, we choose a shorter approach. Applying Fact A.5 and Lemma A.7 we find an invariant measure $\{\nu^{cu}_{\mathrm{loc}}(u_0)\}_{u_0\in\mathrm{U}_o\Gamma}$ for the strong stable lamination of $\psi_t:\mathrm{U}_o\Gamma\circlearrowleft$ which has the property of being contracted by the flow. Lifting this measure to $\partial^2_\infty\Gamma\times\mathbb{R}$ yields a probability μ_o on $\partial_\infty\Gamma$ satisfying (A.3). Indeed, for closed negatively curved manifolds and the Busemann cocycle this procedure is explained for instance by Babillot in [1, Subsection 7.1]. With obvious adaptations the procedure equally applies in our setting.

(15) The constant h^u is the one introduced in Fact A.5.

Remark A.9. — Recall that $h^u \geqslant 0$. Equality (A.3) shows in fact that h^u is positive. Otherwise the probability μ_o would be Γ-invariant but one can see that this is not possible for a non elementary word hyperbolic group.

We finish this subsection by showing that μ_o has no atoms (this property is needed in the proof of Proposition 7.11). The proof presented here is an adaptation of [18, Proposition 4.3].

LEMMA A.10. — *The measure μ_o has no atoms.*

Proof. — Suppose that there exists an atom $y \in \partial_\infty \Gamma$ for μ_o. Since h^u is positive the point y cannot be fixed by an element of Γ, hence

$$1 = \mu_o(\partial_\infty \Gamma) \geqslant \sum_{\gamma \in \Gamma} e^{-h^u c_o(\gamma^{-1}, y)} \mu_o(y). \tag{A.4}$$

To finish we prove the following.

LEMMA A.11. — *There exists a sequence $\gamma_n \longrightarrow \infty$ such that $c_o(\gamma_n^{-1}, y)$ diverges to $-\infty$.*

Proof of Lemma A.11. — Let x be a point in $\partial_\infty \Gamma$ different from y and $\|\cdot\|$ be any norm on $\mathbb{R}^d$. Take a sequence $\gamma_n \longrightarrow \infty$ such that

$$(\gamma_n)_+ \longrightarrow y \quad \text{and} \quad (\gamma_n)_- \longrightarrow x.$$

By taking a subsequence if necessary we may suppose that γ_n converges uniformly to y on compact sets of $\partial_\infty \Gamma \setminus \{x\}$ (cf. Bowditch [7, Lemma 2.11]). Let $B(x) \subset \partial_\infty \Gamma$ be the complement of a small neighbourhood of x in $\partial_\infty \Gamma$ and $b(y) \subset B(x)$ be a small neighbourhood of y. Then we can suppose that the inclusion $\gamma_n \cdot B(x) \subset b(y)$ holds for every n.

By Proposition 5.2 there exists $\varepsilon > 0$ such that for all n one has

$$\xi(B(x)) \subset B_\varepsilon(S_{d-1}(\rho(\gamma_n))).$$

Take a positive c with the following property: for every n and every vector v in the set $B_\varepsilon(S_{d-1}(\rho(\gamma_n)))$ one has

$$\|\rho(\gamma_n) \cdot v\| \geqslant c \|\rho(\gamma_n)\| \|v\|.$$

Let $v \neq 0$ be a vector in $\xi(y)$. We have that $\rho(\gamma_n^{-1}) \cdot v$ belongs to $B_\varepsilon(S_{d-1}(\rho(\gamma_n)))$ hence

$$\rho(\gamma_n^{-1}) \cdot v \longrightarrow 0$$

as $n \longrightarrow \infty$. The divergence $c_o(\gamma_n^{-1}, y) \longrightarrow -\infty$ follows. □

A combination of (A.4) and Lemma A.11 yields the desired contradiction. □

A.2.3. The Bowen–Margulis probability on $\mathrm{U}_o\Gamma$ and $\mathrm{U}_\tau\Gamma$

Recall that $[\cdot,\cdot]_o$ is the Gromov product of the pair $\{c_o, c_o\}$.

Proposition A.12 (Sambarino [50, Theorem 3.2]). — *The number h^u equals the topological entropy h of ψ_t and the measure*

$$e^{-h[\cdot,\cdot]_o}\mu_o \otimes \mu_o \otimes \mathrm{d}t$$

induces a measure on the quotient space $\mathrm{U}_o\Gamma$ proportional to the Bowen–Margulis probability of ψ_t.

Proof. — From explicit computations one can show that

$$e^{-h^u[\cdot,\cdot]_o}\mu_o \otimes \mu_o \otimes \mathrm{d}t$$

equals the product of measures ν^{cu}_{loc} and ν^{ss}_{loc} as in Fact A.6. □

We now state the desired result of this appendix: the analogue of [50, Proposition 4.3]. Provided with Proposition A.12, the same proof applies in our setting.

Proposition A.13 (Sambarino [50, Proposition 4.3]). — *There exists a positive $M = M_{\rho,o}$ such that*

$$Me^{-ht} \sum_{\gamma\in\Gamma_{\mathrm{H}},\ell_{c_o}(\gamma)\leqslant t} \delta_{\gamma_-}\otimes\delta_{\gamma_+} \longrightarrow e^{-h[\cdot,\cdot]_o}\mu_o\otimes\mu_o$$

as $t \longrightarrow \infty$ on $C_c^*(\partial^2_\infty\Gamma)$.

For the flow on $\mathrm{U}_\tau\Gamma$ we obtain analogue results.

Remark A.14. — Let $[\cdot,\cdot]_\tau$ be the Gromov product of the pair $\{c_o, c_\tau\}$ defined in Subsection 8.2. The same arguments of Proposition A.12 and Proposition A.13 apply to obtain that

$$e^{-h[\cdot,\cdot]_\tau}\mu_o \otimes \mu_\tau \otimes \mathrm{d}t$$

induces the Bowen–Margulis probability of the translation flow on $\mathrm{U}_\tau\Gamma$ and that there exists a positive $M' = M'_{\rho,\tau}$ such that

$$M'e^{-ht} \sum_{\gamma\in\Gamma_{\mathrm{H}},\ell_{c_\tau}(\gamma)\leqslant t} \delta_{\gamma_-}\otimes\delta_{\gamma_+} \longrightarrow e^{-h[\cdot,\cdot]_\tau}\mu_o\otimes\mu_\tau$$

as $t \longrightarrow \infty$ on $C_c^*(\partial^2_\infty\Gamma)$.

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