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On the first restricted cohomology of a reductive Lie algebra and its Borel subalgebras


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ON THE FIRST RESTRICTED COHOMOLOGY OF A
REDUCTIVE LIE ALGEBRA AND ITS BOREL
SUBALGEBRAS

by Rudolf TANGE (*)

ABSTRACT. — Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and let \( G \) be a connected reductive group over \( k \). Let \( B \) be a Borel subgroup of \( G \) and let \( g \) and \( b \) be the Lie algebras of \( G \) and \( B \). Denote the first Frobenius kernels of \( G \) and \( B \) by \( G_1 \) and \( B_1 \). Furthermore, denote the algebras of regular functions on \( G \) and \( g \) by \( k[G] \) and \( k[g] \), and similarly for \( B \) and \( b \). The group \( G \) acts on \( k[G] \) via the conjugation action and on \( k[g] \) via the adjoint action. Similarly, \( B \) acts on \( k[B] \) via the conjugation action and on \( k[b] \) via the adjoint action. We show that, under certain mild assumptions, the cohomology groups \( H^1(G_1, k[g]) \), \( H^1(B_1, k[b]) \), \( H^1(G_1, k[G]) \) and \( H^1(B_1, k[B]) \) are zero. We also extend all our results to the cohomology for the higher Frobenius kernels.

Résumé. — Soit \( k \) un corps algébriquement clos de caractéristique \( p > 0 \) et soit \( G \) un groupe réductif connexe sur \( k \). Soit \( B \) un sous-groupe de Borel de \( G \) et soit \( g \) et \( b \) les algèbres de Lie de \( G \) et \( B \). Notons les premiers noyaux de Frobenius de \( G \) et \( B \) par \( G_1 \) et \( B_1 \). De plus, notons les algèbres des fonctions régulières sur \( G \) et \( g \) par \( k[G] \) et \( k[g] \), et de même pour \( B \) et \( b \). Le groupe \( G \) agit sur \( k[G] \) par conjugaison et sur \( k[g] \) par l’action adjointe. De même, \( B \) agit sur \( k[B] \) par l’action de conjugaison et sur \( k[b] \) par l’action adjointe. Nous montrons que, sous certaines hypothèses, les groupes de cohomologie \( H^1(G_1, k[g]) \), \( H^1(B_1, k[b]) \), \( H^1(G_1, k[G]) \) et \( H^1(B_1, k[B]) \) sont nuls. Nous étendons aussi nos résultats à la cohomologie pour les noyaux de Frobenius supérieurs.

Introduction

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), let \( G \) be a connected reductive group over \( k \), and let \( g \) be the Lie algebra of \( G \). Recall that \( g \) is a restricted Lie algebra: it has a \( p \)-th power map \( x \mapsto x^{[p]} : g \to g \), see [3, I.3.1]. In the case of \( G = \text{GL}_n \) this is just the \( p \)-th matrix power. A
$\mathfrak{g}$-module $M$ is called restricted if $(x^{[p]})_M = (x_M)^p$ for all $x \in M$. Here $x_M$ is the endomorphism of $M$ representing $x$.

Recall that an element $v$ of a $\mathfrak{g}$-module $M$ is called a $\mathfrak{g}$-invariant if $x\cdot v = 0$ for all $x \in \mathfrak{g}$. We denote the space of $\mathfrak{g}$-invariants in $M$ by $M^\mathfrak{g}$. The right derived functors of the left exact functor $M \mapsto M^\mathfrak{g}$ from the category of restricted $\mathfrak{g}$-modules to the category of vector spaces over $k$ are denoted by $H^i(G_1, \cdot)$.

Let $k[\mathfrak{g}]$ be the algebra of polynomial functions on $\mathfrak{g}$. If one is interested in describing the algebra of invariants $(k[\mathfrak{g}]/I)^\mathfrak{g}$ for some $\mathfrak{g}$-stable ideal $I$ of $k[\mathfrak{g}]$, then it is of interest to know if $H^1(G_1, k[\mathfrak{g}]) = 0$, because then we have an exact sequence

$$k[\mathfrak{g}]^\mathfrak{g} \to (k[\mathfrak{g}]/I)^\mathfrak{g} \to H^1(G_1, I) \to 0$$

by the long exact cohomology sequence. So, in this case, $(k[\mathfrak{g}]/I)^\mathfrak{g}$ is built up from the image of $k[\mathfrak{g}]^\mathfrak{g}$ in $k[\mathfrak{g}]/I$, and $H^1(G_1, I)$.

The paper is organised as follows. In Section 1 we state some results from the literature that we need to prove our main result. This includes a description of the algebra of invariants $k[\mathfrak{g}]^G$, the normality of the nilpotent cone $\mathcal{N}$, and some lemmas on graded modules over graded rings. In Section 2 we prove Theorems 2.1 and 2.2 which state that, under certain mild assumptions on $p$, $H^1(G_1, k[\mathfrak{g}])$ and $H^1(B_1, k[\mathfrak{b}])$ are zero. In Section 3 we prove Theorems 3.1 and 3.2 which state that, under certain mild assumptions on $p$, $H^1(G_1, k[G])$ and $H^1(B_1, k[B])$ are zero. In Section 4 we extend theses four theorems to the cohomology for the higher Frobenius kernels $G_r$ and $B_r$, $r \geq 2$.

We briefly indicate some background to our results. For convenience we only discuss the $G$-module $k[\mathfrak{g}]$. As is well-known, under certain mild assumptions $k[\mathfrak{g}]$ has a good filtration, see [6] or [11, II.4.22]. So a natural approach to prove that $H^1(G_1, k[\mathfrak{g}]) = 0$ would be to use that $H^1(G_1, \nabla(\lambda)) = 0$ for all induced modules $\nabla(\lambda)$ that show up in a good filtration of $k[\mathfrak{g}]$. However, this isn’t true: even for $p > h$, $h$ the Coxeter number, one can easily deduce counterexamples from [1, Cor. 5.5] (or [11, II.12.15]).(1) It is also easy to see that we cannot have $H^i(G_1, k[\mathfrak{g}]) = 0$ for all $i > 0$: the trivial module $k$ is direct summand of $k[\mathfrak{g}]$, and for $p > h$ we have $H^*(G_1, k) \cong k[\mathcal{N}]$ where the degrees of $k[\mathcal{N}]$ are doubled, see [11, II.12.14].

The idea of our proof that $H^1(G_1, k[\mathfrak{g}]) = 0$ is as follows. Noting that $H^1(G_1, k[\mathfrak{g}])$ is a $k[\mathfrak{g}]^G$-module, we interpret a certain localisation of

(1) This approach does work when proving (the well-known fact) that $H^1(G, k[\mathfrak{g}]) = 0$. 

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\( H^1(G_1, k[\mathfrak{g}]) \) as the cohomology group of the coordinate ring of the generic fiber of the adjoint quotient map \( \mathfrak{g} \to \mathfrak{g}/G \). It is easy to see that this cohomology group has to be zero, so we are left with showing that \( H^1(G_1, k[\mathfrak{g}]) \) is torsion-free over the invariants \( k[\mathfrak{g}]^G \). To prove the latter we use Hochschild’s characterisation of the first restricted cohomology group and a “Nakayama Lemma type result”. The ideas of the proofs of the other main results are completely analogous.

1. Preliminaries

Throughout this paper \( k \) is an algebraically closed field of characteristic \( p > 0 \). For the basics of representations of algebraic groups we refer to [11].

1.1. Restricted representations and restricted cohomology

Let \( G \) be a linear algebraic group over \( k \) with Lie algebra \( \mathfrak{g} \). Let \( G_1 \) be the first Frobenius kernel of \( G \) (see [11, Ch. I.9]). It is an infinitesimal group scheme with \( \dim k[G_1] = p^{\dim(\mathfrak{g})} \). Its category of representations is equivalent to the category of restricted representations of \( \mathfrak{g} \), see the introduction.

Let \( M \) be an \( G_1 \)-module. By [7] (see also [11, I.9.19]) we have

\[
H^1(G_1, M) = \{ \text{restricted derivations : } \mathfrak{g} \to M \}/\{ \text{inner derivations of } M \}.
\]

Here a \textit{derivation} from \( \mathfrak{g} \) to \( M \) is a linear map \( D: \mathfrak{g} \to M \) satisfying

\[
D([x, y]) = x \cdot D(y) - y \cdot D(x)
\]

for all \( x, y \in \mathfrak{g} \). Such a derivation is called \textit{restricted} if

\[
D(x^{[p]}) = (x_M)^{p-1}(D(x))
\]

for all \( x \in \mathfrak{g} \), where \( x_M \) is the vector space endomorphism of \( M \) given by the action of \( x \), and \(-[p] \) denotes the \( p \)-th power map of \( \mathfrak{g} \). An \textit{inner derivation} of \( M \) is a map \( x \mapsto x \cdot u : \mathfrak{g} \to M \) for some \( u \in M \). If \( M \) is restricted, then every inner derivation is restricted. Clearly \( H^1(G_1, M) \) is an \( G \)-module with trivial \( \mathfrak{g} \)-action: If \( D \) is a derivation and \( y \in \mathfrak{g} \), then \([y, D]\) is the inner derivation given by \( D(y) \). Note also that \( H^1(G_1, k[\mathfrak{g}]) \) is a \( k[\mathfrak{g}]^0 \)-module, since the restricted derivations \( \mathfrak{g} \to k[\mathfrak{g}] \) form a \( k[\mathfrak{g}]^0 \)-module and the map \( f \mapsto (x \mapsto x \cdot f) \) from \( k[\mathfrak{g}] \) to the restricted derivations \( \mathfrak{g} \to k[\mathfrak{g}] \) is \( k[\mathfrak{g}]^0 \)-linear.
1.2. Actions of restricted Lie algebras

Let $\mathfrak{g}$ be a restricted Lie algebra over $k$. Following [17] we define an action of $\mathfrak{g}$ on an affine variety $X$ over $k$ to be a homomorphism $\mathfrak{g} \to \text{Der}_k(k[X])$ of restricted Lie algebras, where $\text{Der}_k(k[X])$ is the (restricted) Lie algebra of $k$-linear derivations of $k[X]$. It is easy to see that this includes the case that $X$ is a restricted $\mathfrak{g}$-module. If $\mathfrak{g}$ acts on $X$ and $x \in X$, then we define $\mathfrak{g}_x$ to be the stabiliser in $\mathfrak{g}$ of the maximal ideal $\mathfrak{m}_x$ of $k[X]$ corresponding to $x$. In case $X$ is a closed subvariety of a restricted $\mathfrak{g}$-module, then we have $\mathfrak{g}_x = \{y \in \mathfrak{g} | y \cdot x = 0\}$.

**Lemma 1.1.** — Let $\mathfrak{g}$ be a restricted Lie algebra over $k$ acting on a normal affine variety $X$ over $k$. If $\max_{x \in X} \text{codim}_g \mathfrak{g}_x = \dim X$, then $k[X]^\mathfrak{g} = k[X]^p$.

**Proof.** — By [17, Thm. 5.2(5)] we have $[k(X) : k(X)^\mathfrak{g}] = p^{\dim(X)}$. By [4, Cor. 3 to Thm. V.16.6.4] we have $[k(X) : k(X)^p] = p^{\dim(X)}$. So $k(X)^\mathfrak{g} = k(X)^p$, since we always have $\subseteq$. Clearly, $k(X)^p = \text{Frac}(k[X]^p)$, $k(X)^\mathfrak{g} = \text{Frac}(k[X]^\mathfrak{g})$ and $k[X]^\mathfrak{g}$ is integral over $k[X]^p$. Since $X$ is normal variety, $k[X]^p \cong k[X]$ is a normal ring. It follows that $k[X]^\mathfrak{g} = k[X]^p$. □

1.3. Two lemmas on graded rings and modules

We recall a version of the graded Nakayama lemma which follows from [14, Ch. 13, Lem. 4, Ex. 3, Lem. 3].

**Lemma 1.2** ([14, Ch. 13]). — Let $R = \bigoplus_{i \geq 0} R^i$ be a positively graded ring with $R^0$ a field, let $M$ be a positively graded $R$-module and let $(x_i)_{i \in I}$ be a family of homogeneous elements of $M$. Put $R^+ = \bigoplus_{i > 0} R^i$.

1. If the images of the $x_i$ in $M/R^+M$ span the vector space $M/R^+M$ over $R^0$, then the $x_i$ generate $M$.

2. If $M$ is projective and the images of the $x_i$ in $M/R^+M$ form an $R^0$-basis of $M/R^+M$, then $(x_i)_{i \in I}$ is an $R$-basis of $M$.

**Lemma 1.3.** — Let $R$ be a positively graded ring with $R^0$ a field and let $N$ be a positively graded $R$-module which is projective.

1. Let $M$ be a submodule of $N$ with $(R^+N) \cap M \subseteq R^+M$. Then $M$ is free and a direct summand of $N$.

2. Let $M$ be a positively graded $R$-module, let $\varphi : M \to N$ be a graded $R$-linear map and let $\overline{\varphi} : M/R^+M \to N/R^+N$ be the induced $R^0$-linear map. Assume the canonical map $M \to M/R^+M$ maps $\text{Ker}(\varphi)$ onto $\text{Ker}(\overline{\varphi})$. Then $\text{Im}(\varphi)$ is free and a direct summand of $N$. 
Proof.

(1) — From the assumption it is immediate that the natural map $M/R^+M \to N/R^+N$ is injective. Now choose an $R^0$-basis $(\pi_i)_{i \in I}$ of $M/R^+M$ and extend it to a basis $(\pi_i)_{i \in I \cup J}$ of $N/R^+N$. Let $(x_i)_{i \in I \cup J}$ be a homogeneous lift of this basis to $N$. Then this is a basis of $N$ by Lemma 1.2(2). Furthermore, $(x_i)_{i \in I}$ must span $M$ by Lemma 1.2(2). So $M$ is (graded-) free and has the $R$-span of $(x_i)_{i \in J}$ as a direct complement.

(2) — By (1) it suffices to show that $(R^+N) \cap \text{Im}(\varphi) \subseteq R^+ \text{Im}(\varphi)$. Let $x \in M$ and assume that $\varphi(x) \in R^+N$. Then $x := x + R^+M \in \text{Ker}(\varphi)$. By assumption there exists $x_1 \in \text{Ker}(\varphi)$ such that $x = x_1$. Then $x - x_1 \in R^+M$ and $\varphi(x) = \varphi(x - x_1) \in R^+ \text{Im}(\varphi)$. □

There is also an obvious version of Lemma 1.3 (and of course of Lemma 1.2) for a local ring $R$: simply assume $R$ local, omit the gradings everywhere and replace $R^+$ by the maximal ideal of $R$.

1.4. The standard hypotheses and consequences

In the remainder of this paper $G$ is a connected reductive group over $k$ and $\mathfrak{g}$ is its Lie algebra. Recall that $\mathfrak{g}$ is a restricted Lie algebra, see [3, I.3.1], we denote its $p$-th power map by $x \mapsto x^{[p]}$. The group $G$ acts on $\mathfrak{g}$ and the nilpotent cone $\mathcal{N}$ via the adjoint action and on $G$ via conjugation, and therefore it also acts on their algebras of regular functions: $k[\mathfrak{g}]$, $k[\mathcal{N}]$ and $k[G]$. Fix a maximal torus $T$ of $G$ and let $\mathfrak{t}$ be its Lie algebra. We fix an $\mathbb{F}_p$-structure on $G$ for which $T$ is defined and split over $\mathbb{F}_p$. Then $\mathfrak{g}$ has an $\mathbb{F}_p$-structure and $\mathfrak{t}$ is $\mathbb{F}_p$-defined. Denote the $\mathbb{F}_p$-defined regular functions on $\mathfrak{g}$ and $\mathfrak{t}$ by $\mathbb{F}_p[\mathfrak{g}]$ and $\mathbb{F}_p[\mathfrak{t}]$. We will need the following standard hypotheses, see [10, 6.3, 6.4] or [12, 2.6, 2.9]:

(H1) The derived group $DG$ of $G$ is simply connected,

(H2) $p$ is good for $G$,

(H3) There exists a $G$-invariant non-degenerate bilinear form on $\mathfrak{g}$.

Put $G_x = \{g \in G \mid \text{Ad}(g)(x) = x\}$ and $\mathfrak{g}_x = \{y \in \mathfrak{g} \mid [y, x] = 0\}$. Assuming (H1)-(H3) we have by [12, 2.9] that $\text{Lie}(G_x) = \mathfrak{g}_x$ for all $x \in \mathfrak{g}$. See also [15, Sect. 2.1]. Put $n = \text{dim}(T)$. We call $x \in \mathfrak{g}$ regular if $\text{dim}G_x$ (or $\text{dim} \mathfrak{g}_x$) equals $n$, the minimal value. Under assumptions (H1) and (H3) we have that $d\alpha \neq 0$ for all roots $\alpha$, so restriction of functions defines an isomorphism $k[\mathfrak{g}]^G \cong k[\mathfrak{h}]^W$, see [12, Prop. 7.12]. The set of regular semisimple elements in $\mathfrak{g}$ is the nonzero locus of the regular function $f_{rs}$ on
\( \mathfrak{g} \) which corresponds under the above isomorphism to the product of the differentials of the roots. Note that \( f_{rs} \in \mathbb{F}_p[\mathfrak{g}] \): \( f_{rs} \) is defined over \( \mathbb{F}_p \).

Under assumptions (H1)–(H3) it follows from work of Demazure [5] that \( k[t]^W \) is a polynomial algebra in \( n \) homogeneous elements defined over \( \mathbb{F}_p \), see [10, 9.6 end of proof]. We denote the corresponding elements of \( \mathbb{F}_p[\mathfrak{g}] \) by \( s_1, \ldots, s_n \). Assuming (H1)–(H3) the vanishing ideal of \( \mathcal{N} \) in \( k[\mathfrak{g}] \) is generated by the \( s_i \), see [12, 7.14], and all regular orbit closures are normal, in particular \( \mathcal{N} \) is normal, see [12, 8.5].

We call \( g \in G \) regular if \( G_g := \{ h \in G \mid hgh^{-1} = g \} \) has dimension \( n \), the minimal value. Restriction of functions defines an isomorphism \( k[\mathfrak{g}]^G \cong k[T]^W \), see [19, 6.4]. The set of regular semisimple elements in \( G \) is the nonzero locus of the regular function \( f_{rs} \) on \( G \) which corresponds under the above isomorphism to \( \prod_{\alpha \text{ a root}} (\alpha - 1) \). If \( G \) is semisimple, simply connected, then \( k[\mathfrak{g}]^G \) is a polynomial algebra in the characters \( \chi_1, \ldots, \chi_n \) of the irreducible \( G \)-modules whose highest weights are the fundamental dominant weights. Furthermore, the schematic fibers of the adjoint quotient \( G \to G/G \) are reduced and normal and they are regular orbit closures. See [19] and [8, 4.24]. One can also deduce from (H1)–(H3) that \( \text{Lie}(G_g) = \mathfrak{g}_g := \{ x \in \mathfrak{g} \mid \text{Ad}(g)(x) = x \} \).

2. **The cohomology groups** \( H^1(G_1, k[\mathfrak{g}]) \) and \( H^1(B_1, k[\mathfrak{b}]) \)

Throughout this section we assume that hypotheses (H1)–(H3) from Section 1.4 hold.

**Theorem 2.1.** — \( H^1(G_1, k[\mathfrak{g}]) = 0 \).

**Proof.** — Let \( K \) be an algebraic closure of the field of fractions of \( R := k[\mathfrak{g}]^G \). Since the action of \( \mathfrak{g} \) on \( k[\mathfrak{g}] \) is \( R \)-linear we have \( H^1(G_1, k[\mathfrak{g}]) = H^1((G_1)_R, k[\mathfrak{g}]) \), where \( -_R \) denotes base change from \( k \) to \( R \), see [11, I.1.10]. So, by the Universal Coefficient Theorem [11, Prop. I.4.18], we have

\[ K \otimes_R H^1(G_1, k[\mathfrak{g}]) = H^1((G_1)_K, K \otimes_R k[\mathfrak{g}]) = H^1((G_K)_1, K \otimes_R k[\mathfrak{g}]). \]

For \( i \in \{ 1, \ldots, n \} \) denote the regular function on \( \mathfrak{g}_K \) corresponding to \( s_i \in k[\mathfrak{g}] \) by \( \tilde{s}_i \). Then \( K \otimes_R k[\mathfrak{g}] = K[\mathfrak{g}_K]/(\tilde{s}_1 - s_1, \ldots, \tilde{s}_n - s_n) = K[F] \), where \( F \subseteq \mathfrak{g}_K \) is the fiber of the morphism

\[ x \mapsto (\tilde{s}_1(x), \ldots, \tilde{s}_n(x)) : \mathfrak{g}_K \to \mathbb{A}_K^n \]

over the point \( (s_1, \ldots, s_n) \in \mathbb{A}_K^n \). Let \( f_{rs} \in \mathbb{F}_p[\mathfrak{g}] \cap k[\mathfrak{g}]^G \) be the polynomial function from Section 1.4 with nonzero locus the set of regular semisimple elements in \( \mathfrak{g} \), and let \( \tilde{f}_{rs} \) be the corresponding polynomial function on
Then we have for all \( x \in F \) that \( \tilde{f}_{rs}(x) = f_{rs} \neq 0 \). So \( F \) consists of regular semisimple elements. By [20, Lem. 3.7, Thm. 3.14] this means that \( F = G_K/S \) for some maximal torus \( S \) of \( G_K \). In particular, \( K[F] \) is an injective \( G_K \)-module. But then it is also injective as a \((G_K)_1\)-module, see [11, Rem. I.4.12, Cor. I.5.13b]). So \( K \otimes_R H^1(G_1, k[g]) = H^i((G_K)_1, K[F]) = 0 \) for all \( i > 0 \).

So it now suffices to show that \( H^1(G_1, k[g]) \) has no \( R \)-torsion. We are going to apply Lemma 1.3(2) to the \( R \)-linear map

\[
\varphi = f \mapsto (x \mapsto x \cdot f) : k[g] \to \text{Hom}_k(g, k[g]).
\]

Here the grading of \( \text{Hom}_k(g, k[g]) \) is given by

\[
\text{Hom}_k(g, k[g])^i = \text{Hom}_k(g, k[g]^i).
\]

As explained in [12, 7.13, 7.14] the conditions of [16, Prop. 10.1] are satisfied under the assumptions (H1)–(H3), so \( k[g] \) is a free \( R \)-module. So \( \text{Hom}_k(g, k[g]) \) is also a free \( R \)-module. We have \( k[g]/R^+k[g] = k[N] \), and

\[
\overline{\varphi} = f \mapsto (x \mapsto x \cdot f) : k[N] \to \text{Hom}_k(g, k[N]).
\]

By [12, 6.3,6.4], we have \( \min_{x \in N} \dim g_x = n \) and \( \dim N = \dim g - n \). So from Lemma 1.1 it is clear that the restriction map \( k[g] \to k[N] \) maps the \( g \)-invariants of \( k[g] \) onto those of \( k[N] \). But \( k[g]^0 = \text{Ker}(\varphi) \) and \( k[N]^0 = \text{Ker}(\overline{\varphi}) \). So, by Lemma 1.3(2), \( \text{Im}(\varphi) \) is a direct \( R \)-module summand of \( \text{Hom}_k(g, k[g]) \). In particular, \( \text{Hom}_k(g, k[g]) / \text{Im}(\varphi) \) is isomorphic to an \( R \)-submodule of \( \text{Hom}_k(g, k[g]) \) and therefore \( R \)-torsion-free. From (1.1) in Section 1.1 it is clear that \( H^1(G_1, k[g]) \) is isomorphic to an \( R \)-submodule of \( \text{Hom}_k(g, k[g]) / \text{Im}(\varphi) \), so it is also \( R \)-torsion-free.

Let \( B \) be a Borel subgroup of \( G \) containing \( T \), let \( b \) be its Lie algebra and let \( u \) be the Lie algebra of the unipotent radical \( U \) of \( B \).

**Theorem 2.2.** — \( H^1(B_1, k[b]) = 0 \).

**Proof.** — Consider the restriction map \( k[b]^B \to k[t] \). Under the assumptions (H1)–(H3) \( t \) contains elements which are regular in \( g \). Furthermore, the set of regular semisimple elements in \( g \) is open in \( g \). So the regular semisimple elements of \( g \) in \( b \) are dense in \( b \). Since the union of the \( B \)-conjugates of \( t \) is the set of all semisimple elements in \( b \), by [3, Prop. 11.8], it is also dense in \( b \). This shows that the map \( k[b]^B \to k[t] \) is injective. Furthermore, \( \text{Ad}(g)(x) - x \in u \) for all \( g \in B \) and \( x \in b \) by [3, Prop. 3.17], since \( DB \subseteq U \). So if we extend \( f \in k[t] \) to a regular function \( f \) on \( b \) by \( f(x + y) = f(x) \) for all \( x \in t \) and \( y \in u \), then \( f \in k[b]^B \). So the map
$k[\mathfrak{b}]^B \to k[\mathfrak{t}]$ is surjective, that is, restriction of functions defines an isomorphism

$$k[\mathfrak{b}]^B \cong k[\mathfrak{t}].$$

Extend a basis of $\mathfrak{t}^*$ to (linear) functions $\xi_1, \ldots, \xi_n$ on $\mathfrak{b}$ in the manner indicated above. Then these functions are algebraically independent generators of $k[\mathfrak{b}]^B$, and $k[\mathfrak{b}]$ is a free $k[\mathfrak{b}]^B$-module. Clearly, the vanishing ideal of $\mathfrak{u}$ in $k[\mathfrak{b}]$ is generated by the $\xi_i$. Furthermore, $\min_{x \in \mathfrak{u}} \dim \mathfrak{b}_x = n$, see [12, 6.8].

We can now follow the same arguments as in the proof of Theorem 2.1. Just replace $G, g, N, k[\mathfrak{g}]G$ and the $s_i$ by $B, \mathfrak{b}, \mathfrak{u}, k[\mathfrak{b}]^B$ and the $\xi_i$, and replace $f_{rs}$ by its restriction to $\mathfrak{b}$. □

Remark 2.3. — We have $k[N] = \text{ind}^G_B k[\mathfrak{u}]$. Using [11, Lem. II.12.12a)] and the arguments from [11, II.12.2] it follows that $H^1(G_1, k[N]) = \text{ind}^G_B H^1(B_1, k[\mathfrak{u}])$. From this one can easily deduce examples with $H^1(G_1, k[N]) \neq 0$.

3. The cohomology groups $H^1(G_1, k[G])$ and $H^1(B_1, k[B])$

Assume first that $G = \text{GL}_n$. Put $R = k[\mathfrak{g}]^G$ and $R_1 = R[\det^{-1}]$. Then, using the fact that the $\mathfrak{g}$-action on $k[G]$ is $R_1$-linear, the Universal Coefficient Theorem and Theorem 2.1, we obtain

$$H^1(G_1, k[G]) = H^1((G_1)_{R_1}, k[G]) = R_1 \otimes_R H^1((G_1)_R, k[\mathfrak{g}])$$

$$= R_1 \otimes_R H^1(G_1, k[\mathfrak{g}]) = 0.$$

Similarly, we obtain $H^1(B_1, k[B]) = 0$.

To prove our result for the case of arbitrary reductive $G$ we assume in this section the following:

There exists a central (see [3, 22.3]) surjective morphism $\psi : \tilde{G} \to G$ where $\tilde{G}$ is a direct product of groups of the following types:

1. a simply connected simple algebraic group of type $\neq A$ for which $p$ is good,
2. $\text{SL}_m$ for $p \nmid m$,
3. $\text{GL}_m$,
4. a torus.

Theorem 3.1. — $H^1(G_1, k[G]) = 0$.

Proof. — First we reduce to the case that $G$ is of one of the above four types. Let $\psi : \tilde{G} \to G$ be as above. Then $G$ is the quotient of $\tilde{G}$ by a (schematic) central diagonalisable closed subgroup scheme $\tilde{Z}$, see [11,
II.1.18. Let $N$ be the image of $\tilde{\G}_1$ in $G_1$. Then $N$ is normal in $G_1$ and $G_1/N$ is diagonalisable. So $H^i(G_1, k[G]) = H^i(N, k[G])^{G_1/N}$, by [11, I.6.9(3)]. Furthermore, $H^i(N, k[G]) = H^i(\tilde{\G}_1, k[G])$, by [11, I.6.8(3)], since the kernel of $\tilde{\G}_1 \to N$ is central.

The group scheme $\tilde{Z}$ also acts via the right multiplication action on $k[\tilde{\G}]$ and this action commutes with the conjugation action of $\tilde{\G}$. So $k[G] = k[\tilde{\G}]^G$ is a direct $\tilde{\G}$-module summand of $k[\tilde{\G}]$. So it suffices to show that $H^i(\tilde{\G}_1, k[\tilde{\G}]) = 0$. By the Künneth Theorem we may now assume that $G$ is of one of the above four types.

For $G$ a torus the assertion is obvious, and for $G = \text{GL}_n$ we have already proved the assertion. Now assume that $G$ is of type (1) or (2). Then $G$ satisfies (H1)–(H3) and $G$ is simply connected simple. By [20, 2.15] the centraliser of a semisimple group element is connected, so when the element is also regular, its centraliser is a maximal torus. As in the proof of Theorem 2.1 we are now reduced to showing that $H^1(\tilde{\G}_1, k[\tilde{\G}]) = 0$. By the Künneth Theorem we may now assume that $G$ is of one of the above four types.

Let $B$ be a Borel subgroup of $G$.

**Theorem 3.2.** — $H^1(B_1, k[B]) = 0$.

**Proof.** — This follows by modifying the proof of Theorem 3.1 in the same way as the proof of Theorem 2.1 was modified to obtain the proof of Theorem 2.2. □

**Remark 3.3.** — One can also prove Theorem 3.2 assuming (H1)–(H3). The point is that it is obvious that restriction of functions always defines an isomorphism $k[B]^B \cong k[T]$.

**Remark 3.4.** — We briefly discuss the $B$-cohomology of $k[B]$ and $k[b]$. From [13, Thm. 1.13, Thm. 1.7(a)(ii)] it is immediate that $H^i(B, k[B]) = 0$ for all $i > 0$. Now assume that there exists a central surjective morphism
\( \psi : G \to G \) where \( \tilde{G} \) is a direct product of groups of the types (1)–(4) mentioned before, except that for type (2) we drop the condition on \( p \). Then we deduce from the arguments from the proof of [1, Prop. 4.4] that \( H^i(B, k[\mathfrak{b}]) = 0 \) for all \( i > 0 \) as follows. First we reduce as in the proof of Theorem 3.1 to the case that \( G \) is simple of type (1) or (2) and then we deal with type (2) as in [1]. Now assume \( G \) is of type (1) and let \( I \) be the vanishing ideal of \( B \) in \( k[\tilde{G}] \). As in [1] write
\[
(3.1) \quad \mathfrak{m} = M \oplus \mathfrak{m}^2
\]
where \( \mathfrak{m} \) is the vanishing ideal in \( k[\tilde{G}] \) of the unit element of \( G \) and \( M \cong \mathfrak{g}^* \) as \( G \)-modules. It suffices to show that \( I = I \cap M + I \cap \mathfrak{m}^2 \), since then we get a decomposition analogous to (3.1) for \( k[B] \) and we can finish as in [1]. Let \( f \in I \). Then the \( M \)-component of \( f \) correspond to \( df \in \mathfrak{g}^* \) which vanishes on \( \mathfrak{b} \). This means it corresponds under the trace form of the chosen representation \( \rho : G \to V \) (the adjoint representation for exceptional types) to an element \( x \in \mathfrak{u} \). So the \( M \)-component of \( f \) is \( g \mapsto \text{tr} (\rho(g)df(x)) \) which vanishes on \( B \). But then the \( \mathfrak{m}^2 \)-component of \( f \) must also vanish on \( B \).

4. The cohomology groups for the higher Frobenius kernels

In this section we will generalise the results from the previous two sections to all Frobenius kernels \( G_r, r \geq 1 \).

**Lemma 4.1.** — Let \( G \) be a linear algebraic group over \( k \) acting on a normal affine variety \( X \) over \( k \). If \( \max_{x \in X} \text{codim}_k \mathfrak{g}_x = \dim X \), then \( k[X]^{G_r} = k[X]^{p^r} \) for all integers \( r \geq 1 \).

**Proof.** — Since \( \text{codim}_k \mathfrak{g}_x \leq \text{codim}_k G_x \leq \dim X \) and \( \max_{x \in X} \text{codim}_k \mathfrak{g}_x = \dim X \) we must have that for \( x \in X \) with \( \text{codim}_k \mathfrak{g}_x = \dim X \) the schematic centraliser of \( x \) in \( G \) is reduced. So \( (G_r)_x = (G_x)_r \) and
\[
(G_r : (G_r)_x) := \frac{\dim(k[\tilde{G}_r])}{\dim(k[(G_r)_x])} = \frac{p^r \dim(G)}{p^r \dim(G_x)} = p^r \dim(X).
\]

By [17, Thm. 2.1(5)] we get \( [k(X) : k(X)^{G_r}] = p^r \dim(X) \). By [4, Cor. 3 to Thm. V.16.6.4] and the tower law we have \( [k(X) : k(X)^{p^r}] = p^r \dim(X) \). So \( k(X)^{G_r} = k(X)^{p^r} \), since we always have \( \supseteq \). Clearly, \( k(X)^{p^r} = \text{Frac}(k[X]^{p^r}) \), \( k(X)^{G_r} = \text{Frac}(k[X]^{G_r}) \) and \( k[X]^{G_r} \) is integral over \( k[X]^{p^r} \). Since \( X \) is normal variety, \( k[X]^{p^r} = k[X]^{G_r} \) is a normal ring. It follows that \( k[X]^{G_r} = k[X]^{p^r} \).

\[\square\]
Theorem 4.2. — Let $r$ be an integer $\geq 1$.

1. Under the assumptions of Section 2 we have

\[ H^1(G_r, k[g]) = 0 \text{ and } H^1(B_r, k[b]) = 0. \]

2. Under the assumptions of Section 3 we have

\[ H^1(G_r, k[G]) = 0 \text{ and } H^1(B_r, k[B]) = 0. \]

Proof.

1. Let $(H, M)$ be the group and module in question, i.e. $(G, k[g])$ or $(B, k[b])$. Put $R = k[h]^H$. Let $\varphi$ be the first map in the Hochschild complex of the $H_r$-module $M$, see [11, I.4.14]:

\[ \varphi = f \mapsto (\Delta_M(f) - 1 \otimes f) : M \to k[H_r] \otimes M. \]

Then the induced map $\overline{\varphi} : M/R^+M \to k[H_r] \otimes (M/R^+M)$ is the first map in the Hochschild complex of the $H_r$-module $M/R^+M$ which is $k[N]$ or $k[u]$. So $\text{Ker}(\varphi) = M^H$ and $\text{Ker}(\overline{\varphi}) = (M/R^+M)^H$. Now the proof is the same as that of the corresponding result in Section 2, except that we work with the above map $\varphi$ and instead of Lemma 1.1 we apply Lemma 4.1.

2. Let $(H, M)$ be the group and module in question, i.e. $(G, k[G])$ or $(B, k[B])$. As in the proof of the corresponding result in Section 3 we reduce to the case that $G$ is simple of type (1) or (2). Put $R = k[H]^H$. Fix a maximal ideal $m$ of $R$. Let $\varphi$ be the first map in the Hochschild complex of the $H_r$-module $M_m$. Then the induced map $\overline{\varphi} : M_m/mM_m \to k[H_r] \otimes (M_m/mM_m)$ is the first map in the Hochschild complex of the $H_r$-module $M_m/mM_m$ which is the coordinate ring of the fiber of $H \to H//H$ over the point $m$. So $\text{Ker}(\varphi) = (M_m)^H$ and $\text{Ker}(\overline{\varphi}) = (M/mM)^H$. Now the proof is the same as that of the corresponding result in Section 3, except that we work with the above map $\varphi$ and instead of Lemma 1.1 we apply Lemma 4.1.

Remark 4.3. — For $G$ classical with natural module $V = k^n$ we consider the cohomology groups $H^1(G_r, S^iV)$ and $H^1(G_r, S^i(V^*))$.

Results about these modules can mostly easily be deduced from results on induced modules in the literature. For induced modules one can reduce to $B_r$-cohomology using the following result of Andersen–Jantzen for general $G$. Let $B$ be a Borel subgroup of $G$ with unipotent radical $U$ and let $T$ be a maximal torus of $B$. For $\lambda \in X(T)$, the character group of $T$, we denote by $\nabla(\lambda)$, the $G$-module induced from the 1-dimensional $B$-module given by $\lambda$. We call the roots of $T$ in the opposite Borel subgroup $B^+$ positive. By [11, II.12.2] we have for $\lambda$ dominant.
Below we will always take \( \lambda = \varpi_1 \) the first non-constant diagonal matrix coordinate. First take \( G = \text{GL}_n \). Let \( B \) and \( T \) be the lower triangular matrices and the diagonal matrices. Then the character group \( X(T) \) of \( T \) identifies with \( \mathbb{Z}^n \). Let \( \varepsilon \) be the first standard basis element of \( X(T) \), i.e. the character \( D \mapsto D_{ii} \). Then \( S^iV = \nabla(i\varepsilon_1) \) and \( S^i(V^*) = \nabla(-i\varepsilon_n) \). Replacing \( u^*[s] \) by \( \lambda \otimes u^*[s] \) for \( \lambda = i\varepsilon_1 \) or \( \lambda = -i\varepsilon_n \) in the proof of [11, Lem. II.12.1] and using (4.1) we obtain \( H^1(G_r, S^iV) = H^1(G_r, S^i(V^*)) = 0 \).

Now take \( G = \text{SL}_n \). Then \( S^iV = \nabla(i\varpi_1) \) and \( S^i(V^*) = \nabla(i\varpi_{n-1}) \), where \( \varpi_j \) denotes the \( j \)-th fundamental dominant weight. From [2, Cor. 3.2(a)] we easily deduce that \( H^1(G_r, S^iV) \neq 0 \) if and only if \( H^1(G_r, S^i(V^*)) \neq 0 \) if and only if \( n = 2 \) and \( p^r | i + 2p^s \) for some \( s \in \{0, \ldots, r-1\} \), or \( n = 3 \), \( p = 2 \) and \( 2^r | i - 2^{r-1} \).

For \( G = \text{Sp}_n \), \( n \geq 4 \) even, we deduce using \( S^i(V) = \nabla(i\varpi_1) \) and [2, Cor. 3.2(a)] that \( H^1(G_r, S^iV) \neq 0 \) if and only if \( p = 2 \) and \( i \) is odd.

Now let \( G \) be the special orthogonal group \( \text{SO}_n \), \( n \geq 4 \), as defined in [18, Ex. 7.4.7(3), (4), (6), (7)] (when \( p = 2 \) this is an abuse of notation). Note that \( V \cong V^* \) unless \( n \) is odd and \( p = 2 \). Although the simply connected cover \( \hat{G} \to G \) need not be separable, it still follows from [11, I.6.8(3), I.6.9(3)] that \( H^1(G_r, M) = H^1(\hat{G}_r, M)^{Tr} \) for any \( G \)-module \( M \), and \( H^1(B_r, M) = H^1(\hat{B}_r, M)^{Tr} \) for any \( B \)-module \( M \). So one has to pick out the weight spaces of the weights in \( p^rX(T) \subseteq p^rX(\hat{T}) \). For \( n \geq 8 \) it follows from [2, Cor. 3.2(a)] that \( H^1(\hat{G}_r, \nabla(i\varpi_1)) = 0 \) for all \( i \geq 0 \). For general \( n \geq 4 \) we proceed as follows. From [2, Sect. 2.5–2.7] we deduce that all weights of \( H^1(B_r, i\varpi_1) \) are of the form \( i\varpi_1 + p^s\alpha \) for some \( s \in \{0, \ldots, r-1\} \) and some \( \alpha \) simple or “long” (i.e. there is a shorter root). Since such weights don’t occur in \( p^rX(T) \) for \( \text{SO}_n \), \( n \geq 4 \), we get that \( H^1(B_r, i\varpi_1) = 0 \), and therefore by (4.1) \( H^1(G_r, \nabla(i\varpi_1)) = 0 \) for all \( i \geq 0 \). By [11, II.2.17,18] \( S^i(V^*) \) has a filtration with sections \( \nabla(i\varpi_1), \nabla((i-2)\varpi_1), \ldots \). So \( H^1(G_r, S^i(V^*)) = 0 \) for all \( i \geq 0 \).

The fact that the weights of \( H^1(B_r, i\varpi_1) \) have the form stated above can be seen more directly as follows. First one observes that 1-coycles in the Hochschild complex of a \( U_r \)-module \( M \) can be seen as the linear maps \( D : \text{Dist}^+(U_r) \to M \) with \( D(ab) = aD(b) \) for all \( a \in \text{Dist}(U_r) \) and \( b \in \text{Dist}^+(U_r) \). Here \( \text{Dist}^+(U_r) \) denotes the distributions without constant term, i.e. the distributions \( a \) with \( a(1) = 0 \). Then one shows that, outside type \( G_2 \), \( \text{Dist}(U_r) \) is generated by the \( \text{Dist}(U_{-\alpha,r}) \) with \( \alpha \) simple or long.\(^{(2)}\)

\(^{(2)}\) If \( p \) is not special in the sense of [9], then (also in type \( G_2 \)) \( \text{Dist}(U_r) \) is generated by the \( \text{Dist}(U_{-\alpha,r}) \) with \( \alpha \) simple.
It follows that $H^1(U_r, M)$ is a subquotient of $M \otimes \bigoplus_{0 \leq s < r} u^{-[s]}$, the $\alpha$ simple or long. Now use that, for $M$ a $B_r$-module, $H^1(B_r, M) = H^1(U_r, M)^{Tr}$.

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