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STABILITY OF SOLUTIONS TO COMPLEX MONGE–AMPÈRE FLOWS

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Dedicated to Jean-Pierre Demailly on the occasion of his 60th birthday

ABSTRACT. — We establish a stability result for elliptic and parabolic complex Monge–Ampère equations on compact Kähler manifolds, which applies in particular to the Kähler–Ricci flow.

RÉSUMÉ. — Nous établissons un résultat de stabilité pour les équations de Monge–Ampère complexes elliptiques et paraboliques sur les variétés Kähleriennes compactes, qui s'appliquent en particulier au flot de Kähler–Ricci.

1. Introduction

Several important problems of Kähler geometry (e.g. finding canonical metrics) necessitate to study the existence and regularity of solutions to certain degenerate complex Monge–Ampère equations. The fundamental work of Yau [18] guarantees the existence of smooth solutions to a large class of equations. It has been generalized in various interesting directions, providing weak solutions to several degenerate situations. We refer the reader to [13] for a recent overview.

In this paper we are interested in the stability properties of solutions to such equations. Let X be a compact Kähler manifold of dimension n, θ be a Kähler form and dV a volume form on X. Fix $0 \leqslant f$ a positive density on X and let φ be a θ-plurisubharmonic function solving

$$(1.1) \qquad \mathrm{MA}_\theta(\varphi) = e^\varphi f dV,$$

where $\mathrm{MA}_\theta(\varphi) = (\theta + \mathrm{dd}^c\varphi)^n$ denotes the complex Monge–Ampère measure with respect to the form θ. Throughout this note we always assume that the densities are normalized by $\int_X f\mathrm{d}V = 1$.

If $(f_j\mathrm{d}V)$ is a sequence of Borel measures converging in total variation to $f\mathrm{d}V$ then, it follows from [12] that the corresponding sequence of solutions (φ_j) converges in $L^1(X)$ to φ. Our aim is to establish a quantitative version of this convergence.

Our first main result gives a satisfactory answer for L^p densities with respect to Lebesgue measure, $p > 1$ (see [3, 12, 16] for related results).

THEOREM 1.1. — *Fix $p > 1$ and $0 \leqslant f, g \in L^p(X, \mathrm{d}V)$. If φ, ψ are bounded θ-plurisubharmonic functions on X such that*

$$\mathrm{MA}_\theta(\varphi) = e^\varphi f\mathrm{d}V \; ; \; \mathrm{MA}_\theta(\psi) = e^\psi g\mathrm{d}V,$$

then

$$\|\varphi - \psi\|_\infty \leqslant C\|f - g\|_p^{1/n}.$$

where $C > 0$ depends on $p, n, X, \theta, \mathrm{d}V$ and uniform bounds on $\|f\|_p, \|g\|_p$.

We use some ideas of the proof of 1.1 to establish a stability result for parabolic complex Monge–Ampère flows. We consider the equation

(1.2) $$(\omega_t + \mathrm{dd}^c\varphi_t)^n = e^{\dot\varphi + F(t,\cdot,\varphi)} f\mathrm{d}V,$$

in $X_T := \,]0, T[\, \times X$, where $0 < T < +\infty$ and

- $(\omega_t)_{t\in[0,T]}$ is a smooth family of Kähler forms on X;
- there exists a fixed Kähler form θ such that $\theta \leqslant \omega_t$ for all t;
- $F = F(t, x, r) : \hat{X}_T := [0, T] \times X \times \mathbb{R} \to \mathbb{R}$ is smooth, non-decreasing in the last variable, uniformly Lipschitz in the first and the last variables, i.e. $\exists\, L > 0$ s.t. $\forall\, ((t_1, t_2), x, (r_1, r_2)) \in [0, T]^2 \times X \times \mathbb{R}^2$,

(1.3) $$|F(t_1, x, r_1) - F(t_2, x, r_2)| \leqslant L(|r_1 - r_2| + |t_1 - t_2|),$$

- $0 < f \in \mathcal{C}^\infty(X, \mathbb{R})$.

Our second main result is the following:

THEOREM 1.2. — *Fix $F, G : \hat{X}_T := [0, T] \times X \times \mathbb{R} \to \mathbb{R}$ satisfying the above conditions and fix $p > 1$. Assume that $\varphi : [0, T] \times X \longrightarrow \mathbb{R}$ is a smooth solution to the parabolic equation (1.2) with data (F, f) and $\psi : [0, T[\, \times X \longrightarrow \mathbb{R}$ is a smooth solution to (1.2) with data (G, g). Then*

$$\sup_{X_T} |\varphi - \psi| \leqslant \sup_X |\varphi_0 - \psi_0| + T \sup_{\hat{X}_T} |F - G| + A\, \|g - f\|_p^{1/n},$$

where $A > 0$ is a constant depending on X, θ, n, p, L, a uniform bound on $\varphi_0, \psi_0, \dot\varphi_0, \dot\psi_0$ on X and a uniform bound on $\|f\|_p$ and $\|g\|_p$.

Dédicace. C'est un plaisir de contribuer à ce volume en l'honneur de Jean-Pierre Demailly, dont nous apprécions l'exigence et la générosité. Ses notes de cours [7, 8] ont fortement contribué au développement de la théorie du pluripotentiel en France, nous lui en sommes très reconnaissants.

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2. Preliminaries

2.1. Elliptic complex Monge–Ampère equations

Let (X, θ) be a compact Kähler manifold of dimension n.

A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called *quasi-plurisubharmonic* (quasi-psh for short) on X if it can locally be written $u = \rho + \varphi$, where φ is a plurisubharmonic function and ρ is a smooth function.

A function u is called *θ-plurisubharmonic* (θ-psh for short) if it is quasi-psh on X and $\theta + \mathrm{dd}^c u \geqslant 0$ in the weak sense of currents on X. The set of all θ-psh functions on X is denoted by $\mathrm{PSH}(X, \theta)$.

It follows from the seminal work of Bedford and Taylor [1, 2] that the Monge–Ampère measure $(\theta + \mathrm{dd}^c u)^n =: \mathrm{MA}_\theta(u)$ is well-defined when u is a bounded θ-psh function, and it satisfies several continuity properties.

We refer the readers to [13] for a detailed exposition of global pluripotential theory. In this note we will need the following *comparison principle*.

PROPOSITION 2.1. — *Assume that $u, v \in \mathrm{PSH}(X, \theta) \cap L^\infty(X)$ are such that $e^{-u} \mathrm{MA}_\theta(u) \geqslant e^{-v} \mathrm{MA}_\theta(v)$ on X. Then $u \leqslant v$ on X.*

This result is well known (a proof can be found in [6, Lemma 2.5]). We shall also need the following version of the *domination principle* :

PROPOSITION 2.2. — *Fix a non-empty open subset $D \subset X$ and let u, v be bounded θ-psh functions on X such that for all $\zeta \in \partial D$,*

$$\limsup_{D \ni z \to \zeta} (u - v)(z) \geqslant 0.$$

If $\mathrm{MA}_\theta(u)(\{u < v\} \cap D) = 0$ then $u \geqslant v$ in D.

Proof. — Adding a large constant to both u and v, we can assume that $v \geqslant 0$. For each $\varepsilon > 0$ consider $v_\varepsilon := (1 - \varepsilon)v$. Then $v_\varepsilon \in \mathrm{PSH}(X, \theta)$ and

$v_\varepsilon \leqslant v$, hence $\limsup_{D \ni z \to \partial D} (u(z) - v_\varepsilon(z)) \geqslant 0$. The comparison principle (see [5]) yields

$$\int_{\{u < v_\varepsilon\} \cap D} \theta_{v_\varepsilon}^n \leqslant \int_{\{u < v_\varepsilon\} \cap D} \theta_u^n \leqslant \int_{\{u < v\} \cap D} \theta_u^n = 0.$$

Since $\theta_{v_\varepsilon}^n \geqslant \varepsilon^n \theta^n$, we deduce that $u \geqslant v_\varepsilon$ almost everywhere (with respect to Lebesgue measure) in the open set D, hence everywhere in D. The result follows by letting $\varepsilon \to 0$. $\qquad\square$

2.2. Complex Monge–Ampère flows

We recall the following definition of (sub/super)solution which will be used in this note.

DEFINITION 2.3. — *Let* $\varphi : [0, T[\times X \longrightarrow \mathbb{R}$ *be a function satisfying:*

- *φ is continuous in $X_T := [0, T] \times X$,*
- *for any $x \in X$, the function $\varphi(\cdot, x)$ is $\mathcal{C}^1$ in $[0, T]$ and $\dot{\varphi} = \partial_t \varphi$ its partial derivative in t is continuous in $[0, T] \times X$;*
- *for any $t \in [0, T]$ the function φ_t is bounded and ω_t-plurisubharmonic.*

We say that the function φ is a

- *solution to the equation (1.2) with data (F, f) if for any $t \in]0, T[$,*

$$(\omega_t + dd^c \varphi_t)^n = e^{\dot{\varphi}(t, \cdot) + F(t, \cdot, \varphi_t)} f dV.$$

- *subsolution to the equation (1.2) with data (F, f) if for any $t \in]0, T[$,*

$$(\omega_t + dd^c \varphi_t)^n \geqslant e^{\dot{\varphi}(t, \cdot) + F(t, \cdot, \varphi_t)} f dV.$$

- *supersolution to (1.2) with data (F, f) if for any $t \in]0, T[$,*

$$(\omega_t + dd^c \varphi_t)^n \leqslant e^{\dot{\varphi}(t, \cdot) + F(t, \cdot, \varphi_t)} f dV.$$

All the above inequalities have to be understood in the weak sense of currents: the LHS is a well defined Borel measure since φ_t is a bounded ω_t-psh function (see [1]), while the RHS is a well defined measure which is absolutely continuous with respect to Lebesgue measure.

3. Stability in the elliptic case

3.1. A general semi-stability result

Theorem 1.1 is a consequence of the following more general result which generalizes stability results where the right-hand side does not depend on the unknown function (see [9, 16]).

THEOREM 3.1. — Fix $p > 1$. Assume that $0 \leqslant f, g \in L^p(X, \mathrm{d}V)$ and φ, ψ are bounded θ-plurisubharmonic functions on X such that

$$\mathrm{MA}_\theta(\varphi) \geqslant e^\varphi f \mathrm{d}V \quad and \quad \mathrm{MA}_\theta(\psi) \leqslant e^\psi g \mathrm{d}V.$$

Then there is a constant $C > 0$ depending on p, n, X, θ and a uniform bound on $\log \|f\|_p$ and $\log \|g\|_p$ such that

$$\varphi \leqslant \psi + (C + 2\mathrm{osc}_X \varphi + 2\mathrm{osc}_X \psi) \exp\left(\frac{\mathrm{osc}_X \varphi}{n}\right) \|(g - f)_+\|_p^{1/n}.$$

Proof. — We use a perturbation argument inspired by an idea of Kołodziej [14] (see also [17, proof of Theorem 3.11]) who considered the local case.

For simplicity we normalize θ and $\mathrm{d}V$ so that $\int_X \mathrm{d}V = \mathrm{Vol}(\theta) = 1$, and we denote by $\|f\|_p$ the L^p-norm of f with respect to the volume form $\mathrm{d}V$. We assume that $\|f\|_p, \|g\|_p$ are uniformly bounded away from zero and infinity (i.e. $\log \|f\|_p, \log \|g\|_p$ are uniformly bounded).

If $\|(g - f)_+\|_p = 0$ then $g \leqslant f$ almost everywhere in X. In this case, φ is a subsolution and ψ is a supersolution to the same complex Monge–Ampère equation. Then the comparison principle (Proposition 2.1) yields $\varphi \leqslant \psi$ in X, which proves the result.

We assume in the sequel that $\|(g - f)_+\|_p > 0$. Integrating the inequality $\mathrm{MA}_\theta(\varphi) \geqslant e^\varphi f \mathrm{d}V$, we see that

$$\inf_X \varphi \leqslant -\log\left(\int_X f \mathrm{d}V\right) = 0$$

hence $\sup_X \varphi \leqslant \mathrm{osc}_X \varphi$. Similarly $\sup_X \psi \geqslant 0$, hence $-\inf_X \psi \leqslant \mathrm{osc}_X \psi$.

Set $\varepsilon := e^{\sup_X \varphi/n} \|(g - f)_+\|_p^{1/n}$. If $\varepsilon \geqslant 1/2$ then

$$\sup_X(\varphi - \psi) \leqslant \sup_X \varphi - \inf_X \psi$$

$$\leqslant 2\left(\sup_X \varphi - \inf_X \psi\right) \exp\left(\frac{\sup_X \varphi}{n}\right) \|(g - f)_+\|_p^{1/n},$$

as desired.

So we can assume that $\varepsilon < 1/2$. Hölder inequality yields

$$\int_X \frac{(g-f)_+}{\|(g-f)_+\|_p} dV \leqslant 1.$$

It follows from [15] that there exists a bounded θ-psh function ρ such that

$$\mathrm{MA}_\theta(\rho) = hdV := \left(a + \frac{(g-f)_+}{\|(g-f)_+\|_p} \right) dV, \ \sup_X \rho = 0,$$

where $a \geqslant 0$ is a normalization constant to insure that $\int_X hdV = \int_X dV$. Since $\|h\|_p \leqslant 2$ the uniform estimate of Kołodziej [15] guarantees that

$$-C_1 \leqslant \rho \leqslant 0,$$

where C_1 only depends on p, θ, n.

Observe that one can use here soft techniques and avoid the use of Yau's Theorem (see [10, 13]). Consider now

$$\varphi_\varepsilon := (1-\varepsilon)\varphi + \varepsilon\rho - C_2\varepsilon + n\log(1-\varepsilon),$$

where C_2 is a positive constant to be specified hereafter, and $\varepsilon < 1/2$ is defined as above. Then φ_ε is a bounded θ-psh function, and a direct computation shows that

$$\mathrm{MA}_\theta(\varphi_\varepsilon) \geqslant (1-\varepsilon)^n \mathrm{MA}_\theta(\varphi) + \varepsilon^n \mathrm{MA}_\theta(\rho)$$
$$\geqslant e^{\varphi + n\log(1-\varepsilon)} fdV + e^\varphi (g-f)_+ dV.$$

We choose $C_2 = -\inf_X \varphi$ so $\rho - \varphi \leqslant C_2$ and $\varphi_\varepsilon \leqslant \varphi + n\log(1-\varepsilon) \leqslant \varphi$. Thus

$$\mathrm{MA}_\theta(\varphi_\varepsilon) \geqslant e^{\varphi_\varepsilon}(f + (g-f)_+)dV \geqslant e^{\varphi_\varepsilon} gdV.$$

In other words, φ_ε is a subsolution and ψ is a supersolution to the equation $\mathrm{MA}_\theta(\phi) = e^\phi gdV$. The comparison principle (Proposition 2.1) insures that $\varphi_\varepsilon \leqslant \psi$, hence

$$\varphi - \psi = \varphi_\varepsilon - \psi + \varepsilon(\varphi - \rho) + C_2\varepsilon - n\log(1-\varepsilon)$$
$$\leqslant (C_1 - \inf_X \varphi + \sup_X \varphi + 2n)\varepsilon$$
$$= (C_1 + \mathrm{osc}_X \varphi + 2n) \exp\left(\frac{\sup_X \varphi}{n} \right) \|(g-f)_+\|_p^{1/n}.$$

Since C_1 only depends on p, θ, n, the result follows. □

3.2. The proof of Theorem 1.1

Without loss of generality we normalize f, g and θ so that $\int_X f \mathrm{d}V = \int_X g \mathrm{d}V = \int_X \theta^n = \int_X \mathrm{d}V$. Let ϕ be the unique bounded θ-plurisubharmonic function on X normalized by $\sup_X \phi = 0$ such that $\mathrm{MA}_\theta(\phi) = f \mathrm{d}V$. The existence of ϕ follows from Kołodziej's celebrated work [15] and moreover, $-C \leqslant \phi \leqslant 0$, where C is a uniform constant depending on $X, \theta, p, \|f\|_p$. Then ϕ is a subsolution while $\phi + C$ is a supersolution to the Monge–Ampère equation

$$\mathrm{MA}_\theta(u) = e^u f \mathrm{d}V.$$

It thus follows from the comparison principle (Proposition 2.1) that

$$\phi \leqslant \varphi \leqslant \phi + C.$$

We thus obtain a uniform bound for φ. The same arguments give a uniform bound for ψ as well. 1.1 follows therefore from Theorem 3.1.

Remark 3.2. — In 1.1, one can replace the L^p-norm of $f - g$ by the L^1-norm, at the cost of decreasing the exponent $1/n$ to $1/(n + \varepsilon)$ ($\varepsilon > 0$ arbitrarily small).

Namely under the assumptions of 1.1, for any $\varepsilon > 0$, there exists a constant $C > 0$ which depends on $p, n, X, \theta, \varepsilon$ and a uniform bound on $\|f\|_p$ and $\|g\|_p$ such that

$$\|\varphi - \psi\|_\infty \leqslant C \|f - g\|_1^{1/(n+\varepsilon)}.$$

Indeed repeating the proof with an exponent $1 < r < p$ close to 1, we get a bound in terms of $\|f - g\|_r^{1/n}$. Then observe that if we write $r = (1-t) + tp$, by Hölder-interpolation inequality applied to $h := |f - g| \in L^p(X)$, we have

$$\int_X h^r \mathrm{d}V \leqslant \left(\int_X h \mathrm{d}V \right)^{1-t} \left(\int_X h^p \right)^t,$$

which implies that

$$\|h\|_r \leqslant \|h\|_1^{(1-t)/r} \|h\|_p^{tp/r}.$$

Since $t = (r-1)/(p-1)$ and $1 - t = (p-r)/(p-1)$, we see that the exponent $(1-t)/r = (p-r)/r(p-1)$ is arbitrary close to 1 as $r \to 1$.

4. Stability in the parabolic case

4.1. A parabolic comparison principle

We establish in this section a maximum principle which is classical when the data are smooth. It has been obtained for continuous data in [11], we propose here a different approach which applies to our present setting:

THEOREM 4.1. — *Fix $\varphi : [0, T[\times X \longrightarrow \mathbb{R}$ a subsolution, $\psi : [0, T[\times X \longrightarrow \mathbb{R}$ a supersolution to the parabolic equation (1.2), where $0 \leqslant f \in L^p(X)$ with $p > 1$. Then*

$$\sup_{X_T}(\varphi - \psi) \leqslant \sup_X (\varphi_0 - \psi_0)_+.$$

Recall that $(\varphi_0 - \psi_0)_+ := \sup\{\varphi_0 - \psi_0, 0\}$.

Proof. — Fix $T' < T$, $0 < \varepsilon$. We first assume that $M_0 := \sup_X(\varphi_0 - \psi_0) \leqslant 0$ and we are going prove that $\varphi \leqslant \psi + 2\varepsilon t$ in $X_{T'}$.

Consider $w(t, x) := \varphi(t, x) - \psi(t, x) - 2\varepsilon t$. This function is upper semi-continuous on the compact space $[0, T'] \times X$. Hence w attains a maximum at some point $(t_0, x_0) \in [0, T'] \times X$. We claim that $w(t_0, x_0) \leqslant 0$. Assume by contradiction that $w(t_0, x_0) > 0$, in particular $t_0 > 0$, and set

$$K := \{x \in X; w(t_0, x) = w(t_0, x_0)\}.$$

The classical maximum principle insures that for all $x \in K$,

$$\partial_t \varphi(t_0, x) \geqslant \partial_t \psi(t_0, x) + 2\varepsilon.$$

By continuity of the partial derivatives in (t, x), we can find an open neighborhood D of K such that for all $x \in D$

$$\partial_t \varphi(t_0, x) > \partial_t \psi(t_0, x) + \varepsilon.$$

Set $u := \varphi(t_0, \cdot)$ and $v := \psi(t_0, \cdot)$. Since φ is a subsolution and ψ is a supersolution to (1.2) we infer

$$(\omega_{t_0} + dd^c u)^n \geqslant e^{F(t_0, x, u(x)) - F(t_0, x, v(x)) + \varepsilon}(\omega_{t_0} + dd^c v)^n,$$

in the weak sense of measures in D. Recall that

- u and v are continuous on D,
- F is non-decreasing in r,
- $u(x) > v(x) + \varepsilon t_0$ for any $x \in K$.

Shrinking D if necessary, we can assume that the latter inequality is true in D. We thus get

$$(\omega_{t_0} + \mathrm{dd}^c u)^n \geqslant e^\varepsilon (\omega_{t_0} + \mathrm{dd}^c v)^n.$$

From this we see in particular that $D \neq X$, hence $\partial D \neq \emptyset$.

Consider now $\tilde{u} := u + \min_{\partial D}(v - u)$. Since $v \geqslant \tilde{u}$ on ∂D, Proposition 2.2 yields

$$\int_{\{v < \tilde{u}\} \cap D} e^\varepsilon (\omega_{t_0} + \mathrm{dd}^c v)^n \leqslant \int_{\{v < \tilde{u}\} \cap D} (\omega_{t_0} + \mathrm{dd}^c u)^n$$

$$\leqslant \int_{\{v < \tilde{u}\} \cap D} (\omega_{t_0} + \mathrm{dd}^c v)^n.$$

It then follows that $\tilde{u} \leqslant v$, almost everywhere in D with respect to the measure $(\omega_{t_0} + \mathrm{dd}^c v)^n$, hence everywhere in D by the domination principle (see Proposition 2.2). In particular for all $x \in D$,

$$(4.1) \qquad u(x) - v(x) + \min_{\partial D}(v - u) = \tilde{u}(x) - v(x) \leqslant 0,$$

Since $K \cap \partial D = \emptyset$, we infer $w(t_0, x) < w(t_0, x_0)$, for all $x \in \partial D$, i.e.

$$u(x) - v(x) < u(x_0) - v(x_0) \text{ for all } x \in \partial D,$$

contradicting (4.1). Altogether this shows that $t_0 = 0$, thus $\varphi \leqslant \psi + 2\varepsilon t$ in $X_{T'}$. Letting $\varepsilon \to 0$ and $T' \to T$ we obtain that $\varphi \leqslant \psi$ in X_T.

We finally get rid of the assumption $\varphi_0 \leqslant \psi_0$. If $M_0 := \sup_X(\varphi_0 - \psi_0) > 0$ then $\varphi - M_0$ is a subsolution of the same equation since F is non decreasing in the last variable. Hence $\varphi - M_0 \leqslant \psi$ in X_T. This proves the required inequality. $\qquad \square$

Remark 4.2. — We note for later works that the above proof only requires $t \mapsto \varphi(t, \cdot), \psi(t, \cdot)$ to be $\mathcal{C}^1$ in $]0, T[\times X$. This should be useful in analyzing the smoothing properties of complex Monge–Ampère flows at time zero.

4.2. A parabolic semi-stability theorem

We now establish a technical comparison principle which is a key step in the proof of 1.2. The proof of this result does not require the smoothness assumption on F, G, f, g. We assume in this subsection that

- $F, G : \hat{X}_T := [0, T[\times X \times \mathbb{R} \to \mathbb{R}$ are continuous;
- F, G are non decreasing in the last variable;
- F, G satisfy condition (1.3) with the same constant $L > 0$.

- $0 \leqslant f, g \in L^p(X)$ with $p > 1$.

THEOREM 4.3. — *Assume that $\varphi : [0, T[\times X \longrightarrow \mathbb{R}$ is a subsolution to the parabolic equation (1.2) with data (F, f) and $\psi : [0, T[\times X \longrightarrow \mathbb{R}$ is a supersolution to the parabolic equation (1.2) with data (G, g). Then*

$$\sup_{X_T}(\varphi - \psi) \leqslant \sup_X(\varphi_0 - \psi_0)_+ + T \sup_{\hat{X}_T}(G - F)_+ + A \|(g - f)_+\|_p^{1/n},$$

where $A > 0$ depends on X, θ, n, p and a uniform bound on $\dot{\varphi}$, φ, ψ, and $\sup_{X_T} G(t, x, \sup_{X_T} \varphi)$.

Remark 4.4. — In the second term of the estimate above one can replace $\sup_{\hat{X}_T}(G - F)_+$ by

$$\sup_{[0, T[\times X \times I}(G - F)_+,$$

where $I = [\inf_{X_T} \varphi, \sup_{X_T} \varphi]$ is a compact interval in $\mathbb{R}$.

Proof. — We first assume that $\|(g - f)_+\|_p > 0$. Since $\int_X \mathrm{d}V = \int_X \theta^n = 1$, it follows from [15] that there exists $\rho \in \mathrm{PSH}(X, \theta) \cap \mathcal{C}^0(X)$ such that

$$(4.2) \qquad (\theta + \mathrm{dd}^c \rho)^n = \left(a + \frac{(g - f)_+}{\|(g - f)_+\|_p}\right) \mathrm{d}V$$

normalized by $\max_X \rho = 0$, where $a \geqslant 0$ is a normalizing constant given by

$$a := 1 - \frac{\|(g - f)_+\|_1}{\|(g - f)_+\|_p} \in [0, 1].$$

We moreover have a uniform bound on ρ which only depends on the L^p norm of the density of $(\theta + \mathrm{dd}^c \rho)^n$ which is here bounded from above by 2,

$$(4.3) \qquad \|\rho\|_\infty \leqslant C_0(a + 1) \leqslant 2C_0,$$

where $C_0 > 0$ is a uniform constant depending only on (X, θ, p).

Fix $B, M > 0$. For $0 < \delta < 1$ and $(t, x) \in X_T$ we set

$$\varphi_\delta(t, x) := (1 - \delta)\varphi(t, x) + \delta\rho + n \log(1 - \delta) - B\delta t - Mt.$$

The plan is to choose $B, M > 0$ in such a way that φ_δ be a subsolution for the parabolic equation (1.2) with data (G, g). The conclusion will then follow from the comparison principle (Theorem 4.1).

Observe that for $t \in]0, T[$ fixed, $\varphi_\delta(t, \cdot)$ is ω_t-plurisubharmonic in X and

$$(\omega_t + \mathrm{dd}^c \varphi_\delta(t, \cdot))^n \geqslant (1 - \delta)^n(\omega_t + \mathrm{dd}^c \varphi_t)^n + \delta^n(\theta + \mathrm{dd}^c \rho)^n.$$

Using that φ is a subsolution to (1.2) with density f, we infer

$$(4.4) \ (\omega_t + \mathrm{dd}^c \varphi_\delta(t, \cdot))^n \geqslant e^{\dot{\varphi} + F(t, \cdot, \varphi) + n \log(1 - \delta)} f \mathrm{d}V + \delta^n \frac{(g - f)_+}{\|(g - f)_+\|_p} \mathrm{d}V.$$

Set $m_0 := \inf_{X_T} \varphi$, $m_1 := \inf_{X_T} \dot{\varphi}$ and choose $M := \sup_{\hat{X}_T}(G - F)_+$. Noting that $\varphi \geqslant \varphi_\delta + \delta\varphi$ and recalling that G is non decreasing in the last variable, we obtain

$$\dot{\varphi}(t, x) + F(t, x, \varphi(t, x)) + n \log(1 - \delta)$$
$$\geqslant \dot{\varphi}_\delta(t, x) + \delta\dot{\varphi}(t, x) + G(t, x, \varphi(t, x)) - M + n\log(1 - \delta) + B\delta + M$$
$$\geqslant \dot{\varphi}_\delta(t, x) + \delta\dot{\varphi}(t, x) + G(t, x, \varphi_\delta(t, x) + \delta\varphi(t, x)) + n\log(1 - \delta) + B\delta$$
$$\geqslant \dot{\varphi}_\delta(t, x) + \delta m_1 + G(t, x, \varphi_\delta(t, x) + \delta m_0) + n\log(1 - \delta) + B\delta.$$

The Lipschitz condition (1.3) yields

$$\dot{\varphi}(t, x) + F(t, x, \varphi(t, x)) + n\log(1 - \delta)$$
$$\geqslant \dot{\varphi}_\delta(t, x) + G(t, x, \varphi_\delta(t, x)) + B\delta - L\delta m_0 + \delta m_1 + n\log(1 - \delta).$$

Using the elementary inequality $\log(1 - \delta) \geqslant -2(\log 2)\delta$ for $0 < \delta \leqslant 1/2$, it follows that for $0 < \delta \leqslant 1/2$,

$$B\delta - L\delta m_0 + \delta m_1 + n\log(1 - \delta) \geqslant (B - Lm_0 + m_1 - 2n\log 2)\delta.$$

We now choose $B := Lm_0 - m_1 + 2n\log 2$ so that

$$\dot{\varphi}(t, x) + F(t, x, \varphi(t, x)) + n\log(1 - \delta) \geqslant \dot{\varphi}_\delta(t, x) + G(t, x, \varphi_\delta(t, x)),$$

which, together with (4.4), yields

$$(4.5) \quad (\omega_t + dd^c\varphi_\delta(t, \cdot))^n \geqslant e^{\dot{\varphi}_\delta(t, \cdot) + G(t, \cdot, \varphi_\delta(t, \cdot))} f dV + \delta^n \frac{(g - f)_+}{\|(g - f)_+\|_p}.$$

On the other hand, if we set

$$M_1 := \sup_{X_T} \dot{\varphi}, \ M_0 := \sup_{X_T} \varphi \text{ and } N := \sup_{X_T} G(t, x, M_0),$$

then the properties of G insure

$$\dot{\varphi}_\delta(t, x) + G(t, x, \varphi_\delta(t, x))$$
$$\leqslant (1 - \delta)\sup_{X_T} \dot{\varphi} + \sup_{X_T} G(t, x, (1 - \delta)\varphi(t, x))$$
$$\leqslant (1 - \delta)M_1 + \sup_{X_T} G(t, x, (1 - \delta)M_0)$$
$$\leqslant (1 - \delta)M_1 + M_0 L\delta + N \leqslant N + \max\{LM_0, M_1\}$$

Using (4.5) we conclude that for $0 < \delta < 1/2$,

$$(4.6) \quad (\omega_t + dd^c\varphi_\delta(t, \cdot))^n$$
$$\geqslant e^{\dot{\varphi}_\delta(t, \cdot) + G(t, \cdot, \varphi_\delta(t, \cdot))} \left(f + \delta^n e^{-M_2} \frac{(g - f)_+}{\|(g - f)_+\|_p} \right) dV,$$

where $M_2 := N + \max\{LM_0, M_1\}$.

To conclude that φ_δ is a subsolution, we finally set

$$\delta := \|(g-f)_+\|_p^{1/n} e^{M_2/n}. \tag{4.7}$$

Assume first that $\|(g-f)_+\|_p \leqslant 2^{-n} e^{-M_2}$ so that $\delta \leqslant 1/2$. It follows from (4.6) that

$$(\omega_t + dd^c\varphi_\delta(t,\cdot))^n \geqslant e^{\dot\varphi_\delta(t,\cdot)+G(t,x,\varphi_\delta(t,\cdot))}(f+(g-f)_+)dV$$

$$\geqslant e^{\dot\varphi_\delta(t,\cdot)+G(t,x,\varphi_\delta(t,\cdot))}g dV,$$

hence φ_δ is a subsolution to (1.2) for the data (G,g) in D. The comparison principle (Theorem 4.1) insures that for all $(t,x) \in X_T$,

$$\varphi_\delta(t,x) - \psi(t,x) \leqslant \max_X(\varphi_\delta(0,\cdot) - \psi(0,\cdot)_+).$$

Taking into account the estimates (4.3) and (4.7), we get

$$\varphi(t,x) - \psi(t,x) \leqslant \max_X(\varphi(0,\cdot) - \psi(0,\cdot)_+ + TM + A_1\|(g-f)_+\|_p^{1/n},$$

where

$$A_1 := (M_0 + 2C_0 + 2n\log 2 + BT)e^{M_2/n}.$$

When $\|(g-f)_+\|_p > 2^{-n} e^{-M_2}$, we can choose a constant $A_2 > 0$ so that

$$\varphi(t,x) - \psi(t,x) \leqslant \max_X(\varphi_0 - \psi_0)_+ + A_2 2^{-n} e^{-M_2}.$$

We eventually take $A = \max\{A_1, A_2\}$.

Assume finally that $\|(g-f)_+\|_p = 0$ which means that $g \leqslant f$ almost everywhere in X. In this case we solve the equation (4.2) with the right hand side equal to dV and repeat the same arguments with an arbitrary $\delta > 0$. The conclusion follows by letting $\delta \to 0$. $\qquad\square$

4.3. Proof of Theorem 1.2

The proof of Theorem 1.2 goes by symmetrizing the roles of φ and ψ, and establishing uniform bounds on $\varphi, \psi, \dot\varphi, \dot\psi$ depending on uniform bounds for $\varphi_0, \dot\varphi_0, \psi_0, \dot\psi_0$ and $\|f\|_p, \|g\|_p$.

4.3.1. Bounds on φ

By assumption, we can fix $a, A > 0$ such that

$$a\theta \leqslant \omega_t \leqslant A\theta, \ \forall \ (t,x) \in X_T.$$

Let ρ be the unique bounded normalized θ-psh function $\sup_X \rho = 0$ such that $MA_\theta(\rho) = C_0 f dV$, where $C_0 > 0$ is a uniform normalization constant.

Then ρ is bounded by a constant depending only on the L^p norm of f and on θ, p. We set

$$C_1 := \sup_X (a\rho - \varphi_0) \text{ and } C_2 := \sup_X (\varphi_0 - A\rho).$$

We next introduce the following uniform constants

$$C_3 := \sup_{(t,x)\in[0,T]\times X} F(t, x, \varphi_0(t,x)) \text{ and } C_4 := \inf_{(t,x)\in[0,T]\times X} F(t, x, \varphi_0(t,x)).$$

A direct computation shows that the function defined on X_T by

$$u(t, x) := a\rho - C_1 - \max(C_3 - n \log a - \log C_0, 0)t$$

is a subsolution to the parabolic equation (1.2) with data (f, F). Indeed, for fixed $t \in \,]0, T[$, the Monge–Ampère measure of u_t can be estimated as

$$(\omega_t + \mathrm{dd}^c u_t)^n \geqslant a^n (\theta + \mathrm{dd}^c \rho)^n = a^n C_0 f \mathrm{d}V.$$

Since F is non-decreasing in the last variable and $\varphi_0 \geqslant u_t$ we obtain $C_3 \geqslant F(t, x, \varphi_0) \geqslant F(t, x, u_t)$. Thus

$$e^{\dot{u}_t + F(t,\cdot,u_t)} f \mathrm{d}V \leqslant e^{-\max(C_3 - n \log a - \log C_0, 0) + C_3} f \mathrm{d}V$$
$$\leqslant a^n C_0 f \mathrm{d}V \leqslant (\omega_t + \mathrm{dd}^c u_t)^n.$$

A similar computation shows that the function defined on X_T by

$$v(t, x) := A\rho + C_2 + \max(-C_4 + n \log A + \log C_0, 0)t$$

is a supersolution to the parabolic equation

$$(4.8) \qquad (A\theta + \mathrm{dd}^c v_t)^n = e^{\dot{v}_t + F(t,x,v_t)} f \mathrm{d}V.$$

Since $A\theta \geqslant \omega_t$ we see that φ_t is a subsolution to the equation (4.8). The parabolic comparison principle (Theorem 4.1) therefore yields

$$u \leqslant \varphi \leqslant v.,$$

in X_T.

4.3.2. Bounds on $\dot{\varphi}$

We now provide a uniform bound on $\dot{\varphi}_t$, assuming all data are smooth. We only outline the proof since the arguments are classical. The previous subsection has provided a uniform bound

$$-B_0 \leqslant \varphi(t, x) \leqslant B_0, \ \forall \, (t, x) \in [0, T[\times X.$$

Since ω_t is smooth in t and ω_t is uniformly Kähler we can fix a positive constant B_1 such that

$$-B_1 \omega_t \leqslant \dot{\omega}_t \leqslant B_1 \omega_t.$$

Up to enlarging B_1 we can further assume that

$$-B_1 \leqslant \partial_r F(t, x, r) \leqslant B_1, \ \forall \, (t, x, r) \in [0, T[\times X \times [-B_0, B_0].$$

Set

$$\Delta_t(h) = \Delta_{\omega_t + \mathrm{dd}^c \varphi_t}(h) = n \frac{\mathrm{dd}^c h \wedge (\omega_t + \mathrm{dd}^c \varphi_t)^{n-1}}{(\omega_t + \mathrm{dd}^c \varphi_t)^n},$$

and

$$\mathrm{Tr}_t(\eta) := n \frac{\eta \wedge (\omega_t + \mathrm{dd}^c \varphi_t)^{n-1}}{(\omega_t + \mathrm{dd}^c \varphi_t)^n}.$$

A straightforward computation yields

$$\ddot{\varphi}_t = \Delta_t(\dot{\varphi}_t) + \mathrm{Tr}_t(\dot{\omega}_t) - \partial_t F(t, x, \varphi_t) - \dot{\varphi}_t \partial_r F(t, x, \varphi_t)$$
$$\leqslant \Delta_t(\dot{\varphi}_t) - B_1 \Delta_t(\varphi_t) + C - \dot{\varphi}_t \partial_r F(t, x, \varphi_t),$$

using that φ_t, hence $\partial_t F(t, x, \varphi_t)$, is uniformly bounded on X_T, and that

$$\mathrm{Tr}_t(\dot{\omega}_t) \leqslant B_1 \, \mathrm{Tr}_t(\omega_t) = B_1 n - B_1 \Delta_t(\varphi_t).$$

Consider

$$H(t, x) := \dot{\varphi}_t(x) - B_1 \varphi_t(x) - (C+1)t.$$

It follows from the above computation that

$$\left(\frac{\partial}{\partial t} - \Delta_t \right) H \leqslant -1 - [B_1 + \partial_r F(t, x, \varphi_t)] \dot{\varphi}_t.$$

If H reaches its maximum at time zero, then

$$H \leqslant \sup_X \dot{\varphi}_0 - B_1 \inf_X \varphi_0,$$

hence $\dot{\varphi}_t \leqslant C(T, B_0, B_1) + \sup_X \dot{\varphi}_0$. If H reaches its maximum at (t_0, x_0) with $t_0 > 0$, we obtain at (t_0, x_0),

$$0 \leqslant \left(\frac{\partial}{\partial t} - \Delta_t \right) H \leqslant -1 - [B_1 + \partial_r F(t, x, \varphi_t)] \dot{\varphi}_t,$$

hence $\dot{\varphi}_t(t_0, x_0) \leqslant 0$ since $[B_1 + \partial_r F(t, x, \varphi_t)] \geqslant 0$. Therefore

$$H_{max} = H(t_0, x_0) \leqslant -B_1 \inf_X \varphi_{t_0} \leqslant B_0 B_1,$$

and we obtain an appropriate bound from above for $\dot{\varphi}_t$.

The proof for the lower bound goes along similar lines, considering $G = \dot{\varphi}_t(x) + B_1 \varphi_t(x) + (C'+1)t$.

5. Concluding remarks

5.1. Varying the reference forms

It is certainly interesting to study the stability properties when the reference forms θ, ω_t are varying. For simplicty we address this issue here only in the elliptic case.

THEOREM 5.1. — Fix θ, ω Kähler forms. Fix $p > 1$ and $0 \leqslant f, g \in L^p(X, \mathrm{d}V)$. If φ (resp. ψ) is a bounded θ-plurisubharmonic (resp. ω-plurisubharmonic) function on X such that

$$\mathrm{MA}_\theta(\varphi) = e^\varphi f \mathrm{d}V \text{ and } \mathrm{MA}_\omega(\psi) = e^\psi g \mathrm{d}V,$$

then

$$\|\varphi - \psi\|_\infty \leqslant C \left\{ \|f - g\|_p^{1/n} + d(\omega, \theta) \right\},$$

where $C > 0$ depends on $p, n, X, \omega, \mathrm{d}V$ and uniform bounds on $\|f\|_p, \|g\|_p$.

We use here the following distance on positive forms,

$$d(\omega, \theta) := \inf\{t > 0 \; ; \; e^{-t}\omega \leqslant \theta \leqslant e^t \omega\}.$$

Proof. — Set

$$c = \inf\{t > 0 \; ; \; (1 - t)\omega \leqslant \theta \leqslant (1 + t)\omega\}.$$

Adjusting the constant we can assume that $c \leqslant 1/2$ and $c \simeq d(\omega, \theta)$. Now

$$\psi_c := (1 - c)\psi + n \log(1 - c) + c \inf_X \psi$$

is a θ-psh function whose Monge–Ampère measure can be estimated as

$$\mathrm{MA}_\theta(\psi_c) \geqslant (1 - c)^n e^\psi g \mathrm{d}V \geqslant e^{\psi_c} g \mathrm{d}V.$$

It thus follows from Theorem 3.1 that

$$\psi_c \leqslant \varphi + C\|f - g\|_p^{1/n},$$

for a uniform constant C. Note that the uniform norm of ψ_c is uniformly controlled by $\|\psi\|$ because $c \leqslant 1/2$. From this and the definition of ψ_c we obtain

$$\psi \leqslant \varphi + C'(\|f - g\|_p^n + c),$$

where C' is a uniform constant. Exchanging the roles of φ and ψ yields the conclusion. □

5.2. Big classes

The ideas we have developed so far can also be applied to the more general setting of cohomology classes that are merely big rather than Kähler. We briefly explain the set up for the elliptic stability.

We assume that θ is a smooth form representing a big cohomology class. A θ-psh function φ is no longer bounded on X, but it can have *minimal singularities*. The function

$$V_\theta = \sup\{u \in \mathrm{PSH}(X,\theta) \; ; \; u \leqslant 0\}$$

is an example of θ-psh with minimal singularities. Any other θ-psh function φ with minimal singularities satisfies $\|\varphi - V_\theta\|_{L^\infty(X)} < +\infty$.

The pluripotential theory in big cohomology classes has been developed in [4]. In short, the Bedford–Taylor theory can be developed, replacing X by the *ample locus* of θ, a Zariski open subset in which V_θ is locally bounded.

The a priori estimate of Kołodziej can be extended, as well as 1.1. It suffices indeed to establish the following:

THEOREM 5.2. — *Fix $p > 1$ and assume that $0 \leqslant f, g \in L^p(X, \mathrm{d}V)$, and φ, ψ are θ-psh functions on X with minimal singularities such that*

$$\mathrm{MA}_\theta(\varphi) \geqslant e^\varphi f \mathrm{d}V \; ; \; \mathrm{MA}_\theta(\psi) \leqslant e^\psi g \mathrm{d}V.$$

Then there is a constant $C > 0$ depending on p, n, X, θ such that

$$\varphi \leqslant \psi + \left(C + 2 \sup_X |V_\theta - \varphi| + 2 \sup_X \max(V_\theta - \psi, 0) \right)$$
$$\exp\left(\frac{\sup_X \varphi}{n} \right) \|f - g\|_p^{1/n}.$$

Proof. — Set $\varepsilon := e^{\sup_X \varphi / n} \|f - g\|_p^{1/n}$. If $\varepsilon \geqslant 1/2$ then

$$\sup_X (\varphi - \psi) \leqslant \left(\sup_X |\varphi - V_\theta| + \sup_X \max(V_\theta - \psi, 0) \right)$$
$$\leqslant 2 \left(\sup_X |\varphi - V_\theta| + \sup_X \max(V_\theta - \psi, 0) \right) \exp\left(\frac{\sup_X \varphi}{n} \right) \|f - g\|_p^{1/n},$$

as desired. So we can assume that $\varepsilon < 1/2$. Hölder inequality yields

$$\int_X \frac{|f - g|}{\|f - g\|_p} \mathrm{d}V \leqslant 1.$$

Let ρ be the unique θ-psh function with minimal singularities such that

$$\mathrm{MA}_\theta(\rho) = h\mathrm{d}V := \left(a + \frac{|f - g|}{\|f - g\|_p} \right) \mathrm{d}V, \; \sup_X \rho = 0,$$

where $a \geqslant 0$ is a normalization constant to insure that $\int_X h \mathrm{d}V = \int_X \mathrm{d}V$.

Since $\|h\|_p \leqslant 2$, it follows from [4, Theorem 4.1] that

$$-C_1 \leqslant \rho - V_\theta \leqslant 0,$$

where C_1 only depends on p, θ, n. Consider now

$$\varphi_\varepsilon := (1 - \varepsilon)\varphi + \varepsilon\rho - C_2\varepsilon + n\log(1 - \varepsilon),$$

where C_2 is a positive constant to be specified hereafter. Then φ_ε is a θ-psh function with minimal singularities, and a direct computation shows that

$$\mathrm{MA}_\theta(\varphi_\varepsilon) \geqslant (1 - \varepsilon)^n \mathrm{MA}_\theta(\varphi) + \varepsilon^n \mathrm{MA}_\theta(\rho)$$
$$\geqslant e^{\varphi + n\log(1-\varepsilon)} f \mathrm{d}V + e^\varphi |f - g| \mathrm{d}V.$$

If we choose $C_2 = \sup_X(V_\theta - \varphi)$ then $\rho - \varphi \leqslant C_2$ and $\varphi_\varepsilon \leqslant \varphi + n\log(1-\varepsilon) \leqslant \varphi$. So we can continue the above estimate to arrive at

(5.1) $$\mathrm{MA}_\theta(\varphi_\varepsilon) \geqslant e^{\varphi_\varepsilon}(f + |f - g|)\mathrm{d}V \geqslant e^{\varphi_\varepsilon} g \mathrm{d}V.$$

It follows from (5.1) that φ_ε is a subsolution and ψ is a supersolution of the equation $\mathrm{MA}_\theta(\phi) = e^\phi g \mathrm{d}V$. The comparison principle [4, Proposition 6.3] insures that $\varphi_\varepsilon \leqslant \psi$, hence

$$\varphi - \psi = \varphi_\varepsilon - \psi + \varepsilon(\varphi - \rho) + C_2\varepsilon - n\log(1 - \varepsilon)$$

$$\leqslant \left(C_1 + C_2 + \sup_X(\varphi - V_\theta) + 2n\right)\varepsilon$$

$$= \left(C_1 + \mathrm{osc}_X(\varphi - V_\theta) + 2n\right)\exp\left(\frac{\sup_X \varphi}{n}\right)\|f - g\|_p^{1/n}.$$

The result follows since C_1 only depends on p, θ, n. $\qquad\square$

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