

# ANNALES DE L'INSTITUT FOURIER

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*Annales de l'institut Fourier*, tome 19, n° 2 (1969), p. 339-353

[<http://www.numdam.org/item?id=AIF\\_1969\\_\\_19\\_2\\_339\\_0>](http://www.numdam.org/item?id=AIF_1969__19_2_339_0)

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## UNIFORM APPROXIMATION OF HARMONIC FUNCTIONS

by G. F. VINCENT-SMITH

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### Introduction.

Let  $\omega$  be a bounded open set in Euclidian  $n$ -space ( $n > 1$ ), with closure  $\overline{\omega}$  and frontier  $\omega^*$ . Corollary 1 below gives a necessary and sufficient condition that each continuous real-valued function on  $\overline{\omega}$  harmonic in  $\omega$ , may be uniformly approximated on  $\overline{\omega}$  by functions harmonic in a neighbourhood of  $\overline{\omega}$ . The purpose of this paper is to extend corollary 1 to axiomatic potential theory.

Suppose  $a_p$  is a sequence of points chosen one from each domain in  $\bigcap \overline{\omega}$ . Let  $\Phi_n^{a_p}$  be the elementary harmonic functions relative to  $a_p$  [10, § 1]. Then  $\Phi_n^{a_p}$  is a potential of support  $a_p$ ,  $n = 1, 2, \dots$ . If  $C(\overline{\omega})$  denotes the space of continuous real-valued functions on  $\overline{\omega}$ , then following Deny [9], [10, § 4] and de La Pradelle [16], we consider the following linear function spaces :

$$M = \{f \in C(\overline{\omega}) : f \text{ is harmonic in } \omega\};$$

$$L = \{f \in C(\overline{\omega}) : f \text{ extends to a function harmonic in a neighbourhood } U_f \text{ of } \overline{\omega}\};$$

$$K = \{f \in C(\overline{\omega}) : f \text{ extends to the difference of two potentials with compact support contained in } \bigcap \overline{\omega}\};$$

$$J = \{f \in C(\overline{\omega}) : f \text{ extends to a function in the linear span of the elementary harmonic functions } \Phi_n^{a_p}\}.$$

Then  $J \subset K \subset L \subset M$ , and Deny [10, th. 5] proves the following approximation theorem.

**THEOREM 1.** —  *$J$  is uniformly dense in  $M$  if and only if the sets  $\int \omega$  and  $\int \bar{\omega}$  are effilé (thin) at the same points.*

The points at which  $\int \omega$  is not thin [7, ch. VII, § 1] are precisely the regular points of  $\omega^*$  for the Dirichlet problem [7, ch. VIII, § 6], while the points where  $\int \bar{\omega}$  is not thin are precisely the stable points of  $\omega^*$  for the Dirichlet problem.

Suppose now that  $\omega$  is a relatively compact open subset of a harmonic space  $\Omega$  which satisfies Brelot's axioms 1, 2 and 3, and on which there exists a strictly positive potential. Suppose also that the topology of  $\Omega$  has a countable base of completely determining open sets, that potentials with the same one point support are proportional, and that adjoint potentials with one point support are proportional. De La Pradelle [16, th. 5] proves the following generalisation of theorem 1.

**THEOREM 1'.** —  *$K$  is uniformly dense in  $M$  if and only if the sets  $\int \omega$  and  $\int \bar{\omega}$  are thin at the same points.*

Deny's proof of theorem 1 consists of showing that the same measures on annihilate  $J$  and  $M$ , and the same method is used to prove theorem 1'. In this paper the conditions on  $\Omega$  are relaxed, and the following corollary to theorem 1 is generalised.

**COROLLARY 1.** —  *$L$  is uniformly dense in  $M$  if and only if every regular point of  $\omega^*$  is stable.*

The proof of corollary 1, using elementary harmonic functions, does not adapt to axiomatic potential theory. In example 2 we give a proof which does generalise. This proof is rather satisfying, since it uses Bauer's characterisation of regular points, and the following generalisation of the Stone-Weierstrass theorem [13, th. 5].

**THEOREM 2.** — *Suppose that  $X$  is a compact Hausdorff space, that  $L$  is a linear subspace of  $C(X)$  which contains the*

constant functions, separates the points of  $X$ , and has the weak Riesz separation property, and that  $L$  is contained in the linear subspace  $M$  of  $C(X)$ . Then  $L$  is uniformly dense in  $M$  if and only if  $\partial_L(X) = \partial_M(X)$ .

$L$  is said to have the weak Riesz separation property (R.s.p.) if whenever  $\{f_1, f_2, g_1, g_2\} \subset L$  with  $f_1 \vee f_2 < g_1 \wedge g_2$ , there exists  $h \in L$  with  $f_1 \vee f_2 \leq h \leq g_1 \wedge g_2$ . The Choquet boundary of  $M$  is denoted  $\partial_M(X)$  [15] and Bauer [1, th. 6] shows that in the classical case  $\partial_M(\bar{\omega})$  is precisely the set of regular points of  $\omega^*$ . Brelot [7, ch. VIII, § 1] remarks that this remains true when  $\omega$  is a relatively compact open subset of a harmonic space satisfying Brelot's axioms 1, 2 and 3', and that in this case  $\partial_L(\bar{\omega})$  is precisely the set of stable points of  $\omega^*$ . Using Bauer's results, corollary 1 is an immediate consequence of Theorem 2, both in the classical case, and when  $\omega$  is a relatively compact open subset of a harmonic space satisfying Brelot's axioms 1, 2 and 3'.

If  $\omega$  is a relatively compact open subset of one of the harmonic spaces of Boboc and Cornea [4], which are more general than those of Brelot, then the set of regular points of  $\omega^*$  corresponds not to  $\partial_M(\bar{\omega})$  but to  $\omega^* \cap \partial_W(\bar{\omega})$ , where  $W \subset C(\bar{\omega})$  is the min-stable wedge of continuous functions on  $\bar{\omega}$  superharmonic in  $\omega$ . In this case we need a strengthened form of theorem 2, which, together with this characterisation of regular points, has corollary 1 as a direct consequence. This we supply in theorem 4.

In order to strengthen theorem 2 we consider min-stable wedges  $\mathcal{G} \subset W$  in  $C(X)$ , and a geometric simplex  $(X, \mathcal{G}, L)$ . In theorem 4 we give a sufficient condition that  $L$  be uniformly dense in the space  $M$  of continuous  $W$ -affine functions on  $X$ . This condition is given in terms of the Choquet boundaries  $\partial_W(X)$  and  $\partial_{\mathcal{G}}(X)$ . In lemma 5 a pair of conditions equivalent to this is given. These are of a more analytic nature. Theorem 4 is deduced from proposition 1, which is a characterisation of geometric simplexes. This is proved by repeated use of filtering arguments together with the following form of Dini's theorem.

**THEOREM 3.** — *If  $\{f_i : i \in I\}$  is an upward filtering family in  $C(X)$  and  $g$  is an upper bounded upper semicontinuous*

function such that  $g < \sup \{f_i : i \in I\}$ , then  $g < f_{i_0}$  for some  $i_0 \in I$ .

$f > 0$  ( $\geq 0$ ) will mean that  $f(x) > 0$  ( $\geq 0$ ) for all  $x \in X$ .

### A characterisation of geometric simplexes.

Let  $X$  be a compact Hausdorff space, and let  $\mathcal{G} \subset W$  be min-stable wedges in  $C(X)$ . If  $f \wedge g \in W$  whenever  $f, g \in W$  then  $W$  is said to be *min-stable*. We shall assume that  $\mathcal{G}$  contains a function  $p \geq 1$  and a function  $q < -1$ . The Choquet theory for min-stable wedges has been developed in [11] [5] where proofs of the following results may be found.

The wedge  $W$  induces a partial order  $\prec_W$  on the positive regular Borel measures on  $X$  given by the formula

$$\mu \prec_W \lambda, \quad \lambda(f) \leq \mu(f) \text{ whenever } f \in W.$$

A measure which is maximal for  $\prec_W$  is said to be *W-extremal*. A measure  $\mu$  is *W-extremal* if and only if

$$(1) \quad \mu(g) = \inf \{\mu(f) : g < f \in W\}$$

whenever  $g \in -W$  [5, Th. 1.2]. An extended real-valued function  $g$  on  $X$  is  $\omega$ -concave if the upper integral  $\int^+ g d\mu \leq g(x)$  whenever  $\varepsilon_x \prec_W \mu$ . The function  $g$  is *W-affine* if both  $g$  and  $-g$  are  $W$ -concave. The min-stable wedge of lower bounded extended real-valued lower semicontinuous  $\omega$ -concave functions on  $X$  will be denoted  $\hat{W}$ .

LEMMA 1. — [11, Th. 1] [5, Cor. 1.4 d)]. Each  $f \in \hat{W}$  is the pointwise supremum of an upward filtering family in  $W$ .

A closed subset  $A$  of  $X$  is a *W-face* ( $W$ -absorbent set [5, § 2],  $W$ -extreme set [11, § 2]) if for each  $x \in A$

$$\mu(X \setminus A) = 0 \text{ whenever } \varepsilon_x \prec_W \mu.$$

If  $A$  is a  $W$ -face and  $f \in \hat{W}$  then the function  $f_A^\infty$ , equal to  $f$  on  $A$  and to  $+\infty$  on  $X \setminus A$ , belongs to  $\hat{W}$  [11, § 2]. The  $W$ -faces are ordered by inclusion, and each  $W$ -face contains a minimal  $W$ -face. The measure  $\varepsilon_x$  is  $W$ -extremal if and only if  $x$  belongs to a minimal  $W$ -face. The Choquet boundary

of  $W$  is the union of all minimal  $W$ -faces of  $X$ , and is denoted  $\partial_W(X)$  [5, § 2]. Each  $\mathcal{G}$ -face is a  $W$ -face, so that each minimal  $\mathcal{G}$ -face contains at least one minimal  $W$ -face.

LEMMA 2. — [2, Satz 2] [5, Cor. 2.1] *A function  $f \in \hat{W}$  is positive if and only if it is positive on  $\partial_W(X)$ .*

We say that  $W$  distinguishes the points  $x, y \in X$  if there exists  $f, g \in W$  such that

$$f(x)g(y) \neq f(y)g(x).$$

If  $W$  contains the constant functions, then  $W$  distinguishes  $x$  and  $y$  if and only if  $W$  separates  $x$  and  $y$ . The subspace  $(W - W)/p = \{(f - g)/p : f, g \in W\}$  is a sublattice of  $C(X)$  containing the constant functions.  $(W - W)/p$  separates points of  $X$  if and only if  $W$  distinguishes points of  $X$ . By Stone's theorem,  $W - W$  is uniformly dense in  $C(X)$  if and only if  $W$  distinguishes points of  $X$ . The following lemma is an immediate consequence of [5, Th. 2.1 c)].

LEMMA 3. —  *$W$  distinguishes  $x, y \in \partial_W(X)$  if and only if  $x$  and  $y$  belong to different minimal  $W$ -faces of  $X$ .*

Example 1. — Let  $X = [0, 1] \times [0, 1]$ , and let  $\mathcal{G} = \{f \in C(X) : y \rightsquigarrow f(x, y) \text{ is convex for each } x, \text{ and } x \rightsquigarrow f(x, y) \text{ is affine with } f(1, y) = 2f(0, y) \text{ for each } y\}$ . Then the sets  $A = \{(x, 0) : x \in [0, 1]\}$  and  $B = \{(x, 1) : x \in [0, 1]\}$  are minimal  $\mathcal{G}$ -faces.  $\mathcal{G}$  separates, yet does not distinguish the points of  $A$ . The Choquet boundary

$$\partial_{\mathcal{G}}(X) = A \cup B.$$

The  $\mathcal{G}$ -affine functions are the  $f \in \mathcal{G}$  which are affine in  $y$  for each  $x$ .

LEMMA 4. — *If  $\mathcal{G} \subset W$  are min-stable wedges in  $C(X)$ , and if  $\mathcal{G}$  contains a positive function  $p$  and a negative function  $q$ , then the following conditions are equivalent:*

(i) *For each pair of (disjoint) minimal  $\omega$ -faces  $A_1, A_2$ , there exists a pair of (disjoint)  $\mathcal{G}$ -faces  $B_1, B_2$ , such that  $A_1 \subset B_1$  and  $A_2 \subset B_2$ ;*

(ii) *Same statement as (i) but with  $B_1, B_2$  minimal  $\mathcal{G}$ -faces;*

(iii)  $\partial_W(X) \subset \partial_{\mathcal{G}}(X)$  and  $\mathcal{G}$  distinguishes points of  $\partial_W(X)$  which are distinguished by  $W$ .

*Proof.* — (i)  $\Rightarrow$  (ii). Let  $A$  be a minimal  $W$ -face, and put  $G = \bigcap \{F : F \text{ is an } \mathcal{G}\text{-face and } A \subset F\}$ . Then  $G$  is an  $\mathcal{G}$ -face, and contains a minimal  $\mathcal{G}$ -face  $H$ . Now  $H$  is a  $W$ -face and contains a minimal  $W$ -face  $A'$ . If  $A \cap A' = \emptyset$ , then there exist disjoint  $\mathcal{G}$ -faces  $B, B'$  such that  $A \in B$  and  $A' \in B'$ . Then  $B \cap G$  is an  $\mathcal{G}$ -face properly contained in  $G$ , which contradicts the definition of  $G$ . Therefore  $A = A'$ , so that  $G \subset H$  and  $G$  is a minimal  $\mathcal{G}$ -face. It follows immediately that if  $A_1, A_2$  are disjoint minimal  $W$ -faces, then  $A_1 \subset G_1$  and  $A_2 \subset G_2$ , where  $G_1$  and  $G_2$  are disjoint minimal  $\mathcal{G}$ -faces.

(ii)  $\Rightarrow$  (iii).  $\partial_W(X) = \bigcup \{A : A \text{ is a minimal } W\text{-face}\} \subset \bigcup \{B : B \text{ is a minimal } \mathcal{G}\text{-face}\} = \partial_{\mathcal{G}}(X)$ . Suppose  $W$  distinguishes  $x_1$  and  $x_2 \in \partial_W(X)$ , then by lemma 3 there are disjoint minimal  $W$ -faces  $A_1$  and  $A_2$  with  $x_1 \in A_1$  and  $x_2 \in A_2$ . Therefore there are disjoint minimal  $\mathcal{G}$ -faces  $B_1, B_2$  with  $x_1 \in A_1 \subset B_1$  and  $x_2 \in A_2 \subset B_2$ , and by lemma 3  $\mathcal{G}$  distinguishes  $x_1$  and  $x_2$ .

(iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i). If  $A_1$  and  $A_2$  are disjoint minimal  $W$ -faces, then the points  $x_1 \in A_1$  and  $x_2 \in A_2$  are distinguished by  $W$ . Therefore  $x_1$  and  $x_2$  are distinguished by  $\mathcal{G}$ . Since  $x_1, x_2 \in \partial_W(X) \subset \partial_{\mathcal{G}}(X)$  there are disjoint minimal  $\mathcal{G}$ -faces  $B_1, B_2$  with  $x_1 \in B_1$  and  $x_2 \in B_2$ . Since  $A_1$  is minimal  $A_1 \subset A_1 \cap B_1$ , so that  $A_1 \subset B_1$ . Similarly  $A_2 \subset B_2$ .

If  $L$  and  $M$  are linear subspaces of  $C(X)$ , then we will put

$$\mathcal{L} = \{f_1 \wedge \cdots \wedge f_r : f_i \in L, \quad i = 1 \dots r\}$$

and

$$\mathcal{M} = \{f_1 \wedge \cdots \wedge f_r : f_i \in M, \quad i = 1 \dots r\}.$$

Then  $\mathcal{L}$  and  $\mathcal{M}$  are min-stable wedges in  $C(X)$  and if the functions in  $L$  are  $\mathcal{G}$ -affine then  $\mathcal{L} \subset \mathcal{G}$ .

Suppose  $L$  is a linear subspace of continuous  $\mathcal{G}$ -affine functions on  $X$ . The triple  $(X, \mathcal{G}, L)$  is a *geometric simplex* if given  $f \in -\mathcal{G}$  and  $g \in \mathcal{G}$  with  $f < g$ , then there exists

$h \in L$  with  $f \leq h \leq g$  [5, § 4]. We have assumed that  $p, q \in \mathcal{G}$  with  $p > 0$  and  $q < 0$ , so that  $\alpha p < q$  for some  $\alpha < 0$ . If  $(X, \mathcal{G}, L)$  is a geometric simplex it follows that  $L$  contains an element  $l > 0$ .

**PROPOSITION 1.** —  $(X, \mathcal{G}, L)$  is a geometric simplex if and only if  $L$  has the weak R.s.p.,  $\partial \mathcal{G}(X) \subset \partial \mathcal{L}(X)$  and  $\mathcal{L}$  distinguishes points of  $\partial \mathcal{G}(X)$  which are distinguished by  $\mathcal{G}$ .

*Proof.* — Let  $(X, \mathcal{G}, L)$  be a geometric simplex and suppose that  $\{f_1, f_2, g_1, g_2\} \subset L$  with  $f_1 \vee f_2 < g_1 \wedge g_2$ . Since  $g_1 \wedge g_2 \in \hat{\mathcal{G}}$  there exists a family  $\Lambda = \{h_i \in \mathcal{G} : h_i < g_1 \wedge g_2, i \in I\}$  filtering up to  $g_1 \wedge g_2$ . By Dini's theorem there exists  $h_{i_0} \in \Lambda$  such that  $f_1 \vee f_2 < h_{i_0} < g_1 \wedge g_2$ . Similarly, there exists  $h_{j_0} \in -\mathcal{G}$  such that  $f_1 \vee f_2 < h_{j_0} < h_{i_0} < g_1 \wedge g_2$ . Since  $(X, \mathcal{G}, L)$  is a geometric simplex there exists  $h \in L$  such that

$$f_1 \vee f_2 \leq h_{j_0} \leq h \leq h_{i_0} \leq g_1 \wedge g_2$$

and  $L$  has the weak R.s.p.

Suppose  $x_i \in \partial \mathcal{G}(X)$ ,  $i = 1, 2$ , and  $f_j \in -\mathcal{L}$ ,  $j = 1, 2$ . Then  $f_j \in -\bar{\mathcal{G}}$  and by (1)

$$\begin{aligned} f_j(x_i) &= \inf \{h(x_i) : f_j < h \in \mathcal{G}\}, \\ (2) \quad &= \inf \{g(x_i) : g \in L, f_j < g < h \in \mathcal{G}\}, \end{aligned}$$

since  $(X, \mathcal{G}, L)$  is a geometric simplex. Therefore  $x_i \in \partial \mathcal{L}(X)$ , and  $\partial \mathcal{G}(X) \subset \partial \mathcal{L}(X)$ . If  $\varepsilon > 0$  then by (2) there exists  $g_1, g_2 \in L$  such that

$$|g_j(x_i) - f_j(x_i)| < \varepsilon, \quad i, j = 1, 2.$$

If  $f_1$  and  $f_2$  distinguish  $x_1$  and  $x_2$ , and  $\varepsilon$  is small enough, then  $g_1$  and  $g_2$  distinguish  $x_1$  and  $x_2$ , and the conditions of the proposition are necessary.

Suppose that  $(X, \mathcal{G}, L)$  satisfies the given conditions, and that  $f \in -\mathcal{G}$ ,  $g \in \mathcal{G}$  with  $f < g$ . If  $A$  is a minimal  $\mathcal{G}$ -face, then by lemma 4  $A$  is contained in a minimal  $\mathcal{L}$ -face  $B$ . If  $\alpha$  is the smallest real number such that  $\alpha l \geq f$  on  $B$ , then

$$D = \{x \in B : (\alpha l - f)(x) = 0\} = \{x \in X : (\alpha l - f) \stackrel{\infty}{B} (x) = 0\}$$



is a  $\mathcal{G}$ -face [5, prop. 2.2].  $D$  contains a minimal  $\mathcal{G}$ -face  $A'$ , and by lemma 4,  $A = A'$ . Similarly

$$A \subset \{x \in B : (g - \beta l)(x) = 0\},$$

where  $\beta$  is the greatest real number such that  $\beta l \leq g$  on  $B$ . Since  $l$  is strictly positive,  $\alpha < \beta$ , and if  $\alpha < \gamma < \beta$ , then  $f < \gamma l < g$  on  $B$ . By lemma 1, the function  $(\gamma l)_B^\infty$  is the supremum of an increasing filtering family  $\{f_i \in \mathcal{L} : i \in I\}$ . Since  $f < (\gamma l)_B^\infty$ , it follows from Dini's theorem that  $f < f_{i_0}$  ( $= h_1 \wedge \dots \wedge h_n$ ;  $h_r \in L$ ,  $r = 1, \dots, n$ ) for some  $i_0 \in I$ . Therefore there exists  $h \in L$  with  $f < h$  on  $X$  and  $h < g$  on  $B$ .

Suppose that  $f < h_1 \wedge h_2$  with  $h_1, h_2 \in L$ . Since  $L$  has the weak R.s.p. and contains a positive function, the family  $\{k \in L : k < h_1 \wedge h_2\}$  filters up. Therefore

$$\bar{k} = \sup \{k' \in \mathcal{L} : k' < h_1 \wedge h_2\} = \sup \{k \in L : k < h_1 \wedge h_2\}.$$

Thus  $\bar{k}$  is the supremum of a filtering family of continuous  $\mathcal{L}$ -affine functions and is therefore  $\mathcal{L}$ -affine and lower semicontinuous. Therefore  $\bar{k} \in \hat{\mathcal{G}}$ . It follows from (1) that  $\bar{k} = h_1 \wedge h_2$  on  $\partial_{\mathcal{G}}(X)$ . Since  $\partial_{\mathcal{G}}(X) \subset \partial_{\mathcal{L}}(X)$ , the function  $\bar{k} - f$  is strictly positive on  $\partial_{\mathcal{G}}(X)$ . By lemma 2,  $\bar{k} > f$ . By Dini's theorem there exists  $h \in L$  such that  $f < h < h_1 \wedge h_2$ , and the family  $\mathcal{F} = \{h \in L : f < h\}$  is filtering down.

Therefore the function  $\underline{h} = \inf \{h \in L : f < h\}$  is upper semicontinuous  $\mathcal{L}$ -affine and  $\mathcal{G}$ -affine. If  $A$  is a minimal  $\mathcal{G}$ -face, then there exists  $h \in \mathcal{F}$  with  $h < g$  on  $A$ . Therefore  $\underline{h} < g$  on  $\partial_{\mathcal{G}}(X)$ , and by lemma 2,  $\underline{h} < g$ . By Dini's theorem there exists  $h \in L$  such that  $f < h < g$ . Therefore  $(X, \mathcal{G}, L)$  is a geometric simplex.

We may now extend the density theorem in [13].

**THEOREM 4.** — Suppose that  $\mathcal{G} \subset W$  are min-stable wedges in  $C(X)$ , and that  $\mathcal{G}$  contains a positive function  $p$  and a negative function  $q$ . Let  $M = \{f \in C(X) : f \text{ is } W\text{-affine}\}$  and let  $L \subset C(X)$  be a linear subspace of  $\mathcal{G}$ -affine functions. If  $(X, \mathcal{G}, L)$  is a geometric simplex and if  $\partial_W(X) \subset \partial_{\mathcal{G}}(X)$  and if  $\mathcal{G}$  distinguishes points of  $\partial_W(X)$  which are distinguished by  $W$ , then  $L$  is uniformly dense in  $M$ .

*Proof.* — It follows from proposition 1 that  $\partial_W(X) \subset \partial_{\mathcal{L}}(X)$  and that  $\mathcal{L}$  distinguishes points of  $\partial_W(X)$  distinguished by  $W$ . Therefore  $(X, W, L)$  is a geometric simplex. If  $f \in M$  and  $\varepsilon > 0$ , then by lemma 1 and by Dini's theorem there exist  $h \in -W$ ,  $k \in W$  such that

$$f + \varepsilon q < h < k < f + \varepsilon p.$$

Since  $(X, \omega, L)$  is a geometric simplex, there exists  $g \in L$  such that  $f + \varepsilon q < h \leq g \leq k < f + \varepsilon p$ , and  $L$  is uniformly dense in  $M$ .

Suppose that  $L \subset M$  are linear subspaces of  $C(X)$  containing the constant functions, and that  $L$  has the weak R.s.p. Then  $\mathcal{L}$  and  $\mathcal{M}$  are min-stable wedges,  $\partial_{\mathcal{L}}(X) = \partial_L(X)$  the Choquet boundary of  $L$ , and  $\partial_{\mathcal{M}}(X) = \partial_M(X)$ , the Choquet boundary of  $M$  [15], and  $(X, \mathcal{L}, L)$  is a geometric simplex. Since  $L$  contains the constant functions, points are distinguished by  $\mathcal{L}$  (resp.  $\mathcal{M}$ ) if and only if they are separated by  $L$  (resp.  $M$ ). We have therefore the following corollary to theorem 4.

**COROLLARY 1.** — [13, cor. to th. 5]. *If  $\partial_L(X) = \partial_M(X)$  and  $L$  separates the points of  $\partial_M(X)$  which are separated by  $M$ , then  $L$  is uniformly dense in  $M$ .*

We may replace the conditions in proposition 1 and theorem by a pair of conditions very similar to those used by D. A. Edwards [12].

Suppose we are given wedges  $W_0$  and  $\mathcal{G}_0$  such that the min-stable wedges  $\{f_1 \wedge \cdots \wedge f_r : f_i \in \omega_0, i = 1, \dots, r\}$  and  $\{f_1 \wedge \cdots \wedge f_r : f_i \in \mathcal{G}_0, i = 1, \dots, r\}$  are uniformly dense in  $W$  and  $\mathcal{G}$  respectively. For example, in corollary 1 we could take  $M = W_0$  and  $L = \mathcal{G}_0$ . Since  $\mathcal{G}$  contains a positive element it follows that  $\mathcal{G}_0$  contains a positive element which we may take as  $p$ . We consider the following conditions :

(a) If  $x \in \partial_W(X)$ ,  $\varepsilon > 0$  and  $f_1, f_2 \in \mathcal{G}_0$ , then there exists  $g \in -\mathcal{G}$  such that  $g < f_1 \wedge f_2$  and  $f_1 \wedge f_2(x) < g(x) + \varepsilon$ .

(a') Same as (a), but with  $g \in -\mathcal{G}_0$ .

(b) If  $x_1$  and  $x_2 \in \partial_W(X)$ ,  $\varepsilon > 0$  and  $0 < f \in W_0$ , then there exists  $g \in \mathcal{G}_0$  such that  $|f(x_i) - g(x_i)| < \varepsilon, i = 1, 2$ .

Suppose that  $\mathcal{G}_0$  satisfies condition (a). Then there exists  $\{h_1, \dots, h_n\} \subset -\mathcal{G}_0$  such that  $g \leq h_1 \vee \cdots \vee h_n < f_1 \wedge f_2$ .

Then  $h_i < f_1 \wedge f_2$  and  $f_1 \wedge f_2(x) < h_i(x) + \varepsilon$  for some  $i$  with  $1 \leq i \leq n$ . Therefore (a) implies (a') and since (a') implies (a), the two conditions are equivalent.

LEMMA 5. —  $\partial_W(X) \subset \partial_g(X)$  if and only if  $\mathcal{G}_0$  satisfies condition (a).

*Proof.* — It follows from (1) that  $x \in \partial_g(X)$  if and only if whenever  $f \in \mathcal{G}$  there exists  $g \in -\mathcal{G}$  with  $g < f$  and  $f(x) < g(x) + \varepsilon$ . Therefore the condition is necessary.

If  $\mathcal{G}_0$  satisfies condition (a) then it satisfies (a'). Consider  $x \in \partial_W(X)$ ,  $\varepsilon > 0$  and  $f \in \mathcal{G}$ . If  $\delta > 0$  choose  $\{f_i, \dots, f_n\} \subset \mathcal{G}$  such that  $|f - f_1 \wedge \dots \wedge f_n| < \delta$ . Let

$$c = \min \{f_i(x) : i = 1, \dots, n\}.$$

By condition (a') there exists  $k \in \mathcal{G}_0$  such that  $k(x) = -c$  and  $\{g_1, \dots, g_n\} \subset -\mathcal{G}_0$  such that

$$g_i < (f_i + k) \wedge 0, \quad g_i(x) > -\varepsilon/n, \quad i = 1, \dots, n.$$

Then

$$g_0 = \Sigma \{g_i : i = 1, \dots, n\} \\ < (f_1 + k) \wedge \dots \wedge (f_n + k) = f_1 \wedge \dots \wedge f_n + k,$$

and  $g_0(x) > -\varepsilon$ . Therefore  $g_0 - k = h \in -\mathcal{G}_0$  and  $h < f_1 \wedge \dots \wedge f_n < f + \delta$  with  $h(x) > c - \varepsilon > f(x) - \delta - \varepsilon$ . Choosing  $\delta$  such that  $\delta(1 + p(x)) < \varepsilon$  and then putting  $g = h - \delta p$  it follows that  $g < f$  and  $g(x) > f(x) - 2\varepsilon$ . It follows from (1) that  $x \in \partial_g(X)$  and that  $\partial_W(X) \subset \partial_g(X)$ .

LEMMA 6. —  $\partial_W(X) \subset \partial_g(X)$  and  $\mathcal{G}$  distinguishes points of  $\partial_W(X)$  which are distinguished by  $W$  if and only if  $\mathcal{G}_0$  and  $W_0$  satisfy conditions (a) and (b).

*Proof.* — If  $W$  distinguishes the points  $x_1$  and  $x_2$  of  $\partial_W(X)$ , then there exists  $f \in W$  such that

$$f(x_1)p(x_2) \neq f(x_2)p(x_1).$$

Since  $p \in W$ , we may assume that  $f > 0$ . If  $\mathcal{G}_0$  satisfies condition (b) and  $\varepsilon < 0$ , then there exists  $g \in \mathcal{G}_0$  such that  $|g(x_i) - f(x_i)| < \varepsilon$ ,  $i = 1, 2$ . If  $\varepsilon$  is small enough, then  $g(x_1)p(x_2) \neq g(x_2)p(x_1)$ , and  $\mathcal{G}$  distinguishes  $x_1$  and  $x_2$ .

If  $\mathcal{G}_0$  also satisfies condition (a) then  $\partial_W(X) \subset \partial_{\mathcal{G}}(X)$ , by lemma 4.

Conversely, suppose that  $x_1, x_2 \in \partial_W(X)$ ,  $\varepsilon > 0$  and  $0 < f \in W_0$ . We consider the following cases:

(i)  $f(x_1)p(x_2) = f(x_2)p(x_1)$ . Choose real  $c$  such that  $cp(x_1) = f(x_1)$  and  $cp(x_2) = f(x_2)$ . Then  $cp = g \in \mathcal{G}_0$  and  $|f(x_i) - g(x_i)| = 0 < \varepsilon$ ,  $i = 1, 2$ .

(ii)  $f(x_1)p(x_2) < f(x_2)p(x_1)$ . If  $\partial_W(X) \subset \partial_{\mathcal{G}}(X)$  and  $\mathcal{G}$  distinguishes points of  $\partial_W(X)$  distinguished by  $W$ , then  $\mathcal{G}$  distinguishes  $x_1$  and  $x_2$ , and  $x_1$  belongs to a minimal  $\mathcal{G}$ -face  $A$ . Then the function  $0_A^\infty \in \hat{\mathcal{G}}$ . It follows from lemma 1 that there exists  $k \in \mathcal{G}$  such that  $k(x_1) < 0$  and  $k(x_2) > 0$ . Since  $\mathcal{G}_0$  is a wedge containing  $p$ , there exists  $h \in \mathcal{G}_0$  such that  $h(x_1) = 0$  and  $h(x_2) > 0$ . Define  $g \in \mathcal{G}_0$  by the formula

$$g = \frac{f(x_1)}{p(x_1)} p + \frac{f(x_2)p(x_1) - f(x_1)p(x_2)}{f(x_1)h(x_2)} h$$

Then  $|f(x_i) - g(x_i)| = 0 < \varepsilon$ ,  $i = 1, 2$ , and  $W_0$  and  $\mathcal{G}_0$  satisfy the conditions (a) and (b).

### Application to axiomatic potential theory.

Let  $\omega$  be an open relatively compact MP subset [4, § 2] of a harmonic space which satisfies one of the axiomatic systems [4,  $H_0, \dots, H_4$ ] [3,  $A_1, \dots, A_3$ ]. Let

$W = \{f \in C(\bar{\omega}) : f \text{ is superharmonic in } \omega\}$ ,

$\mathcal{G} = \{f \in C(\bar{\omega}) : f \text{ extends to a function superharmonic in an open neighbourhood } U_f \text{ of } \bar{\omega}\}$ ,

and define  $L$  and  $M$  as in the introduction. Then  $\mathcal{G} \subset W$  are min-stable wedges in  $C(\bar{\omega})$ ,  $M$  is the space of continuous  $W$ -affine functions, and  $L$  is the space of continuous  $\mathcal{G}$ -affine functions on  $\bar{\omega}$ . We suppose that  $\mathcal{G}$  contains a positive function  $p$  and a negative function  $q$ , and distinguishes points of  $\omega^*$ .

LEMMA 7. — If  $A$  is a minimal  $W$ -face of  $\bar{\omega}$ , then  $A \cap \omega^* \neq \emptyset$ .

*Proof.* — The function  $0_A$  belongs to  $\hat{W}$  and is therefore hyperharmonic [4, § 1]. Suppose  $A \cap \omega^* = \emptyset$ , then  $0_A^\infty = p$

is non-negative on  $\omega \setminus A$ , and for any point  $x_0 \in \omega^*$ ,  $\liminf \{(0_A^\omega - p)(x) : x \rightarrow x_0\} = \infty$ . Since  $\omega$  is an MP set,  $0_A^\omega - p > 0$  and therefore  $A = \emptyset$ . Therefore  $A \cap \omega^* \neq \emptyset$ .

We now recall the definitions and some properties of regular and stable points of  $\omega^*$ . If  $f \in C(\omega^*)$  put  $\Phi_f^\omega = \{\nu : \nu \text{ is hyperharmonic in } \omega \text{ and}$

$$\liminf \{\nu(x) : x \in \omega, x \rightarrow x_0\} \geq f(x_0), x \in \omega^*\},$$

put  $\bar{H}_f^\omega = \inf \{\nu : \nu \in \Phi_f^\omega\}$ , and put  $\underline{H}_f^\omega = -\bar{H}_{(-f)}^\omega$ . Since  $(\mathcal{G} - \mathcal{G})|_{\omega^*}$  is uniformly dense in  $C(\omega^*)$  it may be shown as in [7, ch. VIII, § 3] [14] [3, Satz 24], that  $\underline{H}_f^\omega = \bar{H}_f^\omega = H_f^\omega$  whenever  $f \in C(\omega^*)$ . Moreover  $f \rightsquigarrow H_f^\omega$  is a linear map from  $C(\omega^*)$  to the bounded continuous functions on  $\omega$ , which is continuous for the supremum norms. A point  $x_0 \in \omega^*$  is *regular* if  $\lim \{H_f^\omega(x) : x \in \omega, x \rightarrow x_0\} = f(x_0)$  whenever  $f \in C(\omega^*)$ . Since  $(\mathcal{G} - \mathcal{G})|_{\omega^*}$  is dense in  $C(\omega^*)$  and the map  $f \rightsquigarrow H_f^\omega$  is continuous,  $x_0$  is regular if and only if  $\lim \{H_f^\omega(x) : x \in \omega, x \rightarrow x_0\} = f(x_0)$  whenever  $f \in -\mathcal{G}|_{\omega^*}$ .

If  $f \in C(\omega^*)$  then put  $\Psi_f^\omega = \{\nu : \nu \text{ is hyperharmonic in a neighbourhood of } \bar{\omega} \text{ and}$

$$\liminf \{\nu(x) : x \in \bar{\omega}, x \rightarrow x_0\} \geq f(x_0)\},$$

put  $\bar{K}_f^\omega = \inf \{\nu : \nu \in \Psi_f^\omega\}$  and put  $\underline{K}_f^\omega = -\bar{K}_{(-f)}^\omega$ . As in [6, § 2] it may be shown that  $\underline{K}_f^\omega = \bar{K}_f^\omega = K_f^\omega$ , a continuous function on  $\bar{\omega}$ , harmonic in  $\omega$ , whenever  $f \in C(\omega^*)$ . The map  $f \rightsquigarrow K_f^\omega$  is a linear map from  $C(\omega^*)$  to  $C(\bar{\omega})$  continuous for the supremum norms. If  $f(x) = K_f^\omega(x)$  whenever  $f \in C(\omega^*)$  then  $x$  is a *stable* point of  $\omega^*$ . As with regular points,  $x$  is stable if and only if  $f(x) = K_f^\omega(x)$  whenever  $f \in -\mathcal{G}|_{\omega^*}$ .

Suppose that  $F \in -\mathcal{G}$ , and let  $\bar{F}$  be a continuous subharmonic function defined on an open neighbourhood  $U_F$  of  $\bar{\omega}$ , which equals  $F$  on  $\bar{\omega}$ . If  $\bar{\omega} = \bigcap \{\omega_i : i \in I\}$  the intersection of a decreasing filtering family of open subsets of  $U_F$ , then (by an abuse of language)  $\{H_F^{\omega_i} : i \in I\}$  is a decreasing filtering family in  $L$ , and  $K_F = \inf \{H_F^{\omega_i} : i \in I\}$  [6, § 2]. If  $x_0 \in \omega^*$  is stable, then

$$F(x_0) = \inf \{H_F^{\omega_i}(x_0) : i \in I\} \geq \inf \{h(x_0) : F < h \in \mathcal{G}\},$$

so that  $x_0 \in \partial \mathcal{G}(\overline{\omega})$  by (1). Conversely, if  $x_0 \in \partial \mathcal{G}(\overline{\omega}) \cap \omega^*$  and  $F \in -\mathcal{G}$ ,  $G \in \mathcal{G}$  with  $F < G$ , then  $\overline{F}|_{\omega_i} < \overline{G}|_{\omega_i}$  for some  $i \in I$ . Therefore  $F < H_{\mathbb{F}}^{\omega_i} < G$  on  $\omega$ . Therefore  $(\overline{\omega}, \mathcal{G}, L)$  is a geometric simplex [11, prop. 5] [5, p. 521]. It follows that  $F(x_0) = \inf \{g(x_0) : F < g \in \mathcal{G}\} \geq \inf \{H_{\mathbb{F}}^{\omega_i}(x_0) : i \in I\} \geq F(x_0)$ . Therefore  $x_0$  is stable and the following lemma holds.

LEMMA 7. — *The set of stable points of  $\omega^*$  is precisely  $\partial \mathcal{G}(\overline{\omega}) \cap \omega^*$ .*

Example 2. — *The classical case.* Let  $\omega$  be a bounded open subset of  $\mathbb{R}^n$ ,  $n > 1$ . The affine functions on  $\mathbb{R}^n$  are harmonic,  $\partial_M(\overline{\omega})$  is precisely the set of regular points of  $\omega^*$ , while  $\partial_L(\overline{\omega})$  is precisely the set of stable points of  $\omega^*$ . Since  $L$  contains the constant functions, separates the points of  $\overline{\omega}$ , and has the weak R.s.p., the following theorem is an immediate consequence of theorem 2.

THEOREM 5. —  *$L$  is uniformly dense in  $M$  if and only if every regular point of  $X$  is stable.*

We now return to the general case.

THEOREM 6. — *If every regular point of  $\omega^*$  is stable, then  $L$  is uniformly dense in  $M$ .*

Proof. — Suppose  $x_i$  belongs to the minimal  $W$ -face  $A_i$ ,  $i = 1, 2$ . Since  $\mathcal{G}$  distinguishes points of  $\omega^*$  it follows from lemma 3, that  $A_i \cap \omega^*$  is a one point set  $\{y_i\}$ . If  $F \in -\mathcal{G}$  and  $f = F|_{\omega^*}$  then  $\inf \{G : G \in \omega, F < G\} \geq H_f^\omega \geq F$  on  $\omega$ . Since  $y_i \in \partial_W(\overline{\omega})$ ,  $F(y_i) = \inf \{G(y_i) : G \in W, F < G\}$ . Therefore  $\lim \{H_f(x) : x \in \omega, x \rightarrow y_i\} = f(y_i)$ , and  $y_i$  is regular. Therefore  $y_i$  is stable. By lemma 7 there exist minimal  $\mathcal{G}$ -faces  $B_i$ , with  $y_i \in B_i$ ,  $i = 1, 2$ . Since  $A_i \cap B_i \neq \emptyset$  and  $A_i$  is minimal,  $A_i \subset B_i$ . Therefore  $\partial_W(\overline{\omega}) \subset \partial \mathcal{G}(\overline{\omega})$ . If  $\omega$  distinguishes  $x_1$  and  $x_2$  then by lemma 3  $\omega$  distinguishes  $y_1$  and  $y_2$ , and  $y_1 \neq y_2$ . Therefore  $\mathcal{G}$  distinguishes  $y_1$  and  $y_2$  so that  $B_1 \neq B_2$ , and  $\mathcal{G}$  distinguishes  $x_1$  and  $x_2$ . It follows from theorem 4 that  $L$  is uniformly dense in  $M$ .

Boboc and Cornea [5, th. 4.3], with the additional hypothesis that  $\omega$  is weakly determining, show that  $(\overline{\omega}, W, M)$  is a

geometric simplex, and that the set of regular points of  $\omega^*$  is precisely  $\partial_W(\bar{\omega}) \cap \omega^*$ . In this case we have a complete generalisation of theorem 5 to axiomatic potential theory.

**COROLLARY 2.** — *If  $\omega$  is weakly determining, then  $L$  is uniformly dense in  $M$  if and only if every regular point of  $\omega^*$  is stable.*

*Proof.* — If  $x$  is a regular point of  $\bar{\omega}$  then  $x \in \partial_\omega(\bar{\omega})$  [5, th. 4.3].  $(\omega, W, M)$  is a geometric simplex so by proposition 1,  $x \in \partial_M(\bar{\omega})$ . If  $L$  is dense in  $M$ , then  $\mathcal{L}$ -faces are  $\mathcal{M}$ -faces, and  $x$  belongs to a minimal  $\mathcal{L}$ -face  $A$ . Since  $(\bar{\omega}, \mathcal{G}, L)$  is a geometric simplex, it follows from proposition 1 and lemma 4 that  $A$  contains a unique minimal  $\mathcal{G}$ -face  $B$  and a unique minimal  $W$ -face  $C$ . Therefore  $x \in C \subset B$ , so that  $x \in \partial_{\mathcal{G}}(\bar{\omega})$  and  $x$  is stable by lemma 7. The corollary is now an immediate consequence of theorem 6.

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Manuscrit reçu le 9 juin 1969.

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