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Tome 67, n ${ }^{\circ} 5$ (2017), p. 2043-2082.
[http://aif.cedram.org/item?id=AIF_2017__67_5_2043_0](http://aif.cedram.org/item?id=AIF_2017__67_5_2043_0)
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# A REDUCTION OF CANONICAL STABILITY INDEX OF 4 AND 5 DIMENSIONAL PROJECTIVE VARIETIES WITH LARGE VOLUME 

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#### Abstract

We study the canonical stability index of nonsingular projective varieties of general type with either large canonical volume or large geometric genus. As applications of a general extension theorem established in the first part, we prove some optimal results in dimensions 4 and 5 , which are parallel to some well-known results on surfaces and 3 -folds.

Résumé. - Nous étudions l'indice de stabilité canonique d'une variété projective lisse de type général avec un grand volume canonique ou un grand genre géométrique. Comme applications d'un théorème général d'extension établi dans la première partie, nous prouvons des résultats optimaux en dimensions 4 et 5 similaires à certains résultats bien connus sur les surfaces et les variétés de dimension 3 .


## 1. Introduction

Understanding pluricanonical systems of nonsingular projective varieties of general type has been one of the major tasks in birational geometry. By the work of Bombieri [2] for surfaces and that of Hacon-McKernan [11], Takayama [24], Tsuji [26] for higher dimensional varieties, for each $n>0$, there exists an optimal constant $r_{n} \in \mathbb{Z}_{>0}$, such that the pluricanonical map $\varphi_{m, X}$ is birational onto its image for all $m \geqslant r_{n}$ and for all nonsingular projective $n$-folds $X$ of general type. Such a constant $r_{n}$ is, in general, nonexplicit except that $r_{1}=3$, that $r_{2}=5$ by Bombieri and that $27 \leqslant r_{3} \leqslant 61$

[^0]by Iano-Fletcher [13] and Chen-Chen [5, Theorem 1.6 (1)]. Usually the number $r_{n}$ is referred to as the $n$-th canonical stability index.

The existence of $r_{n}$ directly implies the existence of another optimal constant $v_{n}>0$ such that the canonical volume $\operatorname{vol}(X) \geqslant v_{n}$ for all nonsingular projective $n$-folds $X$ of general type, where we recall that the canonical volume " $\operatorname{vol}(X)$ " is defined as the following number

$$
\operatorname{vol}(X)=\lim _{m \in \mathbb{Z}_{>0}} \frac{n!h^{0}\left(X, m K_{X}\right)}{m^{n}}
$$

which is an important birational invariant. We know that $v_{1}=2$ and $v_{2}=1$. By Chen-Chen [ 5 , Theorem $1.6(2)$ ], one has $v_{3} \geqslant \frac{1}{1680}$. Similar to the situation of $r_{n}$, when $n \geqslant 4, v_{n}$ is non-explicit either.

It is also interesting to consider another optimal constant $r_{n}^{+}$so that, for all nonsingular projective $n$-folds $X$ of general type with $p_{g}(X)>0, \varphi_{m, X}$ is birational for all $m \geqslant r_{n}^{+}$. By definition $r_{n}^{+} \leqslant r_{n}$ for any $n>0$. One has $r_{1}^{+}=r_{1}$ and $r_{2}^{+}=r_{2}=5$ according to Bombieri. By Iano-Fletcher [13], Chen [6] and Chen-Chen [5, Corollary 1.7], we know $14 \leqslant r_{3}^{+} \leqslant 18<r_{3}$. Recently, Brown and Kasprzyk [3] found many new canonical 4 -folds from which one deduces that $r_{4}>r_{4}^{+} \geqslant 39$.

In this paper we restrict our interest to varieties of large birational invariants. Early in 1973, Bombieri [2] proved that, for a nonsingular projective surface $S$ with either the canonical volume $\operatorname{vol}(S) \geqslant 3$ or the geometric genus $p_{g}(S) \geqslant 4, \varphi_{m, S}$ is birational for $m \geqslant r_{1}=r_{1}^{+}=3$. Such phenomenon may not be accidental, since the following theorem respectively due to G. Todorov and the first author has been realized:

Theorem 0. - Let $X$ be a nonsingular projective 3-fold of general type satisfying either of the following conditions:
(1) $\operatorname{vol}(X) \geqslant 4355^{3}$ (see Todorov [25]);
(2) $p_{g}(X) \geqslant 4$ (see Chen [6, Theorem $\left.\left.1.2(2)\right]\right)$.

Then $\varphi_{m, X}$ is birational for all $m \geqslant r_{2}=r_{2}^{+}=5$
Note that Theorem $0(1)$ has been improved by the first author [8] in loosing the volume constraint: say, $\operatorname{vol}(X) \geqslant 12^{3}$ is sufficient. One may also refer to Di Biagio [10] and Xu [28] for other relevant results in dimensions 3 and 4.

Hence it is natural to consider the higher dimensional analog of Theorem 0 . In fact such problem was first raised by McKernan in his review to the paper [25]. Anyway one should be basically guided by the following example:

Example 1.1. - By the definition of $r_{n}$, for any $n \geqslant 3$, there is a nonsingular projective $(n-1)$-fold, say $X_{n-1}$, of general type so that $\varphi_{r_{n-1}-1, X_{n-1}}$ is not birational by the definition of $r_{n-1}$. Pick up any smooth projective curve $C$ with $g(C) \geqslant 2$. Take $X_{n}=X_{n-1} \times C$. Then $\varphi_{r_{n-1}-1, X_{n}}$ is not birational and, clearly, $\operatorname{vol}\left(X_{n}\right)$ can be arbitrarily large as long as $g(C)$ is arbitrarily large. Thus $r_{n} \geqslant r_{n-1}$. For the same reason, one knows that $r_{n}^{+} \geqslant r_{n-1}^{+}$.

One of the main observations of this paper is the following result.
Theorem 1.2. - Let $\mathfrak{X}$ be a birationally bounded family (see Definition 3.1) and $m>1$ be an integer. Let $f: X \longrightarrow T$ be a morphism with connected fibers from a nonsingular projective variety $X$ of dimension $n \geqslant 2$ onto a smooth complete curve $T$. Assume that the general fiber $F$ of $f$ is birationally equivalent to an element of $\mathfrak{X}$. Then there exists a constant $C(\mathfrak{X})>0$ such that, whenever $\operatorname{vol}(X)>C(\mathfrak{X})$, the restriction map

$$
H^{0}\left(X, m K_{X}\right) \rightarrow H^{0}\left(F_{1}, m K_{F_{1}}\right) \oplus H^{0}\left(F_{2}, m K_{F_{2}}\right)
$$

is surjective for any two different general fibers $F_{1}$ and $F_{2}$ of $f$.
Theorem 1.2 has been applied to prove the following theorem.
Theorem 1.3. - For any integer $n>3$, there exists a constant $K(n)>$ 0 such that, for all nonsingular projective $n$-folds $X$ with $\operatorname{vol}(X)>K(n)$, the pluricanonical map $\varphi_{m}$ is birational for all

$$
m \geqslant \max \left\{r_{n-1},\left\lfloor 2 \prod_{j=2}^{n-1}\left(1+(n-j) \sqrt[n-j]{\frac{2}{v_{n-j}}}\right)\right\rfloor\right\}
$$

As the 4-dimensional analog of Theorem $0(1)$, the following theorem is obtained by a direct application of Theorem 1.3.

Theorem 1.4. - There is a constant $K(4)>0$. For any smooth projective 4 -folds $X$ with $\operatorname{vol}(X)>K(4), \varphi_{m, X}$ is birational for all $m \geqslant r_{3}$.

For projective varieties of large geometric genus, we prove the following theorem which is parallel to Theorem $0(2)$.

Theorem 1.5. - There exist two constants $L(4)>0$ and $L(5)>0$. For any smooth projective $n$-fold $X$ of general type with $p_{g}(X)>L(n)$ $(n=4,5), \varphi_{m, X}$ is birational for all $m \geqslant r_{n-1}^{+}$.

Remark 1.6. - The effectivity of $K(4), L(4)$ and $L(5)$ relies on the constant $C(\mathfrak{X})$ for certain birationally bounded families $\mathfrak{X}$.

We briefly explain the structure of this paper. In the first part, we mainly prove the key extension theorem (i.e. Theorem 1.2) which is the core of the paper. Then we use Theorem 1.2 to prove a general result like Theorem 1.3. In particular, we obtain Theorem 1.4 in dimension 4 which is parallel to Todorov's theorem for 3-folds. In Section 5, we study nonsingular projective varieties with large geometric genus. Theorem 1.5 generalizes an earlier theorem of the first author for 3 -folds to the case of dimension 4 and 5 . In the last section, we propose some open problems and conjectures and discuss their relations.

## 2. Preliminaries and Notation

### 2.1. Notation

Throughout we adopt the following symbols:

- "~" denotes linear equivalence.
- " $\sim_{\mathbb{Q}}$ " denotes $\mathbb{Q}$-linear equivalence.
- "三" denotes numerical equivalence.
- For $\mathbb{Q}$-divisors $A$ and $B, " A \geqslant B$ " means that $A-B$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor.
- " $\left|M_{1}\right| \succeq\left|M_{2}\right|$ " means, for linear systems $\left|M_{1}\right|$ and $\left|M_{2}\right|,\left|M_{1}\right| \supseteq$ $\left|M_{2}\right|+$ (effective divisor).
- $\operatorname{Fix}|D|$ denotes the fixed part of the complete linear system $|D|$ and $\operatorname{Mov}|D|=|D|-\operatorname{Fix}|D|$ is the moving part.


### 2.2. Convention

(1) For any linear system $|D|$ of positive dimension on a normal projective variety, denote by $\Phi_{|D|}=\Phi_{\mathrm{Mov}|D|}$ the rational map corresponding to $|D|$. In particular, $\varphi_{m, X}=\Phi_{\left|m K_{X}\right|}$ where $K_{X}$ denotes the canonical divisor of $X$.
(2) We say that $|D|$ is not composed of a pencil if $\operatorname{dim} \overline{\Phi_{|D|}(X)}>1$.
(3) A generic irreducible element of $|D|$ means a general member of $\operatorname{Mov}|D|$ when $|D|$ is not composed of a pencil or, otherwise, an irreducible component in a general member of $\operatorname{Mov}|D|$.

### 2.3. Multiplier ideal sheaves and asymptotic multiplier ideals

We mainly refer to Lazarsfeld [19, Part 3]. Here we provide the precise definition for a couple of ordinary forms of multiplier ideal sheaves which will appear in the context.
(1) For an ideal sheaf $\mathfrak{a} \subseteq \mathscr{O}_{X}$ and an effective $\mathbb{Q}$-divisor $D$ on a smooth projective variety $X$. Let $\pi: X^{\prime} \rightarrow X$ be a $\log$ resolution for the pair $(\mathfrak{a}, D)$ such that the inverse ideal $\pi^{-1}(\mathfrak{a})=\mathfrak{a} \cdot \mathscr{O}_{X^{\prime}}=\mathscr{O}_{X^{\prime}}(-E)$ for a divisor $E$ on $X^{\prime}$ and that $E+\pi^{*} D$ has simple normal crossing supports. Then, for any real number $c>0$, the multiplier ideal $\mathscr{J}\left(X, \mathfrak{a}^{c}+D\right)$ is defined to be

$$
\pi_{*} \mathscr{O}_{X^{\prime}}\left(K_{X^{\prime} / X}-\left\lfloor c E+\pi^{*} D\right\rfloor\right)
$$

which is proven to be an ideal sheaf.
Especially, when $\mathfrak{a}=\mathscr{O}_{X}$, we obtain the multiplier ideal

$$
\mathscr{J}(X, D)=\mathscr{J}\left(X, \mathscr{O}_{X}+D\right)
$$

(2) For a line bundle $L$ on $X$ with non-negative Kodaira-Iitaka dimension, define " $\mathfrak{b}$." to be the system of base ideals $\mathfrak{b}_{k}=\mathfrak{b}(|k L|)$ (where $k \in \mathbb{Z}_{>0}$ )-the ideal sheaves corresponding to $\mathrm{Bs}|k L|$. Then, for any positive real number $c>0$, the asymptotic multiplier ideal is defined to be

$$
\mathscr{J}(X,\|c L\|+D)=\mathscr{J}\left(X, \mathfrak{b}_{p}^{\frac{c}{p}}+D\right)
$$

for any sufficiently large and sufficiently divisible integer $p$.

### 2.4. The volume of a line bundle and the local multiplicity

Let $Z$ be a normal projective $n$-fold and $D$ a Cartier divisor on $Z$. The volume of $D$ is defined as:

$$
\operatorname{vol}(D)=\limsup _{m \mapsto \infty} \frac{n!h^{0}(Z, \mathscr{O}(m D))}{m^{n}}
$$

The canonical volume $\operatorname{vol}(Z)$ is defined to be $\operatorname{vol}\left(K_{Z^{\prime}}\right)$, where $Z^{\prime}$ is any smooth projective model of $Z$. We will frequently and tacitly use the following well-known results, where the integer $k$ can be independent of any given point.

Lemma 2.1 (see for instance Lazarsfeld [19, Lemma 10.4.12]). - Let $x \in Z$ be a smooth point. If $\operatorname{vol}(D)>\alpha^{n}$ for some rational number $\alpha>0$, then, for any sufficiently divisible integer $k \gg 0$, there exists an effective divisor $A_{x} \in|k D|$ with $\operatorname{mult}_{x}\left(A_{x}\right)>k \alpha$.

Lemma 2.2 (see for instance Todorov [25, Lemma 2.3]). - Let $x, y$ be two smooth points on $Z$. If $\operatorname{vol}(D)>2 \beta^{n}$ for some rational number $\beta>0$, then, for any sufficiently divisible integer $k \gg 0$, there exists an effective divisor $A_{x, y} \in|k D|$ with $\operatorname{mult}_{x}\left(A_{x, y}\right)>k \beta$ and $\operatorname{mult}_{y}\left(A_{x, y}\right)>k \beta$.

### 2.5. Non-klt locus

Let $D$ be an effective $\mathbb{Q}$-divisor on a nonsingular projective variety $X$. The non-klt locus

$$
\operatorname{Nklt}(X, D)=\operatorname{Supp}\left(\mathscr{O}_{X} / \mathscr{J}(X, D)\right) \subset X
$$

can be endowed with the reduced scheme structure as a sub-scheme of $X$.
Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ and $(X, \Delta)$ a $\log$ canonical pair. For any point $x \in X$, we denote by $\operatorname{LLC}(X, \Delta, x)$ the set of all $\log$ canonical centers passing through $x$. The following result of Kawamata ([15, Section 1]) will be tacitly used in the proof of our main theorem.

Lemma 2.3. - Let $X$ be a nonsingular projective variety and $\Delta \geqslant 0$ an effective $\mathbb{Q}$-divisor on $X$. Assume that $(X, \Delta)$ is log canonical at some point $x \in X$. The following statement hold:
(1) If $W_{1}, W_{2} \in \operatorname{LLC}(X, \Delta, x)$ and $W$ is any irreducible component of $W_{1} \cap W_{2}$ containing $x$, then $W \in \operatorname{LLC}(X, \Delta, x)$. Therefore, if $(X, \Delta)$ is not klt at $x$, then $\operatorname{LLC}(X, \Delta, x)$ has a unique minimal irreducible element, say $V$.
(2) There exists an effective $\mathbb{Q}$-divisor $E$ such that

$$
L L C(X,(1-\epsilon) \Delta+\epsilon E, x)=\{V\}
$$

for all $0<\epsilon \ll 1$.
(3) One may also assume that there is a unique place lying over $V$. If $x$ is general and $D$ is a big divisor, then one can take an effective $\mathbb{Q}$ divisor $E \sim_{\mathbb{Q}} a D$, for some positive rational number $a$, to fulfill (2).

### 2.6. A weak extension theorem

The problem of extending pluricanonical forms is an important subject in birational geometry (see for instance [22, 16]). In this paper, we will always use the following weak version, which corresponds to the special case (i.e. $S=\operatorname{spec}(k)$ ) of Kawamata [16, Theorem A]:

Theorem 2.4. - Let $V$ be a smooth projective variety and $X$ a smooth divisor of $V$. Assume that, for an ample $\mathbb{Q}$-divisor $A$ and an effective $\mathbb{Q}$ divisor $B, K_{V}+X \sim_{\mathbb{Q}} A+B$ and that $X$ is not contained in the support of $B$. Then
(1) the natural restriction map

$$
H^{0}\left(V, m\left(K_{V}+X\right)\right) \rightarrow H^{0}\left(X, m K_{X}\right)
$$

is surjective for any integer $m \geqslant 2$;
(2) the natural restriction map

$$
H^{0}\left(V, m\left(K_{V}+L+X\right)\right) \rightarrow H^{0}\left(X, m K_{X}+\left.m L\right|_{X}\right)
$$

is surjective for any nef divisor $L$ on $X$ and for any integer $m \geqslant 2$.
Remark 2.5. - By [1] or [23], the pair $(V, X)$ has a minimal model ( $V_{0}, X_{0}$ ) since, for $0<\varepsilon_{1} \ll \varepsilon_{2} \ll 1$, one may choose an effective $\mathbb{Q}$-divisor

$$
D_{\varepsilon_{1}, \varepsilon_{2}} \sim_{\mathbb{Q}}\left(1-\varepsilon_{1}\right) X+\left(\varepsilon_{1} X+\varepsilon_{2} A\right)+\varepsilon_{2} B
$$

so that $\left(V, D_{\varepsilon_{1}, \varepsilon_{2}}\right)$ is klt and one has

$$
K_{V}+D_{\varepsilon_{1}, \varepsilon_{2}} \equiv\left(1+\varepsilon_{2}\right)\left(K_{V}+X\right)
$$

Hence Theorem 2.4(1) follows from a direct application of KawamataViehweg vanishing theorem on ( $V_{0}, X_{0}$ ). Similarly one gets Theorem $2.4(2)$ as well.

## 3. Multplier ideals and the restriction of pluricanonical forms

We start with recalling the following definition (see [12]).
Definition 3.1. - We say that a set $\mathfrak{X}$ of varieties is birationally bounded if there is a projective morphism between schemes, say $\tau: Z \rightarrow T$ where $T$ is of finite type, such that for every element $X \in \mathfrak{X}$, there is a closed point $t \in T$ and a birational equivalence $X \rightarrow Z_{t}=\tau^{-1}(t)$.

For a real number $M>0$, define $\mathfrak{X}_{n, M}$ to be the set of smooth projective $n$-folds $X$ of general type with $\operatorname{vol}\left(K_{X}\right) \leqslant M$. One knows that $\mathfrak{X}_{n, M}$ is birationally bounded (see [11, 24, 26]).

Proposition 3.2. - Let $\mathfrak{a}$ be an ideal sheaf on a smooth projective variety $X$ and $L$ an effective $\mathbb{Q}$-divisor on $X$ with $L \not \equiv 0$. Fix a positive
integer $N>0$, there exists a constant $c$, which depends on $L, N$ and $X$, such that

$$
\mathscr{J}\left(X, \mathfrak{a}^{s}+\epsilon D\right)=\mathscr{J}\left(X, \mathfrak{a}^{s}\right)
$$

holds for any positive rational number $s$ whose denominator divides $N$, for any number $\epsilon$ with $0 \leqslant \epsilon<c$ and for any effective $\mathbb{Q}$-divisor $D$ satisfying $D \equiv L$.

Proof. - We first take a $\log$ resolution $\mu: X^{\prime} \rightarrow X$ of $(X, \mathfrak{a})$ and write $\mu^{*}(\mathfrak{a})=\mathscr{O}_{X^{\prime}}\left(-\sum_{i} a_{i} E_{i}\right)$ where $\sum E_{i}$ is a $\mu$-exceptional divisor.

By the birational transformation rule [19, Theorem 9.2.33],

$$
\mathscr{J}\left(X, \mathfrak{a}^{s}\right)=\mu_{*}\left(\mathscr{O}_{X^{\prime}}\left(K_{X^{\prime} / X}\right) \otimes \mathscr{J}\left(X^{\prime}, \mu^{*} \mathfrak{a}^{s}\right)\right)
$$

and

$$
\mathscr{J}\left(X, \mathfrak{a}^{s}+\epsilon D\right)=\mu_{*}\left(\mathscr{O}_{X^{\prime}}\left(K_{X^{\prime} / X}\right) \otimes \mathscr{J}\left(X^{\prime}, \mu^{*} \mathfrak{a}^{s}+\epsilon \mu^{*} D\right)\right)
$$

Hence, it suffices to show that $\mathscr{J}\left(X^{\prime}, \mu^{*} \mathfrak{a}^{s}\right)=\mathscr{J}\left(X^{\prime}, \mu^{*} \mathfrak{a}^{s}+\epsilon \mu^{*} D\right)$.
We have

$$
\begin{aligned}
\mathscr{J}\left(X^{\prime}, \mu^{*} \mathfrak{a}^{s}\right) & =\mathscr{O}_{X^{\prime}}\left(-\sum_{i}\left\lfloor s a_{i}\right\rfloor E_{i}\right) \\
\mathscr{J}\left(X^{\prime}, \mu^{*} \mathfrak{a}^{s}+\epsilon \mu^{*} D\right) & =\mathscr{O}_{X^{\prime}}\left(-\sum_{i}\left\lfloor s a_{i}\right\rfloor E_{i}\right) \otimes \mathscr{J}\left(X^{\prime}, \sum_{i}\left\{s a_{i}\right\} E_{i}+\epsilon \mu^{*} D\right) .
\end{aligned}
$$

Note that, since $s=\frac{t}{N}$ with $t \in \mathbb{Z},\left\{s a_{i}\right\} \leqslant \frac{N-1}{N}$. Therefore, it suffices to prove the following statement:

Let $\sum_{i} b_{i} E_{i}$ be a divisor with simple normal crossing support on a smooth projective variety $X$, where $0<b_{i} \leqslant \frac{N-1}{N}$. Let $L$ be an effective $\mathbb{Q}$-divisor. Fix a very ample divisor $H$ on $X$ and let $M=\left(H^{n-1} \cdot L\right)_{X}$. Then for all effective $\mathbb{Q}$-divisor $D$ with $D \equiv L$ and for all $\epsilon<c=\frac{1}{N M}$, we have $\mathscr{J}\left(X, \sum_{i} b_{i} E_{i}+\epsilon D\right)=\mathscr{O}_{X}$.
Since $D \equiv L$ and $H$ is very ample, $\operatorname{mult}_{x} D \leqslant M$ for any $x \in X$. We shall apply a similar argument to that in the proof of [19, Proposition 9.5.13]. In fact, the above statement is certainly true in dimension 1 . Assume $\operatorname{dim} X=$ $n>1$. For any $x \in X$, we take a smooth divisor $Y \in|H|$ passing through $x$, which is not contained in the support of $D$, so that $\sum_{i} E_{i}+H$ is a simple normal crossing divisor. Note that $\left.H\right|_{Y}$ is still very ample and $\left(\left.H\right|_{Y} ^{n-2}\right.$. $\left.\left.D\right|_{Y}\right)_{Y}=M$. Then, by induction, we have

$$
\mathscr{J}\left(Y,\left.\sum_{i} b_{i} E_{i}\right|_{Y}+\left.\epsilon D\right|_{Y}\right)=\mathscr{O}_{Y}
$$

for all $\epsilon<c$. By the restriction theorem (see [19, Theorem 9.5.1]), we have

$$
\left.\mathscr{J}\left(Y,\left.\sum_{i} b_{i} E_{i}\right|_{Y}+\left.\epsilon D\right|_{Y}\right) \subset \mathscr{J}\left(X, \sum_{i} b_{i} E_{i}+\epsilon D\right)\right|_{Y} .
$$

Thus $\mathscr{J}\left(X, \sum_{i} b_{i} E_{i}+\epsilon D\right)_{x}=\mathscr{O}_{X, x}$ holds for any $x \in X$. Hence

$$
\mathscr{J}\left(X, \sum_{i} b_{i} E_{i}+\epsilon D\right)=\mathscr{O}_{X}
$$

for all $\epsilon<c$.
In practice we need a relative version of the above proposition.
Proposition 3.3. - Let $f: \mathbb{X} \rightarrow T$ be a projective smooth morphism between irreducible smooth varieties. Let $N>0$ be a positive integer and let $\mathfrak{a}$ be an ideal sheaf on $\mathbb{X}$ such that $\mathfrak{a}_{t}=\mathfrak{a} \cdot \mathscr{O}_{X_{t}}$ is non-zero for all $t \in T$. Let $L$ be an effective $\mathbb{Q}$-divisor on $\mathbb{X}$ with $L \not \equiv 0$ such that $X_{t}=f^{-1}(t)$ is not contained in the support of $L$ for each $t \in T$. Then there exists a constant $c$, which depends on $L, N$ and $\mathbb{X}$, such that

$$
\mathscr{J}\left(X_{t}, \mathfrak{a}_{t}^{s}+\epsilon D\right)=\mathscr{J}\left(X_{t}, \mathfrak{a}_{t}^{s}\right)
$$

holds for all positive rational number $s$ whose denominator divides $N$, for all $0 \leqslant \epsilon<c$, for all $t \in T$, and for all effective $\mathbb{Q}$-divisor $D$ on $X_{t}$ with $\left.D \equiv L\right|_{X_{t}}$.

Proof. - Let $\mu: \mathbb{X}^{\prime} \rightarrow \mathbb{X}$ be a $\log$ resolution of $(\mathbb{X}, \mathfrak{a})$ and write $\mu^{*}(\mathfrak{a})=$ $\mathscr{O}_{\mathbb{X}^{\prime}}\left(-\sum_{i} a_{i} E_{i}\right)$ where $\sum E_{i}$ is a $\mu$-exceptional divisor. We note that there exists a Zariski open subset $U$ of $T$ such that $\sum_{i} E_{i}$ remains a simple normal crossing divisor on $X_{t}^{\prime}$ for all $t \in U$ (see [19, proof of Theorem 9.5.35]). Since $h$ is projective, we can fix a $h$-very ample divisor $H$ on $X$. Define $\hat{M}=\left(\left.\left.H\right|_{X_{t}} ^{n-1} \cdot L\right|_{X_{t}}\right)$ and define $C_{1}=\frac{1}{N M}$. As in the proof of Proposition 3.2, $\mathscr{J}\left(X_{t}, \mathfrak{a}_{t}^{s}+\epsilon D\right)=\mathscr{J}\left(X_{t}, \mathfrak{a}_{t}^{s}\right)$ for all $t \in U$ and for all $0 \leqslant \epsilon<C_{1}$.

We then consider the finite number of families over $T \backslash U$. By Noether's induction, we may eventually find the desired positive number $c$ by taking the minimum of thos numbers in $\left\{C_{i}\right\}$.

Proposition 3.4. - Let $X$ be a nonsingular projective variety with positive Kodaira dimension. Assume that, for some integer $m>1$ and for a non-negative rational number $\epsilon<1$, the relation

$$
\mathscr{J}\left(X,\left\|(m-1-\epsilon) K_{X}\right\|+\epsilon D\right) \supseteq \mathscr{J}\left(X,\left\|(m-1) K_{X}\right\|\right)
$$

holds for all effective $\mathbb{Q}$-divisors $D$ with $D \equiv K_{X}$. Then, for all nonsingular projective birational model $X^{\prime}$ of $X$ and for all effective $\mathbb{Q}$-divisor $D^{\prime}$ on
$X^{\prime}$ with $D^{\prime} \equiv K_{X^{\prime}}$, the natural inclusion

$$
\begin{aligned}
H^{0}\left(X^{\prime}, \mathscr{O}_{X^{\prime}}\left(m K_{X^{\prime}}\right) \otimes \mathscr{J}\left(X^{\prime}, \|(m-1-\epsilon) K_{X^{\prime}}\right.\right. & \left.\left.\|+\epsilon D^{\prime}\right)\right) \\
& \subseteq H^{0}\left(X^{\prime}, \mathscr{O}_{X^{\prime}}\left(m K_{X^{\prime}}\right)\right)
\end{aligned}
$$

induces an isomorphism of vector spaces.
Proof. - Since $\mathscr{J}\left(X,\left\|(m-1-\epsilon) K_{X}\right\|+\epsilon D\right) \supseteq \mathscr{J}\left(X,\left\|(m-1) K_{X}\right\|\right)$, we have

$$
\begin{aligned}
H^{0}\left(X, \mathscr{O}_{X}( \right. & \left.\left.m K_{X}\right)\right) \\
& \supseteq H^{0}\left(X, \mathscr{O}_{X}\left(m K_{X}\right) \otimes \mathscr{J}\left(X,\left\|(m-1-\epsilon) K_{X}\right\|+\epsilon D\right)\right) \\
& \supseteq H^{0}\left(X, \mathscr{O}_{X}\left(m K_{X}\right) \otimes \mathscr{J}\left(X,\left\|(m-1) K_{X}\right\|\right)\right) \\
& \supseteq H^{0}\left(X, \mathscr{O}_{X}\left(m K_{X}\right) \otimes \mathscr{J}\left(X,\left\|m K_{X}\right\|\right)\right) \\
& =H^{0}\left(X, \mathscr{O}_{X}\left(m K_{X}\right)\right)
\end{aligned}
$$

by [19, Proposition 11.2.10].
For any smooth birational model $X^{\prime}$ of $X$, we take a common smooth birational model $X^{\prime \prime}$ of both $X$ and $X^{\prime}$, say $\pi: X^{\prime \prime} \rightarrow X$ and $\mu: X^{\prime \prime} \rightarrow X^{\prime}$. For some effective $\mathbb{Q}$-divisor $D^{\prime} \equiv K_{X^{\prime}}$, we take $D^{\prime \prime}=\mu^{*} D^{\prime}+K_{X^{\prime \prime} / X^{\prime}} \equiv$ $K_{X^{\prime \prime}}$. We may assume that $s D^{\prime \prime}$ is an effective divisor for some integer $s>0$. Since $s D^{\prime \prime}=s K_{X^{\prime \prime}}+P^{\prime \prime}$ for some divisor $P^{\prime \prime} \in \operatorname{Pic}^{0}\left(X^{\prime \prime}\right)$, we write $P^{\prime \prime}=$ $\pi^{*}(P)=\pi^{*}(s Q)$ for some divisor $Q \in \operatorname{Pic}^{0}(X)$. Since $h^{0}\left(X^{\prime \prime}, s K_{X^{\prime \prime}}+P^{\prime \prime}\right)=$ $h^{0}\left(X, s\left(K_{X}+Q\right)\right)$, we have $\left|s K_{X^{\prime \prime}}+P^{\prime \prime}\right|=\pi^{*}\left|s\left(K_{X}+Q\right)\right|+s K_{X^{\prime \prime} / X}$. Thus there exists an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $D^{\prime \prime}=\pi^{*}(D)+K_{X^{\prime \prime} / X}$. Note that $D$ is nothing but $\pi_{*}\left(D^{\prime \prime}\right)$. We also note that for any positive integer $N,\left|N K_{X^{\prime \prime}}\right|=\pi^{*}\left|N K_{X}\right|+N K_{X^{\prime \prime} / X}$. Hence,

$$
\begin{aligned}
& \pi_{*}\left(\mathscr{O}_{X^{\prime \prime}}\left(m K_{X^{\prime \prime}}\right) \otimes \mathscr{J}\left(X^{\prime \prime},\left\|(m-1-\epsilon) K_{X^{\prime \prime}}\right\|+\epsilon D^{\prime \prime}\right)\right) \\
& =\pi_{*}\left(\mathscr{O}_{X^{\prime \prime}}\left(m K_{X^{\prime \prime}}\right)\right. \\
& \left.\quad \otimes \mathscr{J}\left(X^{\prime \prime},\left\|(m-1-\epsilon) \pi^{*} K_{X}\right\|+(m-1-\epsilon) K_{X^{\prime \prime} / X}+\epsilon D^{\prime \prime}\right)\right) \\
& =\pi_{*}\left(\mathscr{O}_{X^{\prime \prime}}\left(K_{X^{\prime \prime}}+(m-1) \pi^{*} K_{X}\right) \otimes \mathscr{J}\left(X^{\prime \prime}, \pi^{*}\left\|(m-1-\epsilon) K_{X}\right\|+\epsilon \pi^{*} D\right)\right) \\
& \left.=\mathscr{O}_{X}\left(m K_{X}\right) \otimes \pi_{*}\left(\mathscr{O}_{X^{\prime \prime}}\left(K_{X^{\prime \prime} / X}\right) \otimes \mathscr{J}\left(X^{\prime \prime}, \pi^{*}\left\|(m-1-\epsilon) K_{X}\right\|+\epsilon \pi^{*} D\right)\right)\right) \\
& =\mathscr{O}_{X}\left(m K_{X}\right) \otimes \mathscr{J}\left(X,\left\|(m-1-\epsilon) K_{X}\right\|+\epsilon D\right),
\end{aligned}
$$

where the first equality holds by [19, Proposition 9.2.31] and the definition of asymptotic multiplier ideals, the third equality holds by projection formula, and the last equality holds by the birational transformation rule [19, Theorem 9.2.33].

Hence

$$
\begin{array}{r}
\operatorname{dim} H^{0}\left(X^{\prime \prime}, \mathscr{O}_{X^{\prime \prime}}\left(m K_{X^{\prime \prime}}\right) \otimes \mathscr{J}\left(X^{\prime \prime},\left\|(m-1-\epsilon) K_{X^{\prime \prime}}\right\|+\epsilon D^{\prime \prime}\right)\right) \\
=P_{m}(X)=P_{m}\left(X^{\prime \prime}\right)
\end{array}
$$

and we have the desired isomorphism on $X^{\prime \prime}$.
Similarly, we have

$$
\begin{aligned}
\mu_{*}\left(\mathscr{O}_{X^{\prime \prime}}\left(m K_{X^{\prime \prime}}\right)\right. & \left.\otimes \mathscr{J}\left(X^{\prime \prime},\left\|(m-1-\epsilon) K_{X^{\prime \prime}}\right\|+\epsilon D^{\prime \prime}\right)\right) \\
& =\mathscr{O}_{X^{\prime}}\left(m K_{X^{\prime}}\right) \otimes \mathscr{J}\left(X^{\prime},\left\|(m-1-\epsilon) K_{X^{\prime}}\right\|+\epsilon D^{\prime}\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
P_{m}\left(X^{\prime}\right) & =P_{m}\left(X^{\prime \prime}\right) \\
& =\operatorname{dim} H^{0}\left(X^{\prime}, \mathscr{O}_{X^{\prime}}\left(m K_{X^{\prime}}\right) \otimes \mathscr{J}\left(X^{\prime},\left\|(m-1-\epsilon) K_{X^{\prime}}\right\|+\epsilon D^{\prime}\right)\right),
\end{aligned}
$$

which means that the isomorphism on $X^{\prime}$ holds.
The last equality naively implies the following property:

$$
\begin{equation*}
\mathfrak{b}\left(\left|m K_{X^{\prime}}\right|\right) \subseteq \mathscr{J}\left(X^{\prime},\left\|(m-1-\epsilon) K_{X^{\prime}}\right\|+\epsilon D^{\prime}\right) \tag{3.1}
\end{equation*}
$$

We are done.
Combing the above propositions, we have the following theorem.
Theorem 3.5. - Let $\mathfrak{X}$ be a birationally bounded set of smooth projective varieties. Then there exists a positive constant $c(\mathfrak{X})$ such that, for any nonsingular projective variety $X$ which is birationally equivalent to some element in $\mathfrak{X}$, for all effective $\mathbb{Q}$-divisors $D$ on $X$ with $D \equiv K_{X}$ and for all $0 \leqslant \epsilon<c(\mathfrak{X})$, the isomorphism of vector spaces
$H^{0}\left(X, \mathscr{O}_{X}\left(m K_{X}\right) \otimes \mathscr{J}\left(X,\left\|(m-1-\epsilon) K_{X}\right\|+\epsilon D\right)\right) \cong H^{0}\left(X, \mathscr{O}_{X}\left(m K_{X}\right)\right)$
holds for all $m \geqslant 2$.
Proof. - Let $h: Z \rightarrow T$ be a projective morphism, where $T$ is of finite type, such that each element of $\mathfrak{X}$ is birational to a fiber of $h$. After taking birational modifications of $Z$ and $T$ and taking stratifications of $T$, we may assume that $h$ is a smooth morphism between smooth varieties, say $T=\sqcup_{i} T_{i}$ is of finite type, $T_{i}$ are affine, and $h=\sqcup_{i}\left(h_{i}: Z_{i} \rightarrow T_{i}\right)$.

By [1], we know that the canonical ring

$$
\mathscr{R}_{i}=\bigoplus_{m=0}^{\infty} h_{i *} \mathscr{O}_{Z_{i}}\left(m K_{Z_{i}}\right)
$$

is a finitely generated $\mathscr{O}_{T_{i}}$ graded algebra. Let $s_{j} \in h_{i *} \mathscr{O}_{Z_{i}}\left(m_{j} K_{Z_{i}}\right)(1 \leqslant$ $j \leqslant k)$ be generators of this ring and let $M \geqslant 2$ be a common multiple of all $m_{j}$. Denote by $\widetilde{\mathfrak{b}}_{i}$ the base ideal of $\left|M K_{Z_{i}}\right|$. By Siu's deformation invariance of plurigenera (see for instance [19, Theorem 11.5.1]), $P_{m}\left(Z_{t}\right)$ is a constant for each $m \geqslant 1$ where $Z_{t}$ denotes the fiber of $h_{i}$ over $t \in T_{i}$. Equivalently, as being pointed out in [19, p. 310, (11.26), (11.27)], the natural restriction (as a ring homomorphism between graded rings) $\mathscr{R}_{i} \rightarrow$ $\bigoplus_{m=0}^{\infty} H^{0}\left(Z_{t}, m K_{Z_{t}}\right)$ is surjective in all degrees for any $t \in T$. In particular, the natural map $h_{i *} \mathscr{O}_{Z_{i}}\left(M K_{Z_{i}}\right) \rightarrow H^{0}\left(Z_{t}, M K_{Z_{t}}\right)$ is surjective. We have the following commutative diagram:

where $\theta_{1}$ surjective and so is $\theta_{2}=\theta_{1} \otimes \tilde{\mathfrak{b}}_{i}$. Also $\theta_{3}$ is surjective by the definition of base ideal. Thus $\widetilde{\mathfrak{b}}_{i, t}=\widetilde{\mathfrak{b}}_{i} \otimes_{\mathscr{O}_{Z_{i}}} \mathscr{O}_{Z_{t}}$ is the base ideal of $\left|M K_{Z_{t}}\right|$. Hence $M$ is also a common multiple of the degrees of the generators of the canonical ring of $Z_{t}$.

Now, by the definition of asymptotic multiplier ideal, we have

$$
\begin{aligned}
\mathscr{J}\left(Z_{t},\left\|(m-1) K_{Z_{t}}\right\|\right) & =\mathscr{J}\left(Z_{t}, \mathfrak{b}\left(\left|(m-1) p K_{Z_{t}}\right|\right)^{\frac{1}{p}}\right) \quad \text { for } p \gg 0 \\
& =\mathscr{J}\left(Z_{t}, \mathfrak{b}\left(\left|M K_{Z_{t}}\right|\right)^{\frac{l}{p}}\right) \quad(\text { here },(m-1) p=l M) \\
& =\mathscr{J}\left(Z_{t}, \widetilde{\mathfrak{b}}_{i, t}^{\frac{m-1}{M}}\right) .
\end{aligned}
$$

Proposition 3.3 implies that there exists a constant $\widetilde{C}_{i}$ such that $\mathscr{J}\left(Z_{t},\left\|(m-1) K_{Z_{t}}\right\|+\epsilon D_{t}\right)=\mathscr{J}\left(Z_{t},\left\|(m-1) K_{Z_{t}}\right\|\right)$ holds for all $t \in T_{i}$, for all effective $\mathbb{Q}$-divisor $D_{t}$ on $Z_{t}$ with $D_{t} \equiv K_{Z_{t}}$ and for all $0 \leqslant \epsilon<\widetilde{C}_{i}$. Hence

$$
\begin{aligned}
\mathscr{J}\left(Z_{t},\left\|(m-1-\epsilon) K_{Z_{t}}\right\|+\epsilon D_{t}\right) & \supseteq \mathscr{J}\left(Z_{t},\left\|(m-1) K_{Z_{t}}\right\|+\epsilon D_{t}\right) \\
& =\mathscr{J}\left(Z_{t},\left\|(m-1) K_{Z_{t}}\right\|\right) .
\end{aligned}
$$

Since we only need to consider finitely many strata of $T$, we take $c(\mathfrak{X})=$ $\min \left\{\widetilde{C}_{i}\right\}$, which is positive. Then

$$
\mathscr{J}\left(Z_{t},\left\|(m-1-\epsilon) K_{Z_{t}}+\epsilon D_{t}\right\| \supseteq \mathscr{J}\left(Z_{t},\left\|(m-1) K_{Z_{t}}\right\|\right)\right.
$$

holds for all $m \geqslant 2$, for all $t \in T$, for all effective $\mathbb{Q}$-divisor $D_{t}$ on $Z_{t}$ with $D_{t} \equiv K_{Z_{t}}$ and for all $0 \leqslant \epsilon<c(\mathfrak{X})$. The main statement directly follows from Proposition 3.4.

Corollary 3.6. - Let $\mathfrak{X}$ be a birationally bounded set of smooth projective varieties. Assume $P_{m}(Y) \neq 0$ for all $Y \in \mathfrak{X}$ and $m>1$. Then, for any nonsingular projective variety $X$ which is birationally equivalent to some element in $\mathfrak{X}$, for all effective $\mathbb{Q}$-divisors $D$ on $X$ with $D \equiv K_{X}$ and for all $0 \leqslant \epsilon<c(\mathfrak{X})$, we have

$$
\mathscr{J}(X, \epsilon D) \supseteq \mathfrak{b}\left(\left|m K_{X}\right|\right) .
$$

Proof. - By Theorem 3.5 and Property (3.1) in the proof of Proposition 3.4, we have

$$
\mathscr{J}(X, \epsilon D) \supseteq \mathscr{J}\left(X,\left\|(m-1-\epsilon) K_{X}\right\|+\epsilon D\right) \supseteq \mathfrak{b}\left(\left|m K_{X}\right|\right)
$$

Theorem 3.7. - Let $\mathfrak{X}$ be a birationally bounded family. Let $f: X \longrightarrow$ $T$ be fibration from a nonsingular projective variety onto a smooth complete curve T. Assume that the general fiber $F$ of $f$ is birationally equivalent to an element of $\mathfrak{X}$. Then there exists a constant $c_{1}(\mathfrak{X})>0$ such that, whenever $\operatorname{vol}(X)>c_{1}(\mathfrak{X})$, the restriction map

$$
H^{0}\left(X, m K_{X}\right) \rightarrow H^{0}\left(F, m K_{F}\right)
$$

is surjective for all $m \geqslant 2$.
Proof. - By assumption, $X$ must be of general type since $\operatorname{vol}(X)>0$. For $k \gg 0$, we have $P_{k}(X) \approx \operatorname{vol}(X) \cdot \frac{k^{n}}{n!}$ where $n=\operatorname{dim} X$. On the other hand, there is a constant $M(\mathfrak{X})>0$ such that $\operatorname{vol}(Z)<M(\mathfrak{X})$ for any element $Z \in \mathfrak{X}$. In particular, for the general fiber $F$ of $f, h^{0}\left(F, k K_{F}\right)<$ $M(\mathfrak{X}) \cdot \frac{k^{n-1}}{(n-1)!}$ for $k \gg 0$. Pick general fibers $F_{i}(1 \leqslant i \leqslant s)$ of $f$, we consider the short exact sequence

$$
0 \rightarrow \mathscr{O}_{X}\left(k K_{X}-\sum_{i=1}^{s} F_{i}\right) \rightarrow \mathscr{O}_{X}\left(k K_{X}\right) \rightarrow \oplus_{i} \mathscr{O}_{F_{i}}\left(k F_{i}\right) \rightarrow 0
$$

By comparing dimensions of those cohomological groups in question, we see that $H^{0}\left(X, \mathscr{O}_{X}\left(k K_{X}-\sum_{i=1}^{s} F_{i}\right)\right) \neq 0$ for $k \gg 0$ and

$$
\left\lfloor\frac{k \operatorname{vol}\left(K_{X}\right)}{n M(\mathfrak{X})}\right\rfloor-1<s<\frac{k \operatorname{vol}\left(K_{X}\right)}{n M(\mathfrak{X})} .
$$

Thus, $K_{X} \sim_{\mathbb{Q}} p F+E$, where $E$ is an effective $\mathbb{Q}$-divisor and

$$
p>\left\lfloor\frac{\operatorname{vol}\left(K_{X}\right)}{n M(\mathfrak{X})}\right\rfloor-1
$$

is a rational number.

When $\operatorname{vol}\left(K_{X}\right)>2 n M(\mathfrak{X})$, we can pick $p>1$, then $(m-1) K_{X}-F$ is a big divisor. We denote by $\mathscr{J}=\mathscr{J}\left(\left\|(m-1) K_{X}-F\right\|\right)$ and consider the short exact sequence:
$\left.0 \rightarrow \mathscr{O}_{X}\left(m K_{X}-F\right) \otimes \mathscr{J} \rightarrow \mathscr{O}_{X}\left(m K_{X}\right) \otimes \mathscr{J} \rightarrow \mathscr{O}_{F}\left(m K_{F}\right) \otimes \mathscr{J}\right|_{F} \rightarrow 0$.
By Nadel's vanishing theorem, we have

$$
H^{1}\left(X, \mathscr{O}_{X}\left(m K_{X}-F\right) \otimes \mathscr{J}\right)=0
$$

We then have the surjective map

$$
H^{0}\left(X, \mathscr{O}_{X}\left(m K_{X}\right) \otimes \mathscr{J}\right) \rightarrow H^{0}\left(F,\left.\mathscr{O}_{F}\left(m K_{F}\right) \otimes \mathscr{J}\right|_{F}\right)
$$

It suffices to show that the natural inclusion

$$
H^{0}\left(F,\left.\mathscr{O}_{F}\left(m K_{F}\right) \otimes \mathscr{J}\right|_{F}\right) \subseteq H^{0}\left(F, \mathscr{O}_{F}\left(m K_{F}\right)\right)
$$

is an isomorphism of vector spaces. Next we study the sheaf $\left.\mathscr{J}\right|_{F}$. Since $\left.\left|l(m-1) K_{F}\right| \supseteq\left|l\left((m-1) K_{X}-F\right)\right|\right|_{F}$ for any integer $l>0$, we have

$$
\left.\mathscr{J}\right|_{F} \subseteq \mathscr{J}\left(\left\|(m-1) K_{F}\right\|\right)
$$

(see for instance [18, Section 11.2]).
We pick a sufficiently large and divisible integer $N$ and take $\epsilon=\frac{m}{p+1}$. Then

$$
\begin{aligned}
\left|N\left((m-1) K_{X}-F\right)\right| & =\left|N\left((m-1-\epsilon) K_{X}+(m-1-\epsilon) F+\epsilon E\right)\right| \\
& \supseteq\left|N(m-1-\epsilon)\left(K_{X}+F\right)\right|+N \epsilon E
\end{aligned}
$$

On the other hand, by Theorem 2.4, we know that

$$
\left.\left|N(m-1-\epsilon)\left(K_{X}+F\right)\right|\right|_{F}=\left|N(m-1-\epsilon) K_{F}\right| .
$$

Hence

$$
\left.\mathscr{J}\right|_{F} \supseteq \mathscr{J}\left(F,\left\|(m-1-\epsilon) K_{F}\right\|+\epsilon D_{F}\right),
$$

where $D_{F}=\left.E\right|_{F} \sim_{\mathbb{Q}} K_{F}$ is an effective $\mathbb{Q}$-divisor.
Therefore

$$
\begin{aligned}
& H^{0}\left(F,\left.\mathscr{O}_{F}\left(m K_{F}\right) \otimes \mathscr{J}\right|_{F}\right) \\
& \quad \supseteq H^{0}\left(F, \mathscr{O}_{F}\left(m K_{F}\right) \otimes \mathscr{J}\left(F,\left\|(m-1-\epsilon) K_{F}\right\|+\epsilon D_{F}\right)\right) .
\end{aligned}
$$

As in the proof of Theorem 3.5, there exists a finite set $\mathbb{S}$ of positive integers such that for any $F \in \mathfrak{X}$, the canonical ring of $F$ is generated by some elements whose degree numbers belong to $\mathbb{S}$. Let $N$ be a common multiple of those numbers in $\mathbb{S}$. When $\operatorname{vol}(X)$ is large enough, we can take a sufficiently large integer $p$ so that, for any $2 \leqslant m \leqslant N+1, \epsilon=$ $\frac{m}{p+1} \leqslant \frac{N+1}{p+1}<c(\mathfrak{X})$. Then, by Theorem 3.5, we see that the restriction map $H^{0}\left(X, m K_{X}\right) \rightarrow H^{0}\left(F, m K_{F}\right)$ is surjective for $2 \leqslant m \leqslant N+1$. On
the other hand, for any $k \geqslant N+2$ and $s \in H^{0}\left(F, k K_{F}\right)$, $s$ can be written as a sum of products of elements of degrees between 2 and $N$. Hence the restriction map $H^{0}\left(X, k K_{X}\right) \rightarrow H^{0}\left(F, k K_{F}\right)$ is again surjective and we conclude the statement.

Corollary 3.8. - Under the same condition of Theorem 3.7, there exists another constant $c_{2}(\mathfrak{X})$ such that, whenever $\operatorname{vol}(X)>c_{2}(\mathfrak{X})$, the restrction map

$$
H^{0}\left(X, m K_{X}\right) \rightarrow H^{0}\left(F_{1}, m K_{F_{1}}\right) \oplus H^{0}\left(F_{2}, m K_{F_{2}}\right)
$$

is surjective for two arbitrary different general fibers $F_{1}$ and $F_{2}$ of $f$ and for all $m \geqslant 2$.

Proof. - Note that we have

$$
K_{X} \sim_{\mathbb{Q}} \hat{p}\left(F_{1}+F_{2}\right)+\hat{E}
$$

with $\hat{E} \geqslant 0$ and $\hat{p}$ can be arbitrarily large as long as $\operatorname{vol}(X)$ is sufficiently large. As in the proof of Theorem 3.7, we study $\hat{\mathscr{J}}=\mathscr{J}\left(\|(m-1) K_{X}-\right.$ $\left.F_{1}-F_{2} \|\right)$ instead of considering $\mathscr{J}$. All other argument follows accordingly without any problems. We omit the details and leave it to interested readers.

Remark 3.9. - Corollary 3.8 says that, as soon as $P_{m}(X)>0$ for some $m \geqslant 2$ and $\operatorname{vol}(X)>c_{2}(\mathfrak{X}), \varphi_{m}$ distinguishes different fibers of $f$.

Proof of Theorem 1.2. - Theorem 3.7 and Corollary 3.8 directly imply Theorem 1.2.

## 4. Proof of Theorem 1.3 and Theorem 1.4

Given a projective variety $X$, by convention, we say that a point $x \in X$ is in very general position if $x$ is in a countably dominant subset, which is nothing but the complementary set of a union of countably many closed subsets of $X$.

### 4.1. Point separation principle

Let $X$ be a nonsingular projective variety and $L$ a big divisor on $X$. Assume that $x$ and $y$ are two distinct very general points of $X$. If $D$ is an effective $\mathbb{Q}$-divisor such that $L-D$ is nef and big, that $x, y \in \operatorname{Nklt}(X, D)$
and that either $x$ or $y$ is an isolated lc center of $\operatorname{Nklt}(X, D)$, then Nadel's vanishing theorem implies that $\left|K_{X}+L\right|$ separates $x$ and $y$.

In practice, we are going to apply this principle to the special case with $L=(m-1) K_{X}$ for some integer $m>1$. Write $K_{X} \sim_{\mathbb{Q}} A+E$ where $A$ is an ample $\mathbb{Q}$-divisor, $\operatorname{vol}(X) \approx \operatorname{vol}(A)$ and $E$ is an effective $\mathbb{Q}$-divisor. Suppose we can find an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} a A$ for some rational number $a>0$ such that $x, y \in \operatorname{Nklt}(X, D)$ and that either $x$ or $y$ is an isolated lc center. Then it follows that $\varphi_{m, X}$ is birational for all $m \geqslant\lfloor a\rfloor+2$.

### 4.2. The induction formula according to Takayama

We review some formulae and results, of Takayama [24, Section 5], which will be used in our proof. First of all, we fix the notation following [24, Notation 5.2]. Let $X$ be a nonsingular projective $n$-fold.

- For any integer $d$ with $1 \leqslant d \leqslant n$, assume that $\alpha_{d}>0$ is a constant so that $\operatorname{vol}(Z)>\alpha_{d}^{n-d}$ holds for any sub-variety $Z$ of codimension $d$ passing through very general points.
- Pick any rational number $\varepsilon$ with $0<\varepsilon \ll 1$.
- Non-negative integers $s_{d}, s_{d}^{\prime}, t_{d}$ and $t_{d}^{\prime}$ are defined inductively as follows. Let $s_{1}=0$ and $t_{1}=n \sqrt[n]{2} /(1-\varepsilon)$. Assume that $s_{d}$ and $t_{d}$ are defined already. Set $s_{d}^{\prime}=s_{d}+\varepsilon, t_{d}^{\prime}=t_{d}$ and

$$
\begin{aligned}
& s_{d+1}=\left(1+\frac{\sqrt[n-d]{2}(n-d)}{(1-\varepsilon) \alpha_{d}}\right) s_{d}^{\prime}+\frac{2 \sqrt[n-d]{2}(n-d)}{(1-\varepsilon) \alpha_{d}} \\
& t_{d+1}=\left(1+\frac{\sqrt[n-d]{2}(n-d)}{(1-\varepsilon) \alpha_{d}}\right) t_{d}^{\prime}
\end{aligned}
$$

- For the given constant $\varepsilon$, we may find a birational modification $\mu: X^{\prime} \longrightarrow X$ with $X^{\prime}$ smooth and take the decomposition

$$
\mu^{*}\left(K_{X}\right) \sim_{\mathbb{Q}} A_{\varepsilon}+E_{\varepsilon}
$$

where $A_{\varepsilon}$ is $\mathbb{Q}$-ample and $E_{\varepsilon}$ is an effective $\mathbb{Q}$-divisor. For simplicity we just write $A=A_{\varepsilon}$ and $E=E_{\varepsilon}$ as no confusion is likely. Note also that we may assume $|\operatorname{vol}(A)-\operatorname{vol}(X)|<\varepsilon^{100}$ as $\varepsilon$ is very small.

The following proposition of Takayama provides an effective induction:

Proposition 4.1 (cf. [24, Proposition 5.3]). - Let $x_{1}, x_{2}$ be two different very general points on $X^{\prime}$. The following statement $\left(*_{d}\right)$ holds for every $d$ with $1 \leqslant d \leqslant n$ :
$\left(*_{d}\right)$ There exist a positive constant $a_{d}<s_{d}+t_{d} / \alpha_{0}$, a non-empty subset $I_{d} \subset\{1,2\}$ and an effective $\mathbb{Q}$-divisor $D_{d}$ on $X^{\prime}$ satisfying $D_{d} \sim_{\mathbb{Q}} a_{d} A$ such that
(i) $\left(X^{\prime}, D_{d}\right)$ is $\log$ canonical at $x_{i}$ for all $i \in I_{d}$;
(ii) $\left(X^{\prime}, D_{d}\right)$ is not klt at $x_{i}$ for each $i \in I_{d}$;
(iii) $\left(X^{\prime}, D_{d}\right)$ is not $\log$ canonical at $x_{j}$ for each $j \in\{1,2\} \backslash I_{d}$;
(iv) $\operatorname{codim} \operatorname{Nklt}\left(X^{\prime}, D_{d}\right) \geqslant d$ at $x_{i}$ for all $i \in I_{d}$.

In fact, Takayama's clue of the proof of Proposition 4.1 is:

$$
\left(*_{d}\right) \Longrightarrow\left(*_{d}^{\prime}\right) \Longrightarrow\left(*_{d+1}\right),
$$

where " $\left(*_{d}^{\prime}\right)$ " is the following statement:
$\left(*_{d}^{\prime}\right)$ There exist a positive constant $a_{d}^{\prime}<s_{d}^{\prime}+t_{d}^{\prime} / \alpha_{0}$, a nonempty subset $I_{d}^{\prime} \subset\{1,2\}$ and an effective $\mathbb{Q}$-divisor $D_{d}^{\prime}$ on $X^{\prime}$ with $D_{d}^{\prime} \sim_{\mathbb{Q}} a_{d}^{\prime} A$ such that
(i) $\left(X^{\prime}, D_{d}^{\prime}\right)$ is $\log$ canonical at $x_{i}$ for all $i \in I_{d}^{\prime}$,
(ii) $\left(X^{\prime}, D_{d}^{\prime}\right)$ is not klt at $x_{i}$ for each $i \in I_{d}^{\prime}$,
(iii) $\left(X^{\prime}, D_{d}^{\prime}\right)$ is not $\log$ canonical at $x_{j}$ for each $j \in\{1,2\} \backslash I_{d}^{\prime}$,
and either of the following is true:
(iv-0) codim $\operatorname{Nklt}\left(X^{\prime}, D_{d}^{\prime}\right)>d$ at $x_{i}$ for all $i \in I_{d}^{\prime}$.
(iv-1) $\operatorname{Nklt}\left(X^{\prime}, D_{d}^{\prime}\right)=Z \cup Z_{+}$such that $Z$ is irreducible of codimension $d$, and $x_{i} \in Z$ but $x_{i} \notin Z_{+}$for $i \in I_{d}^{\prime}$. In particular $Z$ is unique.
Since Proposition 4.1 is not strong enough for our purpose, we need to use the following improved version for which a similar inequality was observed by Di Biago [10, Theorem 5.5]. We observed as well that Debarre mentioned this in his survey article (see [9, Theorem 5.2]).

Variant 4.2. - Keep the same notation as in Proposition 4.1. Assume that there is a constant $\widetilde{a}_{d}>0$ and an effective $\mathbb{Q}$-divisor $\widetilde{D}_{d} \sim_{\mathbb{Q}} \widetilde{a}_{d} A$ so that the pair $\left(X^{\prime}, \widetilde{D}_{d}\right)$ satisfies conditions $\left(*_{d}\right):(\mathrm{i}) \sim(i v)$. Then
(1) there exists an effective $\mathbb{Q}$-divisor $\widetilde{D}_{d}^{\prime} \sim_{\mathbb{Q}} \widetilde{a}_{d}^{\prime} A$ for some positive constant

$$
\widetilde{a}_{d}^{\prime}<\widetilde{a}_{d}+\varepsilon
$$

such that $\left(X^{\prime}, \widetilde{D}_{d}^{\prime}\right)$ satisfies conditions $\left(*_{d}^{\prime}\right)$.
(2) there exists an effective $\mathbb{Q}$-divisor $\widetilde{D}_{d+1} \sim_{\mathbb{Q}} \widetilde{a}_{d+1} A$ for some positive constant

$$
\begin{equation*}
\widetilde{a}_{d+1}<\widetilde{a}_{d}\left(1+\frac{\sqrt[n-d]{2}(n-d)}{\alpha_{d}}\right)+\frac{2 \sqrt[n-d]{2}(n-d)}{\alpha_{d}}+\varepsilon \tag{4.1}
\end{equation*}
$$

such that $\left(X^{\prime}, \widetilde{D}_{d+1}\right)$ satisfies conditions $\left(*_{d+1}\right):(\mathrm{i}) \sim(i v)$.
Proof. - Statement (1) follows directly from the proof of [24, Lemma 5.5]. One keeps the same construction for $\widetilde{D}_{d}^{\prime}$ as that for $D_{d}^{\prime}$ by replacing $D_{d}$ with $\widetilde{D}_{d}, a_{d}$ with $\widetilde{a}_{d}$ and so on. In fact, one gets $\widetilde{a}_{d}^{\prime}<\widetilde{a}_{d}+\varepsilon$ just consulting [24, p. 580, l. 25] and [24, p. 581, l. 15].

For Statement (2), we first refer to the proof of [24, Main Lemma 5.6], where we observe that $b=\left\lfloor\widetilde{a}_{d}^{\prime}\right\rfloor+1$ and thus the main consequence of the proof of [24, Main Lemma 5.6] should be:

$$
\left(Z \cdot A^{n-d}\right)>\left(\frac{(1-\varepsilon) \alpha_{d}}{1+b}\right)^{n-d}
$$

Then we refer to the proof of [24, Lemma 5.8], where we may replace $b_{d}(X), D_{d}^{\prime}, D_{d+1}$ and $a_{d+1}$ by $b, \widetilde{D}_{d}^{\prime}, \widetilde{D}_{d+1}$ and $\widetilde{a}_{d+1}$, respectively. The same argument, especially by the definitions of $a_{d+1}$ in [24, p. 583, l. 3-19], implies

$$
\begin{aligned}
\widetilde{a}_{d+1} & <\widetilde{a}_{d}^{\prime}+\frac{\sqrt[n-d]{2}(n-d)(1+b)}{(1-\varepsilon) \alpha_{d}} \\
& =\widetilde{a}_{d}\left(1+\frac{\sqrt[n-d]{2}(n-d)}{\alpha_{d}}\right)+\frac{2 \sqrt[n-d]{2}(n-d)}{\alpha_{d}}+f(\varepsilon)
\end{aligned}
$$

where $f(\varepsilon) \mapsto 0$ (as $\varepsilon \mapsto 0)$. We are done.
By Proposition 4.1, we can find an effective $\mathbb{Q}$-divisor $D_{1}$ with $D_{1} \sim_{\mathbb{Q}}$ $a_{1} A$, such that $\left(X^{\prime}, D_{1}\right)$ satisfies the condition $\left(*_{1}\right)$ where

$$
a_{1}<\frac{\sqrt[n]{2} n}{\alpha_{0}}+\varepsilon
$$

For any integer $m$ satisfying $1 \leqslant m \leqslant n$, Variant $4.2(2)$ inductively implies that one can find a rational number

$$
\begin{equation*}
a_{m}<\left(\frac{\sqrt[n]{2} n}{\alpha_{0}}+2\right) \prod_{j=1}^{m-1}\left(1+\frac{\sqrt[n-j]{2}(n-j)}{\alpha_{j}}\right)-2+\varepsilon \tag{4.2}
\end{equation*}
$$

for the given $\varepsilon \ll 1$ and an effective $\mathbb{Q}$-divisor $D_{m}$ with $D_{m} \sim_{\mathbb{Q}} a_{m} A$ such that $\left(X^{\prime}, D_{m}\right)$ satisfies the condition $\left(*_{m}\right)$.

A direct corollary of Takayama's induction and Variant 4.2 is as follows.

Corollary 4.3. - Let $V_{n}=(\sqrt[n]{2} n+2) \prod_{j=1}^{n-1}\left(1+\frac{\sqrt[n-j]{2}(n-j)}{\nu_{j}}\right)+1$. Then for all $n$-fold $X$ with $\operatorname{vol}(X) \geqslant 1$, the $m$-th pluricanonical map $\varphi_{m}$ is birational for all $m \geqslant V_{n}$. In particular, for all nonsingular projective 3 -folds of general type with $\operatorname{vol}(X) \geqslant 1, \varphi_{38, X}$ is stably birational.

Now we are ready for proving the main theorem. We shall discuss the case of dimension 4 in details while the proof for higher dimensions is simply a redundant generalization.

### 4.3. Proof of Theorem 1.3 (the case of dimension 4)

Let $X$ be a nonsingular projective 4 -fold of general type. Given a number $\varepsilon$ with $0<\varepsilon \ll 1$. Modulo possibly a birational modification, we may and do assume that $K_{X} \sim_{\mathbb{Q}} A+E$ for some ample $\mathbb{Q}$-divisor $A$ with $0<\operatorname{vol}(X)-\operatorname{vol}(A) \ll \varepsilon$ and an effective $\mathbb{Q}$-divisor $E$ on $X$. Assume that $\operatorname{vol}(X)>\alpha_{0}^{4}$ for some rational number $\alpha_{0}>0$. Actually we will be working under certain assumption that $\alpha_{0}$ is sufficiently large. Note that we always have $\alpha_{3} \geqslant 2$ and $\alpha_{2} \geqslant 1$. Pick two very general points $x_{1}, x_{2} \in X$.

By Lemma 2.2, Lemma 2.3 and Proposition 4.1, we may take an effective $\mathbb{Q}$-divisor $D_{0}=D_{0}^{\left(x_{1}, x_{2}\right)} \sim_{\mathbb{Q}} a_{0} A$ with $a_{0}<\frac{4 \sqrt[4]{2}}{\alpha_{0}}+\varepsilon$ so that $\left(X, D_{0}\right)$ satisfies the condition $\left(*_{1}\right)$ of Proposition 4.1: namely, both $x_{1}$ and $x_{2}$ belong to $\operatorname{Nklt}\left(X, D_{0}\right) ;\left(X, D_{0}\right)$ is lc at either $x_{1}$ or $x_{2}$, we may simply assume that $\left(X, D_{0}^{\left(x_{1}, x_{2}\right)}\right)$ is lc at $x_{1}$ modulo an exchanging of indices, and there exists a unique $l c$ center $V_{x_{1}}$ passing through $x_{1}$.

We then consider the Hilbert scheme $\mathscr{H}_{a}$, for a rational number $a<$ $\frac{4 \sqrt[4]{2}}{\alpha_{0}}+\varepsilon$, parametrizing the data

$$
\left\{\left(x_{1}, x_{2}, D_{0}^{\left(x_{1}, x_{2}\right)}\right) \mid x_{1}, x_{2} \in D_{0}^{\left(x_{1}, x_{2}\right)} \sim_{\mathbb{Q}} a A\right\} .
$$

Note that $\mathscr{H}_{a}$ is the subset of $X \times X \times|M A|$, where $M$ is an integer such that $M A$ is a divisor. The set $\mathscr{H}_{a}$ consists of $\left\{\left(x_{1}, x_{2}, D\right)\right\} \in X \times X \times|M A|$ such that $D$ passes through both $x_{1}$ and $x_{2}$. Pull back the universal divisors in $|M A| \times X$ for each $M$, we get the universal divisor $\mathscr{D} \subset \mathscr{H}_{a} \times X$, parametrized by $\mathscr{H}_{a}$. Take a log resolution of the pair $\left(\mathscr{H}_{a} \times X, \mathscr{D}\right)$. Since $\log$ canonical singularities is defined by the discrepancies of a log resolution of a pair, we see easily that, in each component of $\mathscr{H}_{a}$ ( $a$ is a small rational number), there exists a constructible subset parametrizing those datum such that

$$
\left(X, D_{0}^{\left(x_{1}, x_{2}\right)}\right)=\left(X, \frac{a}{M} D\right)
$$

is non-klt at $x_{1}, x_{2}$ and is $\log$ canonical at $x_{1}$ with a unique $\log$ canonical centre $V_{x_{1}}$ passing through $x_{1}$. Then, by the countability of Hilbert schemes $\left\{\mathscr{H}_{a}\right\}$, we see that there exists a rational number $a_{0}<\frac{4 \sqrt[4]{2}}{\alpha_{0}}+\varepsilon$ and an irreducible scheme $U$ parametrizing the data $\left(x_{1}, x_{2}, V_{x_{1}}, D_{0}^{\left(x_{1}, x_{2}\right)}\right)$ with the property: $D_{0}^{\left(x_{1}, x_{2}\right)} \sim_{\mathbb{Q}} a_{0} A ;\left(X, D_{0}^{\left(x_{1}, x_{2}\right)}\right)$ is non-klt at $x_{1}$ and $x_{2} ;$ $\left(X, D_{0}^{\left(x_{1}, x_{2}\right)}\right)$ is lc at $x_{1}$ with unique lc centre $V_{x_{1}}$ passing through $x_{1}$; the natural morphism

$$
\begin{aligned}
U & \rightarrow X \times X \text { defined by } \\
\left(x_{1}, x_{2}, V_{x_{1}}, D_{0}^{\left(x_{1}, x_{2}\right)}\right) & \rightarrow\left(x_{1}, x_{2}\right)
\end{aligned}
$$

is dominant. Shrinking $U$ a little bit, we may obtain the minimal number $r_{0}$ with $0 \leqslant r_{0} \leqslant 3$ such that, for very general pair $\left(x_{1}, x_{2}\right) \in X \times X$, there exists $\left(x_{1}, x_{2}, V_{x_{1}}, D_{0}^{\left(x_{1}, x_{2}\right)}\right)$, parametrized by $U$, admitting $\operatorname{dim} V_{x_{1}}=r_{0}$.

Case 1. - Assume $r_{0} \leqslant 2$.
This means that the pair $\left(X, D_{0}^{\left(x_{1}, x_{2}\right)}\right)$ satisfies the condition $\left(*_{2}\right)$ of Proposition 4.1. Set $\widetilde{a}_{2}=a_{0}$. Repeatedly applying Variant 4.2(2), we may find an effective $\mathbb{Q}$-divisor $\widetilde{D}_{4} \sim \widetilde{a}_{4} A$ such that $\left(X, \widetilde{D}_{4}\right)$ satisfies the condition $\left(*_{4}\right)$, where $\tilde{a}_{4}>0$ is a rational number with:

$$
\begin{align*}
\widetilde{a}_{4} & <2(1+2 \sqrt{2}) \widetilde{a}_{2}+(8 \sqrt{2}+2)+3 \varepsilon  \tag{4.3}\\
& <(8 \sqrt{2}+2)+\frac{8(1+2 \sqrt{2}) \sqrt[4]{2}}{\alpha_{0}}+(4 \sqrt{2}+5) \varepsilon \tag{4.4}
\end{align*}
$$

Clearly one can find a computable constant $K_{1}>0$ such that the right hand side of Inequality (4.4) is upper bounded by $\lceil 8 \sqrt{2}+2\rceil=14$ whenever $\alpha_{0}>K_{1}$. Therefore, when $\operatorname{vol}(X)>K_{1}^{4}, \varphi_{m}$ is birational for all $m \geqslant 15$.

Case 2. - Assume $r_{0}=3$.
Set $a_{1}=a_{0}$. According to Variant 4.2 (2) and Inequality (4.2), we may find an effective $\mathbb{Q}$-divisor $D_{4} \sim a_{4} A$ such that $\left(X, D_{4}\right)$ satisfies the condition $\left(*_{4}\right)$, where

$$
a_{4}<2(1+2 \sqrt{2})\left(1+\frac{3 \sqrt[3]{2}}{\alpha_{1}}\right)\left(2+\frac{4 \sqrt[4]{2}}{\alpha_{0}}\right)-2+h(\varepsilon)
$$

and $h(\varepsilon) \mapsto 0$.
Step 1. - If $\alpha_{1}>1000$, by taking a sufficiently small $\varepsilon$, then we have

$$
4(1+2 \sqrt{2})\left(1+\frac{3 \sqrt[3]{2}}{\alpha_{1}}\right)-2<14
$$

Thus one can find a computable constant $K_{2}>0$ such that $a_{4}<14$ whenever $\alpha_{0}>K_{2}$. So, when $\operatorname{vol}(X)>K_{2}^{4}$ and $\alpha_{1}>1000, \varphi_{m}$ is birational for all $m \geqslant 15$.

Step 2. - Slightly shrinking $U$ if necessary, we may assume that, for each $V_{y}$ parametrized by $U$, the number $\operatorname{vol}\left(V_{y}\right) \leqslant 1000^{3}$. By McKernan's idea [20] and particularly Todorov [25, Lemma 3.2] (see also [25, p. 1328]), we have the surjective morphism $\pi: X^{\prime \prime} \longrightarrow X$ from a nonsingular projective variety $X^{\prime \prime}$ onto $X$ and the following diagram:

where $f: X^{\prime \prime} \longrightarrow B$ is a fibration onto the smooth curve $B$. We organize the argument by distinguishing two cases:

Subcase 2.1. $\pi$ is not birational. - Clearly, passing through a very general point $x_{1}$, there are at least two $l c$ centers, say $V_{x_{1}}, V_{y}$ corresponding to two elements of $U$. According to Lemma 2.3, one may find a rational number $\widetilde{a}_{2}<3 a_{0}+\varepsilon$ and an effective divisor $\widetilde{D}_{2} \sim \widetilde{a}_{2} A$ such that the pair $\left(X, \widetilde{D}_{2}\right)$ satisfies condition $\left(*_{2}\right)$. Parallel to the situation of Case 1 and apply Variant $4.2(2)$ once more, we may find an effective $\mathbb{Q}$-divisor $\widetilde{D}_{4} \sim \widetilde{a}_{4} A$ such that $\left(X, \widetilde{D}_{4}\right)$ satisfies the condition $\left(*_{4}\right)$, where $\widetilde{a}_{4}>0$ is a rational number with:

$$
\begin{align*}
\widetilde{a}_{4} & <2(1+2 \sqrt{2}) \widetilde{a}_{2}+(8 \sqrt{2}+2)+3 \varepsilon  \tag{4.5}\\
& <(8 \sqrt{2}+2)+\frac{24(1+2 \sqrt{2}) \sqrt[4]{2}}{\alpha_{0}}+(12 \sqrt{2}+9) \varepsilon . \tag{4.6}
\end{align*}
$$

One can find a computable constant $K_{3}>0$ such that the right hand side of Inequality (4.6) is upper bounded by $\lceil 8 \sqrt{2}+2\rceil=14$ whenever $\alpha_{0}>K_{3}$. Therefore, when $\operatorname{vol}(X)>K_{3}^{4}, \varphi_{m}$ is birational for all $m \geqslant 15$.

Subcase 2.2. $\pi$ is birational. - For simplicity, we may simply assume $X^{\prime \prime}=X$. By construction, the fiber of $f$ passing through a very general point $x_{1}$ is exactly the center $V_{x_{1}} \subset \operatorname{Nklt}\left(X, D_{0}^{\left(x_{1}, x_{2}\right)}\right)$ with $D_{0} \sim a_{0} A$ and $a_{0}<\frac{4 \sqrt[4]{2}}{\alpha_{0}}+\varepsilon$. By our assumption $\operatorname{vol}\left(V_{x_{1}}\right) \leqslant 1000^{3}$. Now we take $\mathfrak{X}=\mathfrak{X}_{3,1000^{3}}$, which is a birationally bounded family. By Theorem 3.7 and Corollary 3.8, there exists a constant $K_{4}$ such that, whenever $\operatorname{vol}(X)>K_{4}^{4}$, $\varphi_{m, X}$ is birational for all $m \geqslant r_{3}$.

By Chen-Chen [5], we know that $27 \leqslant r_{3} \leqslant 61$. Take $K(4)=$ $\max \left\{K_{1}^{4}, K_{2}^{4}, K_{3}^{4}, K_{4}^{4}\right\}$. Then we have seen that $\varphi_{m, X}$ is birational whenever $\operatorname{vol}(X)>K(4)$. So we have proved the theorem in dimension 4 .

The above argument has the following direct corollary.
Corollary 4.4. - There exists a constant $\hat{K}(4)>0$. For any smooth projective 4-fold $X$ with $p_{g}(X)>0$ and $\operatorname{vol}(X)>\hat{K}(4), \varphi_{m}$ is birational for $m \geqslant \max \left\{15, r_{3}^{+}\right\}$. In particular, $\varphi_{18, X}$ is birational.

Proof. - Note that all argument in Subsection 4.3 follows except Subcase 2.2 where $r_{3}$ should be replaced by $r_{3}^{+}$as the general fiber of $f$ has positive geometric genus.

Since $14 \leqslant r_{3}^{+} \leqslant 18$ as mentioned in Introduction, the statement follows.

### 4.4. Proof of Theorem 1.3 (the general case)

Assume $\operatorname{dim}(X)>4$. The argument is similar in the spirit to that of 4.3. Thus we will omit most of the parallel details.

Given a number $\varepsilon$ with $0<\varepsilon \ll 1$. We still write $K_{X} \sim_{\mathbb{Q}} A+E$ for some ample $\mathbb{Q}$-divisor $A$ with $0<\operatorname{vol}(X)-\operatorname{vol}(A) \ll \varepsilon$ and an effective $\mathbb{Q}$-divisor $E$ on $X$. Pick two very general points $x_{1}, x_{2} \in X$.

By Lemma 2.2, Lemma 2.3 and Proposition 4.1, we may take an effective $\mathbb{Q}$-divisor $D_{0}=D_{0}^{\left(x_{1}, x_{2}\right)} \sim_{\mathbb{Q}} a_{0} A$ with $a_{0}<\frac{n \sqrt[n]{2}}{\alpha_{0}}+\varepsilon$ so that $\left(X, D_{0}\right)$ satisfies the condition $\left(*_{1}\right)$ of Proposition 4.1. We may simply assume that $\left(X, D_{0}^{\left(x_{1}, x_{2}\right)}\right)$ is lc at $x_{1}$ modulo an exchanging of indices and that there exists a unique lc center $V_{x_{1}}$ passing through $x_{1}$.

Similarly we consider the Hilbert scheme $\mathscr{H}_{a}$, for $a<\frac{n \sqrt[n]{2}}{\alpha_{0}}+\varepsilon$, and obtain an irreducible scheme $U$ as in 4.3. We may still assume that the scheme $U$ allows to obtain a constant $r_{0}=\operatorname{dim} V_{x_{1}}$ where $\left(x_{1}, x_{2}, V_{x_{1}}, D_{0}^{\left(x_{1}, x_{2}\right)}\right) \in U$. Clearly $0 \leqslant r_{0} \leqslant n-1$.

Case I. - Assume $r_{0} \leqslant n-2$.
This means that the pair $\left(X, D_{0}^{\left(x_{1}, x_{2}\right)}\right)$ satisfies the condition $\left(*_{2}\right)$ of Proposition 4.1. Set $\widetilde{a}_{2}=a_{0}$. Repeatedly applying Variant 4.2 (2), we may find an effective $\mathbb{Q}$-divisor $\widetilde{D}_{n} \sim \widetilde{a}_{n} A$ such that $\left(X, \widetilde{D}_{n}\right)$ satisfies the condition $\left(*_{n}\right)$, where $\widetilde{a}_{n}>0$ is a rational number with:

$$
\begin{equation*}
\widetilde{a}_{n}<\left(\frac{n \sqrt[n]{2}}{\alpha_{0}}+2\right) \prod_{j=2}^{n-1}\left(1+\frac{\sqrt[n-j]{2}(n-j)}{\alpha_{j}}\right)-2+l(\varepsilon) \tag{4.7}
\end{equation*}
$$

with $l(\varepsilon) \mapsto 0$ as $\varepsilon \mapsto 0$. Clearly one can find a computable constant $\hat{K}_{1}>0$ such that the right hand side of Inequality (4.7) is strictly smaller than $u_{1}=\left\lfloor 2 \prod_{j=2}^{n-1}\left(1+\frac{\sqrt[n-j]{2}(n-j)}{\alpha_{j}}\right)\right\rfloor-1$ whenever $\alpha_{0}>\hat{K}_{1}$. Therefore, when $\operatorname{vol}(X)>\hat{K}_{1}^{4}, \varphi_{m, X}$ is birational for all $m \geqslant u_{1}+1$.

Case II. - Assume $r_{0}=n-1$.
Set $a_{1}=a_{0}$. According to Variant 4.2 (2) and Inequality (4.2), we may find an effective $\mathbb{Q}$-divisor $D_{n} \sim a_{n} A$ such that $\left(X, D_{n}\right)$ satisfies the condition $\left(*_{n}\right)$, where

$$
\begin{equation*}
a_{n}<\left(\frac{\sqrt[n]{2} n}{\alpha_{0}}+2\right) \prod_{j=1}^{n-1}\left(1+\frac{\sqrt[n-j]{2}(n-j)}{\alpha_{j}}\right)-2+\varepsilon \tag{4.8}
\end{equation*}
$$

Step i (The case with $\alpha_{1}>T$ ). - From Inequality (4.8), it is clear that there are two constant $T>0$ and $\hat{K}_{2}>0$ such that, whenever $\alpha_{1}>T$ and $\alpha_{0}>\hat{K}_{2}$, the right side of Inequality (4.8) is strictly smaller than $u_{1}$. Thus, meanwhile, $\varphi_{m, X}$ is birational for all $m \geqslant u_{1}+1$.

Step ii (The case with $\alpha_{1} \leqslant T$ ). - Slightly shrinking $U$ if necessary, we may assume that, for each $V_{y}$ parametrized by $U$, the number $\operatorname{vol}\left(V_{y}\right) \leqslant$ $T^{n-1}$. By Todorov [25, Lemma 3.2] (see also [25, p. 1328]), we have the surjective morphism $\pi: X^{\prime \prime} \longrightarrow X$ from a nonsingular projective variety $X^{\prime \prime}$ onto $X$ and the following diagram:

where $f: X^{\prime \prime} \longrightarrow B$ is a fibration onto the smooth curve $B$. We organize the argument by distinguishing two cases:

Subcase ii. 1 ( $\pi$ is not birational). - Clearly, passing through a very general point $x_{1}$, there are at least two $l c$ centers, say $V_{x_{1}}, V_{y}$ corresponding to two elements of $U$. According to Lemma 2.3, one may find a rational number $\widetilde{a}_{2}<3 a_{0}+\varepsilon$ and an effective $\mathbb{Q}$-divisor $\widetilde{D}_{2} \sim \widetilde{a}_{2} A$ such that the pair $\left(X, \widetilde{D}_{2}\right)$ satisfies condition $\left(*_{2}\right)$. Parallel to the situation of Case I and apply Variant $4.2(2)$ once more, we may find an effective $\mathbb{Q}$-divisor $\widetilde{D}_{n} \sim \widetilde{a}_{n} A$ such that $\left(X, \widetilde{D}_{n}\right)$ satisfies the condition $\left(*_{n}\right)$, where $\widetilde{a}_{n}>0$ is a rational number with (compare Inequality (4.7)):

$$
\begin{equation*}
\widetilde{a}_{n}<\left(\frac{3 n \sqrt[n]{2}}{\alpha_{0}}+2\right) \prod_{j=2}^{n-1}\left(1+\frac{\sqrt[n-j]{2}(n-j)}{\alpha_{j}}\right)-2+h(\varepsilon) \tag{4.9}
\end{equation*}
$$

with $h(\varepsilon) \mapsto 0$. One can find a computable constant $\hat{K}_{3}>0$ such that the right hand side of Inequality (4.9) is strictly smaller than $u_{1}$ whenever $\alpha_{0}>\hat{K}_{3}$. Therefore, when $\operatorname{vol}(X)>\hat{K}_{3}^{4}, \varphi_{m}$ is birational for all $m \geqslant u_{1}+1$.

Subcase ii. 2 ( $\pi$ is birational). - For simplicity, we may simply assume $X^{\prime \prime}=X$. By construction, the fiber of $f$ passing through a very general point $x_{1}$ is exactly the center $V_{x_{1}} \subset \operatorname{Nklt}\left(X, D_{0}^{\left(x_{1}, x_{2}\right)}\right)$ with $D_{0} \sim a_{0} A$ and $a_{0}<\frac{n \sqrt[n]{2}}{\alpha_{0}}+\varepsilon$. By our assumption, $\operatorname{vol}\left(V_{x_{1}}\right) \leqslant T^{n-1}$. Now we take $\mathfrak{X}=\mathfrak{X}_{n, T^{n-1}}$, which is a birationally bounded family. By Theorem 3.7 and Corollary 3.8, there exists a constant $\hat{K}_{4}$ such that, whenever $\operatorname{vol}(X)>\hat{K}_{4}^{n}$, $\varphi_{m, X}$ is birational for all $m \geqslant r_{n-1}$.

Finally take $K(n)=\max \left\{\hat{K}_{1}^{n}, \hat{K}_{2}^{n}, \hat{K}_{3}^{n}, \hat{K}_{4}^{n}\right\}$. What we have proved is that $\varphi_{m, X}$ is birational for

$$
m \geqslant \max \left\{r_{n-1}, u_{1}+1\right\}
$$

whenever $\operatorname{vol}(X)>K(n)$. Since

$$
\left\lfloor 2 \prod_{j=2}^{n-1}\left(1+(n-j) \sqrt[n-j]{\frac{2}{v_{n-j}}}\right)\right\rfloor=u_{1}+1
$$

the main theorem is proved.

## 5. Proof of Theorem 1.5

### 5.1. Convention

Let $Z$ be any nonsingular projective variety of general type over an algebraically closed field $k$ of characteristic 0 . By [1] or [23], we know that the canonical ring $R\left(Z, K_{Z}\right)$ is finitely generated over $k$. One can obtain the morphism $\pi_{Z}: Z \longrightarrow Z_{0}$ possibly after a birational modification of $Z$.

### 5.2. The canonical map $\varphi_{1}$

Let $X$ be a nonsingular projective $n$-fold of general type. Assume $p_{g}(X) \geqslant$ 2. Modulo further birational morphism, we may assume that $\operatorname{Mov}\left|K_{X}\right|$ is base point free. Taking the Stein factorization, we have the following diagram:

where $\varphi_{1}$ is the canonical morphism and $N=p_{g}(X)-1$. Denote by $H \in$ $\operatorname{Mov}\left|K_{X}\right|$ the general member. Then $h^{0}\left(X, K_{X}-H\right)=1$. Set $d=\operatorname{dim} \Gamma$.

### 5.3. Canonical restriction inequalities

Denote by $F$ a general fiber of $f$. Unless $d=n, F$ is a smooth projective variety of dimension $n-d>0$. We then take $d-1$ general hyperplane sections $H_{1}, \ldots, H_{d-1}$ on $s(\Gamma)$. Set the curve $W=H_{1} \cap \cdots \cap H_{d-1}, X_{W}=$ $f^{-1}(W)$ and $X_{H_{i}}=f^{-1}\left(H_{i}\right)$ for each $i$. We may assume that $X_{W}$ and $X_{H_{i}}$ are all nonsingular for each $i$. Clearly we have $X_{H_{i}} \sim H$ for each $i=1, \ldots, d-1$.

Consider the fibration $f_{W}: X_{W} \rightarrow W$ with $F$ a general fiber. First we write $\left.H\right|_{X_{W}}=\sum_{j} F_{j} \equiv a F$ for some smooth fibers $F_{j}$ of $f$, where $a=H_{1}^{d} \geqslant p_{g}(X)-d$ (see for instance [17, Lemma 1.3]). By Theorem 2.4, we have

$$
\begin{equation*}
\left.\left|m\left(K_{X_{W}}+\left.\frac{1}{a} H\right|_{X_{W}}\right)\right|\right|_{F}=\left.\left|m\left(K_{X_{W}}+F\right)\right|\right|_{F}=\left|m K_{F}\right| \tag{5.1}
\end{equation*}
$$

for sufficiently large and divisible integer $m$.
Note also that, $\left.K_{X_{W}} \sim\left(K_{X}+(d-1) H\right)\right|_{X_{W}} \leqslant\left.\left(d K_{X}\right)\right|_{X_{W}}$. Moreover, for a sufficiently large and divisible integer $m$, Theorem $2.4(2)$ implies:

$$
\begin{align*}
& \left.\left|m\left(K_{X}+\left(d-1+\frac{1}{a}\right) H\right)\right|\right|_{X_{W}}  \tag{5.2}\\
& =\left.\left|m\left(K_{X}+\sum_{1 \leqslant i \leqslant d-1} X_{H_{i}}+\frac{1}{a} H\right)\right|\right|_{X_{W}} \\
& =\left.\left.\left|m\left(K_{X}+X_{H_{1}}\right)+m\left(\sum_{2 \leqslant i \leqslant d-1} X_{H_{i}}+\frac{1}{a} H\right)\right|\right|_{X_{H_{1}}}\right|_{X_{W}}  \tag{5.3}\\
& =\left.\left|m\left(K_{X_{H_{1}}}+\left.\left(\sum_{2 \leqslant i \leqslant d-1} X_{H_{i}}+\frac{1}{a} H\right)\right|_{X_{H_{1}}}\right)\right|\right|_{X_{W}}  \tag{5.4}\\
& =\left.\left|m\left(K_{X_{H_{1} \cap \ldots \cap H_{d-2}}}+\left.\left(X_{d-1}+\frac{1}{a} H\right)\right|_{X_{H_{1} \cap \ldots \cap H_{d-2}}}\right)\right|\right|_{X_{W}} \\
& =\left|m\left(K_{X_{W}}+\left.\frac{1}{a} H\right|_{X_{W}}\right)\right| \tag{5.5}
\end{align*}
$$

where $X_{H_{1} \cap \cdots \cap H_{d-2}}=f^{-1}\left(H_{1} \cap \cdots \cap H_{d-2}\right)$. Since $K_{X}-H$ is effective, we conclude from (5.1) and (5.5) that

$$
\left.\left|\frac{m(a d+1)}{a} K_{X}\right|\right|_{F} \succeq\left|m K_{F}\right| .
$$

By the base point free theorem, we have

$$
\begin{aligned}
\operatorname{Mov}\left|\frac{m(a d+1)}{a} K_{X}\right| & =\left|\pi_{X}^{*}\left(\frac{m(a d+1)}{a} K_{X_{0}}\right)\right| \\
\operatorname{Mov}\left|m K_{F}\right| & =\left|\pi_{F}^{*}\left(m K_{F_{0}}\right)\right|
\end{aligned}
$$

Thus

$$
\begin{equation*}
\left.\pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F} \geqslant \frac{a}{1+a d} \pi_{F}^{*}\left(K_{F_{0}}\right) . \tag{5.6}
\end{equation*}
$$

Finally, noting that $\pi_{X}^{*}\left(K_{X_{0}}\right) \geqslant H$, we have

$$
\begin{align*}
\operatorname{vol}(X) & =\left(\pi_{X}^{*}\left(K_{X_{0}}\right)\right)_{X}^{n} \geqslant\left(H^{d} \cdot \pi_{X}^{*}\left(K_{X_{0}}\right)^{n-d}\right)_{X} \\
& =a \cdot\left(\left.\pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right)_{F}^{n-d} \geqslant a \cdot\left(\frac{a}{1+a d}\right)^{n-d} \cdot \pi_{F}^{*}\left(K_{F_{0}}\right)^{n-d} \\
& =a \cdot\left(\frac{a}{1+a d}\right)^{n-d} \cdot \operatorname{vol}(F)  \tag{5.7}\\
& \geqslant \frac{v_{n-d}}{(d+1)^{n-d}} \cdot\left(p_{g}(X)-d\right) \tag{5.8}
\end{align*}
$$

Corollary 5.1. - There exist positive constants $a_{n}$ and $b_{n}$ such that

$$
\operatorname{vol}(X) \geqslant a_{n} p_{g}(X)-b_{n}
$$

holds for any smooth projective $n$-fold $X$ of general type.
Proof. - Keep above settings. When $d=n$, one has $\operatorname{vol}(X) \geqslant 2 p_{g}(X)-$ $2 n$ by [17, Proposition (2.1)]. Thus we have

$$
\begin{equation*}
\operatorname{vol}(X) \geqslant \min _{1 \leqslant d \leqslant n-1}\left\{2 p_{g}(X)-2 n, \frac{v_{n-d}}{(d+1)^{n-d}}\left(p_{g}(X)-d\right)\right\} \tag{5.9}
\end{equation*}
$$

### 5.4. Key reduction steps

We keep the same setting as in Subsection 5.2. We always have an induced fibration $f=f_{W}: X_{W} \rightarrow W$ with the general fiber $F$ a smooth projective $(n-d)$-fold of general type. Assume $d<n$. We may write

$$
\left.\pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{X_{W}}=\sum_{i=1}^{a} F_{i}+E_{W} \equiv a F+E_{W}
$$

where $E_{W}$ is an effective $\mathbb{Q}$-divisor on $X_{W}$, each $F_{i}$ is a smooth fiber of $f$ and $a \geqslant p_{g}(X)-d$. For any integer $m$ with $m>d+\frac{1}{a}$, we may consider the linear system

$$
\begin{equation*}
\left|K_{X}+\left\lceil\left(m-d-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right\rceil+(d-1) H\right| \preceq\left|m K_{X}\right| . \tag{5.10}
\end{equation*}
$$

The $\mathbb{Q}$-divisor $\left.\left(m-d-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}$ is nef and big and after further modification of $X$, we may and do assume that $\pi_{X}^{*}\left(K_{X_{0}}\right)$ has simple normal crossing support. Kawamata-Viehweg vanishing theorem ([14, 27]) implies

$$
\begin{equation*}
\left.\left|m K_{X}\right|\right|_{F} \succeq\left|K_{F}+\left\lceil\left.\left(m-d-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right\rceil\right| . \tag{5.11}
\end{equation*}
$$

Our key theorem will essentially work under the following assumption:
(*) Assume that, on a general fiber $F$, there are $l=n-d-1$ positive integers, say $n_{1}, \ldots, n_{l}$, such that, for each $i=1, \ldots, l$, $\operatorname{dim} \overline{\Phi_{\left|L_{i}\right|}(F)} \geqslant i$ where $L_{i}=\operatorname{Mov} \backslash\left\lfloor\left. n_{i} \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right\rfloor \mid$.

Modulo further necessary birational modifications, we may and do assume that, for each $i,\left|L_{i}\right|$ is base point free. For an integer $m>d+\frac{1}{a}+$ $\sum_{i} n_{i}$, we have

$$
\begin{align*}
\mid K_{F}+ & \left.\left|\left(m-d-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right] \mid  \tag{5.12}\\
& \succeq\left|K_{F}+\left[\left.\left(m-d-\frac{1}{a}-\sum_{i=1}^{l} n_{i}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right]+\sum_{i=1}^{l} L_{i}\right|
\end{align*}
$$

Take $V^{(1)}$ to be a generic irreducible element of $\left|L_{1}\right|$ on $F$. Since $\left|L_{1}\right|$ is base point free, $V^{(1)}$ is smooth. For $j=2, \ldots, l$, take $V^{(j)}$ to be a generic irreducible element of $\left|L_{j}\right|_{V^{(j-1)}} \mid$. Each $V^{(j)}$ is smooth simply due to the base point freeness of $\left|L_{j}\right|$ for each $j$. Set $C=V^{(l)}$, which is a smooth complete curve by our assumption that $\operatorname{dim} \overline{\Phi_{\left|L_{l}\right|}(F)} \geqslant l$.

By the Kawamata-Viehweg vanishing theorem, for each $j=1, \ldots, l$, we have

$$
\begin{align*}
& \left\lvert\, K_{F}+\left[\left.\left(m-d-\frac{1}{a}-\sum_{i=1}^{l} n_{i}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right]\right.+\sum_{i=1}^{l} L_{i}| |_{V^{(j)}}  \tag{5.13}\\
& \succeq \left\lvert\, K_{V^{(j-1)}}+\left[\left.\left(m-d-\frac{1}{a}-\sum_{i=1}^{l} n_{i}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{V^{(j-1)}}\right]\right. \\
&+\left.\sum_{i=j}^{l} L_{i}\right|_{V^{(j-1)}}| |_{V^{(j)}} \\
& \succeq \left\lvert\, K_{V^{(j)}}+\left[\left.\left(m-d-\frac{1}{a}-\sum_{i=1}^{l} n_{i}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{V^{(j)}}\right]\right. \\
&+\left.\sum_{i=j+1}^{l} L_{i}\right|_{V^{(j)}} \mid
\end{align*}
$$

Particularly, when $j=l$, we simply get

$$
\begin{equation*}
\left.\left|m K_{X}\right|\right|_{C} \succeq\left|K_{C}+D_{m}\right| \tag{5.14}
\end{equation*}
$$

where $D_{m}=\left\lceil\left.\left(m-d-\frac{1}{a}-\sum_{i=1}^{l} n_{i}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{C}\right\rceil$ is a divisor on $C$ with

$$
\operatorname{deg}\left(D_{m}\right) \geqslant\left(m-d-\frac{1}{a}-\sum_{i=1}^{l} n_{i}\right)\left(\pi_{X}^{*}\left(K_{X_{0}}\right) \cdot C\right)
$$

Set $\eta=\left(\pi_{X}^{*}\left(K_{X_{0}}\right) \cdot C\right)$ and $\alpha_{m}=\left(m-d-\frac{1}{a}-\sum_{i=1}^{l} n_{i}\right) \eta$. Clearly, since $\pi_{X}^{*}\left(K_{X_{0}}\right)$ is nef and big and $C$ is moving in an algebraic family, we see $\eta>0$.

Theorem 5.2. - Let $m>1$ be an integer. Let $X$ be a nonsingular projective $n$-fold ( $n \geqslant 4$ ) with $p_{g}(X) \geqslant 2$. Keep the same notation as above. Assume $d<n$ and that Assumption (*) holds. Then
(1) under the condition $\alpha_{m}>1$, we have

$$
m \eta \geqslant \operatorname{deg}\left(K_{C}\right)+\left\lceil\left(m-d-\frac{1}{a}-\sum_{i=1}^{l} n_{i}\right) \eta\right\rceil
$$

Consequently one has

$$
\eta \geqslant \frac{\operatorname{deg}\left(K_{C}\right)}{d+\frac{1}{a}+\sum_{i=1}^{l} n_{i}}
$$

(2) $\varphi_{m, X}$ is birational provided that $\alpha_{m}>2$.

This theorem is in fact parallel to the key theorem in dimension 3 (see, for instance, Chen-Chen [4, Subsection 2.10]). So we just explain the central part while omitting those redundant details. We will tacitly use the following lemma in reducing the birationality to lower dimensional case:

Lemma 5.3. - Let $V$ be a smooth projective variety with $p_{g}(V)>0$. Let $Q$ be an effective nef and big $\mathbb{Q}$-divisor on $V$. Assume that $|L|$ is a base point free linear system on $V$. Then $\left|K_{V}+\lceil Q\rceil+L\right|$ distinguishes different generic irreducible elements of $|L|$. In other words, $\Phi_{\left|K_{V}+\lceil Q\rceil+L\right|}$ is birational if and only if so is $\left.\Phi_{\left|K_{V}+\lceil Q\rceil+L\right|}\right|_{Y}$ for each generic irreducible element $Y$ of $|L|$.

Proof. - When $|L|$ is not composed of an irrational pencil, since $\mid K_{V}+$ $\lceil Q\rceil+L|\succeq| L \mid, \Phi_{\left|K_{V}+\lceil Q\rceil+L\right|}$ automatically distinguishes different irreducible elements of $|L|$.

When $|L|$ is composed of an irrational pencil, we may assume that $Q$ has simple normal crossing supports after a further modification of $X$. Pick two different generic irreducible elements, say $Y_{1}$ and $Y_{2}$, of $|L|$. Then $L-Y_{1}-Y_{2}$ is nef. Then it follows from the Kawamata-Viehweg vanishing theorem that one has the surjective map:

$$
H^{0}\left(V, K_{V}+\lceil Q\rceil+L\right) \rightarrow H^{0}\left(Y_{1}, K_{Y_{1}}+\left.\lceil Q\rceil\right|_{Y_{1}}\right) \oplus H^{0}\left(Y_{2}, K_{Y_{1}}+\left.\lceil Q\rceil\right|_{Y_{2}}\right)
$$

where $H^{0}\left(Y_{i}, K_{Y_{i}}+\lceil Q\rceil_{Y_{i}}\right) \neq 0$ for $i=1,2$. Thus Lemma 5.3 holds.

## Proof of Theorem 5.2

Since $p_{g}(X)>0$, frequent applications of Lemma 5.3 tells us that $\varphi_{m, X}$ is birational if and only if $\left.\varphi_{m, X}\right|_{C}$ is birational for any irreducible component $C$ in $\left|L_{l}\right|_{V^{(l-1)}} \mid$. Hence we naturally refer to Relation (5.14).

When $\operatorname{deg}\left(D_{m}\right)>1,\left|K_{C}+D_{m}\right|$ is base point free. Thus Relations (5.10) and (5.14) imply that

$$
\begin{equation*}
m\left(\pi_{X}^{*}\left(K_{X_{0}}\right) \cdot C\right) \geqslant \operatorname{deg}\left(K_{C}\right)+\operatorname{deg}\left(D_{m}\right) \tag{5.15}
\end{equation*}
$$

which proves (1). One may always take a very large number $m^{\prime}$ so that $\operatorname{deg}\left(D_{m^{\prime}}\right)>1$. Then Inequality (5.15) reads $\eta \geqslant \frac{\operatorname{deg}\left(K_{C}\right)}{d+\frac{1}{a}+\sum_{i=1}^{l} n_{i}}$ by eliminating $m^{\prime}$.

When $\operatorname{deg}\left(D_{m}\right)>2,\left|K_{C}+D_{m}\right|$ gives a birational map and so is $\varphi_{m, X}$. So the theorem holds true.

Applying the above method to the extremal case with $d=n$, we obtain the following corollary which might be known to experts.

Corollary 5.4. - Let $X$ be a nonsingular projective $n$-fold ( $n \geqslant 3$ ) such that $\varphi_{1, X}$ is generically finite map. Then $\varphi_{m, X}$ is birational for $m \geqslant$ $n+2$.

Proof. - We are in the situation $d=n$. So $f_{W}: X_{W} \rightarrow W$ is a finite map where $\widetilde{C}=X_{W}$ is a smooth projective curve. By the vanishing theorem, we have

$$
\begin{aligned}
\left.\left|m K_{X}\right|\right|_{\widetilde{C}} & \left.\succeq\left|K_{X}+\left\lceil(m-d) \pi_{X}^{*}\left(K_{X_{0}}\right)\right\rceil+H_{d-1}+\cdots+H_{1}\right|\right|_{\widetilde{C}} \\
& \succeq\left|K_{\widetilde{C}}+\left\lceil(m-d) \pi_{X}^{*}\left(K_{X_{0}}\right) \mid \widetilde{C}\right\rceil\right|
\end{aligned}
$$

Repeatedly using Lemma 5.3, we can see that $\varphi_{m, X}$ is birational if and only if so is $\left.\varphi_{m, X}\right|_{\widetilde{C}}$ for a general member $\widetilde{C} \in\left|X_{W}\right|$.

We have $\left.\operatorname{deg} \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{\widetilde{C}} \geqslant 2$, since $\varphi_{1, X}$ is generically finite. So $\mid K_{\widetilde{C}}+$ $\left\lceil\left.(m-d) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{\widetilde{C}}\right\rceil \mid$ gives a birational map when $m-d \geqslant 2$. Thus $\varphi_{m}$ is birational for all $m \geqslant n+2$.

Corollary 5.5. - Let $X$ be a nonsingular projective $n$-fold ( $n \geqslant 3$ ) with $\operatorname{dim} \overline{\varphi_{1, X}(X)}=n-1$. Then $\varphi_{m, X}$ is birational for $m \geqslant 2 n+1$.

Proof. - Under the assumption, $F$ is a smooth projective curve of genus $\geqslant 2$. The situation fits into the consideration of Theorem 5.2. We have $C=F, n_{i}=0$ for each $i$, and $d=n-1$.

Pick a general fiber $F$ of $f$, we need to study the linear system

$$
\left|K_{F}+\left\lceil\left.\left(m-d-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right\rceil\right|
$$

by virtue of Relation (5.11). According to Inequality (5.6), we have

$$
\left.\operatorname{deg} \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F} \geqslant \frac{2 a}{a(n-1)+1}
$$

noting that $F=F_{0}$ is a smooth complete curve of genus $\geqslant 2$.
Whenever $m \geqslant 2 n+1$, we have

$$
\alpha_{m} \geqslant\left.\left(m-d-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F} \geqslant\left(n+2-\frac{1}{a}\right) \cdot \frac{2 a}{a(n-1)+1}>2 .
$$

Thus, by Theorem 5.2, $\varphi_{m, X}$ is birational when $m \geqslant 2 n+1$.
We need the following simple lemma to treat the case with $d=n-2$.

Lemma 5.6. - Let $F$ be smooth projective surface of general type. Then, for any $\mathbb{Q}$-divisor $Q_{\lambda}$ satisfying $Q_{\lambda} \equiv \lambda \pi_{F}^{*}\left(K_{F_{0}}\right)$ with $\lambda>3, \mid K_{F}+$ $\left\lceil Q_{\lambda}\right\rceil \mid$ gives a birational map.

Proof. - When $\left(K_{F_{0}}^{2}, p_{g}\left(F_{0}\right)\right) \neq(1,2)$, this is due to Chen-Chen [5, Lemma 2.3, Lemma 2.5].

When $\left(K_{F_{0}}^{2}, p_{g}\left(F_{0}\right)\right)=(1,2),\left|K_{F}\right|$ is composed of a rational pencil of genus 2 curves. Modulo a further blow up, we may assume that $\operatorname{Mov}\left|K_{F}\right|$ is base point free. Pick two different generic irreducible elements $C$ and $C_{1}$ of $\left|K_{F}\right|$. Write

$$
\pi_{F}^{*}\left(K_{F_{0}}\right) \sim C+E_{0}
$$

for an effective divisor $E_{0}$ on $F$. Then since

$$
Q_{\lambda}-C-C_{1}-2 E_{0} \equiv(\lambda-2) \pi_{F}^{*}\left(K_{F_{0}}\right)
$$

is nef and big, Kawamata-Viehweg vanishing theorem implies the surjective map:

$$
H^{0}\left(F, K_{F}+\left\lceil Q_{\lambda}-2 E_{0}\right\rceil\right) \longrightarrow H^{0}\left(C, K_{C}+D\right) \oplus H^{0}\left(C_{1}, K_{C_{1}}+D_{1}\right)
$$

where $D=\left.\left(\left\lceil Q_{\lambda}-2 E_{0}-C\right\rceil\right)\right|_{C}$ and $D_{1}=\left.\left(\left\lceil Q_{\lambda}-2 E_{0}-C_{1}\right\rceil\right)\right|_{C_{1}}$. Note that $|C|$ is a free pencil. Clearly we have

$$
\operatorname{deg}(D) \geqslant(\lambda-2)\left(\pi_{F}^{*}\left(K_{F_{0}}\right) \cdot C\right)>0
$$

and, similarly, $\operatorname{deg}\left(D_{1}\right)>0$. So $H^{0}\left(K_{C}+D\right) \neq 0$ and $H^{0}\left(C_{1}, K_{C_{1}}+D_{1}\right) \neq$ 0 . This means that $\left|K_{F}+\left\lceil Q_{\lambda}\right\rceil\right|$ distinguishes different irreducible elements of $|C|$.

Applying Kawamata-Viehweg vanishing theorem, we have the surjective map

$$
H^{0}\left(K_{F}+\left\lceil Q_{\lambda}-E_{0}\right\rceil\right) \longrightarrow H^{0}\left(K_{C}+\left.\left\lceil Q_{\lambda}-E_{0}-C\right\rceil\right|_{C}\right)
$$

where

$$
\operatorname{deg}\left(\left.\left\lceil Q_{\lambda}-E_{0}-C\right\rceil\right|_{C}\right) \geqslant(\lambda-1)\left(\pi_{F}^{*}\left(K_{F_{0}}\right) \cdot C\right)=\lambda-1>2
$$

Thus $\left.\Phi_{\left|K_{F}+\left\lceil Q_{\lambda}-E_{0}\right\rceil\right|}\right|_{C}$ is birational. So is $\Phi_{\left|K_{F}+\left\lceil Q_{\lambda}\right\rceil\right| \cdot}$.
Corollary 5.7. - Let $X$ be a nonsingular projective $n$-fold ( $n \geqslant 3$ ) with $\operatorname{dim} \overline{\varphi_{1, X}(X)}=n-2$. Then $\varphi_{m, X}$ is birational for $m \geqslant 4 n-3$.

Proof. - By assumption, $F$ is a smooth projective surface of general type and $p_{g}(F)>0$. By Inequality (5.6), we have

$$
\left.\pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F} \geqslant \frac{a}{a(n-2)+1} \pi_{F}^{*}\left(K_{F_{0}}\right)
$$

where $a \geqslant p_{g}(X)-d$. Referring to Relation (5.11), it is sufficient to study the linear system

$$
\left|K_{F}+\left\lceil\left.\left(m-n+2-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right\rceil\right| .
$$

Whenever $m \geqslant 4 n-3$, we have

$$
\left.\left(m-d-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F} \equiv \lambda \pi_{F}^{*}\left(K_{F_{0}}\right)+H_{\lambda}
$$

for some rational number $\lambda>3$ and an effective $\mathbb{Q}$-divisor $H_{\lambda}$. By Lemma 5.6, we see that $\left|K_{F}+\left\lceil\left.\left(m-d-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}-H_{\lambda}\right\rceil\right|$ gives a birational map. Hence $\varphi_{m, X}$ is birational when $m \geqslant 4 n-3$.

Remark 5.8. - With the same method, Corollary 5.5 and Corollary 5.7 can be easily improved when $p_{g}(X)$ is large enough:
(1) When $\operatorname{dim} \overline{\varphi_{1, X}(X)}=n-1$ and $p_{g}(X) \geqslant n+2, \varphi_{2 n-1, X}$ is birational.
(2) When $\operatorname{dim} \overline{\varphi_{1, X}(X)}=n-2$ and $p_{g}(X) \geqslant n+3, \varphi_{4 n-7, X}$ is birational.

### 5.5. Proof of Theorem 1.5 (the case $n=4$ )

By Corollary 5.4, Corollary 5.5 and Corollary 5.7, we know that $\varphi_{m, X}$ is birational whenever $m \geqslant 13$ and $d \geqslant 2$. Hence we only need to study the case with $d=1$.

In this case, we have $W=\Gamma$ and $X_{W}=X$. Pick a general fiber $F$. We may write

$$
\pi_{X}^{*}\left(K_{X_{0}}\right) \equiv p F+E
$$

where $E$ is an effective $\mathbb{Q}$-divisor and $p \geqslant p_{g}(X)-1$. For any integer $m>1$, the vanishing theorem implies

$$
\begin{align*}
\left.\left|(m+1) K_{X}\right|\right|_{F} & \left.\succeq\left|K_{X}+\left\lceil(m-1) \pi_{X}^{*}\left(K_{X_{0}}\right)\right\rceil+F\right|\right|_{F} \\
& \succeq\left|K_{F}+\left\lceil\left.(m-1) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right\rceil\right| . \tag{5.16}
\end{align*}
$$

Case 1. - $\operatorname{vol}(F)>12^{3}$. By Theorem $0, \varphi_{5, F}$ is birational. First let us consider the case with $g(\Gamma)>0$. By Chen [7, Lemma 2.5], we have $\left.\pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F} \sim \pi_{F}^{*}\left(K_{F_{0}}\right)$. In particular, $\left.\pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}$ is a Cartier divisor. So Relation (5.16) implies that $\left.\varphi_{6, X}\right|_{F}$ is birational. Hence $\varphi_{m, X}$ is birational for all $m \geqslant 6$. Next let us consider the case with $g(\Gamma)=0$. Suppose $p \geqslant 5$. Then Theorem 2.4 (1) implies that

$$
\left.\left|6 K_{X}\right|\right|_{F} \succeq\left|5\left(K_{X}+F\right)\right|_{F}=\left|5 K_{F}\right|
$$

which means $\varphi_{6, X}$ is birational. In a word, $\varphi_{6, X}$ is birational when $p_{g}(X) \geqslant$ 6 and $\operatorname{vol}(F)>12^{3}$.

Case 2. - $\operatorname{vol}(F) \leqslant 12^{3}$. Take $\mathfrak{X}$ to be set of all those 3 -folds of general type with positive geometric genus and the canonical volume being upper bounded by $12^{3}$. By Theorem 1.2 , there exists a constant $c(\mathfrak{X})>0$ such that, whenever $\operatorname{vol}(X)>c(\mathfrak{X}), \varphi_{m, X}$ is birational for all $m \geqslant r_{3}^{+}$. Now by Corollary 5.1, there exists a constant $L(4)>0$ such that $p_{g}(X)>L(4)$ implies $\operatorname{vol}(X)>c(\mathfrak{X})$.

Since we know $r_{3}^{+} \geqslant 14$ as mentioned earlier, we have actually proved Theorem 1.5 in the case $n=4$.

### 5.6. Proof of Theorem 1.5 (the case $n=5$ )

By Corollary 5.4, Corollary 5.5 and Corollary 5.7, we know that $\varphi_{m, X}$ is birational whenever $m \geqslant 17$ and $d \geqslant 3$. Hence we only need to study the cases $d=1,2$.

Case I. $d=2$. - In this case, $F$ is a smooth projective 3-fold of general type with $p_{g}(F)>0$. By Relation (5.11), we need to study the $\mathbb{Q}$-divisor $Q_{m}=\left.\left(m-d-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}$ where $a \geqslant p_{g}(X)-d$. When $p_{g}(X) \geqslant 20$, we have $a \geqslant 18$. For $m \geqslant 35$, Inequality (5.6) implies

$$
Q_{m} \equiv\left(m-d-\frac{1}{a}\right) \cdot \frac{a}{2 a+1} \pi_{F}^{*}\left(K_{F_{0}}\right)+H_{m}
$$

where $\left(m-d-\frac{1}{a}\right) \cdot \frac{a}{2 a+1}>16$ and $H_{m}$ is an effective $\mathbb{Q}$-divisor. Thus, by Chen-Chen [5, Theorem 8.1], $\left|K_{F}+\left\lceil\left.\left(m-d-\frac{1}{a}\right) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right\rceil\right|$ gives a birational map. Thus $\varphi_{35, X}$ is birational whenever $p_{g}(X) \geqslant 20$.

Case II. $d=1$. - Similar to the discussion in Subsection 5.5, we still have the parallel relation to (5.16):

$$
\begin{equation*}
\left.\left.\left|(m+1) K_{X}\right|\right|_{F} \succeq\left|m K_{X}+F\right|\right|_{F} \succeq\left|K_{F}+\left\lceil\left.(m-1) \pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F}\right\rceil\right| \tag{5.17}
\end{equation*}
$$

Subcase II.1. - $\operatorname{vol}(F)>\hat{K}(4)$ where $\hat{K}(4)$ is the same constant as in Corollary 4.4. When $g(\Gamma)>0$, we have $\left.\pi_{X}^{*}\left(K_{X_{0}}\right)\right|_{F} \sim \pi_{F}^{*}\left(K_{F_{0}}\right)$ by Chen [7, Lemma 2.5]. Then both Relation (5.17) and Corollary 4.4 imply that $\varphi_{19, X}$ is birational. When $g(\Gamma)=0$, we work under the assumption $p_{g}(X) \geqslant 19$. Then the extension theorem 2.4 implies that

$$
\left.\left|19 K_{X}\right|\right|_{F} \succeq\left|18\left(K_{X}+F\right)\right|_{F}=\left|18 K_{F}\right|,
$$

which means $\varphi_{19, X}$ is birational. To make the summary, $\varphi_{19, X}$ is birational when $p_{g}(X) \geqslant 19$ and $\operatorname{vol}(F)>\hat{K}(4)$.

Subcase II.2. - $\operatorname{vol}(F) \leqslant \hat{K}(4)$. Take $\mathfrak{X}$ to be set of all those 4 -folds of general type with positive geometric genus and the canonical volume being upper bounded by $\hat{K}(4)$. By Theorem 1.2 , there exists a constant $c(\mathfrak{X})>0$ such that, whenever $\operatorname{vol}(X)>c(\mathfrak{X}), \varphi_{m, X}$ is birational for all $m \geqslant r_{4}^{+}$. Now by Corollary 5.1, there exists a constant $L(5)>0$ such that $p_{g}(X)>L(5)$ implies $\operatorname{vol}(X)>c(\mathfrak{X})$.

Since we know $r_{4}^{+} \geqslant 39$ as mentioned earlier, we have proved Theorem 1.5 in the case $n=5$.

## 6. Open problems and further discussion

Naturally we would like to propose the following conjectures:
Conjecture 6.1. - For any integer $n \geqslant 5$, there exists a constant $K(n)>0$ such that, for all smooth projective $n$-folds $X$ with $\operatorname{vol}(X)>$ $K(n), \varphi_{m, X}$ is birational for all $m \geqslant r_{n-1}$.

Conjecture 6.2. - For any integer $n \geqslant 6$, there exists a constant $L(n)$ such that, for all smooth projective $n$-folds $X$ with $p_{g}(X)>L(n)$, $\varphi_{m, X}$ is birational for all $m \geqslant r_{n-1}^{+}$.

It is also very interesting to compare two key numbers in the statement of Theorem 1.3. We have the following question which has an affirmative answer for $n=3,4$ :

Question 6.3. - For any $n \geqslant 5$, is it true that

$$
r_{n-1} \geqslant\left\lfloor 2 \prod_{j=2}^{n-1}\left(1+(n-j) \sqrt[n-j]{\frac{2}{v_{n-j}}}\right)\right\rfloor
$$

Clearly the positive answer to Question 6.3 implies that Conjecture 6.1 is true. Recently Gavin Brown kindly informed us of many interesting examples of canonical 4 -folds. In fact, in the paper [3], one can find the following 4-fold:

Example 6.4. - Let $\mathbb{P}=\mathbb{P}\left(a_{10}, a_{12}, a_{17}, a_{33}, a_{37}, a_{55}\right)$ be a weighted projective space, where the weight of the homogenous coordinate $a_{i}$ is $i$. Let $X \subset \mathbb{P}$ be a general hypersurface of degree 165 . Then $X$ has canonical singularities and $K_{X}=\mathscr{O}_{X}(1)$. Hence $X$ has the canonical volume $K_{X}^{4}=1 / 830280$. Since $X$ is general, the defining equation of $X$ can be written as $a_{55}^{3}+\{$ other terms $\}$. On the other hand, it is easy to check that the space $H^{0}\left(X, \mathscr{O}_{X}(93)\right)$ is generated by the variables $a_{10}, a_{12}, a_{17}, a_{33}, a_{37}$. Hence, for the degree- 165 hypersurface $X, \varphi_{93}$ is not birational and thus $r_{4} \geqslant 94$.

In the last part of this paper, we establish a general approach and show that a generalization of Theorem 1.2 would give an affirmative answer to Conjecture 6.1.

We begin with the following definition.
Definition 6.5. - Given a birationally bounded set $\mathfrak{X}$ of smooth projective varieties and given a positive number $c$, we say that a fibraion $f: X \rightarrow T$ between smooth projective varieties satisfies condition $(B)_{\mathfrak{X}, c}$ if
(1) a general fiber $F$ of $f$ is birationally equivalent to an element of $\mathfrak{X}$;
(2) for a general point $t \in T$, there exists an effective $\mathbb{Q}$-divisor $D_{t}$ with $D_{t} \sim_{\mathbb{Q}} \epsilon K_{X}$ for a positive rational number $\epsilon<c$, such that the fiber $F_{t}=f^{-1}(t)$ is an irreducible component of $\operatorname{Nklt}\left(X, D_{t}\right)$.

The following question is an analogue of Theorem 1.4 in higher dimensional base case.

Question 6.6. - Given a birationally bounded set $\mathfrak{X}$ and a positive integer $t$, does there exist a number $c=c(\mathfrak{X}, t)$ such that, for any smooth projective variety $X$ of general type and a fibration $f: X \rightarrow T$ satisfying $(B)_{\mathfrak{X}, c}$ and $\operatorname{dim} T=t$, the restriction map $H^{0}\left(X, m K_{X}\right) \rightarrow H^{0}\left(F_{1}, m K_{F_{1}}\right) \oplus$ $H^{0}\left(F_{2}, m K_{F_{2}}\right)$ is surjective for two arbitrary general fibers $F_{1}$ and $F_{2}$ of $f$ and for all $m \geqslant 2$.

## Remark 6.7.

(1) When $\operatorname{dim} T=1$, a fibration $f: X \rightarrow T$ satisfies property $(B)_{\mathfrak{x}, c}$ if and only if $F$ is birational to an element of $\mathfrak{X}$ and $\operatorname{vol}(X)>M_{c}$ for some positive integer $M_{c}$.
(2) However, when $\operatorname{dim} T \geqslant 2$, to work on the extension problem as Question 6.6, we cannot simply assume that $\operatorname{vol}(X) \gg \operatorname{vol}(F)$. For instance, take $X=C \times V \times F$ and $T=C \times V$, where $C$ is a smooth projective curve with genus $g(C)>1$ and $V$ and $F$ are smooth projective varieties of general type. Let $f: X \rightarrow T$ be the natural projection. For any $m>0$, we can choose certain $V$ and $F$ such that $p_{m}(V)=0$ and $p_{m}(F)>0$. Then, when $g(C)$ is large enough, $\operatorname{vol}(X) \gg \operatorname{vol}(F)$, however the restriction $\operatorname{map} H^{0}\left(X, m K_{X}\right) \rightarrow H^{0}\left(F, m K_{F}\right)$ is never surjective. Hence property (2) in Definition 6.5 is reasonable.

We now apply Takayama and McKernan's induction process once again to obtain the following unconditional result, which has its own flavor.

THEOREM 6.8. - Fix a function $\lambda: \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \rightarrow \mathbb{R}_{>0}$. There exists integers $M_{n-1}>M_{n-2}>\ldots>M_{1}>0$ and $K>0$ such that for any smooth projective $n$-fold $X$ with $\operatorname{vol}(X) \geqslant K$, all pluricanonical map $\varphi_{a}$ of $X$ is birational for $a \geqslant 2$, unless that, after birational modifications, $X$ admits a fibration $f: X \rightarrow Z$ which satisfies $(B)_{\mathfrak{X}_{k, M_{k}^{k}}, \lambda\left(k, M_{k}^{k}\right)}$ for some $1 \leqslant k \leqslant n-1$.

Remark 6.9. - Note that bicanonical map of surfaces has been intensively studied. In particular, Reider [21] proved that the bicanonical map of a smooth projective surface $S$ is always birational if $\operatorname{vol}(S) \geqslant 10$, unless $S$ presents the standard case, namely after birational modification of $S$, there exists a fibration $f: S \rightarrow B$ such that a general fiber of $f$ is a genus 2 curve. We can also work out the explicit (not optimal) bounds for 3 -folds and, once again, it seems rather hard to have an explicit version of the above theorem in dimension $\geqslant 4$.

Proof. - For each $1 \leqslant d \leqslant n-1$, we define the positive numbers $M_{d}$ in the following way. First, take $M_{1}$ to be a positive number such that

$$
2^{n-1}\left(\prod_{i=1}^{n-1}\left(1+\frac{i \sqrt[i]{2}}{M_{1}}\right)-1\right)<1
$$

Inductively, if $M_{d-1}$ is defined, take $M_{d}>M_{d-1}$ such that

$$
\begin{equation*}
2^{n-d}\left(\prod_{i=d}^{n-1}\left(1+\frac{i \sqrt[i]{2}}{M_{d}}\right)-1\right)<\lambda\left(d-1, M_{d-1}^{d-1}\right) \tag{6.1}
\end{equation*}
$$

We then run the effective induction of Takayama and McKernan again to finish the proof of Theorem 6.8.

To initiate the induction, pick two very general points $x_{1}$ and $x_{2}$ of $X$. We first take $D_{1} \sim_{\mathbb{Q}} t_{1} K_{X}$ with $t_{1} \leqslant n 2^{\frac{1}{n}} \operatorname{vol}\left(K_{X}\right)^{-\frac{1}{n}}+\epsilon$ such that $x_{1}$, $x_{2} \in \operatorname{Nklt}\left(X, D_{1}\right)$ and that $\left(X, D_{1}\right)$ is $\log$ canonical at $x_{1}$ with the minimal lc center $V_{1}$ containing $x_{1}$.

The induction goes as follows. Assume $x_{1}, x_{2} \in \operatorname{Nklt}\left(X, D_{d}\right),\left(X, D_{d}\right)$ is lc at $x_{1}$ with $V_{d}$ the minimal lc centre of $\left(X, D_{d}\right)$ containing $x_{1}, \operatorname{dim} V_{d}=$ $n_{d} \leqslant n-d$, and $D_{d} \sim_{\mathrm{Q}} t_{d} K_{X}$. Then,
$(1)_{d}$ either $\operatorname{vol}\left(V_{d}\right) \geqslant M_{n_{d}}^{n_{d}}$ and by Takayama's induction, there exists $D_{d+1} \sim_{Q} t_{d+1} K_{X}$ with

$$
\begin{equation*}
t_{d+1}<\left(1+\frac{n_{d} \sqrt[n_{d}]{2}}{M_{n_{d}}}\right) t_{d}+2 \frac{n_{d} \sqrt[n_{d}]{2}}{M_{n_{d}}}+\epsilon \tag{6.2}
\end{equation*}
$$

such that $x_{1}, x_{2} \in \operatorname{Nklt}\left(X, D_{d+1}\right),\left(X, D_{d+1}\right)$ is lc at $x_{1}$ with $V_{d+1}$ the minimal lc centre of $\left(X, D_{d}\right)$ containing $x_{1}$, and $\operatorname{dim} V_{d+1}=$ $n_{d+1} \leqslant n-d-1 ;$
(2) or $\operatorname{vol}\left(V_{d}\right)<M_{n_{d}}^{n_{d}}$, and by McKernan's lemma (see [25, Lemma 3.2]), we have either
$(2.1)_{d} \operatorname{deg} \pi \geqslant 2$, then there exists $D_{d+1} \sim_{Q} t_{d+1} K_{X}$ with

$$
\begin{equation*}
t_{d+1}<2 t_{d}+\epsilon \tag{6.3}
\end{equation*}
$$

such that $x_{1}, x_{2} \in \operatorname{Nklt}\left(X, D_{d+1}\right),\left(X, D_{d+1}\right)$ is lc at $x_{1}$ with $V_{d+1}$ the minimal lc centre of $\left(X, D_{d}\right)$ containing $x_{1}$, and $\operatorname{dim} V_{d+1}=n_{d+1} \leqslant n-d-1$; or
$(2.2)_{d}$ after birational modifications of $X$, we have a fibration $f_{n_{d}}$ : $X \rightarrow Z$ with a general fiber $F \in \mathfrak{X}_{n_{d}, M_{n}^{n_{d}}}$ and $F$ is a pure log canonical centre for $\left(X, D_{F}\right)$ with $D_{F} \sim_{\mathbb{Q}} t_{d} K_{X}$.
Combining (6.2) and (6.3), we see that we always have

$$
t_{k+1} \leqslant 2\left(1+\frac{n_{k} \sqrt[n_{k}]{2}}{M_{n_{k}}}\right) t_{k}+2^{k} \frac{n_{k} \sqrt[n_{k}]{2}}{M_{n_{k}}}+\epsilon
$$

for $k \geqslant 2$.
The induction stops when arriving at $(2.2)_{d}$ for some $d$, or we run the induction to the end and get $D_{m} \sim_{\mathbb{Q}} t_{m} K_{X}$ such that $\operatorname{dim} V_{m}=0$.

In the first case, we have

$$
\begin{align*}
t_{d}<n \frac{\sqrt[n]{2}}{\sqrt[n]{\operatorname{vol}(X)}} 2^{d-1} \prod_{i=1}^{d-1}(1 & \left.+\frac{n_{i} \sqrt[n_{i}]{2}}{M_{n_{i}}}\right)  \tag{6.4}\\
& +2^{d-1}\left(\prod_{i=1}^{d-1}\left(1+\frac{n_{i} \sqrt[n_{i}]{2}}{M_{n_{i}}}\right)-1\right)+\epsilon
\end{align*}
$$

By (6.1) and the fact that $M_{i}<M_{i+1}$ and $n_{i} \leqslant n-i$, we see that when $\operatorname{vol}(X)$ is large enough,

$$
\begin{aligned}
t_{d} & <2^{d-1}\left(\prod_{i=1}^{d-1}\left(1+\frac{n_{i} \sqrt[n_{i}]{2}}{M_{n_{d-1}}}\right)-1\right) \\
& \leqslant 2^{n-n_{d}-1}\left(\prod_{i=1}^{d-1}\left(1+\frac{n_{i} \sqrt[n_{i}]{2}}{M_{n_{d-1}}}\right)-1\right) \\
& \leqslant 2^{n-n_{d}-1}\left(\prod_{k=n_{d}+1}^{n-1}\left(1+\frac{k \sqrt[k]{2}}{M_{n_{d}+1}}\right)-1\right) \\
& <\lambda\left(n_{d}, M_{n_{d}}^{n_{d}}\right)
\end{aligned}
$$


In the second case, we have

$$
\begin{align*}
t_{n}<n \frac{\sqrt[n]{2}}{\sqrt[n]{\operatorname{vol}(X)}} 2^{n-1} \prod_{i=1}^{n-1}( & \left(1+\frac{n_{i} \sqrt[n_{i}]{2}}{M_{n_{i}}}\right)  \tag{6.5}\\
& +2^{n-1}\left(\prod_{i=1}^{n-1}\left(1+\frac{n_{i} \sqrt[n_{i}]{2}}{M_{n_{i}}}\right)-1\right)+\epsilon
\end{align*}
$$

Once again, when $\operatorname{vol}(X)$ is large enough,

$$
t_{n}<2^{n-1}\left(\prod_{i=1}^{n-1}\left(1+\frac{i \sqrt[i]{2}}{M_{1}}\right)-1\right)<1
$$

Thus, by the point separation principle, we conclude that $\varphi_{a}$ is birational for all $a \geqslant 2$.

Proposition 6.10. - If Question 6.6 has an affirmative answer, then Conjecture 6.1 holds.

Proof. - Assume that Question 6.6 has an affirmative answer, we then define the function

$$
\begin{array}{r}
\lambda: \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \rightarrow \mathbb{R}_{>0} \\
\lambda(a, b)=c\left(\mathfrak{X}_{a, b}, n-a\right)
\end{array}
$$

By Theorem 6.8, there exists integers $M_{n-1}>M_{n-2}>\ldots>M_{1}>0$ and $K>0$ such that for any smooth projective $n$-fold $X$ with $\operatorname{vol}(X) \geqslant K$, the $\varphi_{m}$ is birational for $m \geqslant 2$, unless that, after birational modifications, $X$ admits a fibration $f: X \rightarrow Z$ which satisfies $(B)_{\mathfrak{X}_{k, M_{k}^{k}}, \lambda\left(k, M_{k}^{k}\right)}$ for some $1 \leqslant k \leqslant n-1$. But in the latter case, by the definition of $\lambda$ and the assumption that Question 6.6 has an affirmative answer, the map $\varphi_{m}$ is birational, for all $m \geqslant r_{k}$.

Combining all these together, we see that $\varphi_{a}$ is birational for all $a \geqslant$ $r_{n-1}$.

## Acknowledgement

We are grateful to Gavin Brown who is so kind to send us his paper [3] and to inform of their interesting results. Thanks are also due to Chenyang Xu who helped us a lot in understanding the new proof of Theorem 2.4 using [1]. Meng Chen thanks Weiping Li for the hospitality during his visit
at HKUST in the spring of 2015. Zhi Jiang thanks Fudan University for the support during his visits in Shanghai in the summer of 2015.

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Manuscrit reçu le 12 mai 2016, révisé le 9 décembre 2016, accepté le 24 janvier 2017.

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[^0]:    Keywords: canonical volumes, pluricanonical systems, extension theorems.
    2010 Mathematics Subject Classification: 14E05, 14J35, 14J40.
    (*) The fist author is supported by National Natural Science Foundation of China (\#11571076, \#11231003, \#11421061) and Program of Shanghai Academic Researcher Leader (Grant no. 16XD1400400). The second author is supported by China's Recruitment Program of Global Experts.

