



ANNALES DE L'INSTITUT FOURIER

Roberto LONGO, YOH TANIMOTO & YOSHIMICHI UEDA

Free products in AQFT

Tome 69, n° 3 (2019), p. 1229-1258.

<http://aif.centre-mersenne.org/item/AIF_2019__69_3_1229_0>

© Association des Annales de l'institut Fourier, 2019,

Certains droits réservés.

 Cet article est mis à disposition selon les termes de la licence
CREATIVE COMMONS ATTRIBUTION – PAS DE MODIFICATION 3.0 FRANCE.
<http://creativecommons.org/licenses/by-nd/3.0/fr/>



FREE PRODUCTS IN AQFT

by Roberto LONGO, Yoh TANIMOTO & Yoshimichi UEDA (*)

ABSTRACT. — We apply the free product construction to various local algebras in algebraic quantum field theory.

If we take the free product of infinitely many identical half-sided modular inclusions with ergodic canonical endomorphism, we obtain a half-sided modular inclusion with ergodic canonical endomorphism and trivial relative commutant. On the other hand, if we take Möbius covariant nets with trace class property, we are able to construct an inclusion of free product von Neumann algebras with large relative commutant, by considering either a finite family of identical inclusions or an infinite family of inequivalent inclusions. In two dimensional spacetime, we construct Borchers triples with trivial relative commutant by taking free products of infinitely many, identical Borchers triples. Free products of finitely many Borchers triples are possibly associated with Haag–Kastler net having S-matrix which is nontrivial and non asymptotically complete, yet the nontriviality of double cone algebras remains open.

RÉSUMÉ. — Nous appliquons la construction du produit libre à diverses algèbres locales dans la théorie des champs quantiques algébriques.

Si l'on prend le produit libre d'une infinité d'inclusions modulaires unilatérales identiques avec endomorphisme canonique ergodique, on obtient une inclusion modulaire unilatérale avec endomorphisme canonique ergodique et commutant relatif trivial. D'autre part, si nous prenons des réseaux covariants de Möbius avec la propriété de trace, nous pouvons construire une inclusion d'algèbres libres de von Neumann avec un grand commutant relatif, en considérant soit une famille finie d'inclusions identiques, soit une famille infinie d'inclusions deux à deux inéquivalentes. Dans l'espace-temps bi-dimensionnel, nous construisons des triplets de Borchers avec un commutant relatif trivial en prenant des produits libres d'un nombre infini de triplets de Borchers identiques. Il est possible que les produits libres d'un nombre fini de triplets de Borchers soient associés au réseau de Haag–Kastler avec une matrice S non triviale et non asymptotiquement complète, mais la non-trivialité des algèbres à double cône reste une question ouverte.

Keywords: Algebraic QFT, half-sided modular inclusions, conformal nets, free products.
2010 Mathematics Subject Classification: 81T05, 81T40, 46L54.

(*) R.L. is supported in part by the ERC Advanced Grant 669240 QUEST “Quantum Algebraic Structures and Models”, PRIN-MIUR, GNAMPA-INDAM and Alexander von Humboldt Foundation.

Y.T. is supported by Programma per giovani ricercatori, anno 2014 “Rita Levi Montalcini” of the Italian Ministry of Education, University and Research.

Y.U. is supported in part by the Grant-in-Aid for Challenging Exploratory Research JP16K13762.

1. Introduction

The whole model of quantum field theory (QFT), in the framework of algebraic QFT, can be reconstructed from just a combination of few local algebras and the vacuum state. This was a striking discovery by Wiesbrock and others [2, 20, 22, 38, 39]. In particular, a half-sided modular inclusion (HSMI) of two half-line algebras coming from the (strongly additive) chiral component of a two-dimensional conformal field theory completely remembers the structure of QFT on the lightray. It opened a hope to construct new models of QFT which is in general very difficult, and since then, HSMIs have been used to construct possibly new chiral components [13, 29] as well as to determine the structure of massless two-dimensional QFT [31, 32]. Actually, a HSMI is not equivalent to a chiral component (Möbius covariant net), and in order to carry out the reconstruction, one needs an additional condition: the relative commutant of the algebras must be also large. The contrary case does happen, for example when one considers wedge inclusions of (two-dimensional) fermionic QFT [7] or bosonic interacting QFT [4, 33]. Yet, even if the relative commutant is not large enough (the vacuum vector is not cyclic for it), if it is just nontrivial, one can still construct a Möbius covariant net on a smaller subspace [7]. Constructing HSMIs and determining the relative commutant are not easy tasks, and in all the known cases, the relative commutant have been nontrivial. In the present work, we provide examples of HSMIs with trivial relative commutant, by exploiting the techniques of free product von Neumann algebras.

The study of free products of (possibly type III) von Neumann algebras dates back to the birth of free probability theory initiated by Voiculescu (see e.g. [37]), and their fundamental properties such as central decomposition and type classification in the not necessarily type II_1 setting were recently established in full generality (see [34]). From a family $\{(\mathcal{M}_\kappa, \Omega_\kappa)\}_{\kappa \in K}$ of von Neumann algebras equipped with vector states, one constructs (\mathcal{M}, Ω) where \mathcal{M} is generated by the faithful images of \mathcal{M}_κ and these images are freely independent, or roughly speaking, they do not commute in the highest degree but still have some “relations” provided by Ω . One can actually start with a family of inclusions $(\mathcal{N}_\kappa \subset \mathcal{M}_\kappa, \Omega_\kappa)$ and obtains an inclusion $\mathcal{N} \subset \mathcal{M}$ of free product von Neumann algebras. We will prove that if $(\mathcal{N}_\kappa \subset \mathcal{M}_\kappa, \Omega_\kappa)$ is an *infinite* family of *identical* HSMIs, the resulting inclusion $\mathcal{N} \subset \mathcal{M}$ must have trivial relative commutant.

Before this work, no serious investigation of inclusions of free product von Neumann algebras has been made without assuming the existence of faithful normal conditional expectations. We provide an illustrative example

whereby the relative commutant is nontrivial by exploiting the techniques of AQFT. This is quite an interesting phenomenon, because it seems, at first sight, contrary to the highest degree of non-commutativity of the free product construction. If $\{(\{\mathcal{M}_\kappa(I)\}_{I \subset S^1}, \Omega_\kappa)\}_{\kappa \in K}$ is a family of (local) Möbius covariant nets with vacuum vectors, we can construct a family of von Neumann algebras satisfying the axioms of Möbius covariant net except locality. Then, by taking K to be finite or considering an appropriate family of local algebras with infinite K , we obtain inclusions $\mathcal{N} \subset \mathcal{M}$ of free product von Neumann algebras where $\mathcal{N}' \cap \mathcal{M}$ is a type III₁ factor. The proof utilizes the L^2 -nuclearity condition [10], a physical condition saying that the state space with small energy is not too large. Although the L^2 -nuclearity condition indeed guarantees the nontriviality of the relative commutant, we do not find any explicit, nontrivial elements in it.

We also apply a similar construction to two-dimensional quantum field theory. A two-dimensional Haag–Kastler net can be reconstructed from a single von Neumann algebra, a vacuum state and a representation of translations (a Borchers triple). One can again consider the free product of Borchers triples. Here, if the index set is infinite and all the given triples are identical, we obtain a Borchers triple with trivial relative commutant. On the other hand, if the index set is finite, we are not able to determine the relative commutant. Yet, one can define and compute the two-particle scattering S-matrix, and it turns out to be nontrivial and not asymptotically complete. This case is of particular interest, because it might give an example of Haag–Kastler net with minimal size, namely, the local algebras are trivial below that size.

This paper is organized as follows. In Section 2 we review the fundamental concepts of algebraic QFT, half-sided modular inclusions and free products of von Neumann algebras. In Section 3, we construct a HSMI with trivial relative commutant by taking the free product of infinitely many copies of a standard HSMI. Section 4 introduces the concept of free product of Möbius covariant nets, and this concept enables us to give an example of inclusion of free product von Neumann algebras with large relative commutant. Finally, in Section 5 we consider free products of two-dimensional Borchers triples. When the index set K is infinite and the family consists of identical ones, the relative commutant is shown to be trivial. When K is finite, it remains open, meaning that it might give a two-dimensional Haag–Kastler net which should not have the wedge split property for small distance.

Acknowledgements. We thank Detlev Buchholz, Gandalf Lechner and Mihály Weiner for stimulating discussions.

Y.T. would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme “Operator algebras: subfactors and their applications”, supported by EPSRC Grant Number EP/K032208/1 and partially by a grant from the Simons Foundation, where part of this work was undertaken.

R.L. and Y.T. acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

2. Preliminaries

2.1. Algebraic QFT

2.1.1. One-dimensional chiral components

Let S^1 denote the unit circle and let \mathcal{I} be the collection of proper intervals (i.e., open, connected, non-empty, non-dense subsets) $I \subset S^1$. For any $I \in \mathcal{I}$, we denote by I' the interior of the complement of I in S^1 . The Möbius group $\text{Möb} := \text{SL}(2, \mathbb{R})/\mathbb{Z}_2 \cong \text{PSU}(1, 1)$ acts on S^1 by diffeomorphisms in a natural fashion. Let $R : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \text{Möb}$ denote the rotation subgroup. Let $\mathcal{A} = (\mathcal{A}(I))_{I \in \mathcal{I}}$ be a Möbius covariant net (or a Möbius covariant precosheaf) on S^1 in the sense of [15, Section 2]. Namely, all the $\mathcal{A}(I)$, $I \in \mathcal{I}$, are von Neumann algebras on a fixed Hilbert space \mathcal{H} with the following properties:

- (1) Isotony: $I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.
- (2) Möbius covariance: There exists a unitary representation $U : G \curvearrowright \mathcal{H}$ such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$ for every $g \in G$ and $I \in \mathcal{I}$.
- (3) Positivity of energy: The generator L_0 of the one parameter unitary group $\theta \mapsto U(R(\theta)) = e^{i\theta L_0}$, called the *conformal Hamiltonian*, is positive.
- (4) Existence of vacuum: There exists a unique (up to a scalar) unit vector $\Omega \in \mathcal{H}$, called the *vacuum vector*, such that $U(g)\Omega = \Omega$ for all $g \in \text{Möb}$ and that Ω is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$ and separating for $\bigwedge_{I \in \mathcal{I}} \mathcal{A}(I)$.

A Möbius covariant net \mathcal{A} is said to be *local* if the following property holds:

- (5) Locality: $I_1 \cap I_2 = \emptyset \Rightarrow \mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)'$.

Algebras $\mathcal{A}(I)$ are of type III_1 , unless the Hilbert space is one-dimensional (we call this case *trivial*) [15, Proposition 2.4(ii)]. The Bisognano–Wichmann property $\Delta_I^{it} := \Delta_{\mathcal{A}(I)}^{it} = U(\Lambda_I(-2\pi t))$ holds, where $\Delta_{\mathcal{A}(I)}$ is the modular operator of $\mathcal{A}(I)$ with respect to Ω and Λ_I is the dilation associated with I , see [15, Proposition 2.2(ii)]. In the following, we always stress whether each statement holds with or without locality, so that no confusion arises.

In a nice situation, the whole Möbius covariant net can be reconstructed from a pair of von Neumann algebras. Let us consider another additional property.

- (6) Strong additivity: For $I \in \mathcal{I}$ and I_1, I_2 the two intervals obtained by removing one point from I , it holds that $\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$.

Let \mathcal{H} be a Hilbert space, \mathcal{M} a von Neumann algebra on \mathcal{H} , T a unitary representation of \mathbb{R} on \mathcal{H} with positive spectrum and Ω a vector in \mathcal{H} . We say (\mathcal{M}, T, Ω) is a *one-dimensional Borchers triple* on \mathcal{H} if

- Ω is the unique (up to a scalar) invariant vector under $T(a)$,
- $\text{Ad } T(a)(\mathcal{M}) \subset \mathcal{M}$ for $a \in \mathbb{R}_+$,
- Ω is cyclic and separating for \mathcal{M} .

This notion is related with the following: a *half-sided modular inclusion* (HSMI)⁽¹⁾ is an inclusion $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras and a vector Ω which is cyclic and separating for both of \mathcal{N}, \mathcal{M} such that $\sigma_t^{\mathcal{M}}(\mathcal{N}) \subset \mathcal{N}$ for $t \geq 0$, where $\sigma_t^{\mathcal{M}}$ is the modular automorphism group of \mathcal{M} with respect to Ω .

If (\mathcal{M}, T, Ω) is a one-dimensional Borchers triple, then $\mathcal{N} = \text{Ad } T(1)(\mathcal{M}) \subset \mathcal{M}$ is a half-sided modular inclusion by the Borchers theorem [5, 18]. The uniqueness of Ω implies the ergodicity $\bigcap_{t \geq 0} \sigma_t^{\mathcal{M}}(\mathcal{N}) = \mathbb{C}\mathbf{1}$. Conversely, if $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is a half-sided modular inclusion, then one can construct a representation $T(a)$ of \mathbb{R} with positive spectrum such that $\mathcal{N} = \text{Ad } T(1)(\mathcal{M})$ and $T(2) = J_{\mathcal{N}}J_{\mathcal{M}}$, where $J_{\mathcal{M}}$ and $J_{\mathcal{N}}$ are the modular conjugations of \mathcal{M} and \mathcal{N} with respect to Ω , respectively [2, 38]. Furthermore, $T(a)$ together with the modular group $\Delta_{\mathcal{M}}^{it}$ generates a representation of the translation-dilation group. We remark that $\sigma_t^{\mathcal{M}}$ is ergodic on \mathcal{N} if and only if the canonical endomorphism $\text{Ad } \Gamma$ is ergodic on \mathcal{N} , where $\Gamma = J_{\mathcal{N}}J_{\mathcal{M}} = T(2)$, and in this case, the uniqueness of Ω follows, giving rise to a one-dimensional Borchers triple.

We say that a HSMI $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is *standard* if Ω is cyclic and separating for $\mathcal{N}' \cap \mathcal{M}$. We also say that a one-dimensional Borchers triple is standard

⁽¹⁾The definition here is called +HSMI in [39]. The other case, the inclusion for $t \leq 0$, is called -HSMI and can be treated in a completely parallel way.

if so is the corresponding half-sided modular inclusion. Let us identify S^1 and the one-point compactification of \mathbb{R} by the stereographic projection. There is a one-to-one correspondence between standard half-sided modular inclusions $(\mathcal{N} \subset \mathcal{M}, \Omega)$ and strongly additive local Möbius covariant nets \mathcal{A} by $\mathcal{A}((0, \infty)) = \mathcal{M}, \mathcal{A}((1, \infty)) = \mathcal{N}$ [20, Corollary 1.7].

Actually, even if $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is not standard, one can still construct a local Möbius covariant net on the subspace $\overline{(\mathcal{N}' \cap \mathcal{M})\Omega}$: indeed, one should just define $\mathcal{A}((0, 1)) := \mathcal{N}' \cap \mathcal{M}$ and then $\mathcal{A}((0, \infty))$ by dilation covariance and closure. In this way, one obtains a standard one-dimensional Borchers triple, hence a strongly additive local Möbius covariant net (c.f. [7, Lemma 5.1, Theorem 5.2]).

Therefore, it is an important problem to determine whether the relative commutant $\mathcal{N}' \cap \mathcal{M}$ of a HSMI is nontrivial. Moreover, in all examples constructed so far, the relative commutant was either nontrivial or difficult to identify. One of the main results in this work is to provide examples of half-sided modular inclusion with trivial relative commutant.

2.1.2. Two-dimensional nets

In $(1 + 1)$ -dimensional Minkowski space, a (Poincaré-covariant) *Haag–Kastler net* is a family $\{\mathcal{A}(O)\}_{O \subset \mathbb{R}^2}$ of von Neumann algebras on a fixed Hilbert space \mathcal{H} parametrized by bounded open regions in \mathbb{R}^2 satisfying the following properties:

- (1) Isotony: If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- (2) Locality: If O_1 and O_2 are spacelike separated, then $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute.
- (3) Poincaré covariance: There is a unitary representation U of the proper orthochronous Poincaré group \mathcal{P}_+^\uparrow on \mathcal{H} such that $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$ for every $g \in \mathcal{P}_+^\uparrow$ and open region $O \subset \mathbb{R}^2$.
- (4) Positivity of the energy: The restriction of U to the translation subgroup \mathbb{R}^2 has the joint spectrum included in $\overline{V}_+ = \{(t_0, t_1) \in \mathbb{R}^2 : t_0 \geq |t_1|\}$.
- (5) Existence of the vacuum: There exists a unique (up to scalar) unit vector $\Omega \in \mathcal{H}$, called the *vacuum vector*, such that $U(g)\Omega = \Omega$ for all $g \in \mathcal{P}_+^\uparrow$ and that Ω is cyclic for $\mathcal{A}(O)$ where O is sufficiently large.

Remark 2.1. — Usually one assumes weak additivity and cyclicity of the vacuum for the global algebra $\bigcup_{O \subset \mathbb{R}^2} \mathcal{A}(O)$, and proves the Reeh–Schlieder property $\overline{\mathcal{A}(O)\Omega} = \mathcal{H}$, where O is open. Yet, the weak additivity is not

necessarily a physical requirement, neither is it known whether it follows from the Wightman axioms. In addition, we are also interested in Haag–Kastler nets possibly with minimal size, namely, Ω is not cyclic for $\mathcal{A}(O)$ when O is small. Therefore, we include the Reeh–Schlieder property for sufficiently large O in the axioms.

It is not an easy task to construct a Haag–Kastler net, especially the infinite family of von Neumann algebras $\{\mathcal{A}(O)\}$, but in some situations it can be reduced to a single von Neumann algebra associated with the wedge $W_R = \{(a_0, a_1) : a_1 > |a_0|\}$. By a *two-dimensional Borchers triple* (\mathcal{M}, T, Ω) we mean a triple of a von Neumann algebra \mathcal{M} on \mathcal{H} , a unitary representation T of \mathbb{R}^2 with joint spectrum in \overline{V}_+ and a vector Ω (called the vacuum vector) such that

- Ω is the unique (up to scalar) invariant vector under $T(a)$.
- $\text{Ad } T(a)\mathcal{M} \subset \mathcal{M}$ for $a \in W_R$.
- Ω is cyclic and separating for \mathcal{M} .

It is immediate that if (\mathcal{A}, U, Ω) is a Poincaré covariant Haag–Kastler net, then the triple $(\mathcal{A}(W_R), U|_{\mathbb{R}^2}, \Omega)$ is a Borchers triple. Conversely, if (\mathcal{M}, T, Ω) is a Borchers triple, one can construct a net as follows. Any double cone in two-dimensional spacetime is the intersection of two-wedges $(W_R + a) \cap (W_L + b) =: D_{a,b}$, where W_L is the reflected (left-)wedge. Then one constructs the net first for double cones by $\mathcal{A}(D_{a,b}) := \text{Ad } T(a)(\mathcal{M}) \cap \text{Ad } T(b)(\mathcal{M}')$, and then for a general region O by

$$\mathcal{A}(O) := \left(\bigcup_{D_{a,b} \subset O} \mathcal{A}(D_{a,b}) \right)''.$$

It is easy to show that this family $\mathcal{A}(O)$ satisfies isotony and locality. Furthermore, the representation T can be extended by the Tomita–Takesaki theory to a representation U of the Poincaré group such that (\mathcal{A}, U) is covariant and Ω is still its fixed vector. The requirement that Ω should be cyclic for sufficiently large O needs to be checked in examples. See [25, Section 2].

2.2. The split property and nuclearity conditions

We say that an inclusion $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras is *split* if there is an intermediate type I factor \mathcal{R} , i.e., $\mathcal{N} \subset \mathcal{R} \subset \mathcal{M}$. If \mathcal{N} and \mathcal{M} are type III algebras on a separable Hilbert space \mathcal{H} , the split property has an

immediate consequence on the relative commutant: $\mathcal{N}' \cap \mathcal{M}$ is nontrivial. Indeed, by identifying the intermediate type I factor \mathcal{R} with $\mathcal{B}(\mathcal{K}_1) \otimes \mathbb{C}\mathbf{1}$, this inclusion is unitarily equivalent to $\mathcal{N} \otimes \mathbb{C} \subset \mathcal{B}(\mathcal{K}_1) \otimes \mathcal{M}$, where \mathcal{M} and \mathcal{N} are type III factors, hence we obtain $\mathcal{N}' \cap \mathcal{M} \cong \mathcal{N}' \otimes \mathcal{M}$.

There are several sufficient conditions for the split property in terms of the modular theory. Let Ω be a common cyclic and separating vector for \mathcal{N} and \mathcal{M} and $\Delta_{\mathcal{M}}$ be the modular operator for \mathcal{M} with respect to Ω . We say that the inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$ satisfies the *modular nuclearity condition* if the map

$$\mathcal{N} \ni x \longmapsto \Delta_{\mathcal{M}}^{\frac{1}{4}} x \Omega$$

is nuclear. The modular nuclearity condition implies the split property for $\mathcal{N} \subset \mathcal{M}$ [10, Propositions 1.1, 2.3]. Furthermore, if $\Delta_{\mathcal{M}}^{\frac{1}{4}} \Delta_{\mathcal{N}}^{-\frac{1}{4}}$ is nuclear, where $\Delta_{\mathcal{N}}$ is the modular operator for \mathcal{N} with respect to Ω , the inclusion $\mathcal{N} \subset \mathcal{M}$ is said to satisfy the *L^2 -nuclearity condition*. The L^2 -nuclearity implies the modular nuclearity [11, Proposition 5.3], hence also the split property.

2.2.1. One-dimensional case

Let us turn to conformal nets (\mathcal{A}, U, Ω) . The half-sided modular inclusion $\mathcal{A}((1, \infty)) \subset \mathcal{A}((0, \infty))$ is never split (see the argument of [8, p. 292(b)]). Yet, one can consider inclusions $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ with $\bar{I} \subset \tilde{I}$.

Let us focus on the generator of rotations L_0 . It has a discrete spectrum, as the 2π -rotation is trivial. If $\text{Tr } e^{-sL_0} < \infty$ for some s , we say that the net (\mathcal{A}, U, Ω) satisfies the *trace class property* with s . In this case, the *distal split property* holds: namely, if $I \subset \tilde{I}$ with “conformal distance” $\ell(I, \tilde{I}) > s$, $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ is split [11, Corollary 6.4].

Actually, for most of the examples of *local* Möbius covariant nets, a stronger property holds: $\dim \ker(L_0 - n)$ grows asymptotically as $e^{\alpha n^\nu}$ where $0 < \nu < 1$. In such a case, $e^{-sn} \cdot e^{\alpha n^\nu}$ tends rapidly to 0, hence $\text{Tr } e^{-sL_0} < \infty$ for any $s > 0$.

2.2.2. Two-dimensional case

Let (\mathcal{M}, T, Ω) be a two-dimensional Borchers triple. There is a sufficient condition for Ω to be cyclic for double cone algebras $\mathcal{A}(D_{0,a}) = \mathcal{M} \cap \text{Ad } T(a)(\mathcal{M}')$ [10]: let $\Delta_{\mathcal{M}}$ be the modular operator for \mathcal{M} with respect to Ω . For $a \in W_R$, consider the map

$$\mathcal{M} \ni x \longmapsto \Delta_{\mathcal{M}}^{\frac{1}{4}} U(a) x \Omega.$$

If this map from \mathcal{M} to \mathcal{H} is nuclear, we say that the Borchers triple (\mathcal{M}, U, Ω) satisfies *the modular nuclearity* (for a), and then there is a type I factor \mathcal{R}_a such that $\text{Ad } U(a)(\mathcal{M}) \subset \mathcal{R}_a \subset \mathcal{M}$ [10, Proposition 2.4], [11, Proposition 5.2]. If $\text{Ad } U(a)(\mathcal{M}) \subset \mathcal{M}$ is split for a sufficiently large a , then Ω is cyclic for $\mathcal{A}(D_{0,a})$ [24, Section 2] (this statement is only implicit there).

2.3. Free product of von Neumann algebras

Here we recall the free product construction in von Neumann algebras. Nowadays, free product von Neumann algebras are usually understood in quite an abstract fashion based on free independence, but a more constructive way starting with the notion of free products of Hilbert spaces is appropriate in the context of AQFT. Hence we summarize an older approach to free product von Neumann algebras.

Let K be a possibly infinite index set and $\{\mathcal{H}_\kappa\}_{\kappa \in K}$ a family of Hilbert spaces with distinguished unit vectors $\Omega_\kappa \in \mathcal{H}_\kappa$. With a one-dimensional subspace $\mathbb{C}\Omega$ and a unit vector Ω , the free product (\mathcal{H}, Ω) of these $(\mathcal{H}_\kappa, \Omega_\kappa)$, $\kappa \in K$, is defined by

$$(2.1) \quad \mathcal{H} = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{\kappa_j \neq \kappa_{j+1} \\ 1 \leq j \leq n-1}} \mathcal{H}_{\kappa_1}^\circ \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_{\kappa_n}^\circ \quad \text{with} \quad \mathcal{H}_\kappa^\circ := \mathcal{H}_\kappa \ominus \mathbb{C}\Omega_\kappa.$$

Let us denote by $P_{\mathbb{C}\Omega}$ and $P_{\mathcal{H}_\kappa^\circ}$ the orthogonal projections onto $\mathbb{C}\Omega$ and \mathcal{H}_κ° , respectively. For any given family $\{T_\kappa\}_{\kappa \in K}$ of bounded operators on \mathcal{H}_κ (linear or antilinear), with norm $\|T_\kappa\|$ not greater than 1, such that $T_\kappa \mathcal{H}_\kappa^\circ \subset \mathcal{H}_\kappa^\circ$, we use the symbol

$$\star_{\kappa \in K} T_\kappa := T_\Omega \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{\kappa_j \neq \kappa_{j+1} \\ 1 \leq j \leq n-1}} T_{\kappa_1}^\circ \otimes \cdots \otimes T_{\kappa_n}^\circ \in \mathcal{B}(\mathcal{H}) \quad \text{with} \quad T_\kappa^\circ := T_\kappa \mid_{\mathcal{H}_\kappa^\circ},$$

where $T_\Omega = \mathbb{1}_\mathbb{C}$ if T_κ 's are linear and $T_\Omega = J_\mathbb{C}$ (the complex conjugation) if T_κ 's are antilinear (according to the direct sum decomposition (2.1)) following [36, §1.8]. When $T_\kappa \Omega_\kappa = \Omega_\kappa$ and $T_\kappa^* \Omega_\kappa = \Omega_\kappa$ (a stronger assumption than $T_\kappa \mathcal{H}_\kappa^\circ \subset \mathcal{H}_\kappa^\circ$), the resulting $\star_{\kappa \in K} T_\kappa$ restores T_κ as its restriction to $\mathcal{H}_\kappa = \mathbb{C}\Omega \oplus \mathcal{H}_\kappa^\circ \subset \mathcal{H}$ and hence it is a common extension of the given T_κ . For each $\kappa \in K$ we have two normal representations $\lambda_\kappa : \mathcal{B}(\mathcal{H}_\kappa) \curvearrowright \mathcal{H}$ and $\rho_\kappa : \mathcal{B}(\mathcal{H}_\kappa) \curvearrowleft \mathcal{H}$ acting from the left and the right, respectively, as in [36, §1.2]. More explicitly, let us fix $\kappa \in K$. \mathcal{H} can be rearranged as

follows:

$$\begin{aligned} \mathcal{H} &= \bigoplus_{n \geq 1} \bigoplus_{\substack{\kappa \neq \kappa_2 \\ \kappa_j \neq \kappa_{j+1} \\ 2 \leq j \leq n-1}} (\mathcal{H}_\kappa^\circ \bar{\otimes} \mathcal{H}_{\kappa_2}^\circ \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_{\kappa_n}^\circ) \oplus (\mathcal{H}_{\kappa_2}^\circ \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_{\kappa_n}^\circ) \\ &\cong \mathcal{H}_\kappa \otimes \mathcal{H}'_\kappa, \end{aligned}$$

where the case $n = 1$ is by convention $\mathcal{H}_\kappa^\circ \oplus \mathbb{C}\Omega \cong \mathcal{H}_\kappa$, \mathcal{H}'_κ is a direct sum of $\mathcal{H}_{\kappa_2}^\circ \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_{\kappa_n}^\circ$ where $\kappa \neq \kappa_2$. $\mathcal{B}(\mathcal{H}_\kappa)$ acts naturally on the left component \mathcal{H}_κ , which we denote by λ_κ . By switching the left and the right, we can similarly construct the right action denoted by ρ_κ .

They enjoy the following commutation relation [36, §1.3]: For any $\kappa, \kappa' \in K$ and $T \in \mathcal{B}(\mathcal{H}_\kappa)$ and $T' \in \mathcal{B}(\mathcal{H}_{\kappa'})$ we have

$$\begin{aligned} [\lambda_\kappa(T), \rho_{\kappa'}(T')] &= \delta_{\kappa, \kappa'}(P_{\mathbb{C}\Omega} + P_{\mathcal{H}_\kappa^\circ})\lambda_\kappa([T, T']) \\ &= \delta_{\kappa, \kappa'}\rho_\kappa([T, T'])(P_{\mathbb{C}\Omega} + P_{\mathcal{H}_\kappa^\circ}). \end{aligned}$$

Define the unitary involution $Z : \mathcal{H} \rightarrow \mathcal{H}$ by $Z\Omega = \Omega$ and

$$Z(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_n \otimes \cdots \otimes \xi_1 \quad (\text{in reverse order})$$

for any $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}_{\kappa_1}^\circ \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_{\kappa_n}^\circ \subset \mathcal{H}$. As remarked in [36, §1.7],

$$(2.2) \quad Z\lambda_\kappa(T) = \rho_\kappa(T)Z$$

holds for every $T \in \mathcal{B}(\mathcal{H}_\kappa)$. We will keep this notation throughout this paper. Let $\mathcal{M}_\kappa \subseteq \mathcal{B}(\mathcal{H}_\kappa)$, $\kappa \in K$, be von Neumann algebras such that each Ω_κ is cyclic for \mathcal{M}_κ . Then, it is known, see [36, Proposition 1.9], that

$$(2.3) \quad \left(\bigvee_{\kappa \in K} \lambda_\kappa(\mathcal{M}_\kappa) \right)' = \bigwedge_{\kappa \in K} \lambda_\kappa(\mathcal{M}_\kappa)' = \bigvee_{\kappa \in K} \rho_\kappa(\mathcal{M}'_\kappa)$$

on \mathcal{H} , where only the second equality is nontrivial. Here the symbols \bigvee and \bigwedge denote the operation of generation as von Neumann algebra and the intersection operation, respectively. Remark that Ω is cyclic for $\bigvee_{\kappa \in K} \lambda_\kappa(\mathcal{M}_\kappa)$ by construction. Moreover, by the above commutation relation, we see that Ω is separating for $\bigvee_{\kappa \in K} \lambda_\kappa(\mathcal{M}_\kappa)$, too, when every Ω_κ is separating for \mathcal{M}_κ . The von Neumann algebra $\mathcal{M} = \bigvee_{\kappa \in K} \lambda_\kappa(\mathcal{M}_\kappa)$ is a concrete and standard realization of the so-called *free product von Neumann algebra* of the $(\mathcal{M}_\kappa, \omega_\kappa)$, $\kappa \in K$, where the vector state ω_κ is constructed from Ω_κ for every $\kappa \in K$. The vector state ω constructed from Ω is called the *free product state*.

Abstractly, the *free product* $(\mathcal{M}, \omega) = \star_{\kappa \in K} (\mathcal{M}_\kappa, \omega_\kappa)$ can be formulated by the following four conditions:

- There exist normal injective $*$ -homomorphisms $\lambda_\kappa : \mathcal{M}_\kappa \rightarrow \mathcal{M}$, $\kappa \in K$, whose ranges generate \mathcal{M} as von Neumann algebra.
- $\omega \circ \lambda_\kappa = \omega_\kappa$ for every $\kappa \in K$.
- The $\lambda_\kappa(\mathcal{M}_\kappa)$, $\kappa \in K$, are *freely independent* in (\mathcal{M}, ω) , that is,

$$(2.4) \quad \omega(\lambda_{\kappa_1}(x_1) \cdots \lambda_{\kappa_n}(x_n)) = 0$$

whenever $x_j \in \text{Ker}(\omega_{\kappa_j}) \cap \mathcal{M}_{\kappa_j}$ with $\kappa_j \neq \kappa_{j+1}$ ($1 \leq j \leq n-1$) and $n \geq 1$.

- The GNS representation of \mathcal{M} associated with ω is faithful.

See [36, §1.5]. It is fundamental that the modular automorphism $\sigma_{\mathcal{M}}^t$ associated with ω can be determined by the modular condition (the KMS condition with inverse temperature -1) as

$$(2.5) \quad \sigma_{\mathcal{M}}^t \circ \lambda_\kappa = \lambda_\kappa \circ \sigma_{\mathcal{M}_\kappa}^t$$

for every $\kappa \in K$, where $\sigma_{\mathcal{M}_\kappa}^t$ is the modular automorphism of \mathcal{M}_κ associated with ω_κ , see [3, Lemma 1][17, Theorem 1] (n.b. the proof of the latter essentially uses Voiculescu's computation of the S -operator in [36, Lemma 1.8]). This formula for the modular automorphisms immediately implies that the modular operator $\Delta_{\mathcal{M}}$ and the modular conjugation $J_{\mathcal{M}}$ with respect to Ω are computed as

$$(2.6) \quad \Delta_{\mathcal{M}}^{it} = \star_{\kappa \in K} \Delta_{\mathcal{M}_\kappa}^{it}, \quad J_{\mathcal{M}} = Z(\star_{\kappa \in K} J_{\mathcal{M}_\kappa}) = (\star_{\kappa \in K} J_{\mathcal{M}_\kappa})Z,$$

where $\Delta_{\mathcal{M}_\kappa}$ and $J_{\mathcal{M}_\kappa}$ denote the modular operator and the modular conjugation for \mathcal{M}_κ with respect to Ω_κ .

3. Free products of Half-sided modular inclusions

Let $\mathcal{N}_\kappa \subset \mathcal{M}_\kappa$, $\kappa \in K$, be a (possibly infinite) family of inclusions of nontrivial von Neumann algebras on Hilbert spaces \mathcal{H}_κ . Assume that each inclusion admits a common cyclic and separating unit vector $\Omega_\kappa \in \mathcal{H}_\kappa$. We denote by ω_κ the vector state obtained from Ω_κ .

Let (\mathcal{H}, Ω) be the free product of the $(\mathcal{H}_\kappa, \Omega_\kappa)$, $\kappa \in K$. Consider the free products $(\mathcal{M}, \omega) = \star_{\kappa \in K}(\mathcal{M}_\kappa, \omega_\kappa)$ and $(\mathcal{N}, \omega) = \star_{\kappa \in K}(\mathcal{N}_\kappa, \omega_\kappa)$. As explained in Section 2.3, these pairs are explicitly constructed as $\mathcal{M} = \bigvee_{\kappa \in K} \lambda_\kappa(\mathcal{M}_\kappa) \supseteq \mathcal{N} = \bigvee_{\kappa \in K} \lambda_\kappa(\mathcal{N}_\kappa)$ and $\omega = \langle \Omega, \cdot \Omega \rangle$ on \mathcal{H} and Ω is a common cyclic and separating unit vector for $\mathcal{N} \subseteq \mathcal{M}$.

As in (2.5), the modular automorphisms $\sigma_{\mathcal{M}}^t$ and $\sigma_{\mathcal{N}}^t$ of \mathcal{M} and \mathcal{N} , respectively, associated with ω are written as

$$\sigma_{\mathcal{M}}^t \circ \lambda_\kappa = \lambda_\kappa \circ \sigma_{\mathcal{M}_\kappa}^t, \quad \sigma_{\mathcal{N}}^t \circ \lambda_\kappa = \lambda_\kappa \circ \sigma_{\mathcal{N}_\kappa}^t,$$

where $\sigma_{\mathcal{M}_\kappa}^t$ and $\sigma_{\mathcal{N}_\kappa}^t$ denotes the modular automorphisms of \mathcal{M}_κ and \mathcal{N}_κ , respectively, associated with ω_κ . This description of the modular automorphisms immediately gives the next Lemma.

LEMMA 3.1. — If $\sigma_{\mathcal{M}_\kappa}^t(\mathcal{N}_\kappa) \subset \mathcal{N}_\kappa$ for all $t \geq 0$ for all $\kappa \in K$, then $\sigma_{\mathcal{M}}^t(\mathcal{N}) \subset \mathcal{N}$ for all $t \geq 0$. Hence, when all the $(\mathcal{N}_\kappa \subset \mathcal{M}_\kappa, \Omega_\kappa)$ are half-sided modular inclusions, so is $(\mathcal{N} \subset \mathcal{M}, \Omega)$.

We call the triple $(\mathcal{N} \subset \mathcal{M}, \Omega)$ the *free product of the (given) half-sided modular inclusions* $(\mathcal{N}_\kappa \subset \mathcal{M}_\kappa, \Omega_\kappa)$, $\kappa \in K$, and write $(\mathcal{N} \subset \mathcal{M}, \Omega) = \star_{\kappa \in K} (\mathcal{N}_\kappa \subset \mathcal{M}_\kappa, \Omega_\kappa)$.

The computation of the modular automorphism $\sigma_{\mathcal{M}}^t$ also enables us to prove the following:

LEMMA 3.2. — If $\mathcal{N} = \mathcal{M}$, then $\mathcal{N}_\kappa = \mathcal{M}_\kappa$ for all $\kappa \in K$.

Proof. — Let us fix $\kappa \in K$. By the formula of the modular automorphism $\sigma_{\mathcal{N}}^t$ explained above, there exists a ω -preserving conditional expectation E from \mathcal{N} onto $\lambda_\kappa(\mathcal{N}_\kappa)$. By assumption, we have the restriction of E to $\lambda_\kappa(\mathcal{M}_\kappa) \subset \mathcal{M} = \mathcal{N}$, which gives a faithful normal conditional expectation from $\lambda_\kappa(\mathcal{M}_\kappa)$ onto $\lambda_\kappa(\mathcal{N}_\kappa)$. Hence $E_\kappa := \lambda_\kappa^{-1} \circ E \circ \lambda_\kappa$ gives a faithful normal conditional expectation from \mathcal{M}_κ onto \mathcal{N}_κ . Observe that $\omega_\kappa(E_\kappa(x)) = \omega_\kappa(\lambda_\kappa^{-1}(E(\lambda_\kappa(x)))) = \omega(E(\lambda_\kappa(x))) = \omega(\lambda_\kappa(x)) = \omega_\kappa(x)$ for all $x \in \mathcal{M}_\kappa$. Hence $x\Omega_\kappa \in \mathcal{M}_\kappa\Omega_\kappa \mapsto E_\kappa(x)\Omega_\kappa \in \mathcal{N}_\kappa\Omega_\kappa$ extends to a unique orthogonal projection from $[\mathcal{M}_\kappa\Omega_\kappa]$ onto $[\mathcal{N}_\kappa\Omega_\kappa]$. Since Ω_κ is cyclic for \mathcal{N}_κ , we conclude that $x\Omega_\kappa = E_\kappa(x)\Omega_\kappa$ holds for every $x \in \mathcal{M}_\kappa$. Since Ω_κ is separating for \mathcal{M}_κ , we get $E_\kappa = \text{id}$, that is, $\mathcal{N}_\kappa = \mathcal{M}_\kappa$. \square

The above lemma actually says that if $\mathcal{N}_\kappa \subsetneq \mathcal{M}_\kappa$ for some $\kappa \in K$, then $\mathcal{N} \subsetneq \mathcal{M}$. We also remark that if the centralizers $(\mathcal{M}_\kappa)_{\omega_\kappa} = \mathbb{C}\mathbf{1}$ for all $\kappa \in K$, then $\mathcal{M}_\omega = \mathbb{C}\mathbf{1}$ thanks to [3, Lemma 7] (also see the proof of [35, Lemma 2.1]).

In what follows, we denote by $J_{\mathcal{N}_\kappa}$ and $J_{\mathcal{M}_\kappa}$ the modular conjugation operators for \mathcal{N}_κ and \mathcal{M}_κ , respectively, with respect to Ω_κ . We also denote by $J_{\mathcal{N}}$ and $J_{\mathcal{M}}$ the modular conjugation operators for \mathcal{N} and \mathcal{M} , respectively, constructed by Ω . Set $\Gamma_\kappa := J_{\mathcal{N}_\kappa} J_{\mathcal{M}_\kappa}$ for every $\kappa \in K$ as well as $\Gamma := J_{\mathcal{N}} J_{\mathcal{M}}$. Furthermore, it holds that $\sigma_{\mathcal{M}_\kappa}^{4\pi n}(\mathcal{N}_\kappa) = \text{Ad } \Gamma_\kappa^n(\mathcal{N}_\kappa)$ and $\sigma_{\mathcal{M}}^{4\pi n}(\mathcal{N}) = \text{Ad } \Gamma^n(\mathcal{N})$, see [38, Corollary 4].

PROPOSITION 3.3. — If $\Gamma_\kappa^n \rightarrow P_{\mathbb{C}\Omega_\kappa}$ weakly as $n \rightarrow \infty$ for all $\kappa \in K$, then $\Gamma^n \rightarrow P_{\mathbb{C}\Omega}$ weakly as $n \rightarrow \infty$. In particular, if $\sigma_{\mathcal{M}_\kappa}^t$ is ergodic on \mathcal{N}_κ for every $\kappa \in K$, then $\sigma_{\mathcal{M}}^t$ is ergodic on \mathcal{N} , or equivalently, the canonical endomorphism $\gamma := \text{Ad } \Gamma$ is ergodic, i.e., $\bigcap_{n \geq 1} \gamma^n(\mathcal{M}) = \mathbb{C}\mathbf{1}$.

Proof. — By construction we observe that $J_{\mathcal{N}_\kappa}\Omega_\kappa = J_{\mathcal{M}_\kappa}\Omega_\kappa = \Omega_\kappa$, and thus $\Gamma_\kappa\Omega_\kappa = \Omega_\kappa$. Write $\Gamma_\kappa^\circ := \Gamma_\kappa|_{\mathcal{H}_\kappa^\circ}$ (which is well-defined) and observe by assumption that $(\Gamma_\kappa^\circ)^n \rightarrow 0$ weakly as $n \rightarrow \infty$. By the fact (2.6), we obtain that

$$\begin{aligned} \Gamma &= (\star_{\kappa \in K} J_{\mathcal{M}_\kappa}) Z Z (\star_{\kappa \in K} J_{\mathcal{N}_\kappa}) = \star_{\kappa \in K} \Gamma_\kappa \\ &= \mathbf{1} \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{\kappa_j \neq \kappa_{j+1} \\ 1 \leq j \leq n-1}} \Gamma_{\kappa_1}^\circ \otimes \cdots \otimes \Gamma_{\kappa_n}^\circ. \end{aligned}$$

Since all the Γ_κ° are unitary operators, it is not hard to see that each $(\Gamma_{\kappa_1}^\circ \otimes \cdots \otimes \Gamma_{\kappa_n}^\circ)^n = (\Gamma_{\kappa_1}^\circ)^n \otimes \cdots \otimes (\Gamma_{\kappa_n}^\circ)^n \rightarrow 0$ weakly as $n \rightarrow \infty$, and thus $\Gamma^n \rightarrow P_{\mathbb{C}\Omega}$ weakly as $n \rightarrow \infty$.

The latter assertion follows from the former thanks to [27, Proposition 2.1, Corollary 2.2] and [28, Remark 5.2] and the fact that $\sigma_M^{4\pi n}(\mathcal{N}) = \text{Ad } \Gamma^n(\mathcal{N})$ and $\text{Ad } \Gamma(\mathcal{M}) \subset \mathcal{N}$. \square

Let us remark that the fact that \mathcal{M} is a factor follows from [34, Theorem 3.4], without the assumption of ergodicity.

One can also take the free product of the corresponding one-dimensional Borchers triples. There one obtains an explicit construction of T and the factoriality and the ergodicity follow from the uniqueness of Ω . As we will present the free product of Möbius covariant nets in Section 4 and the construction will be similar, we omit it here.

The next proposition is a general assertion on free products of infinitely many copies of a fixed pair $(\mathcal{M}_0, \omega_0)$ of von Neumann algebra and faithful normal state, and a nontrivial von Neumann subalgebra $\mathcal{N}_0 \subset \mathcal{M}_0$. We emphasize that we do not assume that \mathcal{N}_0 is the range of a faithful normal conditional expectation from \mathcal{M}_0 (indeed, when \mathcal{M}_0 and \mathcal{N}_0 come from a local Möbius covariant net, such a conditional expectation does not exist, see Appendix 6). One of the keys in the proof below is the explicit use of the conditional expectation E_{K_1} for a subset K_1 of K and also it is an important point in the proof below in comparison to [16, Theorem 5.2] that the analytic approximation b of a need not stay in \mathcal{N} . The proof below is motivated by [14, Theorem 3.3].

PROPOSITION 3.4. — *Let \mathcal{M}_0 be a von Neumann algebra equipped with a faithful normal state ω_0 and $\mathcal{N}_0 \subset \mathcal{M}_0$ a nontrivial von Neumann subalgebra. Assume that all $(\mathcal{N}_\kappa \subset \mathcal{M}_\kappa, \omega_\kappa)$, $\kappa \in K$, are copies of $(\mathcal{N}_0 \subset \mathcal{M}_0, \omega_0)$, and moreover that the index set K is infinite. We set $\mathcal{N} := \bigvee_{\kappa \in K} \lambda_\kappa(\mathcal{N}_0) \subset \mathcal{M}$, where $(\mathcal{M}, \omega) = \star_{\kappa \in K} (\mathcal{M}_\kappa, \omega_\kappa)$. Then $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}\mathbf{1}$, that is, the inclusion $\mathcal{N} \subset \mathcal{M}$ is irreducible.*

Proof. — For any $x \in \mathcal{N}' \cap \mathcal{M}$ the new element $x - \omega(x)\mathbf{1}$ still stays in $\mathcal{N}' \cap \mathcal{M}$. Hence it suffices to prove that any $x \in \mathcal{N}' \cap \mathcal{M}$ with $\omega(x) = 0$ must be zero.

For any subset $K_1 \subset K$, we define \mathcal{M}_{K_1} as the von Neumann subalgebra generated by $\{\lambda_\kappa(\mathcal{M}_0)\}_{\kappa \in K_1}$. By the definition of reduced free products, it is easy to see that \mathcal{M}_{K_1} and $\mathcal{M}_{K \setminus K_1}$ are freely independent in (\mathcal{M}, ω) (see [37, Propositions 2.5.5(ii) and 2.5.7]). Clearly, $\sigma_{\mathcal{M}}^t(\mathcal{M}_{K_1}) = \mathcal{M}_{K_1}$ for all $t \in \mathbb{R}$, and hence Takesaki's criterion guarantees that there exists a unique ω -preserving conditional expectation $E_{K_1} : \mathcal{M} \rightarrow \mathcal{M}_{K_1}$.

Let $K_1 \subset K$ be an arbitrary finite subset. Then we can find $\kappa'_1 \in K \setminus K_1$ since K is an infinite set. Since \mathcal{N}_0 is nontrivial, one can also find a non-zero $a \in \mathcal{N}_0$ in such a way that $\omega_0(a) = 0$.

We claim the following: for x and a as above, it holds that

$$(3.1) \quad \lambda_{\kappa'_1}(a)E_{K_1}(x)\Omega \perp E_{K_1}(x)\lambda_{\kappa'_1}(a)\Omega.$$

Indeed, as remarked above, \mathcal{M}_{K_1} and $\mathcal{M}_{K \setminus K_1}$ are freely independent in (\mathcal{M}, ω) (see (2.4)). Remark that $\lambda_{\kappa'_1}(a) \in \mathcal{M}_{K \setminus K_1}$ and $\omega(\lambda_{\kappa'_1}(a)) = \omega_0(a) = 0$. Observe that $\omega(E_{K_1}(x)) = \omega(x) = 0$. Hence we have

$$\begin{aligned} \langle \lambda_{\kappa'_1}(a)E_{K_1}(x)\Omega, E_{K_1}(x)\lambda_{\kappa'_1}(a)\Omega \rangle &= \langle \Omega, E_{K_1}(x^*)\lambda_{\kappa'_1}(a^*)E_{K_1}(x)\lambda_{\kappa'_1}(a)\Omega \rangle \\ &= \omega(E_{K_1}(x^*)\lambda_{\kappa'_1}(a^*)E_{K_1}(x)\lambda_{\kappa'_1}(a)) = 0 \end{aligned}$$

by free independence.

Let $b \in \mathcal{M}_0$ be an arbitrary, $\sigma_{\mathcal{M}_0}$ -analytic element. Then, as $x \in \mathcal{N}' \cap \mathcal{M}$, we have

$$\begin{aligned} 0 &= \lambda_{\kappa'_1}(a)x - x\lambda_{\kappa'_1}(a) \\ &= \lambda_{\kappa'_1}(a)(x - E_{K_1}(x)) + \lambda_{\kappa'_1}(a)E_{K_1}(x) - (x - E_{K_1}(x))\lambda_{\kappa'_1}(a) \\ &\quad - E_{K_1}(x)\lambda_{\kappa'_1}(a) \\ &= \lambda_{\kappa'_1}(a)(x - E_{K_1}(x)) + \lambda_{\kappa'_1}(a)E_{K_1}(x) \\ &\quad - (x - E_{K_1}(x))\lambda_{\kappa'_1}(a - b) - (x - E_{K_1}(x))\lambda_{\kappa'_1}(b) - E_{K_1}(x)\lambda_{\kappa'_1}(a) \end{aligned}$$

and hence

$$\begin{aligned} E_{K_1}(x)\lambda_{\kappa'_1}(a)\Omega - \lambda_{\kappa'_1}(a)E_{K_1}(x)\Omega \\ = \lambda_{\kappa'_1}(a)(x - E_{K_1}(x))\Omega - (x - E_{K_1}(x))\lambda_{\kappa'_1}(a - b)\Omega \\ - (x - E_{K_1}(x))\lambda_{\kappa'_1}(b)\Omega. \end{aligned}$$

By (3.1), the vectors on the left-hand side are orthogonal. Thus we obtain that

$$\begin{aligned} \|\lambda_{\kappa'_1}(a)E_{K_1}(x)\Omega\| \\ \leq \|a\| \|(x - E_{K_1}(x))\Omega\| + 2\|x\| \|\lambda_{\kappa'_1}(a - b)\Omega\| \\ + \|(x - E_{K_1}(x))\lambda_{\kappa'_1}(b)\Omega\|. \end{aligned}$$

Since b is $\sigma_{\mathcal{M}_0}$ -analytic, $\lambda_{\kappa'_1}(b)$ is also $\sigma_{\mathcal{M}}$ -analytic and

$$(x - E_{K_1}(x))\lambda_{\kappa'_1}(b)\Omega = J_{\mathcal{M}}\lambda_{\kappa'_1}(\sigma_{\mathcal{M}_0}^{i/2}(b)^*)J_{\mathcal{M}}(x - E_{K_1}(x))\Omega.$$

Therefore, we get

$$\begin{aligned} \|\lambda_{\kappa'_1}(a)E_{K_1}(x)\Omega\| \\ \leq \|a\| \|(x - E_{K_1}(x))\Omega\| + 2\|x\| \|\lambda_{\kappa'_1}(a - b)\Omega\| \\ + \|\sigma_{\mathcal{M}_0}^{i/2}(b)\| \|(x - E_{K_1}(x))\Omega\|. \end{aligned}$$

Observe that

$$\|\lambda_{\kappa'_1}(a - b)\Omega\|^2 = \omega(\lambda_{\kappa'_1}((a - b)^*(a - b))) = \omega_0((a - b)^*(a - b)) = \|(a - b)\Omega_0\|^2$$

and that

$$\begin{aligned} \|\lambda_{\kappa'_1}(a)E_{K_1}(x)\Omega\|^2 \\ = \omega(E_{K_1}(x)^*\lambda_{\kappa'_1}(a^*a)E_{K_1}(x)) \\ = \omega(\lambda_{\kappa'_1}(a^*a)) \cdot \omega(E_{K_1}(x)^*E_{K_1}(x)) \quad (\text{by free independence}) \\ = \omega_0(a^*a) \cdot \omega(E_{K_1}(x)^*E_{K_1}(x)) \\ = \|a\Omega_0\|^2 \|E_{K_1}(x)\Omega\|^2. \end{aligned}$$

Consequently, we have obtained that

$$\begin{aligned} \|a\Omega_0\| \|E_{K_1}(x)\Omega\| \\ \leq \|a\| \|(x - E_{K_1}(x))\Omega\| + 2\|x\| \|(a - b)\Omega_0\| \\ + \|\sigma_{\mathcal{M}_0}^{i/2}(b)\| \|(x - E_{K_1}(x))\Omega\|. \end{aligned}$$

Taking the limit as $K_1 \nearrow K$ we get

$$(3.2) \quad \|a\Omega_0\| \|x\Omega\| \leq 2\|x\| \|(a - b)\Omega_0\|,$$

because the Jones projection E_{K_1} associated with E_{K_1} converges to the identity operator thanks to $\mathcal{M}_{K_1} \nearrow \mathcal{M}$ as $K_1 \nearrow K$. One can easily see that the linear manifold of all $b\Omega_0$ for $\sigma_{\mathcal{M}_0}$ -analytic elements $b \in \mathcal{M}_0$ is dense in \mathcal{H}_0 . Hence the right-hand side of (3.2) can arbitrarily be small,

since b is arbitrary. Consequently, we get $x = 0$, since a is non-zero and also since Ω_0 and Ω are separating for \mathcal{M}_0 and \mathcal{M} , respectively. \square

The next theorem is one of the main results of this paper, which is now a simple consequence of Proposition 3.4.

THEOREM 3.5. — *Let $(\mathcal{N}_0 \subset \mathcal{M}_0, \Omega_0)$ be a half-sided modular inclusion with ergodic canonical endomorphism. Then the free product $(\mathcal{N} \subset \mathcal{M}, \Omega)$ of infinitely many copies of $(\mathcal{N}_0 \subset \mathcal{M}_0, \Omega_0)$ is a half-sided modular inclusion with ergodic canonical endomorphism and trivial relative commutant.*

Proof. — This is a half-sided modular inclusion due to Lemma 3.1. By Proposition 3.3 the canonical endomorphism of the resulting half-sided modular inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is ergodic. Finally, Proposition 3.4 shows that $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}\mathbf{1}$. \square

In this Theorem and Proposition 3.4, it is important that we take copies of the *identical* inclusion $\mathcal{N}_0 \subset \mathcal{M}_0$. If we remove this condition, we find indeed an infinite family $(\mathcal{N}_\kappa \subset \mathcal{M}_\kappa, \Omega_\kappa)$ whose free product has nontrivial relative commutant in Proposition 4.5. Such an infinite family must be a little elaborated, and it is not enough that the relative commutants $\mathcal{N}'_\kappa \cap \mathcal{M}_\kappa$ get larger as $\kappa \rightarrow \infty$. For example, if one just take a one-dimensional Borchers triple $(\mathcal{M}_0, T_0, \Omega_0)$ and consider the shifted family $(\text{Ad } T_0(a_\kappa))(\mathcal{M}_0) \subset \mathcal{M}_0, \Omega_0)$ where $a_\kappa \rightarrow \infty$, all such inclusions are actually unitarily equivalent to a fixed one $(\text{Ad } T_0(1))(\mathcal{M}_0) \subset \mathcal{M}_0, \Omega_0$ (through unitaries $\Delta_{\mathcal{M}_0}^{it_\kappa}$ with appropriate t_κ , which preserve the vacuum Ω_0) and their free product has the trivial relative commutant by Theorem 3.5. The example in Proposition 4.5 avoids this problem by taking an explicitly non equivalent family. On the other hand, we do not know whether there is a free product HSMI with nontrivial relative commutant, even if the family is finite.

4. Free products of Möbius covariant nets

In this Section, we exploit the techniques in Möbius covariant nets in order to show that inclusions of free product von Neumann algebras may have nontrivial relative commutant.

Let $\{(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)\}_{\kappa \in K}$ be a family of Möbius covariant nets, where K is an index set. Thanks to the Reeh–Schlieder theorem [15, Theorem 2.1(i)], the vacuum vector Ω_κ is a common cyclic and separating vector for all $\mathcal{A}_\kappa(I)$, $I \in \mathcal{I}$. Let $(\mathcal{H}, \Omega) = \bigstar_{\kappa \in K} (\mathcal{H}_\kappa, \Omega_\kappa)$. As in Section 2.3, the free

products $(\mathcal{A}(I), \omega) = \star_{\kappa \in K} (\mathcal{A}_\kappa(I), \omega_\kappa)$, $I \in \mathcal{I}$, are simultaneously constructed as

$$\mathcal{A}(I) := \bigvee_{\kappa \in K} \lambda_\kappa(\mathcal{A}_\kappa(I)), \quad \omega = \langle \Omega, \cdot \Omega \rangle$$

in \mathcal{H} . By construction, $\mathcal{A} := (\mathcal{A}(I))_{I \in \mathcal{I}}$ satisfies the isotony. We remark that Ω is again cyclic and separating for all $\mathcal{A}(I)$, $I \in \mathcal{I}$, that is, Ω plays a role of the vacuum vector for the family \mathcal{A} . Let us consider $U(g) := \star_{\kappa \in K} U_\kappa(g)$ for every $g \in G$, which is well-defined since $U_\kappa(g)\Omega_\kappa = \Omega_\kappa$ and U_κ is a unitary representation. It is not hard to see that $g \mapsto U(g)$ is a unitary representation on \mathcal{H} and fixes Ω . Here is an expected fact.

PROPOSITION 4.1. — *The resulting $(\mathcal{A} = (\mathcal{A}(I))_{I \in \mathcal{I}}, U, \Omega)$ is a Möbius covariant net. It fails to have locality if at least two of the nets \mathcal{A}_κ are nontrivial. If all the given \mathcal{A}_κ are local, then \mathcal{A} is twisted local: namely, the unitary involution Z in Section 2.3 commutes with $U(g)$ and satisfies $Z\mathcal{A}(I)Z = \mathcal{A}(I)'$.*

Proof. — \mathcal{A} trivially satisfies the isotony. As remarked above, Ω plays a role of the vacuum vector. If two of them are nontrivial, say κ_1, κ_2 , then for nonzero elements $x_1 \in \mathcal{A}_{\kappa_1}(I_1), x_2 \in \mathcal{A}_{\kappa_2}(I_2)$ such that $\omega_{\kappa_j}(x_j) = 0$ (but $\omega_{\kappa_j}(x_j^* x_j) \neq 0$), $\lambda_{\kappa_1}(x_1)$ and $\lambda_{\kappa_2}(x_2)$ do not commute, therefore, locality is lost.

By the definition of the representations λ_κ and the construction of $U(g)$, it is plain to see that $U(g)\lambda_\kappa(x)U(g)^* = \lambda_\kappa(U_\kappa(g)xU_\kappa(g)^*)$ for every $x \in \mathcal{M}_\kappa$, $\kappa \in K$ and $g \in G$. Hence we obtain that

$$U(g)\mathcal{A}(I)U(g)^* = \bigvee_{\kappa \in K} \lambda_\kappa(U_\kappa(g)\mathcal{A}_\kappa(I)U_\kappa(g)^*) = \bigvee_{\kappa \in K} \lambda_\kappa(\mathcal{A}_\kappa(gI)) = \mathcal{A}(gI)$$

for every $g \in G$ and $I \in \mathcal{I}$. Thus we have confirmed that \mathcal{A} satisfies the Möbius covariance.

Since the representation U_κ fixes Ω_κ , $U_\kappa(R(\theta))\mathcal{H}_\kappa^\circ = \mathcal{H}_\kappa^\circ$ and \mathcal{H}_κ° decomposes into the eigenspaces of $L_{0,\kappa}$ with strictly positive eigenvalues. By the construction of $U(g)$ we see

$$U(R(\theta)(\xi_1 \otimes \cdots \otimes \xi_n)) = e^{i\theta} \sum_k l_k \xi_1 \otimes \cdots \otimes \xi_n,$$

where ξ_k are eigenvectors of L_{0,κ_k} with the eigenvalue $l_k > 0$. These vectors span the subspace $\mathcal{H}_{\kappa_1}^\circ \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_{\kappa_n}^\circ$ of \mathcal{H} . Then it immediately follows that the generator L_0 of $U(R(\theta))$ is positive. Namely, we have confirmed that \mathcal{A} satisfies the positivity of the energy. It is also clear that Ω is the unique (up to scalar) invariant vector under $U(R(\theta)) = e^{i\theta L_0}$.

Finally, the existence of the vacuum is immediate by the commutation relation for free products (2.3), since we have known that Ω_κ is a common

cyclic and separating vector for all $\mathcal{A}_\kappa(I)$, $I \in \mathcal{I}$. Hence we have proved the first part.

Assume that all \mathcal{A}_κ are local. We first remark that Z commutes with $\Delta_I^{it} := \Delta_{\mathcal{A}(I)}^{it}$ and $U(g)$ for all $I \in \mathcal{I}$, $t \in \mathbb{R}$ and $g \in G$. Thus it suffices, for the twisted locality, to confirm that $\mathcal{A}(I')' = Z\mathcal{A}(I)Z$ for all $I \in \mathcal{I}$. By the locality of the \mathcal{A}_κ , $\kappa \in K$, we have $\mathcal{A}_\kappa(I') = \mathcal{A}_\kappa(I)'$ on \mathcal{H}_κ (Haag duality) for all $\kappa \in K$ and $I \in \mathcal{I}$, see e.g. [15, Proposition 2.9]. Hence, by the facts (2.2),(2.3) for free products, we obtain that

$$(4.1) \quad \begin{aligned} \mathcal{A}(I')' &= \left(\bigvee_{\kappa \in K} \lambda_\kappa(\mathcal{A}_\kappa(I')) \right)' = \bigvee_{\kappa \in K} \rho_\kappa(\mathcal{A}_\kappa(I'))' \\ &= \bigvee_{\kappa \in K} \rho_\kappa(\mathcal{A}_\kappa(I)) = \bigvee_{\kappa \in K} Z\lambda_\kappa(\mathcal{A}_\kappa(I))Z = Z\mathcal{A}(I)Z. \end{aligned}$$

Hence we are done. \square

We may write

$$(\mathcal{A}, U, \Omega) = \star_{\kappa \in K} (\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa) \quad \text{or simply} \quad \mathcal{A} = \star_{\kappa \in K} \mathcal{A}_\kappa$$

and call it the *free product Möbius covariant net* of the given $(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)$, $\kappa \in K$. The next assertion is a simple application of [21, Corollary B].

PROPOSITION 4.2. — *For every $\kappa \in K$ such that \mathcal{H}_κ is not one-dimensional, there is no local subnet ⁽²⁾ of \mathcal{A} with the split property which is larger than \mathcal{A}_κ .*

Proof. — Assume that $\mathcal{B} = (\mathcal{B}(I))_{I \in \mathcal{I}}$ is a subnet with the split property such that $\mathcal{A}_\kappa(I) \subseteq \mathcal{B}(I) \subseteq \mathcal{A}(I)$ for every $I \in \mathcal{I}$. Thanks to e.g. [15, Proposition 3.1], $\mathcal{B}(I)$ must be hyperfinite by the split property and $\mathcal{A}_\kappa(I)$ is of type III₁ by [15, Proposition 2.4(ii)], in particular diffuse, and there is a faithful normal conditional expectation from $\mathcal{A}(I)$ onto $\mathcal{B}(I)$ by the Bisognano–Wichmann property (n.b. the Möbius covariance of \mathcal{B} is provided by the same U) and Takesaki’s theorem [30, Theorem IX.4.2]. Hence [21, Corollary B] shows that $\mathcal{B}(I)$ must sit inside $\mathcal{A}_\kappa(I)$. \square

We conjecture that the above statement still holds under assuming only locality on subnets.

Let us consider the dual Möbius covariant net $\widehat{\mathcal{A}} = (\widehat{\mathcal{A}}(I))_{I \in \mathcal{I}}$ defined by $\widehat{\mathcal{A}}(I) := \mathcal{A}(I)'$ in \mathcal{H} , see [15, §2]. If all the given \mathcal{A}_κ , $\kappa \in K$ are local, then

$$\widehat{\mathcal{A}}(I) = \mathcal{A}(I)' = Z\mathcal{A}(I)Z = \bigvee_{\kappa \in K} \rho_\kappa(\mathcal{A}_\kappa(I)),$$

⁽²⁾ A subnet \mathcal{B} of (\mathcal{A}, U, Ω) is a family of von Neumann subalgebras $(\mathcal{B}(I))_{I \in \mathcal{I}}$ with $\mathcal{B}(I) \subseteq \mathcal{A}(I)$ satisfying the isotony and the Möbius covariance with respect to the same U .

as in (4.1). It is known that the observable net $\mathcal{C} = (\mathcal{C}(I))_{I \in \mathcal{I}}$ with $\mathcal{C}(I) := \mathcal{A}(I) \cap \widehat{\mathcal{A}}(I)$ is local (see [15, Remark 2.8]). However, the observable net turns out to be trivial under a natural requirement.

PROPOSITION 4.3. — *If all the \mathcal{A}_κ are local and two of them are non-trivial, then $\mathcal{C}(I) = \mathbb{C}\mathbf{1}$ for all $I \in \mathcal{I}$.*

Proof. — As we assume the uniqueness of the vacuum, by e.g. [15, Propositions 2.3(ii) and 2.5] there exist $\kappa_1 \neq \kappa_2 \in K$ such that both $\mathcal{A}_{\kappa_j}(I)$, $j = 1, 2$, are type III₁ factors for every $I \in \mathcal{I}$. Let $I \in \mathcal{I}$ be arbitrarily fixed. Since $\mathcal{A}_{\kappa_j}(I')$ is a type III₁ factor, we can choose an isometry $s_j \in \mathcal{A}_{\kappa_j}(I')$ such that $s_j^n(s_j^n)^* \rightarrow 0$ weakly as $n \rightarrow \infty$. For each $j = 1, 2$ we observe that

$$[\lambda_{\kappa_j}(s_j^n)^*, \rho_{\kappa_j}(s_j^n)] = (P_{\mathbb{C}\Omega} + P_{\mathcal{H}_{\kappa_j}^\circ})\lambda_{\kappa_j}(1 - s_j^n(s_j^n)^*) \rightarrow P_{\mathbb{C}\Omega} + P_{\mathcal{H}_{\kappa_j}^\circ}$$

weakly as $n \rightarrow \infty$. Hence $P_{\mathbb{C}\Omega} + P_{\mathcal{H}_{\kappa_j}^\circ}$ falls into $\mathcal{A}(I') \vee \widehat{\mathcal{A}}(I')$ for every $j = 1, 2$. Therefore, we have

$$P_{\mathbb{C}\Omega} = (P_{\mathbb{C}\Omega} + P_{\mathcal{H}_{\kappa_1}^\circ})(P_{\mathbb{C}\Omega} + P_{\mathcal{H}_{\kappa_2}^\circ}) \in \mathcal{A}(I') \vee \widehat{\mathcal{A}}(I') = \mathcal{C}(I)',$$

from which $\mathcal{C}(I)'$ must be $\mathcal{B}(\mathcal{H})$ since Ω is cyclic for it. Hence we are done. \square

Actually, we need only that there exist $\kappa_1 \neq \kappa_2 \in K$ such that both $\mathcal{A}_{\kappa_j}(I)$, $j = 1, 2$, are properly infinite for every $I \in \mathcal{I}$ and the uniqueness of the vacuum (the factoriality of $\mathcal{A}_\kappa(I)$) is not essential.

Now let us move to the existence of inclusions of free product von Neumann algebras with nontrivial relative commutant, which is our second main result. We take finitely many Möbius covariant nets $\{\mathcal{A}_\kappa\}_{\kappa \in K}$ with trace class property for all $s > 0$ and make the free product $\star_{\kappa \in K} \mathcal{A}_\kappa$ of them, which we denote by \mathcal{A} . We lose locality through free product, but the trace class property for sufficiently large s survives.

THEOREM 4.4. — *If K is finite, and if the Möbius covariant nets \mathcal{A}_κ satisfy the trace class property (for some s_κ), then the free product net \mathcal{A} satisfies the distal split property. In particular, for a pair $I \subset \tilde{I}$ such that the split inclusion holds, the relative commutant of the inclusion $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ of free product von Neumann algebras is nontrivial. Moreover, $\mathcal{A}(I)' \cap \mathcal{A}(\tilde{I})$ must be a type III₁ factor.*

Proof. — It is enough to show that $\text{Tr } e^{-sL_0} < \infty$ for some s . We take s such that $\text{Tr } e^{-sL_{0,\kappa}^\circ} < \epsilon < \frac{1}{|K|-1}$, where $L_{0,\kappa}^\circ$ is the restriction of $L_{0,\kappa}$ on \mathcal{H}_κ° : note that $L_{0,\kappa}$ has the eigenvalue 0 with multiplicity 1 by the uniqueness of the vacuum, while the contributions from the other eigenvalues can

arbitrarily be small as s gets larger. And note that K is finite, hence for a sufficiently large s this condition is satisfied for all $\kappa \in K$.

Recall that the free product Hilbert space is given by

$$\mathcal{H} = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{\kappa_j \neq \kappa_{j+1} \\ 1 \leq j \leq n-1}} \mathcal{H}_{\kappa_1}^\circ \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_{\kappa_n}^\circ \quad \text{with} \quad \mathcal{H}_\kappa^\circ := \mathcal{H}_\kappa \ominus \mathbb{C}\Omega_\kappa.$$

Note that, for a fixed n , the Hilbert space is a direct sum of $|K| \cdot (|K|-1)^{n-1}$ tensor products. Accordingly, we can estimate

$$\mathrm{Tr} e^{-sL_0} \leqslant 1 + \sum_n \epsilon^n |K| \cdot (|K|-1)^{n-1} = 1 + \frac{|K|}{|K|-1} \sum_n \epsilon^n (|K|-1)^n < \infty,$$

as we chose $\epsilon < \frac{1}{|K|-1}$.

It follows by [11, Corollary 6.4] that $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ is split if $\ell(I, \tilde{I}) > s$, hence its relative commutant is again a nontrivial type III₁ factor. \square

THEOREM 4.5. — *Let $(\mathcal{A}_0, U_0, \Omega_0)$ be a Möbius covariant net with the trace class property, let K be a countably infinite set. We fix an interval \tilde{I} and take a sequence $I_\kappa \subset \tilde{I}$ such that the inner distances ⁽³⁾ satisfy $\ell(\tilde{I}, I_\kappa) > s_\kappa$ with $\sum_\kappa (\mathrm{Tr} e^{-s_\kappa L_{0,0}} - 1) < 1$, where $L_{0,0}$ is the conformal Hamiltonian for the net \mathcal{A}_0 . Then the inclusion $\mathcal{N} \subset \mathcal{M}$ of the free product von Neumann algebras $\mathcal{N} = \star_{\kappa \in K} (\mathcal{A}_0(I_\kappa), \Omega_0)$, $\mathcal{M} = \star_{\kappa \in K} (\mathcal{A}_0(\tilde{I}), \Omega_0)$ has nontrivial relative commutant.*

Proof. — This time we invoke the L^2 -nuclearity condition, namely, we are going to prove that $T_{\mathcal{M}, \mathcal{N}} := \Delta_{\mathcal{M}}^{\frac{1}{4}} \Delta_{\mathcal{N}}^{-\frac{1}{4}}$ is of trace class, then the inclusion $\mathcal{N} \subset \mathcal{M}$ is split [11, Propositions 5.2, 5.3].

Let $\Delta_{0, \tilde{I}}, \Delta_{0, I_\kappa}$ be the modular operators of $\mathcal{A}_0(\tilde{I}), \mathcal{A}_0(I_\kappa)$ with respect to Ω_0 , respectively. As both $\Delta_{0, \tilde{I}}^{it}, \Delta_{0, I_\kappa}^{it}$ leave Ω_0 invariant, we can naturally define $\Delta_{0, \tilde{I}}^\circ, \Delta_{0, I_\kappa}^\circ$, their restriction on \mathcal{H}_0° . We know from [11, Proposition 3.1] that $\|\Delta_{0, \tilde{I}}^{\frac{1}{4}} \Delta_{0, I_\kappa}^{-\frac{1}{4}}\|_1 = \|e^{-s_\kappa L_{0,0}}\|_1$, where $\|\cdot\|_1$ denotes the trace norm. Again, the eigenvalue 0 of $L_{0,0}$ has multiplicity 1, and by assumption

$$\begin{aligned} 1 > \sum_\kappa (\|e^{-s_\kappa L_{0,0}}\|_1 - 1) &= \sum_\kappa (\mathrm{Tr} e^{-s_\kappa L_{0,0}} - 1) \\ &= \sum_\kappa (\|(\Delta_{0, \tilde{I}}^\circ)^{\frac{1}{4}} (\Delta_{0, I_\kappa}^\circ)^{-\frac{1}{4}}\|_1) =: \epsilon. \end{aligned}$$

⁽³⁾ The inner distance $\ell(\tilde{I}, I_\kappa)$ can arbitrarily be large if I_κ is sufficiently small, see [11, Section 3.1] and $L_{0,0}$ has the eigenvalue 0 with multiplicity 1, hence such a sequence s_κ exists.

We claim the following expression of $T_{\mathcal{M}, \mathcal{N}}$:

$$\begin{aligned} T_{\mathcal{M}, \mathcal{N}} &= \Delta_{\mathcal{M}}^{\frac{1}{4}} \Delta_{\mathcal{N}}^{-\frac{1}{4}} \\ &= \mathbb{1} \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{\kappa_j \neq \kappa_{j+1} \\ 1 \leq j \leq n-1}} (\Delta_{0, \tilde{I}}^{\circ})^{\frac{1}{4}} (\Delta_{0, I_{\kappa_1}}^{\circ})^{-\frac{1}{4}} \bar{\otimes} \cdots \bar{\otimes} (\Delta_{0, \tilde{I}}^{\circ})^{\frac{1}{4}} (\Delta_{0, I_{\kappa_n}}^{\circ})^{-\frac{1}{4}} \end{aligned}$$

To justify this, note that for any analytic $x_j \in \mathcal{M}_{\kappa_j} \cap \text{Ker}(\omega_{\kappa_j})$ with $\kappa_1 \neq \kappa_2 \neq \cdots \neq \kappa_n$, $\lambda_{\kappa_j}(x_j)$ is an analytic element with respect to $\sigma_{\mathcal{M}}$, and moreover that

$$\sigma_{\mathcal{M}}^z(\lambda_{\kappa_j}(x_j)) = \lambda_{\kappa_j}(\sigma_{\mathcal{M}_{\kappa_j}}^z(x_j)), \quad z \in \mathbb{C},$$

which can easily be confirmed by using formula (2.5). Thus we have

$$\begin{aligned} (4.2) \quad \Delta_{\mathcal{M}}^{\frac{1}{4}} ((x_1 \Omega_{\kappa_1}) \otimes \cdots \otimes (x_n \Omega_{\kappa_n})) \\ &= \Delta_{\mathcal{M}}^{\frac{1}{4}} \lambda_{\kappa_1}(x_1) \cdots \lambda_{\kappa_n}(x_n) \Omega \\ &= \lambda_{\kappa_1}(\sigma_{\mathcal{M}_{\kappa_1}}^{-\frac{i}{4}}(x_1)) \cdots \lambda_{\kappa_n}(\sigma_{\mathcal{M}_{\kappa_n}}^{-\frac{i}{4}}(x_n)) \Omega \\ &= (\Delta_{\mathcal{M}_{\kappa_1}}^{\frac{1}{4}} x_1 \Omega_{\kappa_1}) \otimes \cdots \otimes (\Delta_{\mathcal{M}_{\kappa_n}}^{\frac{1}{4}} x_n \Omega_{\kappa_n}). \end{aligned}$$

Observe that the vectors of the form $x \Omega_{\kappa}$ with analytic $x \in \mathcal{M}_{\kappa} \cap \text{Ker}(\omega_{\kappa})$ form a core for the restriction of $\Delta_{\kappa}^{\frac{1}{4}}$ to $\mathcal{H}_{\kappa}^{\circ}$. Thus equation (4.2) still holds for arbitrary $x_j \in \mathcal{M}_{\kappa_j} \cap \text{Ker}(\omega_{\kappa_j})$, $\kappa_1 \neq \kappa_2 \neq \cdots \neq \kappa_n$. Moreover, the same identity also holds when replacing \mathcal{M} with \mathcal{N} . The contractions $T_{\mathcal{M}, \mathcal{N}} : \mathcal{H} \rightarrow \mathcal{H}$ and $T_{\mathcal{M}_{\kappa}, \mathcal{N}_{\kappa}} : \mathcal{H}_{\kappa} \rightarrow \mathcal{H}_{\kappa}$, $\kappa \in K$ are defined as follows [10, Lemma 2.4]:

$$\begin{aligned} T_{\mathcal{M}, \mathcal{N}}(\Delta_{\mathcal{N}}^{\frac{1}{4}} x \Omega) &= \Delta_{\mathcal{M}}^{\frac{1}{4}} x \Omega, & x \in \mathcal{N}, \\ T_{\mathcal{M}_{\kappa}, \mathcal{N}_{\kappa}}(\Delta_{\mathcal{N}_{\kappa}}^{\frac{1}{4}} x_{\kappa} \Omega_{\kappa}) &= \Delta_{\mathcal{M}_{\kappa}}^{\frac{1}{4}} x_{\kappa} \Omega_{\kappa}, & x_{\kappa} \in \mathcal{N}_{\kappa}. \end{aligned}$$

Since $T_{\mathcal{M}_{\kappa}, \mathcal{N}_{\kappa}} \Omega_{\kappa} = \Omega_{\kappa}$, the operator $\star_{\kappa \in K} T_{\mathcal{M}_{\kappa}, \mathcal{N}_{\kappa}}$ is well defined. For any $x_j \in \mathcal{N}_{\kappa_j} \cap \text{Ker}(\omega_{\kappa_j})$, $\kappa_1 \neq \kappa_2 \neq \cdots \neq \kappa_n$ we have

$$\begin{aligned} T_{\mathcal{M}, \mathcal{N}}((\Delta_{\mathcal{N}_{\kappa_1}}^{\frac{1}{4}} x_1 \Omega_{\kappa_1}) \otimes \cdots \otimes (\Delta_{\mathcal{N}_{\kappa_n}}^{\frac{1}{4}} x_n \Omega_{\kappa_n})) \\ &= T_{\mathcal{M}, \mathcal{N}} \Delta_{\mathcal{N}}^{\frac{1}{4}} ((x_1 \Omega_{\kappa_1}) \otimes \cdots \otimes (x_n \Omega_{\kappa_n})) \\ &= \Delta_{\mathcal{M}}^{\frac{1}{4}} ((x_1 \Omega_{\kappa_1}) \otimes \cdots \otimes (x_n \Omega_{\kappa_n})) \\ &= (\Delta_{\mathcal{M}_{\kappa_1}}^{\frac{1}{4}} x_1 \Omega_{\kappa_1}) \otimes \cdots \otimes (\Delta_{\mathcal{M}_{\kappa_n}}^{\frac{1}{4}} x_n \Omega_{\kappa_n}) \\ &= (T_{\mathcal{M}_{\kappa_1}, \mathcal{N}_{\kappa_1}} \Delta_{\mathcal{N}_{\kappa_1}}^{\frac{1}{4}} x_1 \Omega_{\kappa_1}) \otimes \cdots \otimes (T_{\mathcal{M}_{\kappa_1}, \mathcal{N}_{\kappa_1}} \Delta_{\mathcal{N}_{\kappa_n}}^{\frac{1}{4}} x_n \Omega_{\kappa_n}). \end{aligned}$$

This implies that

$$T_{\mathcal{M}, \mathcal{N}} = \star_{\kappa \in K} T_{\mathcal{M}_\kappa, \mathcal{N}_\kappa}, \quad \text{and} \quad (T_{\mathcal{M}_\kappa, \mathcal{N}_\kappa})^\circ = (\Delta_{0, I}^\circ)^{\frac{1}{4}} (\Delta_{0, I_\kappa}^\circ)^{-\frac{1}{4}}.$$

Therefore, we have justified the expression above for $T_{\mathcal{M}, \mathcal{N}}$.

Now its trace norm can be estimated recursively in n : the contribution from $n = 1$ is exactly ϵ above. The contribution from $n + 1$ can be dominated by the contribution from n multiplied by $\sum_\kappa (\Delta_{0, I}^\circ)^{\frac{1}{4}} (\Delta_{0, I_\kappa}^\circ)^{-\frac{1}{4}}$, say, from the right (namely, any sequence $\kappa_1, \dots, \kappa_{n+1}$ can be obtained from a sequence $\kappa_1, \dots, \kappa_n$ by adding an element κ_{n+1} to the right). Therefore, the total sum is less than $1 + \sum_n \epsilon^n = \frac{1}{1-\epsilon} < \infty$. \square

We emphasize that these Theorems contrast quite sharply with Proposition 3.4.

5. Free products of Borchers triples

As we recalled, to any two-dimensional Haag–Kastler net, one can associate a two-dimensional Borchers triple. The simplest one is the so-called free field net which we will review below, but there are more important interacting nets ranging from the constructive QFT [19] to the recent operator-algebraic constructions [1, 24, 33]. Note also that there are two-dimensional nets with trivial double cone algebras up to a given size [26], although they are subnets of the free field net.

Given a family of Borchers triples, one can promote it to the free product. By arguments parallel to those of Proposition 4.1 and Theorem 3.5, it is straightforward to show the following.

PROPOSITION 5.1. — *Let $\{(\mathcal{M}_\kappa, T_\kappa, \Omega_\kappa)\}_{\kappa \in K}$ be a family of two-dimensional Borchers triples on Hilbert spaces $\{\mathcal{H}_\kappa\}_{\kappa \in K}$. Set*

$$\begin{aligned} (\mathcal{H}, \Omega) &:= \star_{\kappa \in K} (\mathcal{H}_\kappa, \Omega_\kappa), \\ (\mathcal{M}, \omega) &:= \star_{\kappa \in K} (\mathcal{M}_\kappa, \omega_\kappa) \\ T(a) &:= \star_{\kappa \in K} T_\kappa(a). \end{aligned}$$

Then (\mathcal{M}, T, Ω) is again a two-dimensional Borchers triple. Furthermore, if K is an infinite index set, and all these Borchers triples are identical, then the relative commutant $\text{Ad } T(a)(\mathcal{M})' \cap \mathcal{M}$ is trivial for any $a \in W_R$.

Yet, if the index set K is finite, or if $(\mathcal{M}_\kappa, T_\kappa, \Omega_\kappa)$'s are not identical, $\text{Ad } T(a)(\mathcal{M})' \cap \mathcal{M}$ might be nontrivial, and if so, one could construct a Haag–Kastler net.

5.1. Two-dimensional Borchers triples of the massive free field

In order to study closely the (possible) physical properties of the resulting Borchers triple, we present explicitly the simplest Borchers triple, coming from the free massive field. Our notations follow those of [23] (except for $z(\psi)$, which is antilinear in ψ in our notation). The one-particle Hilbert space with the mass $m > 0$ is given by $\mathcal{H}_1 := L^2(\mathbb{R}, d\theta)$ and the translation group acts by $(T_1(a)\psi)(\theta) = e^{i p_m(\theta) \cdot a} \psi(\theta)$, where $p_m(\theta) := (m \cosh \theta, m \sinh \theta)$, and for the Lorentz metric we use the convention $a \cdot b = a_0 b_0 - a_1 b_1$. We introduce the symmetrized Hilbert space $\mathcal{H} := \bigoplus_{n=1}^{\infty} P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto the symmetric subspace.

Let z^\dagger and z be the creation and annihilation operator, namely, $(z^\dagger(\psi)\Psi)_n = \sqrt{n} P_n(\psi \otimes \Psi_{n-1})$ for vectors with finite particle number $\Psi = (\Psi_n), \Psi_n \in P_n \mathcal{H}^{\otimes n}$. The annihilation operator is the adjoint $z(\psi) = z^\dagger(\psi)^*$. The (real) free field ϕ is defined by

$$\phi(f) := z^\dagger(f^+) + z(J_1 f^-), \quad f^\pm(\theta) = \frac{1}{2\pi} \int d^2 a f(a) e^{\pm i p_m(\theta) \cdot a},$$

where f is a test function of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ and $J_1 \psi(\theta) = \overline{\psi(\theta)}$. Our von Neumann algebra is

$$\mathcal{M} := \{e^{i\phi(f)} : \text{supp } f \subset W_R\}''.$$

The translation on the full space is the second quantized representation $T(a) := \Gamma(T_1(a))$ and there is the Fock vacuum vector $\Omega \in \mathcal{H}$. Then (\mathcal{M}, T, Ω) is a Borchers triple and satisfies the modular nuclearity condition for any $a \in W_R$ [12, Section 4].

5.2. Attempts towards the intersection

Let $(\mathcal{M}_\kappa, T_\kappa, \Omega_\kappa)$, $\kappa = 1, 2$ be two copies of the free field Borchers triple. We will put the index κ to all the objects to distinguish these copies. We saw in Section 4 that the trace class condition and the L^2 -nuclearity condition in Möbius covariant nets are preserved under finite free products, at the cost of a finite distance. Let us take two copies of the massive free field and consider their free product. Below are some standard ways to prove the nontriviality of the intersection $\text{Ad } T(a)(\mathcal{M})' \cap \mathcal{M}$, which unfortunately do not work.

- The trace class property does not make sense in two dimensions.

- The L^2 -nuclearity condition amounts to compute the operator $\Delta_{\mathcal{M}}^{\frac{1}{4}} \Delta_{\mathcal{N}}^{-\frac{1}{4}}$, where $\mathcal{N} = \text{Ad } T(a)(\mathcal{M})$. In the two-dimensional situation this is a multiplication operator (of θ 's), hence it cannot be nuclear.
- At first sight, there could be two ways to attempt to prove modular nuclearity. One is, by noting that the tensor product net satisfies the modular nuclearity condition, to consider the map

$$\lambda_{\kappa_1}(x_1)\lambda_{\kappa_2}(x_2)\cdots\lambda_{\kappa_n}(x_n) \longmapsto x_1 \otimes x_2 \otimes \cdots \otimes x_n.$$

We are unable to carry this out because the map cannot be shown to be continuous in norm. The other is to mimic the proof of [24]. This is obstructed as there is no symmetry in, e.g. vectors such as $\lambda_{\kappa_1}(z_1^\dagger(f^+))\lambda_{\kappa_2}(z_2^\dagger(g^+))\Omega$ considered as a two-variable function, and the core analytic continuation cannot be obtained (c.f. [24, Section 4]).

- Note that \mathcal{M} is full and hence not hyperfinite [34, Theorem 4.1]. Therefore, when we consider the inclusions $\text{Ad } T(a)(\mathcal{M}) \subset \mathcal{M}$, the split property must fail for small a .
- Even though each $(\mathcal{M}_\kappa, T_\kappa, \Omega_\kappa)$ has the split property, namely there are type I factors $\mathcal{R}_{\kappa,a}$ such that $\text{Ad } T_\kappa(a)(\mathcal{M}_\kappa) \subset \mathcal{R}_{\kappa,a} \subset \mathcal{M}$, one cannot directly obtain an intermediate type I factor from those $\mathcal{R}_{\kappa,a}$, c.f. [33, Section 4.2], since no type I free product factors can arise (see [34, Theorem 4.1]).

5.3. Two-particle S-matrix

Although we do not know whether there is a corresponding Haag–Kastler net, it is still possible to define the two-particle S-matrix of a Borchers triple. We follow the approach of [6, Section 3][23, Section 4]. For simplicity, we take again two copies of the free field Borchers triple $(\mathcal{M}_\kappa, T_\kappa, \Omega_\kappa)$, $\kappa = 1, 2$. As the free product von Neumann algebra is generated by the components $\mathcal{M}_\kappa = \{e^{i\phi_\kappa(f)} : \text{supp } f \subset W_R\}''$, we can construct the asymptotic fields using unbounded fields ϕ_κ as follows.

For a test function f , its velocity support is defined by $\Gamma(f) := \{(1, p_1/\omega^m(p)) \in \mathbb{R}^2 : p \in \text{supp } \tilde{f}\}$, where $\tilde{f}(p) := \frac{1}{2\pi} \int d^2a e^{ip \cdot a} f(a)$ and $\omega^m(p) = \sqrt{p_1^2 + m^2}$ (ω^m is a function of $p = (p_0, p_1) \in \mathbb{R}^2$ but depends only on p_1). We say $\Gamma(g) \prec \Gamma(f)$ if $\Gamma(f) - \Gamma(g) \subset W_R$. If $\Gamma(g) \prec \Gamma(f)$, then it follows that $\text{supp } f^+ - \text{supp } g^+ \subset \mathbb{R}_+$. We also introduce a family

of time-depending functions

$$f_t(a) := \frac{1}{2\pi} \int d^2 p \tilde{f}(p) e^{i((p_0 - \omega^m(p))t - p \cdot a)}.$$

Let us denote $f_\kappa^+ = \phi_\kappa(f)\Omega$, by identifying \mathcal{H}_κ as subspaces $\mathbb{C}\Omega \oplus \mathcal{H}_\kappa^\circ$ of \mathcal{H} and $\phi'_\kappa(g) := Z\phi_\kappa(g)Z$. These operators can act on \mathcal{H} naturally, by extending the actions λ_κ and $\rho_{\kappa'}$ to unbounded operators. It holds that $\phi_\kappa(f)\Omega = f_\kappa^+ = f_{\kappa,t}^+ = \phi_\kappa(f_t)\Omega$ and $\phi'_\kappa(g)\Omega = g_\kappa^+ = g_{\kappa,t}^+ = \phi'_\kappa(g_t)\Omega$.

Let f and g be test functions as above and we assume further that their Fourier transforms \tilde{f}, \tilde{g} are supported in some neighborhoods of the mass shell $\{(\sqrt{p_1^2 + m^2}, p_1) \in \mathbb{R}^2 : p_1 \in \mathbb{R}\}$. Now we can define the outgoing two-particle states by

$$\begin{aligned} (g_{\kappa'}^+ \times f_\kappa^+)_\text{out} &:= \lim_{t \rightarrow \infty} \phi'_{\kappa'}(g_t) \phi_\kappa(f_t) \Omega, \\ (f_\kappa^+ \times g_{\kappa'}^+)_\text{out} &:= \lim_{t \rightarrow \infty} \phi_\kappa(f_t) \phi'_{\kappa'}(g_t) \Omega. \end{aligned}$$

As $\Gamma(g) \prec \Gamma(f)$, $\phi_\kappa(f_t)$ and $\phi'_{\kappa'}(g_t)$ almost commute as $t \rightarrow \infty$ (see [6, Section 3] for the precise statement) and we get the bosonic statistics $(g_{\kappa'}^+ \times f_\kappa^+)_\text{out} = (f_\kappa^+ \times g_{\kappa'}^+)_\text{out}$. The incoming two-particle states are similar: by exchanging the roles of ϕ_\bullet and ϕ'_\bullet and by noting that $\phi'_\kappa(f_t)$ and $\phi_{\kappa'}(g_t)$ almost commute as $t \rightarrow -\infty$:

$$\begin{aligned} (g_{\kappa'}^+ \times f_\kappa^+)_\text{in} &:= \lim_{t \rightarrow -\infty} \phi_{\kappa'}(g_t) \phi'_\kappa(f_t) \Omega, \\ (f_\kappa^+ \times g_{\kappa'}^+)_\text{in} &:= \lim_{t \rightarrow -\infty} \phi'_\kappa(f_t) \phi_{\kappa'}(g_t) \Omega, \end{aligned}$$

and the two-particle S-matrix is defined by ⁽⁴⁾

$$S(g_{\kappa'}^+ \times f_\kappa^+)_\text{out} = (g_{\kappa'}^+ \times f_\kappa^+)_\text{in}.$$

It is easy to explicitly compute S . Indeed, when $\kappa = \kappa'$, the operators act on \mathcal{H}_κ and Z has no effect, therefore, the field operators are just the free field and we obtain $(g_\kappa^+ \times f_\kappa^+)_\text{out} = (g_\kappa^+ \times f_\kappa^+)_\text{in}$.

The case $\kappa = 1, \kappa' = 2$ is more interesting. In the definition of $(g_{\kappa'}^+ \times f_\kappa^+)_\text{out}$ above, the right-hand side is the product of operators $\phi_\kappa(f), \phi'_{\kappa'}(g)$, acting from the left and the right, respectively. Note also that $\langle \Omega_\kappa, \phi_\kappa(f)\Omega_\kappa \rangle = 0 = \langle \Omega_{\kappa'}, \phi_{\kappa'}(g)\Omega_{\kappa'} \rangle$. Therefore, we simply get

$$(g_{\kappa'}^+ \times f_\kappa^+)_\text{out} = \phi'_{\kappa'}(g) \phi_\kappa(f) \Omega = f^+ \otimes g^+ \in \mathcal{H}_{\kappa,1} \otimes \mathcal{H}_{\kappa',1} \subset \mathcal{H}_\kappa^\circ \otimes \mathcal{H}_{\kappa'}^\circ,$$

⁽⁴⁾ Actually there are several definitions of S-matrix. Our S-matrix is a partial isometry on the Hilbert space of the model, while others are defined on the symmetric Fock space, see e.g. [24, Section 6]. As the incoming and outgoing two-particle spaces are orthogonal in our models, the latter definition would give the zero operator.

where $\mathcal{H}_{\kappa,1}, \mathcal{H}_{\kappa',1}$ are the one-particle Hilbert spaces. Note that $\text{supp } f^+ - \text{supp } g^+ \subset \mathbb{R}_+$, and as $\mathcal{H}_{\kappa,1} = \mathcal{H}_{\kappa',1} = L^2(\mathbb{R})$, hence they can be regarded as a two-variable function $\psi(\theta_1, \theta_2)$ with $\text{supp } \psi \subset \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1 \geq \theta_2\}$. It is also immediate that any L^2 -function with such support can be obtained as the limit of linear combinations of scattering states. As for the incoming state, we exchange the roles of ϕ_\bullet and ϕ'_\bullet , and

$$(g_{\kappa'}^+ \times f_\kappa^+)_{\text{in}} = \lim_{t \rightarrow -\infty} \phi_{\kappa'}(g_t) \phi'_\kappa(f_t) \Omega = g^+ \otimes f^+ \in \mathcal{H}_{\kappa',1} \otimes \mathcal{H}_{\kappa,1},$$

again with $\text{supp } f^+ - \text{supp } g^+ \subset \mathbb{R}_+$. Such two-particle scattering states span the subspace of two-variable functions $\psi(\theta_1, \theta_2), \text{supp } \psi \subset \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1 \leq \theta_2\}$. The action of S is then simply the extension of $S(g^+ \otimes f^+) = f^+ \otimes g^+$.

To summarize, let us denote

$$\begin{aligned} \mathcal{H}_{\kappa,\kappa'}^{\text{out}} &= \{\psi(\theta_1, \theta_2) \in \mathcal{H}_\kappa^\circ \otimes \mathcal{H}_{\kappa'}^\circ : \text{supp } \psi \subset \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1 \geq \theta_2\}\} \\ \mathcal{H}_{\kappa,\kappa'}^{\text{in}} &= \{\psi(\theta_1, \theta_2) \in \mathcal{H}_\kappa^\circ \otimes \mathcal{H}_{\kappa'}^\circ : \text{supp } \psi \subset \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1 \leq \theta_2\}\} \end{aligned}$$

We have

$$\begin{aligned} S_{\kappa,\kappa} &= \mathbb{1} \text{ on } \mathcal{H}_{\kappa,2} \subset \mathcal{H}_\kappa^\circ, & \kappa = 1, 2, \\ S_{\kappa,\kappa'} : \mathcal{H}_{\kappa,\kappa'}^{\text{out}} &\ni \psi(\theta_1, \theta_2) \longmapsto \psi(\theta_2, \theta_1) \in \mathcal{H}_{\kappa,\kappa'}^{\text{in}}, & \kappa \neq \kappa'. \end{aligned}$$

Especially, the S-matrix is *nontrivial*. Furthermore, even if the model had an associated Haag–Kastler net and n -particle asymptotic states were constructed, the model would *not be asymptotically complete*, i.e. the full asymptotic spaces should not contain the orthogonal complement of the asymptotic (outgoing) two-particle space $\mathcal{H}_{1,2} \oplus \mathcal{H}_{2,2} \oplus \mathcal{H}_{1,2}^{\text{out}} \oplus \mathcal{H}_{2,1}^{\text{out}}$ in the full two-particle space $\mathcal{H}_2 := \mathcal{H}_{1,2} \oplus \mathcal{H}_{2,2} \oplus \mathcal{H}_{1,1} \otimes \mathcal{H}_{2,1} \oplus \mathcal{H}_{2,1} \otimes \mathcal{H}_{1,1}$, but only a proper subspace of it. Indeed, it is enough to note that the orthogonal complement above contains vectors which cannot be interpreted as asymptotic 3- or more particle vectors. This is clear because, e.g. there are vectors $\psi(\theta_1, \theta_2) \in \mathcal{H}_{1,1} \otimes \mathcal{H}_{2,1} \cap (\mathcal{H}_{1,2}^{\text{out}})^\perp$ whose support is close enough to the point $(0, 0)$, so that the total energy does not exceed $3m$, and hence cannot be considered as a 3- or more particle state.

6. Appendix. Absence of conditional expectation between local algebras in Möbius covariant nets

Let (\mathcal{A}, U, Ω) be a local Möbius covariant net on S^1 . It follows that $\mathcal{A}(I)$'s are factors (from the uniqueness of the vacuum). Let $I_1 \subsetneq I_2$ be

an inclusion of intervals on S^1 . We show that there is no normal faithful conditional expectation from $\mathcal{A}(I_2)$ onto $\mathcal{A}(I_1)$.

To show this by contradiction, let E be such a conditional expectation. Take a faithful normal state ψ on $\mathcal{A}(I_2)$ which is invariant under E (one can take a faithful normal state on $\mathcal{A}(I_1)$ and extend it to $\mathcal{A}(I_2)$ by E). We may assume that it is a vector state represented by a cyclic and separating vector Ψ thanks to the fact that every representation of a type III factor is standard.

Now, by Takesaki's theorem [30, Theorem IX.4.2], $\mathcal{K} := \overline{\mathcal{A}(I_1)\Psi}$ is a proper subspace. As $\mathcal{A}(I_1)$ is a factor, and since $\mathcal{A}(I_1) \vee (\mathcal{A}(I_1)' \cap \mathcal{A}(I_2)) = \mathcal{A}(I_2)$ (conormality, [20, Theorem 1.6]), $\mathcal{A}(I_1)$ and $\mathcal{A}(I_1)' \cap \mathcal{A}(I_2)$ are in a position of tensor product: this follows from the fact that $E(\mathcal{A}(I_1)' \cap \mathcal{A}(I_2)) = \mathbb{C}\mathbf{1}$ by factoriality, and that $\mathcal{A}(I_1) \vee (\mathcal{A}(I_1)' \cap \mathcal{A}(I_2)) = \mathcal{A}(I_2)$ spans the whole Hilbert space from Ψ . This is impossible, because $\mathcal{A}(I_1)' \cap \mathcal{A}(I_2)$ contains another local algebra $\mathcal{A}(I_3)$ where I_3 is an interval which has a boundary point in common with I_1 (I_1 and I_2 can have either one or no point of boundary in common, and either case one can find I_3). On the other hand, it is well known (e.g. [8, p. 292, footnote]) that $\mathcal{A}(I_1)$ and $\mathcal{A}(I_3)$ are not in the position of tensor product (split), hence we got a contradiction.

Even in higher dimensions, if one assumes the split property and that local algebras are of type III_1 , which are quite generic (see e.g. [9]), the following argument excludes the possibility that there is a conditional expectation from $\mathcal{A}(O_2)$ onto $\mathcal{A}(O_1)$ for all local regions $O_1 \subset O_2$: by composing two such expectations, we may assume that $O_1 \Subset O_2$, and there is an intermediate type I factor \mathcal{R} such that $\mathcal{A}(O_1) \subset \mathcal{R} \subset \mathcal{A}(O_2)$. Now the expectation restricts to \mathcal{R} , which is impossible by a similar argument as above, since $\mathcal{R} = \mathcal{A}(O_1) \vee (\mathcal{R} \cap \mathcal{A}(O_1)')$ because \mathcal{R} is a type I factor, while $\mathcal{A}(O_1)$ is of type III_1 .

BIBLIOGRAPHY

- [1] S. ALAZZAWI & G. LECHNER, “Inverse Scattering and Local Observable Algebras in Integrable Quantum Field Theories”, *Commun. Math. Phys.* **354** (2017), no. 3, p. 913–956, <https://arxiv.org/abs/1608.02359>.
- [2] H. ARAKI & L. ZSIDÓ, “Extension of the structure theorem of Borchers and its application to half-sided modular inclusions”, *Rev. Math. Phys.* **17** (2005), no. 5, p. 491–543, <https://arxiv.org/abs/math/0412061>.
- [3] L. BARNETT, “Free product von Neumann algebras of type III”, *Proc. Am. Math. Soc.* **123** (1995), no. 2, p. 543–553, <http://www.ams.org/journals/proc/1995-123-02/S0002-9939-1995-1224611-7/S0002-9939-1995-1224611-7.pdf>.
- [4] M. BISCHOFF & Y. TANIMOTO, “Integrable QFT and Longo-Witten endomorphisms”, *Ann. Henri Poincaré* **16** (2015), no. 2, p. 569–608, <https://arxiv.org/abs/1305.2171>.

- [5] H.-J. BORCHERS, “The CPT-theorem in two-dimensional theories of local observables”, *Commun. Math. Phys.* **143** (1992), no. 2, p. 315-332, <http://projecteuclid.org/euclid.cmp/1104248958>.
- [6] H.-J. BORCHERS, D. BUCHHOLZ & B. SCHROER, “Polarization-free generators and the S -matrix”, *Commun. Math. Phys.* **219** (2001), no. 1, p. 125-140, <https://arxiv.org/abs/hep-th/0003243>.
- [7] H. BOSTELMANN, G. LECHNER & G. MORSELLA, “Scaling limits of integrable quantum field theories”, *Rev. Math. Phys.* **23** (2011), no. 10, p. 1115-1156, <https://arxiv.org/abs/1105.2781>.
- [8] D. BUCHHOLZ, “Product states for local algebras”, *Commun. Math. Phys.* **36** (1974), p. 287-304, <http://projecteuclid.org/euclid.cmp/1103859773>.
- [9] D. BUCHHOLZ, C. D’ANTONI & K. FREDENHAGEN, “The universal structure of local algebras”, *Commun. Math. Phys.* **111** (1987), no. 1, p. 123-135, <http://projecteuclid.org/euclid.cmp/1104159470>.
- [10] D. BUCHHOLZ, C. D’ANTONI & R. LONGO, “Nuclear maps and modular structures. I. General properties”, *J. Funct. Anal.* **88** (1990), no. 2, p. 233-250, [https://dx.doi.org/10.1016/0022-1236\(90\)90104-S](https://dx.doi.org/10.1016/0022-1236(90)90104-S).
- [11] ———, “Nuclearity and Thermal States in Conformal Field Theory”, *Commun. Math. Phys.* **270** (2007), no. 1, p. 267-293, <https://arxiv.org/abs/math-ph/0603083>.
- [12] D. BUCHHOLZ & G. LECHNER, “Modular nuclearity and localization”, *Ann. Henri Poincaré* **5** (2004), no. 6, p. 1065-1080, <https://arxiv.org/abs/math-ph/0402072>.
- [13] P. CAMASSA, R. LONGO, Y. TANIMOTO & M. WEINER, “Thermal States in Conformal QFT. II”, *Commun. Math. Phys.* **315** (2012), no. 3, p. 771-802, <https://arxiv.org/abs/1109.2064>.
- [14] Y. DABROWSKI, K. J. DYKEMA & K. MUKHERJEE, “The simplex of tracial quantum symmetric states”, *Stud. Math.* **225** (2014), no. 3, p. 203-218, <https://arxiv.org/abs/1401.4692>.
- [15] C. D’ANTONI, R. LONGO & F. RĂDULESCU, “Conformal nets, maximal temperature and models from free probability”, *J. Oper. Theory* **45** (2001), no. 1, p. 195-208, <https://arxiv.org/abs/math/9810003>.
- [16] K. DYKEMA & K. MUKHERJEE, “KMS quantum symmetric states”, *J. Math. Phys.* **58** (2017), no. 1, Art. ID 012103, 12 pages, <https://arxiv.org/abs/1609.01225>.
- [17] K. J. DYKEMA, “Factoriality and Connes’ invariant $T(\mathcal{M})$ for free products of von Neumann algebras”, *J. Reine Angew. Math.* **450** (1994), p. 159-180, <https://arxiv.org/abs/funct-an/9302005>.
- [18] M. FLORIG, “On Borchers’ theorem”, *Lett. Math. Phys.* **46** (1998), no. 4, p. 289-293, <https://dx.doi.org/10.1023/A:1007546507392>.
- [19] J. GLIMM & A. JAFFE, *Quantum physics. A functional integral point of view*, second ed., Springer, 1987, <https://books.google.com/books?id=VSjjBwAAQBAJ>, xxii+535 pages.
- [20] D. GUIDO, R. LONGO & H.-W. WIESBROCK, “Extensions of conformal nets and superselection structures”, *Commun. Math. Phys.* **192** (1998), no. 1, p. 217-244, <https://arxiv.org/abs/hep-th/9703129>.
- [21] C. HOUDAYER & Y. UEDA, “Asymptotic structure of free product von Neumann algebras”, *Math. Proc. Camb. Philos. Soc.* **161** (2016), no. 3, p. 489-516, <https://dx.doi.org/10.1017/S0305004116000396>.
- [22] R. KÄHLER & H.-W. WIESBROCK, “Modular theory and the reconstruction of four-dimensional quantum field theories”, *J. Math. Phys.* **42** (2001), no. 1, p. 74-86, <https://dx.doi.org/10.1063/1.1327597>.
- [23] G. LECHNER, “Polarization-free quantum fields and interaction”, *Lett. Math. Phys.* **64** (2003), no. 2, p. 137-154, <https://arxiv.org/abs/hep-th/0303062>.

- [24] ———, “Construction of quantum field theories with factorizing S -matrices”, *Commun. Math. Phys.* **277** (2008), no. 3, p. 821-860, <https://arxiv.org/abs/math-ph/0601022>.
- [25] ———, “Algebraic constructive quantum field theory: Integrable models and deformation techniques”, in *Advances in Algebraic Quantum Field Theory*, Springer, 2015, <https://arxiv.org/abs/1503.03822>, p. 397-448.
- [26] G. LECHNER & R. LONGO, “Localization in nets of standard spaces”, *Commun. Math. Phys.* **336** (2015), no. 1, p. 27-61, <https://dx.doi.org/10.1007/s00220-014-2199-2>.
- [27] R. LONGO, “Solution of the factorial Stone-Weierstrass conjecture. An application of the theory of standard split W^* -inclusions”, *Invent. Math.* **76** (1984), no. 1, p. 145-155, <https://eudml.org/doc/143121>.
- [28] ———, “Simple injective subfactors”, *Adv. Math.* **63** (1987), no. 2, p. 152-171, [http://dx.doi.org/10.1016/0001-8708\(87\)90051-X](http://dx.doi.org/10.1016/0001-8708(87)90051-X).
- [29] ———, “Notes for a quantum index theorem”, *Commun. Math. Phys.* **222** (2001), no. 1, p. 45-96, <https://arxiv.org/abs/math/0003082>.
- [30] M. TAKESAKI, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, vol. 125, Springer, 2003, <https://books.google.com/books?id=-4GyR1VlQz4C>, xxii+518 pages.
- [31] Y. TANIMOTO, “Construction of Wedge-Local Nets of Observables Through Longo-Witten Endomorphisms”, *Commun. Math. Phys.* **314** (2012), no. 2, p. 443-469, <https://arxiv.org/abs/1107.2629>.
- [32] ———, “Noninteraction of Waves in Two-dimensional Conformal Field Theory”, *Commun. Math. Phys.* **314** (2012), no. 2, p. 419-441, <https://arxiv.org/abs/1107.2662>.
- [33] ———, “Construction of two-dimensional quantum field models through Longo-Witten endomorphisms”, *Forum Math. Sigma* **2** (2014), Art. ID e7, 31 pages, <https://arxiv.org/abs/1301.6090>.
- [34] Y. UEDA, “Factoriality, type classification and fullness for free product von Neumann algebras”, *Adv. Math.* **228** (2011), no. 5, p. 2647-2671, <http://dx.doi.org/10.1016/j.aim.2011.07.017>.
- [35] ———, “On type III₁ factors arising as free products”, *Math. Res. Lett.* **18** (2011), no. 5, p. 909-920, <http://dx.doi.org/10.4310/MRL.2011.v18.n5.a8>.
- [36] D. VOICULESCU, “Symmetries of some reduced free product C^* -algebras”, in *Operator algebras and their connections with topology and ergodic theory (Băstien, 1983)*, Lecture Notes in Mathematics, vol. 1132, Springer, 1985, <http://dx.doi.org/10.1007/BFb0074909>, p. 556-588.
- [37] D. VOICULESCU, K. J. DYKEMA & A. NICÀ, *Free random variables*, CRM Monograph Series, vol. 1, American Mathematical Society, 1992, <https://books.google.com/books?id=r7BJAwAAQBAJ>, vi+70 pages.
- [38] H.-W. WIESBROCK, “Half-sided modular inclusions of von-Neumann-algebras”, *Commun. Math. Phys.* **157** (1993), no. 1, p. 83-92, <https://projecteuclid.org/euclid.cmp/1104253848>.
- [39] ———, “Modular intersections of von Neumann algebras in quantum field theory”, *Commun. Math. Phys.* **193** (1998), no. 2, p. 269-285, <https://www-sfb288.math.tu-berlin.de/pub/Preprints/preprint193.ps.gz>.

Manuscrit reçu le 19 juin 2017,
révisé le 6 février 2018,
accepté le 13 mars 2018.

Roberto LONGO
Dipartimento di Matematica
Università di Roma “Tor Vergata”
Via della Ricerca Scientifica 1,
I-00133 Roma (Italy)
longo@mat.uniroma2.it

Yoh TANIMOTO
Dipartimento di Matematica
Università di Roma Tor Vergata
Via della Ricerca Scientifica 1,
I-00133 Roma (Italy)
hoyt@mat.uniroma2.it

Yoshimichi UEDA
Graduate School of Mathematics,
Nagoya University
Furocho, Chikusaku
Nagoya, 464-8602 (Japan)
ueda@math.nagoya-u.ac.jp