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Akane Nakamura

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AUTONOMOUS LIMIT OF THE 4-DIMENSIONAL PAINLEVï¿½-TYPE EQUATIONS AND DEGENERATION OF CURVES OF GENUS TWO

by Akane NAKAMURA (*)

Abstract. — In recent studies, 4-dimensional analogs of the Painlevï¿½ equations were listed and there are 40 types. The aim of the present paper is to geometrically characterize these 40 Painlevï¿½-type equations. For this purpose, we study the autonomous limit of these equations and degeneration of their spectral curves. The spectral curves are 2-parameter families of genus two curves and their generic degeneration are one of the types classified by Namikawa and Ueno. Liu's algorithm enables us to find the degeneration types of the spectral curves for our 40 types of integrable systems. This result is analogous to the following fact; the families of the spectral curves of the autonomous 2-dimensional Painlevï¿½ equations P_{I}, P_{II}, P_{IV}, $P_{\mathrm{III}}^{D_8}$, $P_{\mathrm{III}}^{D_7}$, $P_{\mathrm{III}}^{D_6}$, P_{V} and P_{VI} define elliptic surfaces with the singular fiber at $H=\infty$ of the Dynkin types $E_8^{(1)}$, $E_7^{(1)}$, $E_6^{(1)}$, $D_8^{(1)}$, $D_7^{(1)}$, $D_6^{(1)}$, $D_5^{(1)}$ and $D_4^{(1)}$, respectively.

Keywords: integrable system, Painlevï¿½-type equations, isospectral limit, spectral curve, hyperelliptic curve, degeneration of curves.
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RÉSUMÉ. — Dans des ï¿½tudes rï¿½centes, des analogues en 4 dimensions des ï¿½quations de Painlevï¿½ ont ï¿½tï¿½ rï¿½pertoriï¿½s et il existe 40 types. Le but du prï¿½sent article est de caractï¿½riser gï¿½omï¿½triquement ces 40 ï¿½quations de type Painlevï¿½. A cet effet, nous ï¿½tudions la limite autonome de ces ï¿½quations et la dï¿½gï¿½nï¿½rescence de leurs courbes spectrales. Les courbes spectrales sont des familles ï¿½ 2 paramï¿½tres de courbes de genre deux et leur dï¿½gï¿½nï¿½rescences gï¿½nï¿½riques sont d'un des types classï¿½s par Namikawa et Ueno. L'algorithme de Liu nous permet de trouver le types de dï¿½gï¿½nï¿½rescence de courbes spectrales pour nos 40 types de systï¿½mes intï¿½grables. Ce rï¿½sultat est analogue au fait suivant; les familles des courbes spectrales des ï¿½quations de Painlevï¿½ autonomes bidimensionnelles P_{I}, P_{II}, P_{IV}, $P_{\mathrm{III}}^{D_8}$, $P_{\mathrm{III}}^{D_7}$, $P_{\mathrm{III}}^{D_6}$, P_{V} et P_{VI} dï¿½finissent des surfaces elliptiques avec une fibre singuliï¿½re ï¿½ $H = \infty$ des types Dynkin $E_8^{(1)}$, $E_7^{(1)}$, $E_6^{(1)}$, $D_8^{(1)}$, $D_7^{(1)}$, $D_6^{(1)}$, $D_5^{(1)}$ et $D_4^{(1)}$, respectivement.

1. Introduction

The Painlevï¿½ equations are 8 types of nonlinear second-order ordinary differential equations with the Painlevï¿½ property[1] which are not solvable by elementary functions. The Painlevï¿½ equations have various interesting features; they can be derived from isomonodromic deformation of certain linear equations, they are linked by degeneration process, they have affine Weyl group symmetries, they can be derived from reductions of soliton equations, and they are equivalent to nonautonomous Hamiltonian systems. Furthermore, their autonomous limits are integrable systems solvable by elliptic functions. We call such systems the autonomous Painlevï¿½ equations.

Various generalization of the Painlevï¿½ equations have been proposed by focusing on one of the features of the Painlevï¿½ equations. The main two directions of generalizations are higher-dimensional analogs and difference analogs. We treat the 4-dimensional analogs in this paper. The eight types of the Painlevï¿½ equations are called the 2-dimensional Painlevï¿½ equations in this context, and the higher-dimensional analogs are $2n$-dimensional Painlevï¿½-type equations ($n = 1, 2, \dots$).

Recently, classification theory of linear equations up to Katz's operations have been developed. Since these operations of linear equations leave isomonodromic deformation equations invariant [12], we can make use of classification theory of linear equations to classify the Painlevï¿½-type

[1] A differential equation is said to have the Painlevï¿½ property if its general solution has no critical singularities that depend on the initial values.

equations[2]. As for the 4-dimensional case, classification and derivation of the Painlevï¿½-type equations from isomonodromic deformation have been completed [21, 23, 22, 25, 51]. According to their result, there are 40 types of 4-dimensional Painlevï¿½-type equations. Among these 40 types of equations, some of the equations coincide with equations already known from different contexts. Such well-known equations along with new equations are all organized in a unified way: isomonodromic deformation and degeneration.

There are problems to their classification. They say that there are "at most" 40 types of 4-dimensional Painlevï¿½-type equations. There is no guarantee that these 40 equations are actually distinct. Some Hamiltonians of these equations look very similar to each other. Since the appearance of Hamiltonians or equations may change significantly by changes of variables, we can not classify equations by their appearances. Intrinsic or geometrical studies of these equations may be necessary to distinguish the Painlevï¿½-type equations. The aim of this paper is to initiate geometrical studies of 4-dimensional Painlevï¿½-type equations starting from these concrete equations.

Let us first review how the 2-dimensional Painlevï¿½ equations are geometrically classified. Okamoto initiated the studies of the space of initial conditions of the Painlevï¿½ equations [42]. He constructed the rational surfaces, whose points correspond to the germs of the meromorphic solutions of the Painlevï¿½ equations by resolving the singularities of the differential equations. Sakai extracted the key features of the spaces of initial conditions and classified what he calls the "generalized Halphen surfaces" with such features [49]. The classification of the 2-dimensional difference Painlevï¿½ equations correspond to the classification of generalized Halphen surfaces, and 8 types of such surfaces give the Painlevï¿½ differential equations. Such surfaces are distinguished by their anticanonical divisors. For the autonomous 2-dimensional Painlevï¿½ equations, the spaces of initial conditions are elliptic surfaces and their anticanonical divisors are one of the Kodaira types [50]. We can say that the 2-dimensional autonomous Painlevï¿½ equations are characterized or distinguished by the corresponding Kodaira types.

[2] In this paper we use the term the Painlevï¿½-type equations synonymously with isomonodromic deformation equations, despite the fact that Painlevï¿½ has never investigated isomonodromy. We prefer to use the term the Painlevï¿½-type equations rather than the Schlesinger-type equations in order to express that our aim is to understand various higher-dimensional analogs of the Painlevï¿½ equations.

It is expected to carry out a similar study for the 4-dimensional Painlevï¿½-type equations. However, a straightforward generalization of the 2-dimensional cases seems to contain many difficulties. One of the reason is that the classification of 4-folds is much more difficult than that of surfaces. We want to avoid facing such difficulties by considering the spectral curves of the autonomous limit of these Painlevï¿½-type equations. While the geometry is made simple considerably in the autonomous limit, important characteristics of the original non-autonomous equations are retained.

Integrable systems are Hamiltonian systems on symplectic manifolds (M^{2n}, ω, H) with n functions $f_1 = H, \dots, f_n$ in involution $\{f_i, f_j\} = 0$. The regular level sets of the momentum map $F = (f_1, \dots, f_n)\colon M \to \mathbb{C}^n$ are the Liouville tori. The image under F of critical points are called the bifurcation diagram, and it is studied for characterizing integrable systems [6]. However, studying the bifurcation diagrams may become complicated for higher dimensional cases. When an integrable system is expressed in a Lax form, it is not difficult to determine the discriminant locus of spectral curves. We can often find correspondence between the bifurcation diagrams and the discriminant locus of spectral curves [5]. Therefore, we mainly study the degeneration of spectral curves in this paper.

Let $\{H_i\}_{i=1,\dots,g}$ be a set of functionally independent invariants of an integrable system (M, ω, H).

In the 2-dimensional case, the following holds.

Theorem (cf. Theorem 4.1). — *Each autonomous 2-dimensional Painlevï¿½ equation defines an elliptic surface, whose general fibers are spectral curves of the system. The Kodaira types of the singular fibers at $H_{\mathrm{J}} = \infty$ are listed as follows.*

Hamiltonian	H_{VI}	H_{V}	$H_{\mathrm{III}(\mathrm{D}_6)}$	$H_{\mathrm{III}(\mathrm{D}_7)}$	$H_{\mathrm{III}(\mathrm{D}_8)}$	H_{IV}	H_{II}	H_{I}
Kodaira type	I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	IV^*	III^*	II^*
Dynkin type	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$

Table 1.1. The singular fiber at $H_{\mathrm{J}} = \infty$ of spectral curve fibrations of the autonomous 2-dimensional Painlevï¿½ equations.

It is well-known that the Dynkin's types in the above diagram appear in the configurations of vertical leaves of the Okamoto's spaces of initial conditions [42].

For the autonomous 4-dimensional Painlevï¿½-type equations, general invariant sets $\bigcap_{i=1,2} H_i^{-1}(h_i)$ are the 2-dimensional Liouville tori. Such Liouville tori are the Jacobian varieties of the corresponding spectral curves.

Instead of studying the degenerations of 2-dimensional Liouville tori, we study the degenerations of the spectral curves of genus 2. As an analogy of the above theorem for the 2-dimensional Painlevï¿½ equations, we find the following:

Main theorem (cf. Theorem 4.5). — *The spectral curves of the autonomous 4-dimensional Painlevï¿½-type equations have the following types of generic degenerations as in Table 4.6 in Section 4.*

The table shows, for example, that generic degeneration of the spectral curves of the autonomous matrix Painlevï¿½ equations $H_{\mathrm{VI}}^{\mathrm{Mat}}$, $H_{\mathrm{V}}^{\mathrm{Mat}}$, $H_{\mathrm{III}(\mathrm{D}_6)}^{\mathrm{Mat}}$, $H_{\mathrm{III}(\mathrm{D}_7)}^{\mathrm{Mat}}$ and $H_{\mathrm{III}(\mathrm{D}_8)}^{\mathrm{Mat}}$ are $\mathrm{I}_0-\mathrm{I}_0^*-1$, $\mathrm{I}_0-\mathrm{I}_1^*-1$, $\mathrm{I}_0-\mathrm{I}_2^*-1$, $\mathrm{I}_0-\mathrm{I}_3^*-1$ and $\mathrm{I}_0-\mathrm{I}_4^*-1$ in Namikawa–Ueno's notation. Those of $H_{\mathrm{IV}}^{\mathrm{Mat}}$, $H_{\mathrm{II}}^{\mathrm{Mat}}$ and $H_{\mathrm{I}}^{\mathrm{Mat}}$ are $\mathrm{I}_0-\mathrm{IV}^*-1$, $\mathrm{I}_0-\mathrm{III}^*-1$ and $\mathrm{I}_0-\mathrm{II}^*-1$. Therefore, generic degeneration of the spectral curves of the autonomous 4-dimensional matrix Painlevï¿½ equations have one additional elliptic curve to those counterpart of the 2-dimensional systems.

There are various ways to have integrable systems in general. To identify equations from different origins is often not easy; equations may change their appearance by transformations. It is hoped that such intrinsic geometrical studies will be helpful for such identification problems. One possible approach using the degeneration of the Painlevï¿½ divisors[(3)] is proposed in [39]. This direction will be investigated further in a forthcoming paper.

In recent years, Rains and his collaborators [43, 46, 48] brought about crucial developments in the Painlevï¿½-type (difference) equations using noncommutative geometry. In their theory, the anticanonical divisors of rational surfaces, determining the Poisson structures, are one of the key ingredients. The relation between the anticanonical divisors in their work and the generic degeneration of spectral curves will be explained in future work.

Contents

The organization of this paper is as follows. In Section 2, after summarizing preliminaries, we review the classification of the 4-dimensional Painlevï¿½-type equations. In Section 3, we consider the autonomous limit of these 40 equations. In Section 4, we study the generic degeneration of

(3) The Painlevï¿½ divisors introduced by Adler–Moerbeke[3, 4] compactify affine Liouville tori.

the spectral curves to characterize these integrable systems. In Appendix A.1, we list integrals of the autonomous 4-dimensional Painlevï¿½-type equations. In Appendix A.3, we list the dual graph of the singular fibers appeared in our table.

2. Classification of 4-dimensional Painlevï¿½-type equations

In this section, we review some of the recent progresses in classification of the 4-dimensional Painlevï¿½-type equations, and introduce notation we use in this paper. The contents of this section is a summary of the other papers [21, 23, 22, 25, 51] and references therein.

The Painlevï¿½ equations were found by Painlevï¿½ through his classification of the second order algebraic differential equations with the "Painlevï¿½ property". However, a straightforward application of Painlevï¿½'s classification method to higher-dimensional cases seems to face difficulties(4). Therefore other properties which characterize the Painlevï¿½ equations become important for further generalization. The Painlevï¿½ equations can be expressed as Hamilton systems [37, 42].

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H_{\mathrm{J}}}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H_{\mathrm{J}}}{\partial q}.$$

(4) Works of Chazy [7] and Cosgrove [8, 9] are famous in this direction.

We list the Hamiltonian functions for all the 2-dimensional Painlevï¿½ equations for later use.

$$
\begin{aligned}
t(t-1)H_{\mathrm{VI}}\begin{pmatrix}\alpha,\beta\\ \gamma,\epsilon\end{pmatrix}; t; q, p\Big) &= q(q-1)(q-t)p^2 \\
&\quad + \{\epsilon q(q-1) - (2\alpha+\beta+\gamma+\epsilon)q(q-t) \\
&\quad + \gamma(q-1)(q-t)\}\, p + \alpha(\alpha+\beta)(q-t),
\end{aligned}
$$

$$
\begin{aligned}
tH_{\mathrm{V}}\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}; t; q, p\right) &= p(p+t)q(q-1) + \beta pq + \gamma p - (\alpha+\gamma)tq, \\
H_{\mathrm{IV}}\left(\alpha,\beta; t; q, p\right) &= pq(p-q-t) + \beta p + \alpha q, \\
tH_{\mathrm{III}}(D_6)\left(\alpha,\beta; t; q, p\right) &= p^2q^2 - (q^2 - \beta q - t)p - \alpha q, \\
tH_{\mathrm{III}}(D_7)\left(\alpha; t; q, p\right) &= p^2q^2 + \alpha qp + tp + q, \\
tH_{\mathrm{III}}(D_8)\left(t; q, p\right) &= p^2q^2 + qp - q - \frac{t}{q}, \\
H_{\mathrm{II}}\left(\alpha; t; q, p\right) &= p^2 - (q^2+t)p - \alpha q, \\
H_{\mathrm{I}}\left(t; q, p\right) &= p^2 - q^3 - tq.
\end{aligned}
$$

The Painlevï¿½ equations have another important aspect initiated by R. Fuchs [11]. Namely, they can be derived from (generalized) isomonodromic deformation of linear equations [19].

Furthermore, these eight types of the Painlevï¿½ equations are linked by processes called degenerations. In fact, $H_{\mathrm{I}}, \cdots, H_{\mathrm{V}}$ can be derived from H_{VI} through degenerations.

$$
\begin{tikzcd}
& & H_{\mathrm{III}}(D_6) \arrow[r] \arrow[ddr] & H_{\mathrm{III}}(D_7) \arrow[r] \arrow[ddr] & H_{\mathrm{III}}(D_8) \\
H_{\mathrm{VI}} \arrow[r] & H_{\mathrm{V}} \arrow[ur] \arrow[dr] & & & \\
& & H_{\mathrm{IV}} \arrow[r] & H_{\mathrm{II}} \arrow[r] & H_{\mathrm{I}}
\end{tikzcd}
$$

2.1. The classification of Fuchsian equations and isomonodromic deformation

If we fix the number of accessory parameters and identify the linear equations that transform into one another by Katz's operations (addition and middle convolutions) [10, 20], we have only finite types of linear equations. We explain the notion of spectral type used in this paper. We follow the notion used in Oshima [44] for Fuchsian linear equations, Kawakami–Nakamura–Sakai [25] for unramified linear equations and Kawakami [21] for ramified equations. For the classification of linear equations, we need to discern the types of linear equations. We review the local normal forms of linear equations, and introduce symbols to express such data. We study linear systems of first-order equations

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = A(x)Y. \tag{2.1}$$

We first consider the Fuchsian case where

$$A(x) = \sum_{i=1}^{n} \frac{A_i}{x - t_i}.$$

We assume that each matrix A_i is diagonalizable. The equation can be transformed into

$$\frac{\mathrm{d}\hat{Y}(x)}{\mathrm{d}x} = \frac{T^{(i)}}{x - t_i}\hat{Y}(x), \quad T^{(i)}\text{: diangonal}$$

by a local transformation $Y = P(x)\hat{Y}$. We express the multiplicity of the eigenvalues by a non-increasing sequence of numbers.

Example 2.1. — When $T^{(i)} = \mathrm{diag}(a, a, a, b, b, c)$, we write the multiplicity as 321.

Collecting such multiplicity data for all the singular points, the spectral type of the linear equation is defined as the $n+1$-tuples of partitions of m,

$$\underbrace{m_1^1 m_2^1 \dots m_{l_1}^1}, \underbrace{m_1^2 \dots m_{l_2}^2}, \dots, \underbrace{m_1^n \dots m_{l_n}^n}, \underbrace{m_1^\infty \dots m_{l_\infty}^\infty},$$
$$\left(\sum_{j=1}^{l_i} m_j^i = m \text{ for } 1 \leqslant i \leqslant n \text{ or } i = \infty\right),$$

where m is the size of matrices.

Theorem 2.2 (Kostov [32]). — *Irreducible Fuchsian equations with two accessory parameters result in one of the four types by successive additions and middle convolutions:*

- $11, 11, 11, 11$
- $111, 111, 111 \qquad 22, 1111, 1111 \qquad 33, 222, 111111.$

Remark 2.3. — Note that only the equation of the type $11, 11, 11, 11$ has four singular points and the other three types have three singular points. The three of the singular points can be fixed at $0, 1, \infty$ by a Möbius transformation. Thus the three equations with only three singularities do not admit the continuous deformation of position of singularities. Among these 4 types, only linear equation of type $11, 11, 11, 11$ admit isomonodromic deformation, and it gives the sixth Painlevï¿½ equation H_{VI}.

The Katz's operations are important for studying Painlevï¿½-type equations, because the following theorem holds.

THEOREM 2.4 (Haraoka–Filipuk [12]). — *Isomonodromic deformation equations are invariant under Katz's operations.*

Remark 2.5. — Katz's operations that do not change the type of the linear equation induce the corresponding Bï¿½cklund transformations on the isomonodromic deformation equation. In fact, all of the $D_4^{(1)}$-type affine Weyl group symmetry that P_{VI} possesses can be derived from Katz's operations and the Schlesinger transformations on the linear equations [38].

2.2. The classification of Fuchsian linear equations and isomonodromic deformation

The starting point of the classification of the 4-dimensional Painlevï¿½-type equation is the following result.

THEOREM 2.6 (Oshima [44]). — *Irreducible Fuchsian equations with four accessory parameters result in one of the following 13 types by successive additions and middle convolutions*(5)*:*

- $11, 11, 11, 11, 11$
- $21, 21, 111, 111$ $31, 22, 22, 1111$ $22, 22, 22, 211$
- $211, 1111, 1111$ $221, 221, 11111$ $32, 11111, 11111$
  $222, 222, 2211$ $33, 2211, 111111$ $44, 2222, 22211$
  $44, 332, 11111111$ $55, 3331, 22222$ $66, 444, 2222211$.

Remark 2.7. — Note that the equation of type $11, 11, 11, 11, 11$ has five singular points, and the next three types have four singular points, and the rest nine types have three singular points. The equation of type $11, 11, 11, 11, 11$ has two singularities to deform after fixing three of the singularities to $0, 1, \infty$. The next three types of equations with four singularities have one singularity to deform after fixing three of the singularities to $0, 1, \infty$. The nine equations with only three singularities do not admit continuous isomonodromic deformation.

The invariance of the isomonodromic deformation equation by Katz' operations is guaranteed by Haraoka–Filipuk [12]. Sakai derived explicit Hamiltonians of the above four equations with four accessory parameters.

(5) These operations are called the Katz' operations [10, 20]

THEOREM 2.8 (Sakai [51]). — *There are four 4-dimensional Painlevï¿½-type equations governed by Fuchsian equations.*

- *The Garnier system in two variables (11,11,11,11,11).*
- *The Fuji–Suzuki system (21,21,111, 111).*
- *The Sasano system (31,22,22,1111).*
- *The Sixth Matrix Painlevï¿½ equation of size 2 (22,22,22,211).*

In this paper, we call these 4 equations derived from Fuchsian equation the "source equations".

2.3. Degeneration scheme of 4-dimensional Painlevï¿½-type equations

Other 4-dimensional Painlevï¿½-type equations we consider are derived from these source equations by degeneration process. The degenerations corresponding to unramified linear equations are treated in Kawakami–Nakamura–Sakai [25]. The degenerations corresponding to ramified linear equations are treated in Kawakami [21, 23, 22]. There are 40 types of the 4-dimensional Painlevï¿½-type equations with 16 partial differential equations corresponding to the differential Garnier equations and 24 ordinary differential equations. Among 24 ordinary differential equations, 8 types are the matrix Painlevï¿½ equations. According to [45], the 7 types with the source equation $H^{D_6}_{\mathrm{Ss}}$ correspond to the symmetric q-difference Garnier equations, and the 9 types with the source equation $H^{A_5}_{\mathrm{FS}}$ correspond to the nonsymmetric q-difference Garnier equations in Rains [46].

As shown in Diagram 2.2 [22], there are 4 series of degeneration diagram corresponding to 4 "source equations". Explicit forms of the Hamiltonians and the Lax pairs can be found in [21, 23, 22, 25].

Remark 2.9. — The names of these Hamiltonians are temporal. As Sakai [49] labeled the 2-dimensional systems by the types of the anticanonical divisors of the compactified spaces of initial conditions, it may be natural to label the 4-dimensional systems from geometrical characterization.

Remark 2.10. — While Kawakami–Nakamura–Sakai [25] studied the degeneration from Fuchsian types, the theory of unramified non-Fuchsian linear equations has developed. Hiroe and Oshima classified all the unramified linear equations with 4 accessory parameters up to some transformations [14, Theorem 3.29]. Yamakawa proved that analogous theorem of Haraoka–Filpuk [12] holds for unramified non-Fuchsian equations [57].

		Fuchsian	Non-Fuchsian	#
Garnier	PDE	$H_{\text{Gar}}^{1+1+1+1+1}$	$H_{\text{Gar}}^{2+1+1+1}$ H_{Gar}^{3+1+1} H_{Gar}^{2+2+1} $H_{\text{Gar}}^{2+\frac{3}{2}+1}$ $H_{\text{Gar}}^{\frac{3}{2}+\frac{3}{2}+1}$ $H_{\text{Gar}}^{\frac{5}{2}+2}$ H_{Gar}^{4+1} H_{Gar}^{3+2} H_{Gar}^{5} $H_{\text{Gar}}^{\frac{7}{2}+1}$ $H_{\text{Gar}}^{\frac{3}{2}+3}$ $H_{\text{Gar}}^{\frac{3}{2}+1+1+1}$ $H_{\text{Gar}}^{\frac{5}{2}+1+1}$ $H_{\text{Gar}}^{\frac{5}{2}+\frac{3}{2}}$ $H_{\text{Gar}}^{\frac{9}{2}}$	16
	ODE	$H_{\text{FS}}^{A_5}$	$H_{\text{FS}}^{A_4}$ $H_{\text{FS}}^{A_3}$ $H_{\text{NY}}^{A_5}$ $H_{\text{NY}}^{A_4}$ $H_{\text{Suz}}^{2+\frac{3}{2}}$ $H_{\text{KFs}}^{2+\frac{4}{3}}$ $H_{\text{KFs}}^{\frac{3}{2}+\frac{4}{3}}$ $H_{\text{KFs}}^{\frac{4}{3}+\frac{4}{3}}$	9
		$H_{\text{Ss}}^{D_6}$	$H_{\text{Ss}}^{D_5}$ $H_{\text{Ss}}^{D_4}$ $H_{\text{KSs}}^{2+\frac{3}{2}}$ $H_{\text{KSs}}^{2+\frac{4}{3}}$ $H_{\text{KSs}}^{2+\frac{5}{4}}$ $H_{\text{KSs}}^{\frac{3}{2}+\frac{5}{4}}$	7
Matrix	ODE	$H_{\text{VI}}^{\text{Mat}}$	$H_{\text{V}}^{\text{Mat}}$ $H_{\text{III}(D_6)}^{\text{Mat}}$ $H_{\text{III}(D_7)}^{\text{Mat}}$ $H_{\text{III}(D_8)}^{\text{Mat}}$ $H_{\text{IV}}^{\text{Mat}}$ $H_{\text{II}}^{\text{Mat}}$ $H_{\text{I}}^{\text{Mat}}$	8

Table 2.1. The list of the 4-dimensional Painlevï¿½-type equations

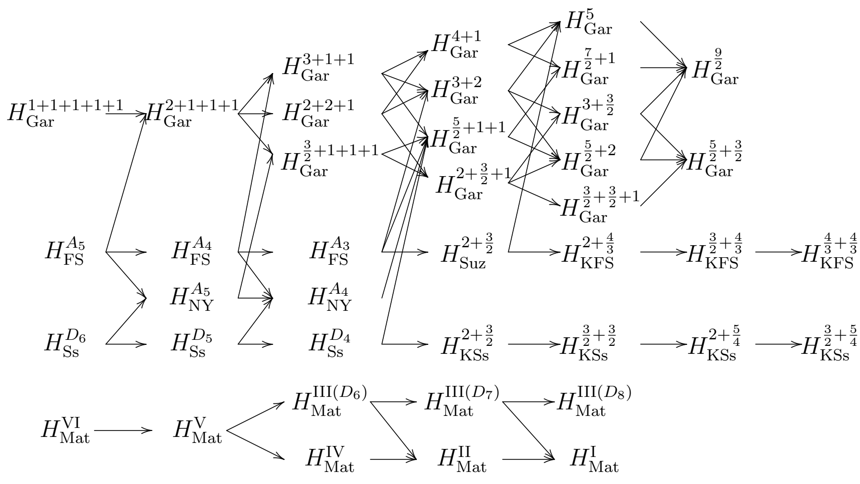

Diagram 2.2.

By comparing the results, all the unramified equations with 4 accessory parameters come from the 4 Fuchsian source equations by degenerations, and the list of 4-dimensional Painlevï¿½-type equations corresponding to the unramified linear equations is complete. To the author's knowledge, it is still an open question whether linear equations of ramified type with 4 accessory parameters can be reduce to this list of 40 types. Rains' work [47] might be conclusive in this direction.

Remark 2.11. — According to the classification of the Lax pairs by Rains [47], there are 40 types of 4-dimensional families which admit continuous deformations, comprising of 16 differential Garnier equations, 7 symmetric q-difference Garnier equations, 9 nonsymmetric q-Garnier equations and 8 matrix Painlevï¿½ equations. The numbers match with Kawakami's list [22].

Remark 2.12. — Some of these 40 equations look similar to each other. For instance, H. Chiba pointed out that $H^{4+1}_{\mathrm{Gar},t_1}$ and $\widetilde{H}^{\mathrm{Mat}}_{\mathrm{II}}$ look almost the same after a symplectic transformations.(6)

$$H^{4+1}_{\mathrm{Gar},t_1} = p_1^2 - \left(q_1^2 + t_1\right) p_1 + \kappa_1 q_1 + p_1 p_2 + p_2 q_2 \left(q_1 - q_2 + t_2\right) + \theta_0 q_2,$$
$$\widetilde{H}^{\mathrm{Mat}}_{\mathrm{II}} = p_1^2 - \left(\frac{q_1^2}{4} + t\right) p_1 - \left(\theta_0 + \frac{\kappa_2}{2}\right) q_1 + p_1 p_2 + p_2 q_2 \left(q_1 - q_2\right) + \theta_0 q_2.$$

One of the key motivations of the present paper is to geometrically distinguish such cases. We will show in Section 4 that the types of generic degeneration of spectral curves are different(7).

Remark 2.13. — Some of the equations in the list, such as the Noumi–Yamada systems, had been known from different context. See [25] for the references for other derivations.

3. Autonomous limit of Painlevï¿½ type equations

In the previous section, we saw that there are 40 types of 4-dimensional Painlevï¿½-type equations. In this section, we consider the autonomous limit of these 40 equations by taking the isospectral limit of the isomonodromic deformation equations. Using the Lax pair, we obtain two functionally independent invariants for each system. Therefore, the autonomous limit of 4-dimensional Painlevï¿½-type equations are integrable in Liouville's sense.

(6) The following transformation to the $H^{\mathrm{Mat}}_{\mathrm{II}}$ in [25] yields $\widetilde{H}^{\mathrm{Mat}}_{\mathrm{II}}$: $p_1 \to 2p_1, q_1 \to \frac{q_1}{2}, p_2 \to -q_2, q_2 \to p_2, \kappa_1 \to -\theta_0 - \kappa_2$.

(7) It is hoped to prove a theorem stating "If the types of generic degeneration of the spectral curves are different, the corresponding space of initial conditions are not isomorphic, or, do not admit biregular map from one another."

3.1. Integrable system and Lax pair representation

Let us recall the definition of integrability in Liouville's sense. A Hamiltonian system is a triple (M, ω, H), where (M, ω) is a symplectic manifold and H is a Hamiltonian function on M; $\iota_{X_H}\omega = \mathrm{d}H$. A function f Poisson commutes with the Hamiltonian H, that is $\{f, H\} = 0$, if and only if f is constant along integrable curve of the Hamiltonian vector field X_H. Such a function f is called a conserved quantity or first integral. A Hamiltonian system is (completely) integrable in Liouville's sense if it possesses $n := \frac{1}{2}\dim M$ independent integrals of motion, $f_1 = H$, $f_2, \ldots, f_n$, which are pairwise in involution with respect to the Poisson bracket; $\{f_i, f_j\} = 0$ for all i, j. This definition of integrability is motivated by Liouville's theorem. Let (M, ω, H) be a real integrable system of dimension $2n$ with integral of motion $f_1, \ldots, f_n$, and let $c \in \mathbb{R}^n$ be a regular value of $f = (f_1, \ldots, f_n)$. Liouville's theorem states that any compact component of the level set $f^{-1}(c)$ is a torus. The complex Liouville theorem is also known [4].

Many integrable systems are known to have Lax pair expressions:

$$\frac{\mathrm{d}A(x)}{\mathrm{d}t} + [A(x), B(x)] = 0, \tag{3.1}$$

where $A(x)$ and $B(x)$ are m by m matrices and x is a spectral parameter. From this differential equation, $\operatorname{tr}\left(A(x)^k\right)$ are conserved quantities of the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\left(A(x)^k\right) = \operatorname{tr}\left(k\,[B(x), A(x)]\,A(x)^{k-1}\right) = 0.$$

Therefore, the eigenvalues of $A(x)$ are all conserved quantities since the coefficients of the characteristic polynomial are expressible in terms of these traces. In fact, the Lax pair is equivalent to the following isospectral problem:

$$\begin{cases} A(x) = YA_0(x)Y^{-1}, \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = B(x)Y, \end{cases}$$

where $A_0(x)$ is a matrix satisfying $\frac{\mathrm{d}A_0(x)}{\mathrm{d}t} = 0$. The curve defined by the characteristic polynomial is called the spectral curve:

$$\det\left(yI_m - A(x)\right) = 0.$$

3.2. Isomonodromic deformation to isospectral deformation

The isomonodromic problems have the following forms:

$$\begin{cases} \dfrac{\partial Y}{\partial x} = A(x,t)Y, \\ \dfrac{\partial Y}{\partial t} = B(x,t)Y, \end{cases}$$

and the deformation equation is expressed as

$$\frac{\partial A(x,t)}{\partial t} - \frac{\partial B(x,t)}{\partial x} + [A(x,t), B(x,t)] = 0. \tag{3.2}$$

We find the similarities in isospectral and isomonodromic problems; the only difference is the existence of the term $\frac{\partial B}{\partial x}$ in isomonodromic deformation equation. In fact, we can consider isospectral problems as the special limit of isomonodromic problem with a parameter δ. We restate the isomonodromic problem as follows[(8)]:

$$\begin{cases} \delta \dfrac{\partial Y}{\mathrm{d}x} = A(x,\tilde{t})Y, \\ \dfrac{\partial Y}{\partial t} = B(x,\tilde{t})Y, \end{cases}$$

where $\tilde{t}$ is a variable which satisfies $\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} = \delta$. The integrability condition $\frac{\partial^2 Y}{\partial x \partial t} = \frac{\partial^2 Y}{\partial t \partial x}$ is equivalent to the following:

$$\frac{\partial A(x,\tilde{t})}{\partial t} - \delta \frac{\partial B(x,\tilde{t})}{\partial x} + [A(x,\tilde{t}), B(x,\tilde{t})] = 0. \tag{3.3}$$

The case when $\delta = 1$ is the usual one[(9)]. When $\delta = 0$, the term $\delta \frac{\partial B}{\partial x}$ drops off from the deformation equation and we have a Lax pair in a narrow sense[(10)]. The deformation equation 3.2 with δ is solved by a Hamiltonian $H(\delta)$. When $\delta = 1$, the Hamiltonian $H(1)$ is equal to the original Hamiltonian of the isomonodromic problem. Therefore, $H(\delta)$ is a slight modification of the Hamiltonian. When $\delta = 0$, $H(0)$ is a conserved quantity of the system.

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(8) Here δ plays the role of λ in Deligne's lambda connections [52]. In some literature such as Levin–Olshanetsky–Zotov [34], it is customary to use κ instead of δ.

(9) Adams–Harnad–Hurtubise [2] and Adams–Harnad–Previato [1] studied finite dimensional integrable systems by embedding them into rational coadjoint orbits of loop algebras. Harnad [13] further generalized their theory as applicable to the isomonodromic systems. Such nonautonomous isomonodromic systems are obtained by identifying the time flows of the integrable system with parameters determining the moment map.

(10) In other words, we mean a Lax pair in the sense of integrable systems. The isomonodromic problems are often called as Lax pairs, but they do not give first integrals.

Remark 3.1. — Isomonodromic equations are flows on moduli space of connections [18]. Isospectral limit correspond to $\lambda \to 0$ limit of moduli of λ-connections [52] to moduli of Higgs bundles. The Painlevï¿½-type equations become the Hitchin systems [15] at the limit.

Taking the isospectral limit of 8 types of 2-dimensional Painlevï¿½ equations, we can state the following classically-known result.

PROPOSITION 3.2. — *As the autonomous limits of 2-dimensional Painlevï¿½ equations, we obtain 8 types of integrable systems with a first integral for each system*(11)*.*

Proof. — We take the second Painlevï¿½ equation as an example to demonstrate a proof. Proofs of the other equations are similar.

$$\begin{cases} \delta \dfrac{\partial Y}{\partial x} = A(x,\tilde{t})Y, & A(x,\tilde{t}) = \left(A_\infty^{(-3)}(\tilde{t})x^2 + A_\infty^{(-2)}(\tilde{t})x + A_\infty^{(-1)}(\tilde{t})\right), \\ \dfrac{\partial Y}{\partial t} = B(x,\tilde{t})Y, & B(x,\tilde{t}) = \left(A_\infty^{(-3)}(\tilde{t})x + B_1(\tilde{t})\right), \end{cases}$$

where

$$\hat{A}_\infty^{(-3)}(\tilde{t}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{A}_\infty^{(-2)}(\tilde{t}) = \begin{pmatrix} 0 & 1 \\ -p+q^2+\tilde{t} & 0 \end{pmatrix},$$

$$\hat{A}_\infty^{(-1)}(\tilde{t}) = \begin{pmatrix} -p+q^2+\tilde{t} & q \\ (p-q^2-\tilde{t})q-\kappa_2 & p-q^2 \end{pmatrix}, \quad \hat{B}_1(\tilde{t}) = \begin{pmatrix} q & 1 \\ p-q^2-\tilde{t} & 0 \end{pmatrix},$$

$$A_\infty^{(-i)} = U^{-1}\hat{A}_\infty^{(-i)}U \quad \text{for } i=1,2,3, \quad B_1 = U^{-1}\hat{B}_1 U, \quad U = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

The deformation equation (3.3) is equivalent to the following differential equations.

$$\frac{\mathrm{d}q}{\mathrm{d}t} = 2p - q^2 - \tilde{t}, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = 2pq + \delta - \kappa_1, \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = 0.$$

The first two equations are equivalent to the Hamiltonian system

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H_{\mathrm{II}}(\delta)}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H_{\mathrm{II}}(\delta)}{\partial q},$$

with the Hamiltonian $H_{\mathrm{II}}(\delta) := p^2 - (q^2+\tilde{t})p + (\kappa_1 - \delta)q$. When $\delta = 1$, it is the usual Hamiltonian of H_{II}. Moreover, when $\delta \neq 0$, we can normalize to the $\delta = 1$ case. Taking the limit $\delta \to 0$, we obtain an autonomous system with a Hamiltonian $H_{\mathrm{II}}(0) = p^2 - (q^2+\tilde{t})p + \kappa_1 q$. Since it is an autonomous system, the Hamiltonian is a first integral. The dimension of

(11) These first integrals are the autonomous Hamiltonians. They are rational in the phase variables q, p.

the phase space is two, the number of first integrals is half of the dimension. Therefore, the autonomous second Painlevï¿½ equation is integrable in Liouville's sense. The Lax pair[12] and the spectral curve of the autonomous second Painlevï¿½ equation are

$$\begin{aligned}&\frac{\mathrm{d}A(x)}{\mathrm{d}t}+[A(x),B(x)]=0,\\&\det(yI-A(x))=y^2-(x^2+\tilde{t})y-\kappa_1x-H_{\mathrm{II}}(0)=0.\end{aligned} \qquad \square$$

Remark 3.3. — For the 2-dimensional cases, parameters of the Painlevï¿½ equations can be thought as roots of affine root systems, and δ corresponds to the null root [49, 50].

3.3. The autonomous limit of the 4-dimensional Painlevï¿½-type equations

We can also consider such autonomous limit for higher dimensional Painlevï¿½-type equations. From the coefficients of the spectral curves, we obtain first integrals.

THEOREM 3.4. — *As the autonomous limits of 4-dimensional Painlevï¿½-type equations, we obtain 40 types of integrable systems with two functionally independent first integrals for each system*[13].

Proof. — One of the simplest 4-dimensional Painlevï¿½-type equation is the first matrix Painlevï¿½ equation [21]. The linear equation is given by

$$\begin{aligned}&\frac{\mathrm{d}A(x)}{\mathrm{d}t}+[A(x),B(x)]=0,\\&A(x)=\left(A_0x^2+A_1x+A_2\right),\quad B(x)=A_0x+B_1.\end{aligned}$$

where

$$A_0=\begin{pmatrix}O_2&I_2\\O_2&O_2\end{pmatrix},\qquad A_1=\begin{pmatrix}O_2&Q\\I_2&O_2\end{pmatrix},\qquad A_2=\begin{pmatrix}-P&Q^2+\tilde{t}I_2\\-Q&P\end{pmatrix},$$

$$B_1=\begin{pmatrix}O_2&2Q\\I_2&O_2\end{pmatrix},\qquad O_2=\begin{pmatrix}0&0\\0&0\end{pmatrix},\qquad I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$Q=\begin{pmatrix}q_1&u\\-q_2/u&q_1\end{pmatrix},\qquad P=\begin{pmatrix}p_1/2&-p_2u\\(p_2q_2-\kappa_2)/u&p_1/2\end{pmatrix}.$$

---

[12] We rewrite $A(x)=A(x,\tilde{t})$ and $B(x)=B(x,\tilde{t})$.
[13] These invariants are rational in the phase variables q_1,p_1,q_2,p_2.

The spectral curve is defined by the characteristic polynomial of the matrix $A(x)$;

$$\det\left(yI_4 - A(x)\right) = y^4 - \left(2\,x^3 + 2\tilde{t}x + h\right) y^2 + x^6 + 2\tilde{t}x^4 + hx^3 + \tilde{t}^2x^2 + \left(\tilde{t}h - \kappa_2^2\right) x + g.$$

The explicit forms of h and g are

$$\begin{aligned} h &:= H_{\mathrm{I}}^{\mathrm{Mat}} = \operatorname{tr}\left(P^2 - Q^3 - \tilde{t}Q\right) \\ &= -2p_2\left(p_2q_2 - \kappa_2\right) + \frac{p_1^2}{2} - 2q_1\tilde{t} - 2q_1\left(q_1^2 - q_2\right) + 4q_1q_2, \\ g &:= G_{\mathrm{I}}^{\mathrm{Mat}} \\ &= q_2\left(p_1p_2 + 3q_1^2 - q_2 + \tilde{t}\right)^2 - \kappa_2p_1\left(p_1p_2 + 3q_1^2 - q_2 + \tilde{t}\right) - 2\kappa_2^2q_1. \end{aligned}$$

Since h and g are coefficient of the spectral curve, they are invariants of the autonomous system. We can also check that h and g are conserved by direct computation:

$$\dot{h} = X_hh = \{h,h\} = 0, \quad \dot{g} = X_hg = \{g,h\} = 0,$$

where X_h is the Hamiltonian vector field associated to the Hamiltonian h. The Poisson bracket in the above equations is defined by

$$\{F,G\} := \sum_{i=1}^{2}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}\right).$$

Since

$$\operatorname{rank}\begin{pmatrix} \dfrac{\partial h}{\partial q_1} & \dfrac{\partial h}{\partial p_1} & \dfrac{\partial h}{\partial q_2} & \dfrac{\partial h}{\partial p_2} \\ \dfrac{\partial g}{\partial q_1} & \dfrac{\partial g}{\partial p_1} & \dfrac{\partial g}{\partial q_2} & \dfrac{\partial g}{\partial p_2} \end{pmatrix} = 2$$

for the general value of (q_1, p_1, q_2, p_2), we have two functionally independent invariants of the system. Thus the autonomous Hamiltonian system with the Hamiltonian $H_{\mathrm{I}}^{\mathrm{Mat}}$ is integrable in Liouville's sense.

From similar direct computations, we obtain the desired results for all the rest of 4-dimensional Painlevï¿½-type equations. We list functions in involution for the ramified types in Appendix A.1. These spectral curves and conserved quantities can be calculated from the data in the papers [21, 23, 22, 25, 51]. The only troublesome part is to find appropriate modifications of the Hamiltonians in the presence of δ. The other parts are straightforward. □

Remark 3.5. — It is an interesting problem to study the invariant surfaces defined by $H^{-1}(c_1)\cap G^{-1}(c_2) \subset \mathbb{C}^4$ for $c_1, c_2 \in \mathbb{C}$, where H and G are functionally independent invariants of the system. As in the case of other integrable systems [4], these Liouville tori can be completed into Abelian surfaces by adjoining the Painlevï¿½ divisors [39].

4. Degeneration of spectral curves

This section is the main part of this paper.

We study generic degeneration of spectral curves of the autonomous 4-dimensional Painlevï¿½-type equations, aiming to characterize these systems.

4.1. Genus 1 fibration and Tate's algorithm

Before discussing the genus 2 cases, corresponding to the autonomous limit of 4-dimensional Painlevï¿½-type equations, we discuss the genus 1 cases, corresponding to the autonomous 2-dimensional Painlevï¿½ equations.

Let us recall some of the basics we need. We can construct the Kodaira–Nï¿½ron model of an elliptic curve E over $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[h])$. The possible types of singular fibers of elliptic surfaces were classified by Kodaira [30, 31]. Tate's algorithm provides a way to determine the Kodaira type of singular fibers without actually resolving the singularities [55].

We consider an elliptic curve E over $\mathbb{A}^1$ with a section in the Weierstrass form:

$$y^2 = x^3 + a(h)x + b(h), \qquad a(h), b(h) \in \mathbb{C}[h]. \tag{4.1}$$

We may assume that for $a(h)$ and $b(h)$, the polynomials $l(h)$ such that $l(h)^4 | a(h),\ l(h)^6 | b(h)$ are only constants. Otherwise, we may divide both sides of the equations by $l(h)^6$ and replace x, y by $x/l(h)^2, y/l(h)^3$ if necessary. Let X_1 be the affine surface defined by the equation (4.1):

$$X_1 = \left\{(x, y, h) \in \mathbb{A}^2 \times \mathbb{A}^1 \;\middle|\; y^2 = x^3 + a(h)x + b(h)\right\}.$$

A general fiber of the projection $\varphi_1 \colon X_1 \to \mathbb{A}^1$ is an affine part of an elliptic curve. Let n be the minimal positive integer satisfying $\deg a(h) \leqslant 4n$ and $\deg b(h) \leqslant 6n$. Dividing equation (4.1) by h^{6n} and replacing $\tilde{x} = x/h^{2n}$, $\tilde{y} = y/h^{3n}$, $\widetilde{H} = 1/h$, we obtain the "∞-model":

$$\tilde{y}^2 = \tilde{x}^3 + \bar{a}(\widetilde{H})\tilde{x} + \bar{b}(\widetilde{H}), \quad \bar{a}(\widetilde{H}), \bar{b}(\widetilde{H}) \in \mathbb{C}[\widetilde{H}] \tag{4.2}$$

where $\bar{a}(\widetilde{H}) = a(h)/h^{4n}$, $\bar{b}(\widetilde{H}) = b(h)/h^{6n}$ are polynomials in h. Let X_2 be the affine surface defined by equation (4.2). Let $\overline{X_1}$ and $\overline{X_2}$ be the projectivized surfaces in $\mathbb{P}^2 \times \mathbb{A}^1$; $\overline{X_i} \subset \mathbb{P}^2 \times \mathbb{A}^1$, $\overline{\varphi_i}\colon \overline{X_i} \to \mathbb{A}^1$. We glue $\overline{X_1}$ and $\overline{X_2}$ by identifying (x, y, h) and $(\tilde{x}, \tilde{y}, \widetilde{H})$ by the equations above. Let us denote the surface obtained this way by W. We call W the Weierstrass model of the elliptic curve (4.1). The surface W has a morphism $\phi\colon W \to \mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$. After the minimal resolution of the singular points of W, we obtain a nonsingular surface X. This nonsingular projective surface X together with the fibration $\phi\colon X \to \mathbb{P}^1$ is called the Kodaira–Nï¿½ron model of the elliptic curve E over $\mathbb{A}^1$.

The singular fibers of an elliptic surface are classified by Kodaira. The Kodaira type of the elliptic surface X can be computed from the equation (4.1) using Tate's algorithm. From the Weierstrass form equation (4.1), we can associate two quantities: $\Delta := 4a^3 + 27b^2$, $j := 4a^3/\Delta$. Here, Δ is the discriminant of the cubic $x^3 + a(h)x + b(h)$ and j is the j-invariant. The Kodaira types of the singular fibers are determined as in Table 4.1 by the order of Δ and j which we denote $\mathrm{ord}_v(\Delta)$, $\mathrm{ord}_v(j)$.

Kodaira	Dynkin	ord(Δ)	ord(j)	Kodaira	Dynkin	ord(Δ)	ord(j)
I_0	-	0	$\geqslant 0$	I_0^*	$D_4^{(1)}$	6	$\geqslant 0$
I_m	$A_{m-1}^{(1)}$	m	$-m$	I_m^*	$D_{4+m}^{(1)}$	$6+m$	$-m$
II	-	2	$\geqslant 0$	IV*	$E_6^{(1)}$	8	$\geqslant 0$
III	$A_1^{(1)}$	3	$\geqslant 0$	III*	$E_7^{(1)}$	9	$\geqslant 0$
IV	$A_2^{(1)}$	4	$\geqslant 0$	II*	$E_8^{(1)}$	10	$\geqslant 0$

Table 4.1. Tate's algorithm and Kodaira types

4.1.1. Elliptic surface associated with the spectral curves

We introduce the main subject of this article, fibration of spectral curves associated with integrable Lax equations. Let us consider a $2n$-dimensional integrable system with a Lax pair. The spectral curve is parametrized by n functionally independent first integrals $H_1, \ldots, H_n$.

Theorem 4.1. — *Each elliptic surface whose general fiber is a spectral curve of the autonomous 2-dimensional Painlevï¿½ equation has the following singular fiber at $H = \infty$.*

Proof. — First, let us consider the first Painlevï¿½ equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = 6q^2 + t.$$

Hamiltonian	H_{VI}	H_{V}	$H_{\mathrm{III(D_6)}}$	$H_{\mathrm{III(D_7)}}$	$H_{\mathrm{III(D_8)}}$	H_{IV}	H_{II}	H_{I}
Kodaira	I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	IV^*	III^*	II^*
Dynkin	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$

Table 4.2. The singular fiber at $H=\infty$ of spectral curve fibrations of the autonomous 2-dimensional Painlevï¿½ equations.

The first Painlevï¿½ equation has a Lax form

$$\frac{\partial A}{\partial t}-\delta\frac{\partial B}{\partial x}+[A,B]=0,$$
$$A(x)=\begin{pmatrix}-p & x^2+qx+q^2+\tilde{t}\\ x-q & p\end{pmatrix},\quad B(x)=\begin{pmatrix}0 & x+2q\\ 1 & 0\end{pmatrix}.$$

The spectral curve associated with its autonomous equation is defined by $\det(yI_2-A(x))=0$. This is equivalent to the following elliptic curve

$$y^2=x^3+\tilde{t}x+H_{\mathrm{I}},\qquad H_{\mathrm{I}}=p^2-q^3+\tilde{t}q.$$

Let us write H_{I} as h for short. We consider this curve as an elliptic curve E over an affine line $\mathbb{A}^1$.

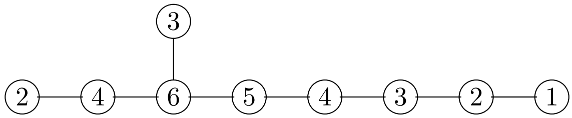
By changing variables as $\widetilde{H}=h^{-1},\ \tilde{x}=h^{-2}x,\ \tilde{y}=h^{-3}y$, obtain the "$\infty$-model";

$$\tilde{y}^2=\tilde{x}^3+\tilde{t}\widetilde{H}^4\tilde{x}+\widetilde{H}^5. \tag{4.3}$$

Thus we get the Weierstrass model $\varphi\colon W\to\mathbb{P}^1$. The Kodaira–Nï¿½ron model $\phi\colon X\to\mathbb{P}^1$ of elliptic curve E is obtained from W by the minimal desingularization. The Kodaira-type of singular fiber at $h=\infty$ can be computed using the equation (4.3). The discriminant of the cubic $\tilde{x}^3+\tilde{t}\widetilde{H}^4\tilde{x}+\widetilde{H}^5$ and the j-invariant are

$$\Delta=4\left(\tilde{t}\widetilde{H}^4\right)^3+27\left(\widetilde{H}^5\right)^2=\widetilde{H}^{10}\left(27+4\tilde{t}^3\widetilde{H}^2\right),$$
$$j=\frac{4}{\Delta}\left(\tilde{t}\widetilde{H}^4\right)^3=\frac{4\tilde{t}^3\widetilde{H}^{12}}{\widetilde{H}^{10}(27+2\tilde{t}^3\widetilde{H}^2)}=\widetilde{H}^2\frac{4\tilde{t}^3}{27+4\tilde{t}^3\widetilde{H}^2}.$$

Therefore, the order of zero of Δ and j at $\widetilde{H}=0$ is $\mathrm{ord}_\infty(\Delta)=10$, $\mathrm{ord}_\infty(j)=2$. Using Tate's algorithm, we find that the elliptic surface $X\to\mathbb{P}^1$ has the singular fiber of type II^*. In Dynkin's notation, this fiber is of type $\mathrm{E}_8^{(1)}$. Let us express the other two zeros of the discriminant Δ by h_+ and h_-. Since h_+ and h_- are both simple zeros of Δ, the Kodaira type of the fibers at h_+ and h_- are I_1, from Tate's algorithm.



The dual graph of the singular fiber of the Kodaira type II^* (Dynkin type $E_8^{(1)}$). The numbers in circles denote the multiplicities of components in the reducible fibers.

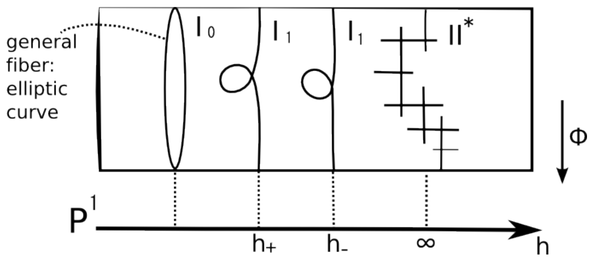


Figure 4.1. The elliptic surface associated to the spectral curves of the autonomous P_{I}.

The spectral curves associated to other autonomous 2-dimensional Painlevï¿½ equations are also curves of genus one for the general values of the Hamiltonians. It is well known that curves with genus one can always be transformed into the Weierstrass normal form. With the aid of computer programs, we can transform the spectral curves of autonomous 2-dimensional Painlevï¿½ equations into the Weierstrass normal form(14). Once the spectral curves are in the Weierstrass form, we can construct the Weierstrass model. After the minimal desingularizations, we obtain the elliptic surfaces. We apply Tate's algorithm to find the Kodaira types of the singular fibers of these elliptic surfaces. □

Remark 4.2. — For the special values of the parameters and the constant $\tilde{t}$ of the equations, the order of zeros or poles of Δ and j changes so that the Kodaira type changes. This corresponds to the situations when the equations have the special solutions. In this paper, we concentrate on the situations when the parameters are generic.

(14) Magma, Sage and Maple serve this purpose. Magma even calculates Kodaira types from the equations.

4.1.2. The Liouville tori and elliptic surface

For the autonomous 2-dimensional Painlevï¿½-type equations, the Liouville tori are elliptic curves. Therefore, elliptic surfaces are naturally associated to these integrable systems as Hamiltonian fibrations. We think of the time variable $\tilde{t}$ as a constant.

Theorem 4.3. — *Each elliptic surface associated to the autonomous 2-dimensional Painlevï¿½ equation as the Hamiltonian fibration has the following singular fiber at $h = \infty$.*

Hamiltonian	H_{VI}	H_{V}	$H_{\mathrm{III(D_6)}}$	$H_{\mathrm{III(D_7)}}$	$H_{\mathrm{III(D_8)}}$	H_{IV}	H_{II}	H_{I}
Kodaira type	I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	IV^*	III^*	II^*
Dynkin type	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$

Proof. — Let us first consider the easiest case: the first Painlevï¿½ equation. We write $h = H_{\mathrm{I}}$ for short. The Hamiltonian of the first Painlevï¿½ equation is $h = p^2 - (q^3 + \tilde{t}q)$. We view it as an elliptic curve over $\mathbb{A}^1$:

$$\left\{(q,p,h) \in \mathbb{A}^2_{(q,p)} \times \mathbb{A}^1_h \;\middle|\; p^2 = q^3 + \tilde{t}q + h\right\} \to \mathbb{A}^1_h.$$

As in the case of the spectral curve fibration, we can construct the Kodaira–Nï¿½ron model of the elliptic surface from the equation. Replacing $\tilde{q} = q/h^2$, $\tilde{p} = p/h^3$, $\widetilde{H} = 1/h$, we obtain the ∞-model:

$$\tilde{p}^2 = \tilde{q}^3 + \tilde{t}\widetilde{H}^4\tilde{q} + \widetilde{H}^5.$$

After compactification and the minimal desingularization, we obtain a regular elliptic surface whose general fiber at h is the elliptic curve defined by $p^2 = q^3 + \tilde{t}q + h$. The discriminant and the j-invariant are:

$$\Delta = 4\left(\tilde{t}\widetilde{H}^4\right)^3 + 27\left(\widetilde{H}^5\right)^2 = \widetilde{H}^{10}(27 + 4\tilde{t}^3\widetilde{H}^2),$$
$$j = \frac{4(\tilde{t}\widetilde{H}^4)^3}{4\widetilde{H}^{10}(27 + 4\tilde{t}^3\widetilde{H}^2)} = \widetilde{H}^2\frac{\tilde{t}^3}{27 + 4\tilde{t}^3\widetilde{H}^2}.$$

Thus the order of zero of Δ and j at $\widetilde{H} = 0$ are $\mathrm{ord}_\infty(\Delta) = 10, \mathrm{ord}_\infty(j) = 2$. It follows from Tate's algorithm that the singular fiber at $h = \infty$ of the autonomous P_{I}-Hamiltonian fibration is of type II^*, or $E_8^{(1)}$ in the Dynkin's notation.

We use computer programs to transform the other Hamiltonians into the Weierstrass normal form. The rest of the proof is similar to the case of H_{I}. □

The agreement of the singular fibers at $h = \infty$ of the spectral curve fibrations and the Liouville torus fibrations is not a coincidence. Liouville tori are related to the Jacobian varieties of the spectral curves, and taking the Jacobians are isomorphism in genus 1 cases by Abel's theorem. It might be natural to study the fibration of the Liouville tori, but we have to deal with families of 2-dimensional Abelian varieties to study the autonomous 4-dimensional Painlevï¿½-type equations. Therefore studying the degeneration of the Liouville tori become harder compared to the cases of the 2-dimensional Painlevï¿½ equations. On the other hand, we only need to deal with genus 2 curves to study spectral curve fibrations of the 4-dimensional autonomous Painlevï¿½ type equations. Thus, studying the degeneration of spectral curves is the main object of this paper.

Using blowing-up process, Okamoto resolved the singularities of 2-dimensional Painlevï¿½ differential equations and constructed the "spaces of initial conditions" [42]. While he deals with singularities of the systems of differential equations, we deal with spectral curves or Hamiltonians themselves for autonomous cases.

A space of initial conditions can be characterized by a pair (X, D) of a rational surface X and the anti-canonical divisor D of X. Each irreducible component of D is a rational curve and, in the case of the Painlevï¿½ equations, is called as a vertical leaf [49]. The intersection diagram of D is given by that of the certain root lattice listed above.

Remark 4.4. — The spaces of initial conditions are also considered for several cases of 4-dimensional Garnier systems and Noumi–Yamada systems [28, 29, 53, 54].

If we restrict our attention to the autonomous cases, the geometrical studies are much simpler. The autonomous 2-dimensional Painlevï¿½ equations constructed from the spaces of initial conditions were studied by Sakai [50].

4.2. Degeneration of genus 2 curves and Liu's algorithm

We apply a similar method as in the previous subsection to the 40 types of the autonomous 4-dimensional Painlevï¿½-type equations. While genus of spectral curves of the autonomous 2-dimensional Painlevï¿½ equations is 1, genus of the autonomous 4-dimensional Painlevï¿½-type equations is 2. As we use Tate's algorithm to determine the fibers of the Nï¿½ron models for the genus 1 curves, we use Liu's algorithm for the genus 2 curves. While the spectral curves of the autonomous 2-dimensional Painlevï¿½ equations are 1-parameter families parameterized by the Hamiltonian, those of the 4-dimensional Painlevï¿½-type equations are 2-parameters families parametrized by two first integrals.

The spectral curve is now 2-parameter family of genus two curves

$$F(w,x,y,h_1,h_2)=0,$$

where h_1 and h_2 are the functionally independent conserved quantities of the system and $F(w,x,y,h_1,h_2)\in\mathbb{C}[w,x,y,h_1,h_2]$. Let us consider a generic line in $\mathrm{Spec}(\mathbb{C}[h_1,h_2])$

$$ah_1+bh_2=c,$$

where $a,b,c\in\mathbb{C}$ are generic. Upon replacing $\tilde{a}=-a/b$, $\tilde{b}=c/b$,

$$h_2=\tilde{a}h_1+\tilde{b}$$

we obtain a one-parameter family of spectral curves

$$\tilde{F}(w,x,y,h_1):=F(w,x,y,h_1,\tilde{a}h_1+\tilde{b})=0.$$

We study degeneration of this one-parameter family of spectral curves assuming $\tilde{a}$ and $\tilde{b}$ are generic.

4.2.1. Liu's algorithm

We summarize studies on the degeneration of genus two curves. The numerical classification of the fibers in pencils of genus 2 curves are given by Ogg [41] and Iitaka [17]. Namikawa and Ueno [40] completed the geometrical classification of such fibers (and added a few missing types in [41] and [17]). There are 120 types in Namikawa–Ueno's classification, while there are only 10 types in Kodaira's classification of the fibers in pencils of genus 1 curves(15). Liu gave an algorithm similar to Tate's algorithm for genus 2 curves [35, 36]. Using Liu's algorithm, we can determine the Namikawa–Ueno type of singular fibers from explicit equations of pencils of genus 2 hyperelliptic curves in the Weierstrass form.

	genus of spectral curve	types of singular fibers in pencils	algorithm
2-dim. Painlevï¿½	1	Kodaira	Tate's algorithm
4-dim. Painlevï¿½	2	Namikawa–Ueno	Liu's algorithm

(15) Kodaira type I_n $(n\geqslant 1)$ and I_n^* $(n\geqslant 1)$ are counted as 1 type, respectively.

4.3. Generic degeneration of the spectral curves of the autonomous 4-dimensional Painlevï¿½-type equations

Let us state the main theorem of this paper.

THEOREM 4.5. — *The spectral curves of the autonomous 4-dimensional Painlevï¿½-type equations have the following types of generic degenerations as in Table 4.6.*

Remark 4.6. — Let us explain the notations used in Table 4.6. The Hamiltonians are the Hamiltonians of the 4-dimensional Painlevï¿½-type equations. The explicit forms of the non-autonomous counterparts can be found in [21, 23, 22, 24, 26, 27, 51]. The spectral types indicate the type of corresponding linear equations. Such notations are explained in Section 2.1 and Appendix A.2. “Namikawa–Ueno type” means the types of degeneration of genus 2 curves in Namikawa–Ueno [40]. When the fiber contains components expressible by the Kodaira-type, we also write its Dynkin's name in the column noted “Dynkin”. The column named “stable” tells us the type of the stable model [35]. The “Φ” indicates the group of connected components of the Nï¿½ron model of the Jacobian $J(C)$. The symbol (n) means the cyclic group with n elements[16]. We also write Ogg's type written in “On pencils of curves of genus two” [41]. Ogg uses the notation “Kod” to express Kodaira-type and do not distinguish them, while Namikawa and Ueno does. Ogg's type might be helpful to see the rough classification. For example, all 8 types of matrix Painlevï¿½ equations have the same Ogg's type 14. The column “monodromy” means 5 monodromy types in Namikawa–Ueno [40]. Elliptic types are those with finite degrees of monodromy, while parabolic types have infinite degrees. Elliptic[1] are those with stable model “I” in Liu's notation. We abbreviate as “ell[1]”. We summarize such correspondences in Table 4.3. The column named “page” indicate the page number of Namikawa–Ueno's paper where some data of the corresponding type can be found.

Proof of Theorem 4.5. — Let us take Gar $\frac{9}{2}$, the most degenerated Garnier system, to demonstrate our computation. The Lax pair is given by

$$\frac{\mathrm{d}A(x)}{\mathrm{d}t_i} + [A(x), B_i(x)] = 0, \quad i = 1, 2$$

[16] When $n = 0$, (n) is the trivial group.

Type	Order of monodromy	stable type (Liu's notation [35])
elliptic[1]	finite	I
elliptic[2]	finite	V
parabolic[3]	infinite	II, VI
parabolic[4]	infinite	III,VII
parabolic[5]	infinite	IV

Table 4.3. Namikawa–Ueno's elliptic and parabolic types and Liu's stable model types.

Hamiltonian	the Hamiltonian of the Painlevï¿½-type equation [21, 23, 22, 25, 51]
spectral type	the spectral type of the corresponding linear equation [21, 25, 44], Appendix A.2
monodromy	5 types of monodromy (elliptic[1],[2], Parabolic[3],[4],[5]) as in Namikawa–Ueno [40]
Namikawa–Ueno	the type of fiber in the minimal model following the notation in Namikawa–Ueno [36, 40]
Dynkin	Dynkin type (when the fiber contains Kodaira-type component)
stable	the type of stable model of the fiber [35]
Φ	the group of connected components of the Nï¿½ron model of the Jacobian $J(C)$ [36]
Ogg	the type of fiber in the minimal model following the notation in Ogg [41]
page	the page number of the fiber in Namikawa–Ueno's paper [40]

Table 4.4.

where

$$\begin{aligned} A(x) &= A_0x^3 + A_1x^2 + A_2x + A_3, \\ B_1(x) &= A_0x^2 + A_1x + B_{10} = \frac{A(x)}{x} + C_1 - \frac{A_3}{x}, \\ B_2(x) &= -A_0x + B_{20}, \end{aligned}$$

for

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & p_1 \\ 1 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} q_2 & p_1^2 + p_2 + 2\tilde{t}_1 \\ -p_1 & -q_2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} q_1 - p_1 q_2 & p_1^3 + 2p_1p_2 - q_2^2 + \tilde{t}_1 p_1 - \tilde{t}_2 \\ -p_2 + \tilde{t}_1 & -q_1 + p_1 q_2 \end{pmatrix},$$

$$B_{10} = \begin{pmatrix} q_2 & p_1^2 + 2p_2 + \tilde{t}_1 \\ -p_1 & -q_2 \end{pmatrix},\ B_{20} = \begin{pmatrix} 0 & -2p_2 \\ -1 & 0 \end{pmatrix},\ C_1 = \begin{pmatrix} 0 & p_2 - \tilde{t}_1 \\ 0 & 0 \end{pmatrix}.$$

The characteristic polynomial

$$\det(yI_2 - A(x)) = 0$$

is expressed as

$$y^2 = x^5 + 3\tilde{t}_2 x^3 - \tilde{t}_1 x^2 + (2\tilde{t}_2^2 - h_1)x + h_2 - \tilde{t}_1\tilde{t}_2,$$

where $h_1 = H^{9/2}_{\mathrm{Gar},\tilde{t}_1}$, $h_2 = H^{9/2}_{\mathrm{Gar},\tilde{t}_2}$. Note that it is already in the Weierstrass form. We consider the degeneration along a line $h_2 = ah_1 + b$, where a and b are generic constants.

$$y^2 = x^5 + 3\tilde{t}_2 x^3 - \tilde{t}_1 x^2 + (2\tilde{t}_2^2 - h_1)x + ah_1 + b - \tilde{t}_1\tilde{t}_2,$$

In order to see the degeneration at $h_1 = \infty$, we introduce $\tilde{x} = x/h_1,\ \tilde{y} = y/h_1^3,\ \widetilde{H} = 1/h_1$.

$$\tilde{y}^2 = \widetilde{H}x^5 + 3\tilde{t}_2\widetilde{H}^3x^3 - \tilde{t}_1\widetilde{H}^4x^2 - \widetilde{H}^4x + 2\tilde{t}_2^2\widetilde{H}^5x + \widetilde{H}^5(a + b\widetilde{H} - \tilde{t}_1\tilde{t}_2\widetilde{H}).$$

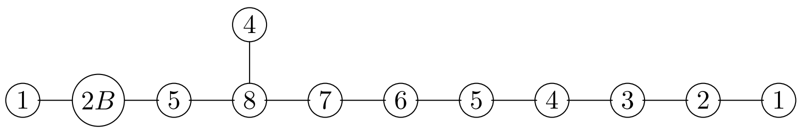
The Igusa invariants of the quintic can be calculated as follows.

$$J_2 = -5\widetilde{H}^5 + \frac{67}{4}\tilde{t}_2^2\widetilde{H}^6,\quad J_4 = \frac{15}{8}\widetilde{H}^{10} + O(\widetilde{H}^{11}),\quad J_6 = \frac{5}{16}\widetilde{H}^{15} + O(\widetilde{H}^{16}),$$

$$J_8 = -\frac{325}{256}\widetilde{H}^{20} + O(\widetilde{H}^{21}),\quad J_{10} = -\frac{1}{16}\widetilde{H}^{25} + O(\widetilde{H}^{26}).$$

Since $5 \cdot \mathrm{ord}_\infty J_{2i} - i \cdot \mathrm{ord}_\infty J_{10} = 0$ for $i \leqslant 5$, the stable model has smooth fiber (type(I) in Liu's notation of stable curves) at $\widetilde{H} = 0$. With further computation, we find that generic degeneration is of type VII* in Namikawa–Ueno's notation. This type is type 22 in Ogg's notation [41].

VII*: $H^{\frac{9}{2}}_{\mathrm{Gar},\tilde{t_1}}$



The numbers in circles denote the multiplicities of components in the reducible fibers. All curves are (-2)-curves except the one expressed as "B", which is a (-3)-curve.

We can transform the spectral curves of the other autonomous 4-dimensional Painlevï¿½ equations into the Weierstrass normal form[17]. We restrict the 2-parameter families of genus 2 curves to a generic line on the base and apply Liu's algorithm to find the generic degeneration. □

Remark 4.7. — In this paper, we proposed a possible clue to characterize the integrable systems studying the degeneration of spectral curves. The Namikawa–Ueno types and the monodromy matrices of the generic singular fibers are given. Our plan in the future work is to understand the phase spaces as the relative compactified Jacobian of the spectral curve fibration by studying the discriminant locus of the base space and combine with the results on monodromies of the singular fibers.

Hamiltonian spectral type	monodromy stable	Namikawa–Ueno Dynkin	Φ	Ogg	page
$H_{\mathrm{VI}}^{\mathrm{Mat}}$ 22,22,22,211	ell[2] V	$\mathrm{I}_0 - \mathrm{I}_0^* - 1$ $\mathrm{I}_0 - D_4 - 1$	$(2)^2$	14	p. 159
$H_{\mathrm{V}}^{\mathrm{Mat}}$ (2)(11),22,22	par[3] VI	$\mathrm{I}_0 - \mathrm{I}_1^* - 1$ $\mathrm{I}_0 - D_5 - 1$	(4)	14	p. 170
$H_{\mathrm{III(D_6)}}^{\mathrm{Mat}}$ (2)(2),(2)(11)	par[3] VI	$\mathrm{I}_0 - \mathrm{I}_2^* - 1$ $\mathrm{I}_0 - D_6 - 1$	$(2)^2$	14	p. 170
$H_{\mathrm{III(D_7)}}^{\mathrm{Mat}}$ $(2)(2), (11)_2$	par[3] VI	$\mathrm{I}_0 - \mathrm{I}_3^* - 1$ $\mathrm{I}_0 - D_7 - 1$	(4)	14	p. 170
$H_{\mathrm{III(D_8)}}^{\mathrm{Mat}}$ $(2)_2, (11)_2$	par[3] VI	$\mathrm{I}_0 - \mathrm{I}_4^* - 1$ $\mathrm{I}_0 - D_8 - 1$	$(2)^2$	14	p. 170
$H_{\mathrm{IV}}^{\mathrm{Mat}}$ ((2))((11)),22	ell[2] V	$\mathrm{I}_0 - \mathrm{IV}^* - 1$ $\mathrm{I}_0 - E_6 - 1$	(3)	14	p. 160
$H_{\mathrm{II}}^{\mathrm{Mat}}$ (((2)))(((11)))	ell[2] V	$\mathrm{I}_0 - \mathrm{III}^* - 1$ $\mathrm{I}_0 - E_7 - 1$	(2)	14	p. 162
$H_{\mathrm{I}}^{\mathrm{Mat}}$ $(((((11)))))_2$	ell[2] V	$\mathrm{I}_0 - \mathrm{II}^* - 1$ $\mathrm{I}_0 - E_8 - 1$	0	14	p. 160

Table 4.5. The generic degeneration of the spectral curves of the autonomous 4-dimensional matrix Painlevï¿½-type equations

[17] We used Maple's command "Weierstrassform" implemented by van Hoeij.

Hamiltonian spectral type	monodromy stable	Namikawa–Ueno Dynkin	Φ	Ogg	page
$H_{\mathrm{Gar}}^{1+1+1+1+1}$ 11,11,11,11,11	ell[1] II	I_{0-0-0}^{*} -	$(2)^4$	33	p. 155
$H_{\mathrm{Gar}}^{2+1+1+1}$ (1)(1),11,11,11	par[3] II	I_{1-0-0}^{*} -	$(4)\times(2)^2$	33	p. 171
$H_{\mathrm{Gar}}^{3/2+1+1+1}$ (2)(1),(1)(1)(1)	par[3] II	I_{2-0-0}^{*} -	$(2)^4$	33	p. 171
H_{Gar}^{2+2+1} (1)(1),(1)(1),11	par[4] III	I_{1-1-0}^{*} -	$(4)^2$	33	p. 180
$H_{\mathrm{Gar}}^{3/2+2+1}$ $(1)_2,(1)(1),11$	par[4] III	I_{1-2-0}^{*} -	$(4)\times(2)^2$	33	p. 180
$H_{\mathrm{Gar}}^{3/2+3/2+1}$ $(1)_2,(1)_2,11$	par[4] III	I_{2-2-0}^{*} -	$(2)^4$	33	p. 180
H_{Gar}^{3+1+1} ((1))((1)),11,11	ell[2] V	$\mathrm{I}_0^{*}-\mathrm{IV}^{*}-(-1)$ $D_4-E_6-(-1)$	$(6)\times(2)$	29a	p. 161
$H_{\mathrm{Gar}}^{5/2+1+1}$ (((1)(1)))(((1)))	ell[2] V	$\mathrm{I}_0^{*}-\mathrm{III}^{*}-(-1)$ $D_4-E_7-(-1)$	$(2)^3$	29a	p. 162
H_{Gar}^{4+1} (((1)))(((1))),11	par[3] II	$\mathrm{III}^{*}-\mathrm{II}_0^{*}$ $E_7-\mathrm{II}_0^{*}$	(8)	23	p. 178
$H_{\mathrm{Gar}}^{7/2+1}$ $(((((1)))))_2,11$	par[3] II	$\mathrm{II}^{*}-\mathrm{II}_0^{*}$ $E_8-\mathrm{II}_0^{*}$	(4)	25	p. 176
H_{Gar}^{3+2} ((1))((1)),(1)(1)	par[3] VI	$\mathrm{IV}^{*}-\mathrm{I}_1^{*}-(-1)$ $E_6-D_5-(-1)$	(12)	29a	p. 175
$H_{\mathrm{Gar}}^{5/2+2}$ $(((1)))_2,(1)(1)$	par[3] VI	$\mathrm{III}^{*}-\mathrm{I}_1^{*}-(-1)$ $E_7-D_5-(-1)$	$(4)\times(2)$	29a	p. 177
$H_{\mathrm{Gar}}^{3/2+3}$ $(1)_2,((1))((1))$	par[3] VI	$\mathrm{IV}^{*}-\mathrm{I}_2^{*}-(-1)$ $E_6-D_6-(-1)$	$(6)\times(2)$	29a	p. 175
$H_{\mathrm{Gar}}^{5/2+3/2}$ $(((1)))_2,(1)_2$	para[3] VI	$\mathrm{III}^{*}-\mathrm{I}_2^{*}-(-1)$ $E_7-D_6-(-1)$	$(2)^3$	29a	p. 177
H_{Gar}^{5} ((((1))))((((1))))	ell[1] I	$\mathrm{IX}-3$ -	(5)	21	p. 157
$H_{\mathrm{Gar}}^{9/2}$ $(((((((1)))))))_2$	ell[1] I	VII^{*} -	(2)	22	p. 156

Table 4.6. The generic degeneration of the spectral curves of the autonomous 4-dimensional Garnier equations

Hamiltonian spectral type	monodromy stable	Namikawa–Ueno Dynkin	Φ	Ogg	page
$H_{\mathrm{FS}}^{A_5}$ 21,21,111,111	par[3] II	II_{3-0} -	(12)	41	p. 171
$H_{\mathrm{FS}}^{A_4}$ (11)(1),21,111	par[5] VI	II_{3-1} -	(13)	41	p.183
$H_{\mathrm{FS}}^{A_3}$ $(1)_2, 21, 111$	par[5] IV	II_{3-2} -	(14)	41	p.183
$H_{\mathrm{Suz}}^{\frac{3}{2}+2}$ $(11)(1), (1)_2 1$	par[5] IV	II_{3-3} -	(15)	41	p. 183
$H_{\mathrm{KFS}}^{\frac{3}{2}+\frac{3}{2}}$ $(1)_3, (11)(1)$	par[5] IV	II_{3-4} -	(16)	41	p. 183
$H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{3}{2}}$ $(1)_3, (1)_2 1$	par[5] IV	II_{3-5} -	(17)	41	p. 183
$H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}}$ $(1)_3, (1)_3$	par[5] IV	II_{3-6} -	(18)	41	p. 183
$H_{\mathrm{NY}}^{A_5}$ (2)(1),111,111	par[4] III	II_{3-1} 0	$(6)\times(2)$	41a	p. 182
$H_{\mathrm{NY}}^{A_4}$ ((11))((1)),111	par[3] II	$\mathrm{IV}^* - \mathrm{II}_3$ $E_6 - \mathrm{II}_3$	(10)	41b	p. 175
$H_{\mathrm{Ss}}^{D_6}$ 31,22,22,1111	par[3] VI	$\mathrm{I}_2 - \mathrm{I}_0^* - 0$ $A_1 - D_4 - 0$	$(2)^3$	2	p. 171
$H_{\mathrm{Ss}}^{D_5}$ (111)(1),22,22	par[4] VII	$\mathrm{I}_2 - \mathrm{I}_1^* - 0$ $A_1 - D_5 - 0$	$(4)\times(2)$	2	p. 180
$H_{\mathrm{Ss}}^{D_4}$ (2)(2),(111)(1)	par[4] VII	$\mathrm{I}_2 - \mathrm{I}_2^* - 0$ $A_1 - D_6 - 0$	$(2)^4$	2	p. 180
$H_{\mathrm{KSs}}^{\frac{3}{2}+2}$ $(1)_2 11, (2)(2)$	par[4] VII	$\mathrm{I}_2 - \mathrm{I}_3^* - 0$ $A_1 - D_7 - 0$	$(4)\times(2)$	2	p. 180
$H_{\mathrm{KSs}}^{\frac{4}{3}+2}$ $(1)_3 1, (2)(2)$	par[4] VII	$\mathrm{I}_2 - \mathrm{I}_4^* - 0$ $A_1 - D_8 - 0$	$(2)^3$	2	p. 180
$H_{\mathrm{KSs}}^{\frac{5}{4}+2}$ $(1)_4, (2)(2)$	par[4] VII	$\mathrm{I}_2 - \mathrm{I}_5^* - 0$ $A_1 - D_9 - 0$	$(4)\times(2)$	2	p. 180
$H_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}}$ $(1)_4, (2)_2$	par[4] VII	$\mathrm{I}_2 - \mathrm{I}_6^* - 0$ $A_1 - D_{10} - 0$	$(2)^3$	2	p. 180

Table 4.7. The generic degeneration of the spectral curves of the autonomous 4-dimensional Fuji–Suzuki and Sasano equations

Appendix

A.1. Conserved quantities

The autonomous 4-dimensional Painlevï¿½-type equations have two functionally independent first integrals. In this subsection, we list these first integrals for the ramified equations[18]. One of the reason is that the other first integrals than the Hamiltonians have long expressions. Writing them for "less-degenerated" systems take huge spaces. But they are easily computable from data in the previous paper [25]. We only give first integrals for autonomous version of equations in Kawakami [21, 23, 22]. We list Hamiltonians H's with δ,[19] and the other invariants G's.

There are 5 ramified cases from the degeneration of A_5 Fuji–Suzuki system[20].

$$
\begin{aligned}
H_{\mathrm{FS}}^{A_3} &= H_{\mathrm{III}}(D_6)(-\theta_2^\infty, \delta + \theta_1^0 + \theta_1; \tilde{t}; q_1, p_1) \\
&\quad + H_{\mathrm{III}}(D_6)(\theta_2^0 - \theta_1^0, \theta_2^0 - \theta_1^0 - \theta^1; \tilde{t}; q_2, p_2) - \frac{1}{\tilde{t}} p_1 p_2 (q_1 q_2 + \tilde{t}), \\
G_{\mathrm{FS}}^{A_3} &= \theta_2^0 \tilde{t} \left(H_{\mathrm{III}}(D_6)(-\theta_2^\infty, \theta_1 + \theta_1^0; \tilde{t}; q_1, p_1) - p_1 p_2 \right) \\
&\quad + \left(q_1 q_2 - \tilde{t} \right) \left(\theta_2^\infty p_2 - \left(\theta_1^0 - \theta_2^0 \right) p_1 \right. \\
&\qquad\qquad \left. + p_1 p_2 \left(\theta_1 + \theta_1^0 - \theta_2^0 + (p_1 - 1) q_1 - (p_2 - 1) q_2 \right) \right), \\
H_{\mathrm{KFS}}^{\frac{3}{2}+2} &= H_{\mathrm{III}}(D_7)(-\theta_1^0; \tilde{t}; q_1, p_1) + H_{\mathrm{III}}(D_7)(\theta_2^0 - \theta_1^0; \tilde{t}; q_2, p_2) \\
&\quad + \frac{1}{\tilde{t}} \left(p_2 q_1 (p_1 (q_1 + q_2) + \theta_2^\infty) - q_1 \right), \\
G_{\mathrm{KFS}}^{\frac{3}{2}+2} &= (q_2 - q_1) \left(\left(\theta_2^0 - \theta_1^0 \right) p_1 p_2 q_1 - \theta_2^\infty p_2^2 q_1 + p_1 p_2^2 q_1 q_2 + p_1^2 p_2 q_1^2 + p_1 q_1 \right. \\
&\qquad\qquad \left. + p_1 p_2 \tilde{t} \right) + \theta_2^0 \left(\tilde{t} H_{\mathrm{III}}(D_7)(-\theta_1^0; \tilde{t}; q_1, p_1) - q_1 + p_1 p_2 q_1^2 \right) \\
&\quad - \theta_2^\infty \tilde{t} H_{\mathrm{III}}(D_7)(-\theta_1^0; \tilde{t}; q_1, p_2 + p_1 p_2 q_1^2 (\theta_2^0 - \theta_2^\infty), \\
H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{3}{2}} &= H_{\mathrm{III}}(D_7)(\theta_1^\infty; \tilde{t}; q_1, p_1) + H_{\mathrm{III}}(D_7)(\delta - \theta_1^\infty; \tilde{t}; q_2, p_2) \\
&\quad - \frac{1}{\tilde{t}} p_1 q_1 p_2 q_2 - \left(\frac{p_2}{q_1} + p_1 + p_2 \right), \\
G_{\mathrm{KFS}}^{\frac{4}{3}+\frac{3}{2}} &= \frac{1}{q_1} \left(p_2 \left(\theta_1^\infty + p_1 q_1 - p_2 q_2 \right) \left(\tilde{t} - p_1 q_1^2 q_2 \right) + (p_1 - p_2) q_2 q_1^2 - \tilde{t} \right),
\end{aligned}
$$

[18] We do not write first integrals of the Garnier equations here, since the first integrals are just autonomous limit of two Hamiltonians. Such Hamiltonians are listed by Kimura [27], Kawamuko [26] and Kawakami [22].

[19] Hamiltonians for the case $\delta = 0$ is the first integral.

[20] Although we obtain the Garnier equations of ramified types from certain degenerations of A_5-type Fuji–Suzuki system, we exclude the Garnier systems.

$$\begin{aligned}
H_{\mathrm{KFS}}^{\frac{3}{2}+\frac{3}{2}} &= H_{\mathrm{III}}(D_7)(\theta_1^\infty - \theta_2^\infty; \tilde{t}; q_1, p_1) + H_{\mathrm{III}}(D_7)(\delta - \theta_1^\infty; \tilde{t}; q_2, p_2) \\
&\quad - \frac{1}{\tilde{t}} p_1 q_1 p_2 q_2 - (p_1 p_2 + p_1 + p_2), \\
G_{\mathrm{KFS}}^{\frac{3}{2}+\frac{3}{2}} &= \left(p_1 p_2 q_1 - p_2^2 q_2 + \theta_1^\infty p_2 - 1\right)\left(-p_1\left(q_1 q_2 - \tilde{t}\right) + \theta_2^\infty q_2\right) \\
&\quad - p_2\left(q_1 q_2 - \tilde{t}\right), \\
H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}} &= \frac{1}{\tilde{t}}\left(p_1^2 q_1^2 + \delta q_1 p_1 - q_1 - \frac{\tilde{t}}{q_1}\right) + H_{\mathrm{III}}(D_8)(\tilde{t}; q_2, p_2) \\
&\quad + \frac{1}{\tilde{t}}\left(-p_1 q_1 p_2 q_2 + \frac{q_1 q_2}{\tilde{t}} + q_1 + q_2\right), \\
G_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}} &= \frac{1}{\tilde{t} q_1 q_2}\left((p_1 q_1 - p_2 q_2)\left(p_1 p_2 q_2^2 q_1^2 \tilde{t} + p_1 q_2 q_1^2 \tilde{t} + q_2^2 q_1^2\right)\right. \\
&\qquad\qquad \left. + \tilde{t}^2\left(p_1 q_1^2 - p_2 q_2^2 - q_2\right)\right).
\end{aligned}$$

There are also 4 systems that are ramified derived from D_6-Sasano system.

$$\begin{aligned}
H_{\mathrm{KSs}}^{\frac{3}{2}+2} &= H_{\mathrm{III}}(D_7)(\theta_0 + 2\theta_2^\infty; \tilde{t}; q_1, p_1) + H_{\mathrm{III}}(D_7)(-\theta_0; \tilde{t}; q_2, p_2) \\
&\quad + \frac{1}{\tilde{t}}(2p_2 q_1(p_1 q_1 - \theta_0 - \theta_1^\infty) - q_1), \\
G_{\mathrm{KSs}}^{\frac{3}{2}+2} &= p_1^2 p_2^2 q_1^4 - p_1^2 q_1^3 - 2p_1^2 p_2^2 q_2 q_1^3 - 2p_1 p_2^2 \theta_0 q_1^3 + p_1^2 p_2 \theta_0 q_1^3 - 2p_1 p_2^2 \theta_1^\infty q_1^3 \\
&\quad + \tilde{t} p_1 p_2^2 q_1^2 + p_1^2 p_2^2 q_2^2 q_1^2 + p_2^2 \theta_0^2 q_1^2 - p_1 p_2 \theta_0^2 q_1^2 + p_2^2 (\theta_1^\infty)^2 q_1^2 + p_1^2 q_2 q_1^2 \\
&\quad + p_1 \theta_0 q_1^2 + p_1 p_2^2 q_2 \theta_0 q_1^2 - p_1^2 p_2 q_2 \theta_0 q_1^2 + p_1 \theta_1^\infty q_1^2 + 2p_1 p_2^2 q_2 \theta_1^\infty q_1^2 \\
&\quad + 2p_2^2 \theta_0 \theta_1^\infty q_1^2 - p_1 p_2 \theta_0 \theta_1^\infty q_1^2 - p_1 \theta_2^\infty q_1^2 - 2p_1 p_2^2 q_2 \theta_2^\infty q_1^2 \\
&\quad + p_1 p_2 \theta_0 \theta_2^\infty q_1^2 + p_2^2 q_2 \theta_0^2 q_1 - p_1 p_2 q_2 \theta_0^2 q_1 - \tilde{t} p_1 q_1 - 2\tilde{t} p_1 p_2^2 q_2 q_1 \\
&\quad - \tilde{t} p_2^2 \theta_0 q_1 + p_1 p_2^2 q_2^2 \theta_0 q_1 + \tilde{t} p_1 p_2 \theta_0 q_1 + p_1 q_2 \theta_0 q_1 - \tilde{t} p_2^2 \theta_1^\infty q_1 \\
&\quad + p_2^2 q_2 \theta_0 \theta_1^\infty q_1 + 2p_1 p_2^2 q_2^2 \theta_2^\infty q_1 - p_2 \theta_0^2 \theta_2^\infty q_1 + 2p_1 q_2 \theta_2^\infty q_1 \\
&\quad + 2p_2^2 q_2 \theta_0 \theta_2^\infty q_1 - 2p_1 p_2 q_2 \theta_0 \theta_2^\infty q_1 + \theta_0 \theta_2^\infty q_1 + 2p_2^2 q_2 \theta_1^\infty \theta_2^\infty q_1 \\
&\quad - p_2 \theta_0 \theta_1^\infty \theta_2^\infty q_1 + \theta_1^\infty \theta_2^\infty q_1 + \tilde{t} p_1 p_2^2 q_2^2 + p_2^2 q_2^2 (\theta_2^\infty)^2 + \tilde{t} p_2 (\theta_2^\infty)^2 \\
&\quad + q_2 (\theta_2^\infty)^2 - p_2 q_2 \theta_0 (\theta_2^\infty)^2 + \tilde{t} p_1 q_2 + \tilde{t} p_2^2 q_2 \theta_0 - \tilde{t} p_1 p_2 q_2 \theta_0 \\
&\quad + \tilde{t} p_2^2 q_2 \theta_1^\infty - p_2 q_2 \theta_0^2 \theta_2^\infty + p_2^2 q_2^2 \theta_0 \theta_2^\infty + 2\tilde{t} p_2 \theta_0 \theta_2^\infty + q_2 \theta_0 \theta_2^\infty \\
&\quad + \tilde{t} p_2 \theta_1^\infty \theta_2^\infty, \\
H_{\mathrm{KSs}}^{\frac{4}{3}+2} &= H_{\mathrm{III}}(D_7)(\theta_0 + 2\theta_2^\infty; \tilde{t}; q_1, p_1) + H_{\mathrm{III}}(D_7)(-\theta_0; \tilde{t}; q_2, p_2) \\
&\quad - \frac{1}{\tilde{t}}\left(2p_2 q_1 + q_1 + \tilde{t} p_2\right),
\end{aligned}$$

$$
\begin{aligned}
G_{\mathrm{KSs}}^{\frac{4}{3}+2} &= -3\theta_0^2\theta_1^\infty p_2q_2 - \theta_0(\theta_1^\infty)^2p_2q_2 + 3\theta_0\theta_1^\infty p_2^2q_2^2 + \theta_0\theta_1^\infty p_2q_1 \\
&\quad + 2\theta_0\theta_1^\infty p_1p_2q_1q_2 - 2\theta_0^3p_2q_2 + 2\theta_0^2p_2^2q_2^2 + 2\theta_0^2p_2q_1 \\
&\quad + 3\theta_0^2p_1p_2q_1q_2 - \theta_0p_1p_2q_1^2 - 3\theta_0p_1p_2^2q_1q_2^2 - \theta_0p_1^2p_2q_1^2q_2 \\
&\quad - 3\theta_0p_2^2q_1q_2 - 3\theta_0p_1q_1q_2 + (\theta_1^\infty)^2p_2^2q_2^2 - 2\theta_1^\infty p_1p_2^2q_1q_2^2 \\
&\quad - 2\theta_1^\infty p_2^2q_1q_2 - 2\theta_1^\infty p_1q_1q_2 - \theta_0p_1p_2q_2\tilde{t} + p_1p_2^2q_2^2\tilde{t} + p_2^2q_2\tilde{t} \\
&\quad + p_1q_2\tilde{t} + p_2^2q_1^2 + p_1q_1^2 + p_1^2p_2^2q_1^2q_2^2 + p_1^2q_1^2q_2 \\
&\quad + 2p_1p_2^2q_1^2q_2 - 2\theta_0p_2\tilde{t} - \theta_1^\infty p_2\tilde{t} + 3\theta_0\theta_1^\infty q_2 + 2\theta_0^2q_2 - 2\theta_0q_1 \\
&\quad + (\theta_1^\infty)^2q_2 - \theta_1^\infty q_1, \\
H_{\mathrm{KSs}}^{\frac{5}{4}+2} &= H_{\mathrm{III}}(D_8)(\tilde{t};q_1,p_1) + H_{\mathrm{III}}(D_7)(-\theta_0,\tilde{t};q_2,p_2) + 2\frac{p_2}{q_1} \\
&\quad - \frac{(\theta_0+1-\delta)p_1q_1}{\tilde{t}} + \frac{1}{q_1} - p_2, \\
G_{\mathrm{KSs}}^{\frac{5}{4}+2} &= \frac{1}{q_1^2}\Big(p_2^2\left(p_1q_2q_1^2\left(2\tilde{t}-\theta_0q_1q_2\right) + p_1^2q_2^2q_1^4 - \theta_0q_2q_1\tilde{t} - q_2^2q_1^3 + \tilde{t}^2\right) \\
&\qquad - p_2q_1^2\left(\theta_0p_1^2q_2q_1^2 + \theta_0p_1\left(\tilde{t}-\theta_0q_1q_2\right) - \theta_0q_2q_1 + \tilde{t}\right) \\
&\qquad + q_1^3q_2(p_1^2q_1-1) + q_1^2p_1\left(\tilde{t}-\theta_0q_1q_2\right)\Big), \\
H_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}} &= \frac{1}{\tilde{t}}\left(p_1^2q_1^2 + \delta q_1p_1 - q_1 - \tilde{t}/q_1\right) + H_{\mathrm{III}}(D_8)(\tilde{t};q_2,p_2) - 2\frac{q_1q_2}{\tilde{t}^2} \\
&\quad + \frac{q_1+q_2}{\tilde{t}}, \\
G_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}} &= \frac{q_1^2q_2^2}{\tilde{t}^2} - \frac{p_1^2q_1^2\tilde{t}}{q_2} + q_1 + q_2 + \frac{\tilde{t}^2}{q_1q_2} \\
&\quad - \frac{1}{4q_1\tilde{t}}\left(2p_2q_2+1\right)\left(\tilde{t}\left(2p_2q_2+1\right)\left(\tilde{t}-p_1^2q_1^3\right) + 4p_1q_2q_1^3\right).
\end{aligned}
$$

We have three ramified systems of matrix Painlevï¿½ equations. $G^{\mathrm{Mat}}_{\mathrm{III}(D_7)}$ and $G^{\mathrm{Mat}}_{\mathrm{III}(D_8)}$ are too long and $H^{\mathrm{Mat}}_{\mathrm{I}}$ is already written in the main part of this paper. So we skip writing first integrals of autonomous matrix Painlevï¿½ equations.

A.2. Local data of linear equations

For the classification of linear equations, we need to discern the types of linear equations. In this subsection, we explain the notation to express spectral types for ramified equations, which is introduced by Kawakami [21]. The Fuchsian case is explained in 2.1.

Now we explain the way to obtain the formal canonical form around each non-Fuchsian singular point. We also explain that these canonical forms can be expressed by the refining sequences of partition. Let us assume that the coefficient matrix $A(x)$ of the equation has a singularity at the origin, and that $A(x)$ is expanded in the Laurent series as follows:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \left(\frac{A^0}{x^{r+1}} + \frac{A^1}{x^r} + \cdots\right) Y. \tag{A.4}$$

Here, A^j $(j = 0, 1, \dots)$ are $m \times m$ matrices. We assume that A^0 is diagonalizable.

With an appropriate choice of the gauge matrix, we can assume that A^0 is diagonal and that its eigenvalues are $t_1^0, \dots, t_m^0$. When $r = 0$, then the origin is a regular singular point. Let us assume that $r > 0$. If $t_i^0 \neq t_j^0$ $(1 \leqslant i \leqslant l,\ l+1 \leqslant j \leqslant m)$, then a gauge transformation by a formal power series $Y = P(x)Z$ $(P(x) = I + P_1 x + P_2 x^2 + \cdots)$ leads to the following form:

$$\frac{\mathrm{d}Z}{\mathrm{d}x} = \left(\frac{B^0}{x^{r+1}} + \frac{B^1}{x^r} + \cdots\right) Z.$$

Here, we can transform B^i into the following form:

$$B^i = \begin{pmatrix} B^i_{11} & O \\ O & B^i_{22} \end{pmatrix},\ B^i_{11} \in M_l(\mathbb{C}),\ B^i_{22} \in M_{m-l}(\mathbb{C}).$$

With successive application of this process, the equation (A.4) is formally decomposed to direct sum of equations whose leading terms have only one eigenvalues respectively. When the leading term of the block is diagonalizable, that is when it is scalar matrix, then this part can be canceled by a gauge transformation by a scalar function, so that the equation is reduced to a equation with smaller r.

Remark A.8. — When A^0 is not diagonalizable, in order to decompose the system into equations of smaller sizes, we need to take an appropriate covering $x = \xi^k$. In that case, the transformation matrix $P(x)$ is a Puiseux series in x. The equations with this property are called ramified. When we do not need to take coverings, that is when $k = 1$, we say that the equations are unramified.

A.2.1. Unramified non-Fuchsian case

When the equation (A.4) is unramified, it can be transformed into the following form:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \left(\frac{T_0}{x^{r+1}} + \frac{T_1}{x^r} + \cdots + \frac{T_r}{x} + \cdots\right) Y.$$

We can assume that T_j's are diagonal matrices and that $T_0 = A_0$. Furthermore, we can eliminate the regular terms by an appropriate diagonal matrix with formal power series components. Thus, the equation (A.4) can be transformed into the following form by gauge transformation of a formal power series:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \left(\frac{T_0}{x^{r+1}} + \frac{T_1}{x^r} + \cdots + \frac{T_r}{x}\right) Y. \tag{A.5}$$

If we write the diagonal components of T_i as t_j^i $(j = 1, \dots, m)$, then a canonical form around the origin can be described by the following data:

$$\begin{array}{cccc} \multicolumn{4}{c}{\overbrace{\hphantom{t_1^0 \quad t_1^1 \quad \dots \quad t_1^r}}^{x=0}} \\ t_1^0 & t_1^1 & \dots & t_1^r \\ \vdots & \vdots & & \vdots \\ t_m^0 & t_m^1 & \dots & t_m^r \end{array}.$$

We write down such formal canonical forms for each singular point, and put them together. This kind of table is called the Riemann scheme of the linear equation. As we can see from the procedure to obtain the canonical form, the leftmost column splits into several groups as equivalence class of values. In the second column from the left, these groups splits further, and so on, we get a nested columns.

We describe such nesting structure by refining sequences of partitions of m and call it the spectral type of the singular point. We line up such spectral types of each singular point, and separate them by commas. We call it the spectral type of the equation. In such a case,

$$\exp\left(-\frac{T_0}{rx^r} + \cdots - \frac{T_{r-1}}{x}\right) x^{T_r}$$

is the fundamental solution matrix for the formal canonical form (A.5). The degree r of the polynomial is called the Poincarï¿½ rank of the singular point. When the singular point is of regular type, the Poincarï¿½ rank is 0. When the equation is ramified, this part is a polynomial in $x^{-1/k}$, and the Poincarï¿½ rank is non-integer rational number. If the thing we want to express is just the Poincarï¿½ ranks at each singularity, we attach Poincarï¿½ rank plus 1 to each singularity, and line them up, and separate them by + signs. When the equation is unramified, Poincarï¿½ rank plus 1 is as same as numbers of columns appeared in refining sequences of partitions at each singularities.

Example A.9. — For instance, let us consider the following normal form:

$$\frac{\mathrm{d}\hat{Y}(x)}{\mathrm{d}x}=\left\{\frac{1}{x^3}\begin{pmatrix}a&0&0&0\\0&a&0&0\\0&0&b&0\\0&0&0&b\end{pmatrix}+\frac{1}{x^2}\begin{pmatrix}c&0&0&0\\0&c&0&0\\0&0&d&0\\0&0&0&e\end{pmatrix}+\frac{1}{x}\begin{pmatrix}f&0&0&0\\0&g&0&0\\0&0&h&0\\0&0&0&i\end{pmatrix}\right\}\hat{Y}.$$

We align the diagonal entries as $\left\{\begin{smallmatrix}a&a&b&b\\c&c&d&e\\f&g&h&i\end{smallmatrix}\right.$. We express the degeneracy of the eigenvalues by $\left\{\begin{smallmatrix}22\\211\\1111\end{smallmatrix}\right\}$. Each row expresses a partition of the matrix size m. The partitions in the lower rows are a refinement of a partition in the upper rows.

In order to express the degeneracy of the eigenvalues briefly, we use parentheses. Firstly, write the finest partition of m in the lowest row, which expresses the degeneracy of the eigenvalue of $T^{(i)}$. Secondly, put the numbers that are grouped together in the second lowest partition in parentheses. We continue this process until the highest row.

Example A.10. — The local data of the example above can be expressed concisely using the parentheses.

$$\overbrace{\begin{matrix}a&c&f\\a&c&g\\b&d&h\\b&e&i\end{matrix}}^{x=0}=\left\{\begin{matrix}a&a&b&b\\c&c&d&e\\f&g&h&i\end{matrix}\right.\longrightarrow\left\{\begin{matrix}22\\211\\1111\end{matrix}\right\}\longrightarrow((11))((1)(1)),$$

level 2	((11))((1)(1))
level 1	(11)(1)(1)
level 0	1111

The way to restore the degeneracy of eigenvalues from the symbol ((11))((1)(1)) is as follows.

- Add the numbers in the outermost parenthesis → 22
- Add the numbers in the inner parenthesis →211
- Write the numbers in the innermost parentheses →1111

We express the types of linear equations by aligning such data for each singular point.

A.2.2. Ramified case

We have the following formal normal form at each singularities (Hukuhara [16], Levelt [33] and Turrittin [56]). Let us assume that $x = 0$ is an irregular singular point. Then, there exist a positive integer q, rational numbers with the common denominator q such that $r_0 < r_1 < \cdots < r_{n-1} < r_n = -1$, diagonal matrices $T_0, \dots, T_n$, a transformation $z = F(x^{1/q})$ in the class of formal series in $x^{1/q}$, such that the transformed system have the following form;

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (T_0 x^{r_0} + \cdots + T_{n-1} x^{r_{n-1}} + T_n x^{-1}) z. \tag{A.6}$$

Let us assume that the diagonal matrix T_k has t_i^k for $i = 1, \dots, m$ as diagonal components. We express the local data by the following table:

$$\overbrace{\begin{matrix} t_1^0 & t_1^1 & \dots & t_1^n \\ \vdots & \vdots & & \vdots \\ t_m^0 & t_m^1 & \dots & t_m^n \end{matrix}}^{x=0\ \left(\frac{n}{q}\right)}.$$

Let us introduce a way to express such local data compactly with examples.[21] Let us assume the following normal form;

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \left\{ c_0 \Omega x^{-\frac{8}{3}} + c_1 \Omega^2 x^{-\frac{7}{3}} + c_2 I_3 x^{-2} + c_3 \Omega x^{-\frac{5}{3}} + c_4 \Omega^2 x^{-\frac{4}{3}} + c_5 I_3 x^{-1} \right\} z,$$

where $\Omega = \mathrm{diag}(1, \omega, \omega^2)$ and ω is a primitive third root of unity. This normal form can be expressed as

$$\overbrace{\begin{matrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_0\omega & c_1\omega & c_2 & c_3\omega & c_4\omega^2 & c_5 \\ c_0\omega^2 & c_1\omega & c_2 & c_3\omega^2 & c_4\omega & c_5 \end{matrix}}^{x=0\ \left(\frac{5}{3}\right)}.$$

Two systems in the lower rows $\frac{\mathrm{d}z_2}{\mathrm{d}x} = (c_0\omega x^{-\frac{8}{3}} + \dots) z_2$ and $\frac{\mathrm{d}z_3}{\mathrm{d}x} = (c_0\omega x^{-\frac{8}{3}} + \dots) z_3$ can be obtained by the first row $\frac{\mathrm{d}z_1}{\mathrm{d}x} = (c_0 x^{-\frac{8}{3}} + \dots) z_1$ upon replacement $x^{\frac{1}{3}} \mapsto \omega x^{\frac{1}{3}} \mapsto \omega^2 x^{\frac{1}{3}}$. Since we have 3 copies of the first equation, we express the local data as $(((((1)))))_3$.

Example A.11. — We show some other examples.

$$\overbrace{\begin{matrix} \alpha & \beta \\ -\alpha & \beta \\ 0 & \gamma \\ 0 & \epsilon \end{matrix}}^{x=0\ \left(\frac{1}{2}\right)} \to (1)_2 11, \qquad \overbrace{\begin{matrix} \alpha & \beta \\ \alpha\omega & \beta \\ \alpha\omega^2 & \beta \\ 0 & \gamma \end{matrix}}^{x=0\ \left(\frac{1}{2}\right)} \to (1)_3 1,$$

[21] Kawakami [21] devised such notation.

$$\begin{array}{cc} \multicolumn{2}{c}{\underbrace{x=0}} & \left(\frac{1}{2}\right) \\ \alpha & \beta & \\ \alpha & \beta & \to (2)_2, \\ -\alpha & \beta & \\ -\alpha & \beta & \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\underbrace{x=0}} & \left(\frac{1}{2}\right) \\ \alpha & \beta & \\ \alpha & \gamma & \to (11)_2. \\ -\alpha & \beta & \\ -\alpha & \gamma & \end{array}$$

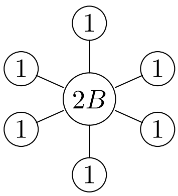
A.3. The dual graphs of the singular fibers

We list the dual graphs of singular fibers appeared in the table. The numbers in circles indicate multiplicities of the components. We adopt, as in Ogg [41], the following symbol for component Γ of singular fibers (Table A.8). K_X is the canonical divisor of surface X. The matrices next to or below the dual graphs are the monodromy [40].

Symbol	Genus	Γ^2	$\Gamma \cdot K_X$
A	1	-1	1
B	0	-3	1
C	1	-2	2
D	0	-4	2
none	0	-2	0

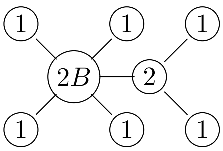
Table A.8. Components of singular fibers

I^*_{0-0-0}: $H^{1+1+1+1+1}_{\mathrm{Gar}}$



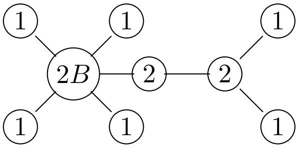
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

I^*_{1-0-0}: $H^{2+1+1+1}_{\mathrm{Gar}}$



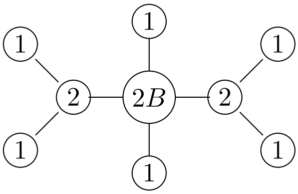
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

I^*_{2-0-0}: $H^{\frac{3}{2}+1+1+1}_{\mathrm{Gar}}$



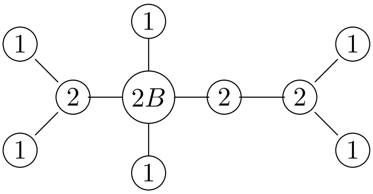
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

I^*_{1-1-0}: H^{2+2+1}_{Gar}



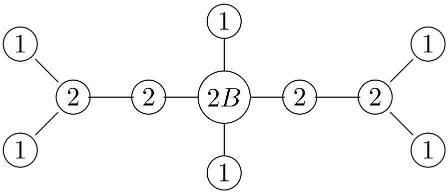
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

I^*_{1-2-0}: $H^{\frac{3}{2}+2+1}_{\mathrm{Gar}}$



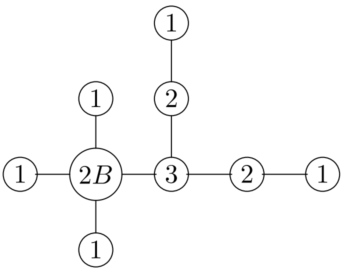
$$\begin{pmatrix} -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

I^*_{2-2-0}: $H^{\frac{3}{2}+\frac{3}{2}+1}_{\mathrm{Gar}}$



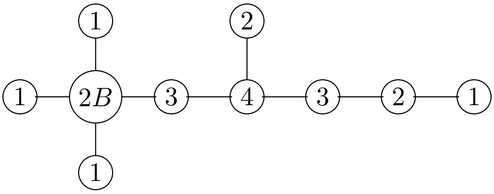
$$\begin{pmatrix} -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{I}^*_0 - \mathrm{IV}^* - (-1)$: H^{3+1+1}_{Gar}



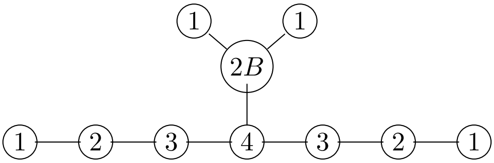
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{I}_0^* - \mathrm{III}^* - (-1)$: $H_{\mathrm{Gar}}^{\frac{5}{2}+1+1}$



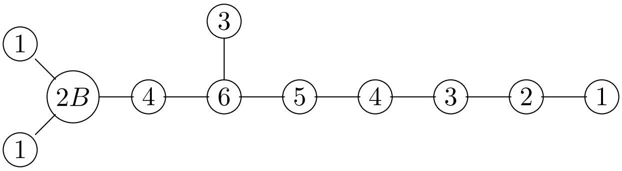
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{III}^* - \mathrm{II}_0^* - (-1)$: H_{Gar}^{4+1}



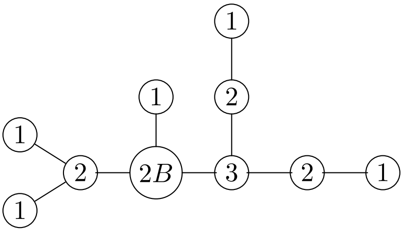
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{II}^* - \mathrm{II}_0^* - (-1)$: $H_{\mathrm{Gar}}^{\frac{7}{2}+1}$



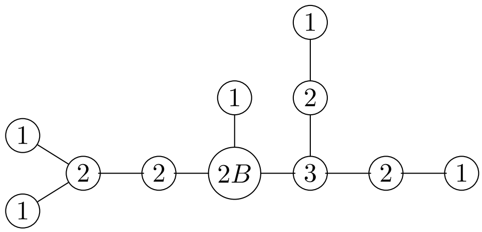
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{IV}^* - \mathrm{I}_1^* - (-1)$: H_{Gar}^{3+2}



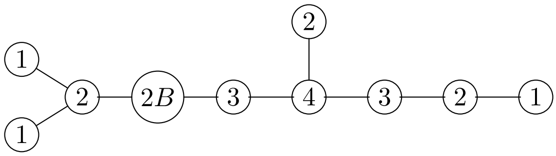
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{IV}^* - \mathrm{I}_2^* - (-1)$: $H_{\mathrm{Gar}}^{3+\frac{3}{2}}$



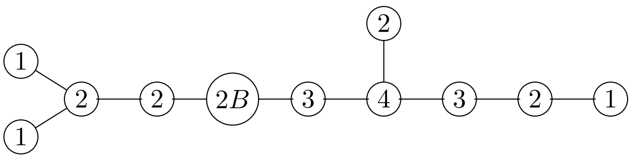
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{III}^* - \mathrm{I}_1^* - (-1)$: $H_{\mathrm{Gar}}^{\frac{5}{2}+2}$



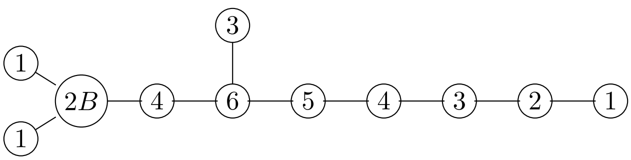
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{III}^* - \mathrm{I}_2^* - (-1)$: $H_{\mathrm{Gar}}^{\frac{5}{2}+\frac{3}{2}}$



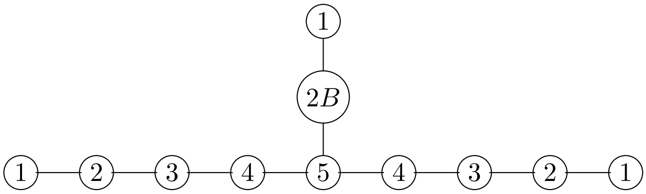
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{II}^* - \mathrm{II}_0^*$: $H_{\mathrm{Gar}}^{\frac{7}{2}+1}$



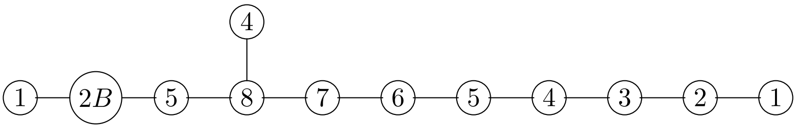
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{IX}-3\colon H^{5}_{\mathrm{Gar}}$



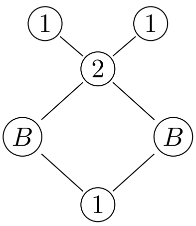
$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\mathrm{VII}^*\colon H^{\frac{9}{2}}_{\mathrm{Gar}}$



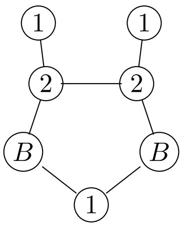
$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

para[3], $\mathrm{II}_{3-0}\colon H^{A_5}_{\mathrm{FS}}$



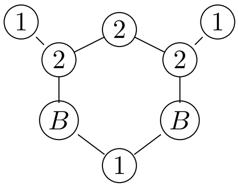
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

para[5], $\mathrm{II}_{3-1}\colon H^{A_4}_{\mathrm{FS}}$



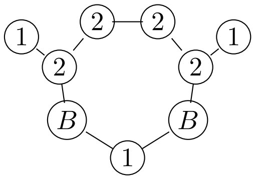
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

para[5], II_{3-2}: $H_{\mathrm{FS}}^{A_3}$



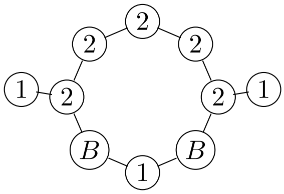
$$\begin{pmatrix} -1 & 0 & -2 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

para[5], II_{3-3}: $H_{\mathrm{Suz}}^{\frac{3}{2}+2}$



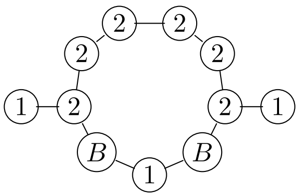
$$\begin{pmatrix} -1 & 0 & -3 & 0 \\ 1 & 1 & 3 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

para[5], II_{3-4}: $H_{\mathrm{KFS}}^{\frac{3}{2}+\frac{3}{2}}$



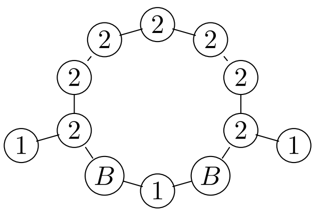
$$\begin{pmatrix} -1 & 0 & -4 & 0 \\ 1 & 1 & 4 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

para[5], II_{3-5}: $H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{3}{2}}$



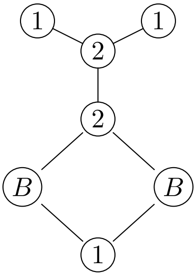
$$\begin{pmatrix} -1 & 0 & -5 & 0 \\ 1 & 1 & 5 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

para[5], II_{3-6}: $H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}}$



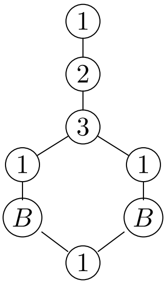
$$\begin{pmatrix} -1 & 0 & -6 & 0 \\ 1 & 1 & 6 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

para[4], II_{3-1}: $H_{\mathrm{NY}}^{A_5}$



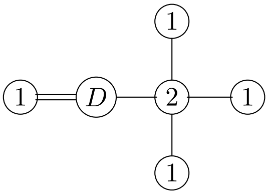
$$\begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{IV}^* - \mathrm{II}_3$: $H_{\mathrm{NY}}^{A_4}$



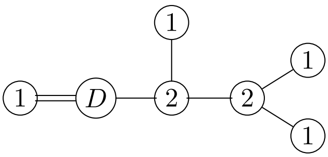
$$\begin{pmatrix} -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{I}_2 - \mathrm{I}_0^* - 0$: $H_{\mathrm{Ss}}^{D_6}$



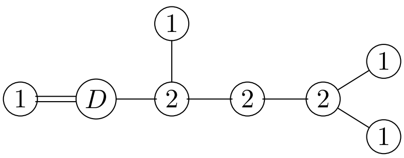
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{I}_2 - \mathrm{I}_1^* - 0$: $H_{\mathrm{Ss}}^{D_5}$



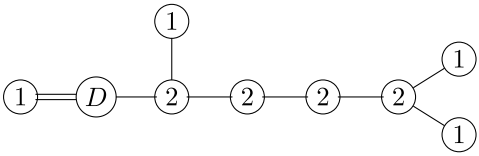
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{I}_2 - \mathrm{I}_2^* - 0$: $H_{\mathrm{Ss}}^{D_4}$



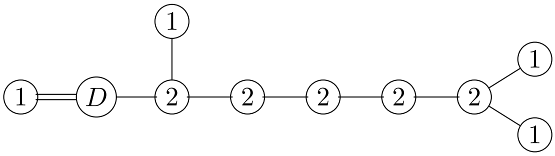
$$\begin{pmatrix} -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{I}_2 - \mathrm{I}_3^* - 0$: $H_{\mathrm{KSs}}^{\frac{3}{2}+2}$



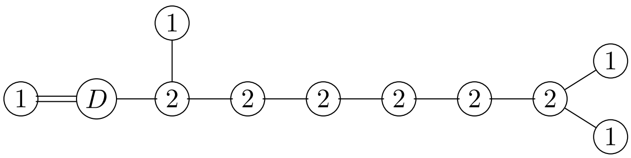
$$\begin{pmatrix} -1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{I}_2 - \mathrm{I}_4^* - 0$: $H_{\mathrm{KSs}}^{\frac{4}{3}+2}$



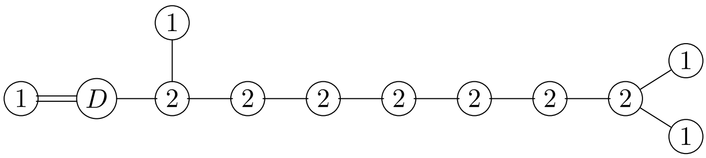
$$\begin{pmatrix} -1 & 0 & -4 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{I}_2 - \mathrm{I}_5^* - 0$: $H_{\mathrm{KSs}}^{\frac{5}{4}+2}$



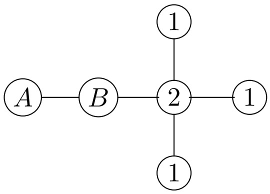
$$\begin{pmatrix} -1 & 0 & -5 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{I}_2 - \mathrm{I}_6^* - 0$: $H_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}}$



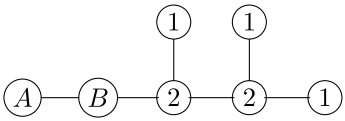
$$\begin{pmatrix} -1 & 0 & -6 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{I}_0^* - 1$: $H_{\mathrm{VI}}^{\mathrm{Mat}}$



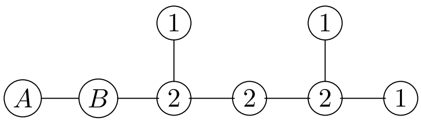
$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{I}_1^* - 1$: $H_{\mathrm{V}}^{\mathrm{Mat}}$



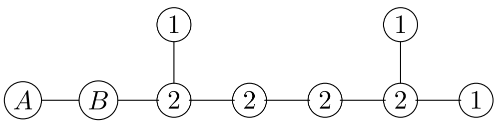
$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{I}_2^* - 1$: $H_{\mathrm{III}(D_6)}^{\mathrm{Mat}}$



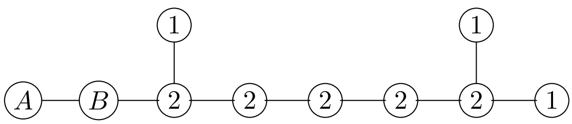
$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & -2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{I}_3^* - 1$: $H_{\mathrm{III}(D_7)}^{\mathrm{Mat}}$



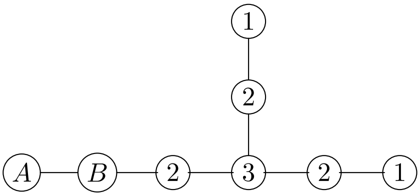
$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & -3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{I}_4^* - 1$: $H_{\mathrm{III}(D_8)}^{\mathrm{Mat}}$



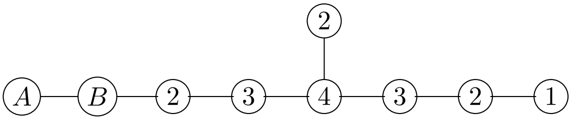
$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & -4\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{IV}^* - 1$: $H_{\mathrm{IV}}^{\mathrm{Mat}}$



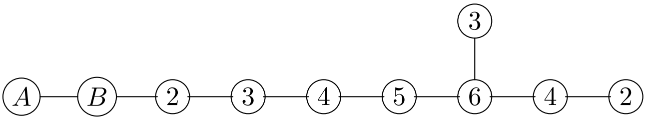
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{III}^* - 1$: $H_{\mathrm{II}}^{\mathrm{Mat}}$



$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{II}^* - 1$: $H_{\mathrm{I}}^{\mathrm{Mat}}$



$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Akane NAKAMURA
Dept. of Mathematics,
Josai University,
1-1 Keyakidai,
Sakado,
Saitama 350-0295 (Japan)
a-naka@josai.ac.jp