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## Densities on locally compact abelian groups

# DENSITIES

# ON LOCALLY COMPACT ABELIAN GROUPS

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## Introduction.

A density on a locally compact Abelian (LCA) group G is a system of measures on the compact quotients of G, which measures satisfy a simple compatibility requirement and have uniformly bounded variation norms. We give a precise definition in the first section. The set of densities on G forms a natural Banach algebra D(G) which coincides with M(G), the measure algebra of G, when G is compact but not otherwise.

A natural equivalent formulation of the notion of density is the following : A density is a continuous functional on the space of all continuous periodic functions on G, whose restriction to the Banach space of functions of any one fixed period is a bounded linear functional.

We may define, in a natural way, the extension of a density to a linear functional on the space of finite linear combinations of continuous periodic functions, but this extension is not necessarily bounded when this space is given the uniform norm. The main purpose of this paper is to establish necessary and sufficient conditions on the group G that all densities on G actually define bounded linear functionals on this space and hence on its closure, the space of semi-periodic functions.

Giving such conditions is equivalent to answering the following question :

When is the algebra of densities really just a sub-algebra of the algebra of all bounded measures on some compactification of G ? We show that if this is the case, then the semi-periodic compactification, $\overline{G}^p$, will serve, where $\overline{G}^p$ is defined as the compact dual of the discrete group generated by the (continuous) periodic characters on G.

Our main result, Theorem 3.10, is that the density algebra extends to a measure algebra precisely when the sum of each two periodic characters is a periodic function. In this case the extension is an isometry. This condition is equivalent to the condition that no two compact quotients of G are independent, in a certain technical sense. An equivalent condition in terms of the structure of G is that either G be totally disconnected or that G have no $R^n$ part and each discrete quotient of G be of bounded order. In this latter case $\overline{G}^p$ is identical to the Bohr compactification. In either of these cases D(G) is isometrically isomorphic to $M(\overline{G}^p)$. However, if G admits a density which is not induced by a measure on $\overline{G}^p$ then the set of densities which are so induced is neither closed nor dense in D(G).

In the first section we present preliminary results and definitions. The second section is concerned primarily with periodic functions and the associated compact quotients. In the third section we consider principally extensions of densities to measures on $\overline{G}^p$. In this section we also consider briefly the problem of decomposition of densities and obtain results which are not analogous to those for measures. A short fourth section touches on uniform distribution, the subject where the notion of densities largely originated.

There are several natural problems concerning densities, some of which we hope to consider in further papers. For example, what is a non-trivial condition on a given density that it extend to a measure on the semi-periodic compactification of G ? Also, there is a naturally defined closed ideal $D^1(G)$ of absolutely continuous densities that bears roughly the same relation to D(G) that $L^1(G)$ bears to the measure algebra M(G) and we hope to investigate its theory elsewhere.

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## 1. Preliminary definitions and results.

We will be dealing with locally compact Abelian (LCA) groups throughout this paper and unless the context clearly indicates otherwise we will assume all groups LCA groups.

We will assume the standard results on LCA groups and almost-periodic functions. As sources we cite [5] and [7]. We will attempt to use standard notation where possible. In particular $G^\wedge$ will denote the dual group of G. If A is a normed linear space we will denote the space of bounded linear functionals on A by A*.

When we refer to a measure on an LCA group G, then unless the context clearly indicates otherwise, we will mean a regular countably additive *finite* measure on the Borel sets of G. By M(G) we will denote the convolution algebra of finite measures on G.

We will often be dealing with several isomorphic spaces at once. When no confusion seems likely we will identify an element of one space with its image in another without changing notation.

DEFINITION. — *We say that a closed subgroup* H *of the group* G *is of compact index if* G/H *is compact.*

If $H_1$ and $H_2$ are subgroups of compact index and if $H_1 \subseteq H_2$ then there is a natural continuous homomorphism $\varphi$ from $G/H_1$ onto $G/H_2$. We let $\{G_\alpha \mid \alpha \in A\} \equiv \{G/H_\alpha \mid \alpha \in A\}$ be the system of compact quotients of G. If $G_\alpha$ is a compact quotient of $G_\beta$ we write $G_\beta > G_\alpha$. We let $M(G_\alpha)$ be the set of regular countably additive measures on the Borel sets of $G_\alpha$ whose variation norm is finite.

DEFINITION. — *A density* $\mu$ *on an LCA group* G *is a system of measures on the compact quotients of* G *satisfying*

1) *If* $G_\alpha$ *is a quotient of* $G_\beta$ *and* E *is a Borel set in* $G_\alpha$ *then* $\mu_\alpha(E) = \mu_\beta(\varphi^{-1}(E))$ *where* $\varphi : G_\beta \longrightarrow G_\alpha$ *is the natural homomorphism.*

2) $\|\mu\| \underset{\text{def.}}{=} \sup\{\|\mu_\alpha\| \mid \alpha \in A\} < \infty$.

That is, $\mu$ is a member of $\prod_{\alpha \in A} M(G_\alpha)$ satisfying 1) and 2).

We will refer to condition 1 as the *compatibility* condition. It is equivalent to 1').

$$\int (f \circ \varphi) \, d\mu_\beta = \int f \, d\mu_\alpha$$

for each Borel function $f$ on $G_\alpha$.

There is clearly a relation between the notion of density and that of martingale ; indeed, by stretching the notion of martingale sufficiently, every density is a martingale, but there seems to be no profit in this point of view for the problems which concern us here.

One of the simplest examples of a density is Haar density $\lambda$ given by taking $\lambda_\alpha$ as Haar measure (normalized to be of mass 1) on $G_\alpha$. The compatibility is easy to verify since $\lambda_\beta(\varphi^{-1}(E))$ is a normalized translation invariant measure on the Borel sets, E, of $G_\alpha$ and hence must coincide with $\lambda_\alpha$. An elaboration of Haar density in the case $G = Z$ may be found in several papers, e.g. Buck [3].

DEFINITION. — *If* $\mu = \{\mu_\alpha | \alpha \in A\}$ *and* $v = \{v_\alpha | \alpha \in A\}$ *are two densities in G we define a density* $\mu * v$ *by* $(\mu * v)_\alpha = \mu_\alpha * v_\alpha$, *where* $\mu_\alpha * v_\alpha$ *denotes the convolution product of* $\mu_\alpha$ *and* $v_\alpha$.

We must show that $\mu * v$ satisfies the compatibility condition. Boundedness is clear. We let $f$ be a Borel function on $G_\alpha$, we let $G_\alpha < G_\beta$ and we let $\varphi : G_\beta \longrightarrow G_\alpha$ be the natural homomorphism. Then

$$\int f \, d(\mu_\alpha * v_\alpha) = \iint f(x + t) \, d\mu_\alpha(t) \, dv_\alpha(x)$$

$$= \iint f(x + \varphi(t)) \, dv_\alpha(x) \, d\mu_\beta(t)$$

$$= \iint f(\varphi(x) + \varphi(t)) \, dv_\beta(x) \, d\mu_\beta(t)$$

$$= \iint (f \circ \varphi)(x + t) \, dv_\beta(x) \, d\mu_\beta(t)$$

$$= \int (f \circ \varphi) \, d(\mu_\beta * v_\beta)$$

It is clear that the densities on the LCA group G form a commutative Banach algebra under convolution. We will denote this algebra by D(G).

## 2. Periodic functions.

We denote by C(G) the space of bounded continuous complex valued functions on G with the supremum norm.

DEFINITION. – *We say that $f \in C(G)$ is periodic if $f$ is constant on the cosets of some subgroup H of compact index and we then say $f$ is of period H. (We note that the choice of period for $f$ is in general not unique.)*

If $f$ is of period H we may identify $f$ with $\tilde{f}$, an element of C(G/H), and conversely, where $\tilde{f}(x + H) = f(x)$ is the desired correspondence.

We will denote the normed space of periodic functions under the supremum norm by P(G). In general, P(G) is not a vector space. We will denote the Banach space of periodic functions of period $H_\alpha$, with the supremum norm, by $P_{H_\alpha}(G)$ or by $P_\alpha(G)$ where no confusion is likely.

We shall see in article 3 that it is useful to consider densities as functionals on P(G). We shall denote the present article to developing the necessary results concerning P(G).

We must first consider some simple relations between periodic and almost periodic functions.

DEFINITION. – *If $f \in C(G)$ we define $f_x$, the translate of $f$ by x, by $f_x(y) = f(y - x)$. The orbit of $f$, orb $f$, is defined by*

$$\text{orb } f = \{ f_x \mid x \in G \}.$$

If the orbit of $f$ is conditionally compact, $f$ is called *almost-periodic*. We denote the Banach algebra of almost-periodic functions on G by AP(G).

LEMMA 2.1. – *Let $f \in C(G)$. Then $f \in P(G)$ if and only if the orbit of $f$ is compact.*

*Proof.* – We suppose first that $f \in P(G)$ with period H and consider $f$ as a function $f^\sim$ on G/H. Then orb $f$ is homeomorphic to orb $f^\sim$. But the orbit of a continuous function on a compact group is

compact. Next we suppose $\theta = \text{orb } f$ is compact. We make orb $f$ into a group by defining $f_s f_t = f_{s+t}$, and consider the homomorphism $\varphi : G \longrightarrow \theta$ given by $\varphi(t) = f_t$. The homomorphism $\varphi$ is continuous since $f \in AP(G)$ and so $f$ is surely uniformly continuous. Hence $\ker \varphi = \{t \,|\, f_t = f\}$ is a subgroup of compact index and $f$, being constant on its cosets, is periodic. This completes the proof of Lemma 2.1.

Hence we see that a periodic function on G is almost periodic.

We recall some standard results about almost periodic functions and in Lemmas 2.2 and 2.3 give without proof some simple consequences.

AP(G) is isometrically isomorphic to $C(\overline{G})$ where $\overline{G}$ denotes the Bohr compactification of G. That is, $\overline{G}$ is the compact dual of $G^\wedge$ when $G^\wedge$ is given the discrete topology.

Where no confusion can arise we will identify functions in AP(G) with functions in $C(\overline{G})$. For example, we will speak of expanding $f \in AP(G)$ in terms of characters of $\overline{G}$.

If $f \in AP(G)$, then $f$ has a Fourier expansion $f \approx \sum\limits_{n=1}^{\infty} \alpha_n \chi_n$ where $\chi_n \in G^\wedge$ and convergence is in the $L^2(G)$ norm. Moreover, there exists a sequence of trigonometric polynomials, $\sum\limits_{n=1}^{m_j} \beta_{nj} \chi_n \xrightarrow[\,j\,]{} f$ where convergence is in the supremum norm. Furthermore, $\lim\limits_j \beta_{nj} = \alpha_n$.

LEMMA 2.2. — *Let* $f \in P(G)$ *be of period* H. *Then*

i) *The function f has a Fourier expansion,* $f \approx \sum\limits_{n=1}^{\infty} a_n \chi_n$, *where each* $\chi_n \in G^\wedge$ *is of period* H *and*

$$\|f - \sum_{n=1}^{k} a_n \chi_n \| \longrightarrow 0 \quad \text{in} \quad L_2(G/H).$$

*If f is also of period* K, *the same Fourier expansion serves ; indeed*

$$\|f - \sum_{n=1}^{k} a_n \chi_n \|_{L_2(G/H)} = \|f - \sum_{n=1}^{k} a_n \chi_n \|_{L_2(G/K)}$$

$$= \|f - \sum_{n=1}^{k} a_n \chi_n \|_{L_2(\overline{G})}.$$

ii) *There exists a sequence* $\left\{ \sum\limits_{n=1}^{m_j} b_{nj} \chi_n \right\}_j$ *of trigonometric poly-*
*nomials of period* H, *tending to f in the supremum norm. The charac-*
*ters appearing in this sequence can be taken to be those appearing in*
*the Fourier expansion that have non-zero coefficients. Of course, in*
*general, we cannot take* $b_{nj} = a_n$ ; *but we must have* $b_{nj} \xrightarrow[j]{} a_n$ .

DEFINITION. – *Let* $G/H_\alpha = G_\alpha$ *be a compact quotient of G. Let*
$\mu_{H_\alpha}$ *be Haar measure in* $\overline{H}_\alpha$ , *the closure of* $H_\alpha$ *in* $\overline{G}$ . *Then if f is in*
AP(G) *we define* $f_\alpha$ , *the projection of f onto* $P_\alpha(G)$ , *by letting*
$f_\alpha(x) = \int_{\overline{G}} f(x - t)d\mu_{H_\alpha}(t)$ . *This defines* $f_\alpha \in C(\overline{G})$ *which we then*
*identify with* $f_\alpha \in$ AP(G).

LEMMA 2.3. – *Let* $f \in$ AP(G). *Let* $G_\alpha$ *be a compact quotient*
*of G. Then*

1) $\|f_\alpha\|_\infty \leqslant \|f\|_\infty$

2) *If* $f \in P_\alpha$ *then* $f_\alpha = f$ .

3) *If* $f \approx \sum\limits_{n=1}^\infty a_n \chi_n$ *then*

$$f_\alpha \approx \sum_E a_n \chi_n , \text{ where } E = \{n \mid \chi_n \in G_\alpha^\wedge\}.$$

We see that each $f \in$ P(G) has a Fourier expansion. We shall see
that it is important to consider the set of characters that occur
(with non-zero coefficients) in this expansion and also the group
generated by this set.

DEFINITION. – *If g is an element of a group G we denote by* [g]
*the subgroup generated (algebraically) by g with the similar notation*
[$\{g_\alpha \mid \alpha \in A\}$] *for the subgroup generated by a set. If* [g] *is discrete*
*as a subset of the* LCA *group G we say that g is a discrete element of G.*

We first have the following useful characterization of periodic
functions.

LEMMA 2.4. − *Let $f \in C(G)$. Then $f$ is periodic if and only if the group generated by the characters appearing in its Fourier expansion is discrete as a subgroup of $G^\wedge$. In particular, if $\chi \in G^\wedge$ then $\chi$ is periodic if and only if $[\chi]$ is discrete.*

*Proof.* − If $f \in P(G)$ and $f$ is of period $H_\alpha$ then $f \in C(G/H_\alpha)$.

Hence $f \approx \sum_{n=1}^{\infty} a_n \chi_n$ where $\{\chi_n\} \subset (G/H_\alpha)^\wedge$. Since $(G/H_\alpha)$ is compact, $(G/H_\alpha)^\wedge$ is discrete as a subgroup of $G^\wedge$.

Conversely, if $f \approx \sum_{n=1}^{\infty} a_n \chi_n$ where $[\{\chi_n\}]$ is discrete as a subgroup of $G^\wedge$, then $f$ is of period $[\{\chi_n\}]^\perp$. That is, $f$ is of period $H_\alpha$ where $(G/H_\alpha)^\wedge = [\{\chi_n\}]$.

This completes the proof of Lemma 2.4.

DEFINITION. − *We will denote by $\pi(G)$ the set of periodic characters of G.*

The following corollary of Lemma 2.4 is now immediate :

LEMMA 2.5. − *If $f_1$ and $f_2$ are periodic where $f_1 \approx \sum_{j=1}^{\infty} a_j \chi_j$ and $f_2 \approx \sum_{j=1}^{\infty} b_j \chi_j$, then $f_1 + f_2$ is periodic if and only if $[\{\chi_j | a_j + b_j \neq 0\}]$, the group generated by $\{\chi_j | a_j + b_j \neq 0\}$, is discrete.*

DEFINITION. − *We will call the vector space of finite linear combinations of periodic functions the span of the periodic functions and denote it by $C_p(G)$. We define similarly the span of the periodic characters of G and denote this space by $C_\pi(G)$.*

DEFINITION. − *We will call the closure of $C_p(G)$ the space of semiperiodic functions. By Lemma 2.2, we see that*

$$(C_p(G))^- = (C_\pi(G))^- .$$

The following well-known result is proved in [5 : Theorem 2.3.2].

LEMMA 2.5. – *Let* $g \in G$. *Then* $[g]$ *is either discrete or has compact closure.*

This leads immediately to the following useful observation.

LEMMA 2.6. – *If* $\chi$ *is a periodic character of infinite order and* $\chi'$ *is a non-periodic character, then* $\chi\chi'$ *is periodic.*

*Proof.* – By Lemma 2.5, $[\chi']$ has compact closure, $V_0$ , say. But if $V_1$ is a compact neighborhood of 0, we see that, for all large enough $n$, $\chi^n \notin V_1$ . Hence, for large enough $m$, $(\chi\chi')^m \notin V_0$ . Hence, again by Lemma 2.5, $[\chi\chi']$ is discrete. This completes the proof of Lemma 2.6.

We quote a useful result from Berg, Rajagopalan and Rubel [2].

LEMMA 2.7. – *Let* G *be an* LCA *group and let* $G_0$ *denote the identity component of* G. *Let* $\chi$ *be a character of* G *and let* H *denote the kernel of* $\chi$. *If* $\chi \notin G_0^\perp$ *then* $G/H = T$, *the circle group in the usual topology, and* $\chi$ *is periodic. If* $\chi \in G_0^\perp$ *then* $G/H$ *is discrete and in this case* $\chi$ *is periodic if and only if the range of* $\chi$ *is finite.*

This immediately leads to the following :

LEMMA 2.8. – G *is totally disconnected if and only if the periodic characters of* G *are precisely those of finite range.*

We now see that if G is not totally disconnected then products of periodic characters separate the points of G.

LEMMA 2.9. – *If* G *is not totally disconnected then* $[\pi(G)] = G^\wedge$.

*Proof.* – If G is not totally disconnected then G has, by Lemma 2.8, at least one periodic character $\chi_1$ of infinite range. Indeed, we may take $\chi_1$ to be any character that does not annihilate the connected component of the identity of G. But if $\chi_2$ is a non-periodic character then by Lemma 2.6, $\chi_1\chi_2$ is periodic. Hence $\chi_2 = \chi_1^{-1}(\chi_1\chi_2)$ and so $[\pi(G)] = G^\wedge$ and our proof is complete.

We now have our first principal result :

THEOREM 2.10. – *Suppose the sum of each two periodic characters of* G *is periodic. Then either*

1) G *is totally disconnected or*

2) G *has no* $R^n$ *part and each discrete quotient of* G *is of bounded order ; equivalently* $G^\wedge$ *has a compact open subgroup of bounded order.*

*In the first case the periodic characters are those of finite order. In the second case all characters are periodic. Conversely, if either 1) or 2) holds then any finite linear combination of periodic characters is periodic.*

*Proof.* — Suppose $\chi_1$, $\chi_2$ discrete implies $[\chi_1, \chi_2]$ discrete. Then in particular $[\chi_1, \chi_2]$ is discrete and hence periodic characters form a group. If G is totally disconnected then by Lemma 2.8 we see that the periodic characters are those of finite range. It is clear by Lemma 2.4 that in this case $C_\pi(G) \subset P(G)$, and our theorem holds for this case. If G is not totally disconnected, then by Lemma 2.9 we see that $\pi(G) = G^\wedge$.

For the remainder of the proof we consider only the case where all characters are periodic.

We write $H = K \times R^n$ where K is compact, as an open closed subgroup of $G^\wedge$. If $n \geqslant 1$ it is clear that there exist discrete $\chi_1$, $\chi_2$ such that $[\chi_1, \chi_2]$ is not discrete ; hence $n = 0$. It is clear that K must contain only elements of finite order.

But if K contains only elements of finite order, an easy category argument shows that the order of elements of K is bounded. If K is a compact open subgroup of $G^\wedge$ of bounded order, we see that $[\chi_1, \dots, \chi_n]$ is discrete for any $\chi_1, \dots, \chi_n \in G^\wedge$. Indeed, if $[\chi_1, \dots, \chi_n]$ were not discrete, then $K \cap [\chi_1, \dots, \chi_n]$ would be an infinite group ; but $K \cap [\chi_1, \dots, \chi_n]$ is a finitely generated Abelian group of bounded order and hence finite.

This completes the proof of Theorem 2.10.

The following immediate corollary to Theorem 2.10 is of independent interest :

COROLLARY 2.11. — *Suppose that* G *is an LCA group such that whenever* $g_1$ *and* $g_2$ *are discrete then* $[g_1, g_2]$ *is discrete. Then for any n, if* $g_1, \dots, g_n$ *are discrete it follows that* $[g_1, \dots, g_n]$ *is discrete.*

*Remark.* – The hypothesis of local compactness in Corollary 2.11 cannot be dropped. Indeed, let G be the subgroup of $R^2$, in the relative topology, generated by $g_1 = (1,0)$, $g_2 = (0,1)$ and $g_3 = (1,\sqrt{2})$. Each of the elements $g_1$, $g_2$, $g_3$ is discrete and the subgroup generated by each pair is discrete, but $[g_1, g_2, g_3]$ is not discrete.

It is important to note that a group may have the property that all finite linear combinations of periodic characters are periodic and yet sums of periodic functions are not necessarily periodic. That is, $C_\pi(G) \subset P(G)$, but $C_p(G) \not\subset P(G)$.

*Example.* – Let $G = Z_2 \times Z$ where $Z_2$ is the 2-adic integers and $Z$ is the integers. G is totally disconnected. We consider now the subgroups of compact index $H_1 = [\tilde{1} \times 1]$ and $H_2 = [\tilde{2} \times 1]$, where $\tilde{1}$ is the generator of the 2-adics and 1 is the generator of Z. Then $G/H_1 \approx G/H_2 \approx Z_2$. But $H_1 \cap H_2 = \tilde{0} \times 0$.

Now consider $f_1 \approx \sum_{i=1}^\infty a_{i1}\chi_{i1}$, where $\{\chi_{i1}\}$ is the set of characters of G which are characters of $G/H_1$ and all $a_{i1}$ are non-zero, and $f_2 \approx \sum_{i=1}^\infty a_{i2}\chi_{i2}$, similarly made up of characters of $G/H_2$.

Then $f_1 + f_2$ cannot be periodic because $[\{\chi_{i1}\} \cup \{\chi_{i2}\}]$ separates points and hence has no annihilator except $\tilde{0} \times 0$.

LEMMA 2.12. – *Let $g_1$, $g_2$ be discrete elements of G and suppose that $[g_1, g_2]$ is not discrete. Then for any non-zero integers $m$, $n$ with $m \neq n$ the group $[(g_1 g_2^n)^s, (g_1 g_2^m)^r]$ is not discrete.*

*Proof.* – A routine application of the division algorithm shows us that $[g_1, g_2]/[(g_1 g_2^n)^s, (g_1 g_2^m)^r]$ is finite.

We now arrive at an important technical theorem which will play a vital role in article 3.

THEOREM 2.13. – *Suppose G admits two periodic characters with a non-periodic sum. Then there exist periodic characters $\chi_1$ and $\chi_2$ such that $\chi_1 + \chi_2$ is not periodic, but for each integer n, $\chi_1\chi_2^n$ is periodic. Moreover if $\chi' \in [\chi_1\chi_2^n]$ and $\chi'' \in [\chi_1\chi_2^m]$ (where m and n are unequal non-zero integers) then $\chi' + \chi''$ is not periodic.*

*Proof.* — We must consider two cases. In the first case the periodic characters do not form a group. Then there must exist $h$, a periodic character of infinite order, and $k$ a non periodic character. Let $\chi_1 = h^{-1}$ and $\chi_2 = h^2 k$. Then $\chi_1 \chi_2^n = h^{2n-1} k^n$ is periodic by Lemma 2.6, but $k \in [\chi_1 \chi_2]$ and hence $\chi_1 + \chi_2$ is not periodic.

In the second case the periodic characters form a group. Then, by the arguments of Theorem 2.10, there exist two periodic characters with a non periodic sum if and only if the reals are a direct summand of the open subgroup. In this case the reals are a subgroup of the dual group and hence $\chi_1 = 1$ and $\chi_2 = \pi$ will serve.

The last assertion is clear by Lemma 2.9.

## 3. Extensions of densities to measures and decompositions of densities.

We first characterize densities as certain functionals on the periodic functions. To this end, when G is compact we freely identify M(G) with the dual space of C(G), writing $\mu(f) = \int f \, d\mu$, and we exploit the natural analogue for densities.

LEMMA 3.1. — *Let $\mu$ be a density on G. Then for each subgroup $H_\alpha$ of compact index, $\mu$ defines a member of $(P_\alpha(G))^*$. That is, $\mu$ defines a bounded linear functional on the space of periodic functions which are of period $H_\alpha$. Indeed, for $f$ of period $H_{\alpha_1}$ we define $\mu(f)$ by*

$$\mu(f) = \mu_{\alpha_1}(f) = \int_{G/H_{\alpha_1}} f(x) \, d\mu_{\alpha_1}(x)$$

where we consider the integrand $f$ as a function on $G/H_{\alpha_1}$ . Then

$\sup\limits_{\alpha \in A} \sup\limits_{f \in P_\alpha(G)} \| \mu(f) \| / \| f \| = \| \mu \|$ . Conversely any set of linear functionals $\{ \mu_\alpha \mid \alpha \in A \}$, where $\mu_\alpha$ is a linear functional in $P_\alpha(G)$, satisfying $\sup\limits_{\alpha} \| \mu_\alpha \| = M$ defines a density whose norm is M.

*Proof.* — The only point which requires any attention is that of showing that $\mu(f)$ is well defined ; that is, that $\mu(f)$ is independent of the choice of period for $f$.

If $f$ is of period $H_{\alpha_1}$ we define $f_{\alpha_1} \in C(G/H_{\alpha_1})$ by

$$f_{\alpha_1}(x + H_{\alpha_1}) = f(x).$$

We note that $\|f_{\alpha_1}\| = \|f\|$. Then we may define $\mu(f)$ by

$$\mu(f) = \mu_{\alpha_1}(f_{\alpha_1}) = \int_{G/H_{\alpha_1}} f_{\alpha_1}(x) d\mu_{\alpha_1}(x).$$

We now assume $f$ is of periods $H_{\alpha_1}$ and $H_{\alpha_2}$. Then $f$ is of period $H_{\alpha_3}$ where $H_{\alpha_3}$ is the closed subgroup generated by $H_{\alpha_1}$ and $H_{\alpha_2}$. Then $G_{\alpha_3}$ is a compact quotient of $G_{\alpha_1}$ and of $G_{\alpha_2}$ and we let $\varphi_1$ and $\varphi_2$ denote the natural homomorphisms. But then

$$\mu_{\alpha_1}(f_{\alpha_1}) = \mu_{\alpha_1}(f_{\alpha_3} \circ \varphi_1) = \mu_{\alpha_3}(f_{\alpha_3}) = \mu_{\alpha_2}(f_{\alpha_2} \circ \varphi_2) = \mu_{\alpha_2}(f_{\alpha_2}).$$

The converse is clear ; each measure $\mu_\alpha$ on the Borel sets of $G_\alpha$ is uniquely determined by a bounded linear functional on $C(G_\alpha)$ and each $f_\alpha \in C(G_\alpha)$ lifts to $f \in P(G)$ by $f(x) = f_\alpha(x + H_\alpha)$. The compatibility condition implies that $\mu$ is well defined as a functional on $P(G)$.

This completes the proof of Lemma 3.1.

LEMMA 3.2. — *Let* $\mu \in D(G)$. *Let* $f_i$, $i = 1, 2, \ldots, n$ *be in* $P(G)$. *Then*

$$\left| \sum_{i=1}^n \mu(f_i) \right| \leqslant (2^{n+1} - 1) \|\mu\| \cdot \left\| \sum_{i=1}^n f_i \right\|_\infty$$

*Proof.* — *Suppose* $\|\mu\| = 1$ *and* $\left\| \sum_{i=1}^n f_i \right\|_\infty = M$. *For* $\varepsilon > 0$ *we may write*

$$f_i = \sum_j \beta_{ij} \chi_{ij} + \varepsilon_i$$

*where* $\|\varepsilon_i\| < \varepsilon$ *and both* $\chi_{ij}$ *and* $\varepsilon_i$ *have period* $H_i$, *the period of* $f_i$. *Then*

$$\left| \sum \mu(f_i) \right| \leqslant \left| \sum_i \left( \sum_j \beta_{ij} \mu(\chi_{ij}) \right) \right| + n\varepsilon.$$

We now write

$$\sum_i \sum_j \beta_{ij} \chi_{ij} = \sum_1 + \sum_1'$$

where $\sum_1$ is the sum over those $\chi_{ij}$ which have period $H_1$ and $\sum_1'$ is the sum over the remainder. Now by Lemma 2.3,

$$\| \sum_1 \| \leqslant M + n\varepsilon$$

since $\sum_1$ is the projection of $\sum \sum \beta_{ij} \chi_{ij}$ onto $P_1$. Hence

$$|\mu\left(\sum_1\right)| \leqslant M + n\varepsilon.$$

Now we consider $\| \sum_1' \|$. We have

$$\| \sum_1' \| \leqslant 2(M + n\varepsilon).$$

We now write

$$\sum_1' = \sum_2 + \sum_2' ,$$

where $\sum_2$ is summed over those $\chi_{ij}$ that occur in $\sum_1'$ and have period $H_2$ and $\sum_2'$ is the remainder. We have

$$\| \sum_2 \| \leqslant 2(M + n\varepsilon)$$

so that

$$|\mu\left(\sum_2\right)| \leqslant 2(M + n\varepsilon)$$

and

$$\| \sum_2' \| \leqslant 4(M + n\varepsilon).$$

Proceeding step by step in this manner we finally get

$$|\sum \mu(f_i)| \leqslant (1 + 2 + \cdots + 2^n)(M + n\varepsilon) + n\varepsilon$$
$$= (2^{n+1} - 1)(M + n\varepsilon) + n\varepsilon$$

from which the lemma follows.

THEOREM 3.3. — *A density on* G *is equivalent to a continuous functional on* P(G) *which is linear in* $P_\alpha(G)$ *for each period* $H_\alpha$.

*Proof.* — We let $\mu$ be a density on G. Then $\mu$ is a continuous linear functional on $P_\alpha(G)$ for each period $H_\alpha$. If $f_1 \in P(G), f_2 \in P(G)$ and $\|f_1 - f_2\|_\infty < \varepsilon$ then $|\mu(f_1) - \mu(f_2)| \leqslant 4\|\mu\|\varepsilon$ by Lemma 3.2. Hence $\mu$ is continuous on P(G).

Suppose $\mu$ is a continuous linear functional on $P_\alpha(G)$ for each $\alpha$, and $\mu$ is continuous on P(G). It is then clear by a simple argument that $\|\mu\|_\alpha$ is uniformly bounded.

This completes the proof of Theorem 3.3.

We will now freely identify densities with their corresponding functionals. Moreover, when there will be no confusion, we will use the same notation even when we enlarge the domain.

LEMMA 3.4. — *Let* $\mu \in D(G)$. *For* $\{f_i\} \subset P(G)$ *we define*

$$\mu(f_1 + \cdots + f_n) = \mu(f_1) + \cdots + \mu(f_n).$$

*Then* $\mu$ *defines a (possibly unbounded) linear functional on* $C_p(G)$.

*Proof.* — All that requires proof is that the functional is well defined. But by Lemma 3.2, if $\sum_{i=1}^{n} f_i = \sum_{j=1}^{m} g_j$ where $f_i$ and $g_j$ are in P(G) then $\sum_{i=1}^{n} \mu(f_i) - \sum_{j=1}^{m} \mu(g_j) = 0$.

DEFINITION. — *If* G *is an* LCA *group we define a compactification of* G *to be a compact group* K *such that there is a continuous homomorphism* $\varphi$ *from* G *onto a dense subset of* K. *We may consider each compactification of* G *as given in the following way : Let* $\Gamma^0$ *be a subgroup of* $G^\wedge$. *If we put* $\Gamma^0$ *in the discrete topology and take* H *as the compact dual of* $\Gamma^0$ *then* H *will be a compactification of* G.

This presentation of H as a dual group makes it clear that C(H) as a Banach algebra is isometric to a subalgebra of C(G) ; indeed C(H) will be a subalgebra of AP(G).

We do not require that the homomorphism $\varphi$ be one to one. For example, any compact quotient $G_\alpha$ is admissable as a compactification of G.

Of particular interest to us is the following compactification :

DEFINITION. — *We define* $\overline{G}^P$, *the semiperiodic compactification of G as the compact dual of* $[\pi(G)]$, *the group generated by* $\pi(G)$, *when* $[\pi(G)]$ *is given the discrete topology.*

We observe that $(C_p(G))^-$, the space of semiperiodic functions, is a subspace of $C(\overline{G}^P)$. Since $\pi(G) \subset C_p(G)$ it is clear that $\overline{G}^P$ is the smallest compactification which will serve.

LEMMA 3.5. — *If G is not totally disconnected, then* $\overline{G}^P = \overline{G}$ ; *that is, the semiperiodic compactification is the Bohr compactification. On the other hand, if G is totally disconnected then* $\overline{G}^P$ *is the compact dual of the group of characters of finite range when this group is given the discrete topology.*

*Proof.* — These two cases are clear by Lemmas 2.9 and 2.8 respectively.

If G is not totally disconnected then it is clear that the homomorphism of G into $\overline{G}^P$ is one-to-one.

If G is totally disconnected the situation is more complicated. We may have $\overline{G}^P = \overline{G}$, as in the case where each element of G is of fixed order $n$. On the other hand, we may have $\overline{G}^P = 0$, as in the case where G is divisible.

Each compact quotient of G is a quotient of $\overline{G}^P$. Indeed, if $G_\alpha$ is a compact quotient of G, then $G_\alpha^\wedge$ is a subgroup of $(\overline{G}^P)^\wedge = [\pi]$.

Any bounded measure $\widetilde{\mu}$ on G induces a density $\mu$. Since $\widetilde{\mu} \in (C(G))^*$, the measure $\widetilde{\mu}$ is surely continuous on P(G) and linear on each $P_\alpha(G)$. Hence we define the density $\mu$ by

$$\mu(f) = \widetilde{\mu}(f) = \int_G f d\widetilde{\mu} \quad \text{for} \quad f \in P(G).$$

However, unless G is compact no bounded measure on G can induce Haar density.

The appropriate place to consider measures which induce densities is $\overline{G}^P$. We note that any measure $\widetilde{\mu}$ on $\overline{G}^P$ induces uniquely a density $\mu$ on G. Indeed, since $C_p(G) \subset C(\overline{G}^P)$, if we define $\mu(f) = \widetilde{\mu}(f)$ for $f \in P(G)$ we have defined the density. Alternatively, one may

consider $\overline{G}^p$ as a compact group such that each compact quotient, $\overline{G}_\alpha$, is a quotient of $\overline{G}^p$.

We note that if the extension of a density $\mu$ to a linear functional on $C_p(G)$ is bounded, then there is a measure $\widetilde{\mu}$ on $\overline{G}^p$ which induces the density $\mu$. Indeed, since $C_p(G) \subset C(\overline{G}^p)$ we simply extend the bounded linear functional on $C_p(G)^-$ to $C(\overline{G}^p)$.

We will put the previous remarks in the form of a lemma.

LEMMA 3.6. — *Any measure $\widetilde{\mu} \in M(\overline{G}^p)$ induces a density $\mu \in D(G)$. If $\mu \in D(G)$ there exists a measure $\widetilde{\mu} \in M(\overline{G}^p)$ inducing the density $\mu$ if and only if the extension of $\mu$ to a functional on $C_p(G)$ is bounded.*

It is clear that $\overline{G}^p$ is the proper group to consider. Indeed, if H is another compactification and $H^\wedge \supset \pi(G)$, then $\overline{G}^p$ is a quotient of H and hence any measure on H induces a measure on $\overline{G}^p$. Hence a density induced by a measure on H is induced by a measure on $\overline{G}^p$. If $H^\wedge \not\supset \pi(G)$, say $\chi \in \pi(G) \sim H^\wedge$, then measures on H do not induce non-trivial measures on $[\chi]^\wedge$ which is a compact quotient of G.

Since a density is well defined as a functional on periodic functions we can now define the Fourier-Stieltjes transform of a density.

DEFINITION. — *Let $\mu$ be a density. Then we define*

$$\mu^\wedge : \pi(G) \longrightarrow C \ by$$

$$\mu^\wedge(\chi) = \int \overline{\chi} d\mu = \int_{G_\alpha} \overline{\chi}_\alpha d\mu_\alpha$$

*for $\chi$ of period $\alpha$.*

We call the (clearly bounded) function $\mu^\wedge$ the *Fourier-Stieltjes transform* of $\mu$.

If $\mu$ is induced by a measure $\widetilde{\mu}$ on $\overline{G}^p$ then $\mu^\wedge(\chi) = (\widetilde{\mu})^\wedge(\chi)$ for $\chi \in \pi(G)$, where $(\widetilde{\mu})^\wedge$ is the usual Fourier-Stieltjes transform of a measure. Hence if $\pi(G)$ is a group we see that $\widetilde{\mu}^\wedge = \mu^\wedge$. It is easy to see that

$$(\mu * v)^\wedge = (\mu^\wedge)(v^\wedge) .$$

Many of the standard results for Fourier-Stieltjes transforms of measures have immediate analogues for densities.

For example, we call a function $v : \pi(G) \longrightarrow C$ *quasi positive definite* if

$$\sum_{i=1}^{n} v(\chi_i \chi_j^{-1}) \xi_i \overline{\xi_j} \geqslant 0$$

for every finite collection $\xi_1, \ldots, \xi_n$ of complex numbers and every finite collection $\chi_1, \ldots, \chi_n$ of characters of a common period.

We call a density *positive* if it is positive as a functional on periodic functions.

Then we have Bochner's theorem ; that is a function

$$v : \pi(G) \longrightarrow C$$

is the Fourier-Stieltjes transform of a positive density if and only if $v$ is quasi positive definite.

However the lack of a general decomposition theorem for densities (see Theorem 3.13) detracts from the usefulness of this characterization.

We will not explore Fourier-Stieltjes transforms further in this paper.

We now characterize those groups in which sums of periodic characters are periodic, in terms of the lattice structure of the set of compact quotients.

DEFINITION. — *We say that two subgroups* $H_1$, $H_2$ *of compact index are incident if there is a proper subgroup H of compact index such that* $(H_1 \cup H_2) \subseteq H$. *We say that* $H_1$ *and* $H_2$ *are independent if there is no one subgroup H of compact index which is incident to them both. If* $H_1$ *and* $H_2$ *are independent subgroups we will also say that* $G/H_1$ *and* $G/H_2$ *are independent.*

We will use the following dual characterization, whose proof is clear.

LEMMA 3.7. — *The compact quotients* $G/H_1$ *and* $G/H_2$ *are independent if and only if for each nontrivial* $\chi_1 \in (G/H_1)^\wedge$ *and* $\chi_2 \in (G/H_2)^\wedge$ *there exists no compact quotient G/H such that* $\chi_1$ *and* $\chi_2$ *are each in* $(G/H)^\wedge$. *That is* $[\chi_1, \chi_2]$ *is not discrete.*

LEMMA 3.8. – *Each finite linear combination of periodic charac-*
*ters of G is periodic if and only if no pair of compact quotients of G*
*is independent.*

*Proof.* – If there is a nonperiodic $f \in C_\pi(G)$, where $f = \sum\limits_{n=1}^{p} a_n \chi_n$,
then $[\chi_1, \ldots, \chi_p]$ is not discrete. But then by Corollary 2.11 there
exist $\chi_1'$, $\chi_2'$ in $\pi(G)$ such that $\chi_1' + \chi_2'$ is not periodic. Then $[\chi_1']^\wedge$,
$[\chi_2']^\wedge$ are the desired independent quotients, since $[\chi_1'^p, \chi_2'^q]$ is not
discrete for non-zero $p$ and $q$.

The converse is immediate.

LEMMA 3.9. – *Given a set* $\{G_\alpha \,|\, \alpha \in A'\}$ *of pairwise independent*
*compact quotients of G and measures* $\{\mu_{G_\alpha} \,|\, \alpha \in A'\}$ *such that* $\mu_{G_\alpha}(G_\alpha)$
*is independent of* $\alpha$ *and such that* $\sup\limits_{\alpha} \|\mu_{G_\alpha}\| < \infty$, *there is a density*
$\mu$ *that extends* $\{\mu_{G_\alpha}\}$ *in the sense that* $\mu_{G_\alpha} = \mu_\alpha$ *for each* $\alpha \in A'$ *and*
$\|\mu\| = \sup\{\|\mu_{G_\alpha}\| \,|\, \alpha \in A'\}$. *Indeed, we may define* $\mu$ *by* $\mu(\chi) = \mu_{G_\alpha}(\chi)$,
*if* $\chi \in G_\alpha^\wedge$ *for some* $\alpha \in A'$ *and* $\mu(\chi) = 0$ *otherwise.*

*Proof.* – We must show that $\mu$ defines a linear functional on
$C(G_\beta)$ for every compact quotient $G_\beta$, which is uniformly bounded
over all $\beta$. If $G_\beta^\wedge \cap G_\alpha^\wedge = 1$ for all $\alpha \in A'$, then $\|\mu_{G_\beta}\| = |\mu(1)|$. Sup-
pose that $G_\beta^\wedge \cap G_{\alpha_1}^\wedge \neq 1$ for some $\alpha_1 \in A'$. Then $G_\beta^\wedge \cap G_\alpha^\wedge = 1$ for all
other $\alpha \in A'$ by the pairwise independence of the $G_\alpha$. It is clear that
we may restrict our attention to trigonometric polynomials. Write
such a polynomial $f \in C(G_\beta)$ as

$$f = \sum_E a_i \chi_i + \sum_{E'} b_i \chi_i, \quad \text{where} \quad E = \{\chi_i \,|\, \chi_i \in G_{\alpha_1}^\wedge\}$$

and $E'$ is the remainder. Then

$$\mu(f) = \mu\left(\sum_E a_i \chi_i\right).$$

But by Lemma 2.3,

$$\|f\| \geqslant \|\sum_E a_i \chi_i\|.$$

Hence,

$$(|\mu(f)|/\|f\|) \leqslant \|\mu_{G_{\alpha_1}}\| .$$

This proves our lemma.

We now come to our main theorem.

THEOREM 3.10. — *The LCA group G has the property that every density extends to a bounded linear functional on the space of semi-periodic functions (that is, every density is induced by a measure on $\overline{G}^p$) if and only if the sum of each pair of periodic characters of G is periodic. In this case, the extension is an isometry and $D(G) = M(\overline{G}^p)$. If there exist two periodic characters of G with a nonperiodic sum then the set of densities induced by measures on $G^p$ is neither closed nor dense in $D(G)$.*

*Proof.* — We suppose first that the sum of each two periodic characters of G is periodic. Then by Theorem 2.10 we see that $C_\pi(G) \subset P(G)$ and $\pi(G)$ is a group. Thus, since $\overline{G}^p$ is the dual of $\pi(G)$, we see that $P(G)$ is dense in $C(\overline{G}^p)$. Hence $\mu$ extends uniquely and isometrically to a functional on $C(\overline{G}^p)$, and thus $D(G) = M(\overline{G}^p)$.

Now we suppose that there exist two periodic characters with a non-periodic sum. Then by Theorem 2.13, there are two periodic characters, $\chi_1$ and $\chi_2$ such that $\{[\chi_1\chi_2^n]^\wedge \mid n = 0, \pm 1, \pm 2, \ldots\}$ is a pairwise independent set of compact quotients of G. We now choose a Fourier series on $[-\pi, \pi]$ which has uniformly bounded partial sums, but is not absolutely convergent, such that

$$f(\theta) \approx \sum a_n e^{in\theta}, \quad \sum |a_n| = \infty, \quad a_0 = 0 .$$

We may choose $a_n = \dfrac{1}{n}$ for $n$ odd and 0 for $n$ even, for example, so that $f$ is a step function.

We now define $\mu_n$ on $[\chi_1\chi_2^n]^\wedge$ by $\mu_n(g) = (sgn\, a_n)\gamma_n$ where $\gamma_n$ is the coefficient of $\chi_1\chi_2^n$ in the Fourier expansion of $g$ on $[\chi_1\chi_2^n]^\wedge$. In other words $\mu_n = (sgn\, a_n)\overline{\chi}_1\overline{\chi}_2^n\,\lambda_n$ where $\lambda_n$ is Haar measure on $[\chi_1\chi_2^n]^\wedge$. Then $\|\mu_n\| \leqslant 1$ and $\int d\mu_n = 0$ for each $n$. By Lemma 3.8 we may extend $\{\mu_n\}$ to $\mu \in D(G)$ so that $\|\mu\| = 1$. But it is clear that $\mu$ is not bounded in $C_p(G)$. For if we define $F_n \in C_p(G)$ by

$$F_n(x) = \sum_{j=-n}^{n} a_n \chi_1 \chi_2^n(x)$$

we see that

$$\| F_n(x) \|_\infty \leqslant \| \sum_{j=-n}^{n} a_n e^{in\theta} \| \leqslant M. \text{ But } \mu(F_n) = \sum_{j=-n}^{n} |a_n|.$$

Hence $\mu$ is not bounded on $C_p(G)$ and therefore $\mu$ is not induced by a measure in $\overline{G}^p$.

Let us now denote the space of densities which are induced by measures on $\overline{G}^p$ by $\widetilde{M}(\overline{G}^p)$. If we choose $v \in D(G)$ such that $\| v - \mu \| < \dfrac{1}{2}$, then $|v(F_n)| \geqslant \sum_{j=-n}^{n} |\eta_n a_n|$ where $|\eta_n - 1| < \dfrac{1}{2}$. Hence $v$ is not bounded on $C_p(G)$ and so $(\widetilde{M}(\overline{G}^p))^- \underset{\neq}{\subset} D(G)$.

If we require $\rho_n = b_n (sgn\ a_n) \overline{\chi}_1 \overline{\chi}_2^n \lambda_n$ where $b_n \longrightarrow 0$ we can find $\rho \in D(G)$ so that $\rho \in (\widetilde{M}(\overline{G}^p))^-$. If we choose the $b_n$ approaching 0 sufficiently slowly we will still have $|\rho(G_n)| \longrightarrow \infty$ and hence $\rho \notin \widetilde{M}(\overline{G}^p)$. Thus $\widetilde{M}(\overline{G}^p)$ is not closed.

This completes the proof of Theorem 3.10.

In particular we have

COROLLARY 3.11. — $M(\overline{G}) = D(G)$ *if and only if G has no* $R^n$ *part and each discrete quotient of G is of bounded order.*

Theorem 3.10 also immediately implies that each density on G extends to a measure on $\overline{G}^p$ if and only if $P(G)$ is dense in $C_p(G)$.

In the case where $G = Z$, the integers, the first author presented in [1] a construction for $M(\overline{G}^p)$ by means of matrices. The term density was not used ; however, it is easy to recognize that each matrix corresponds to a density.

Theorem 3.10 has a curious implication. We recall that even if $C_\pi(G) \subset P(G)$ we may have $C_p(G) \not\subset P(G)$. Hence it would seem reasonable that there might exist some family of periodic functions with a periodic span other than the family of periodic characters which would be dense in $C_p(G)$ even if $C_\pi(G) \not\subset P(G)$. Theorem 3.10 tells us that this is impossible.

Theorem 3.10, unfortunately, does not tell us precisely which densities do extend to measures in the case where $D(G) \neq M(\overline{G^p})$.

We might hope to attack this problem by means of positivity of a density. For example, it is clear that a density which is positive as a functional on $C_p(G)$ extends to a measure on $\overline{G^p}$.

DEFINITION. — *We say a density $\mu$ is real if $\mu(f)$ is real for each real $f \in P(G)$. We say a density $\mu$ is positive if for each nonnegative real $f \in P(G)$ we have $\mu(f) \geqslant 0$.*

It is clear that if a real density is induced by a measure on $\overline{G^p}$ then $\mu$ decomposes into the difference of positive densities. However, a positive density is not necessarily positive or even bounded on $C_p(G)$. Indeed, we show that every real density on R can be written as the difference of two positive densities ; yet we know that densities on R do not necessarily extend to measures on $\overline{R}$.

A more surprising fact is that on the cylinder group, R x T, real densities do not necessarily decompose into the difference of positive densities.

THEOREM 3.12. — *Each real density on R is the difference of two positive densities.*

*Proof.* — We may assume that $\mu(1) \geqslant 0$. We choose $H = [h]$, a subgroup of compact index of R. Now we consider $P_{\{rh\}}$, the vector space of periodic functions whose period can be written as $rh$ for some rational $r$. Then the density $\mu$ is clearly a bounded linear functional on $P_{\{rh\}}$ and hence restricted to $P_{\{rh\}}$ can be written as $\mu_h^+ - \mu_h^-$ where $\| \mu_h^+ \|$, $\| \mu_h^- \| \leqslant \| \mu \|$. (Indeed, $\mu$ restricted to $P_{\{rh\}}$ is a measure on the compact dual of the group of rational multiples of $h$ when that group is given the discrete topology). We now put on the extra constraint $\mu_h^+(1) = \mu(1)$, $\mu_h^-(1) = 0$ and still get $\| \mu_h^+ \|$, $\| \mu_h^- \| \leqslant 2 \| \mu \|$.

We now select a maximal set of reals $\{h_\alpha\}$ such that no $h_\alpha$ is a rational multiple of any $h_\beta$ for $\beta \neq \alpha$.

It is clear that $P = \underset{\alpha}{\cup} P_{\{rh_\alpha\}}$ expresses P(R) as union of vector spaces which are disjoint, except for the constants. It is also clear that if $f \in P_{\{rh_\alpha\}}$, $g \in P_{\{rh_\beta\}}$ and if $f$ and $g$ are not constant then $f + g \notin P$.

For $f \in P_{\{rh_\alpha\}}$ we now write $\mu(f) = \mu_h^+(f) - \mu_h^-(f)$ where $\mu_h^+(1) = \mu(1)$ and $\mu_h^-(1) = 0$. Since each periodic $f$ belongs to $P_{\{rh_\alpha\}}$ for some $h_\alpha$ we see that $\mu^+$ is a bounded linear functional on $P_\gamma$ for each real $\gamma$ with norm uniformly bounded over all $\gamma$. Hence $\mu^+$ and $\mu^-$ are the desired densities.

We cannot always find such decompositions.

THEOREM 3.13. — *There is a real density on the cylinder group* $G = R \times T$ *that is not the difference of two positive densities.*

*Proof.* — We let $\{\alpha_n\}$ $n = 1,2,3, \ldots$ be a sequence of rationally independent real numbers, and let $\{\beta_n\}$ be a sequence of distinct elements of T. For $f$ in P(G), we take $\mu(f)$ as given by

$$\mu(f) = \sum_{n=1}^{\infty} \lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^{m} f(j\alpha_n, \beta_n)(-1)^j.$$

We will now explain this expression by showing that $\mu(f)$ is defined as a finite sum for all $f \in P(G)$, that $|\mu(f)| \leqslant 2\|f\|_\infty$, and that $\mu(f_1 + f_2) = \mu(f_1) + \mu(f_2)$ whenever $f_1$ and $f_2$ have the same period, so that $\mu$ is consequently defined as a density.

We suppose now that we are given a subgroup H of compact index with $K = G/H$, and that $f$ is a continuous periodic function of period H. For the moment, we suppose that $f$ is a finite linear combination of characters with period H. Then there are two integers $n_1$ and $n_2$, *depending only on* H, such that if

$$A_n(f) = \lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^{m} f(j\alpha_n, \beta_n)(-1)^j.$$

then $A_n(f)$ exists for all values of $n$, and $A_n(f) = 0$ except possibly for $n = n_1, n_2$. Clearly, then, $|A_n(f)| \leqslant \|f\|_\infty$, and consequently $|\mu(f)| \leqslant 2\|f\|_\infty$. To this end, we may suppose that $f = \chi$ is a periodic character with period H. Then $[\chi]$ is a discrete subgroup of $G^\wedge$. We may write $\chi = (n, \theta)$ where $n \in R = R^\wedge$ and $\theta \in Z = T^\wedge$. If $n$ is the trivial character, then $A_n(\chi) = 0$, as a direct calculation shows. Supposing then that $n \neq 0$, we have

$$A_n(\chi) = \theta(\beta_n) \lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^{m} e^{2\pi i n j \alpha_n} (-1)^j,$$

so that $A_n(\chi) = 0$ unless $\exp(2\pi i n \alpha_n + \pi i) = 1$, or equivalently, there is an integer $p_n$ such that $n = (2p_n - 1)/\alpha_n$. Because of the rational independence of the $\alpha_n$, this can hold for at most one $\alpha_n$. We now exclude, depending only on H, all but two of the $\alpha_n$. Now $\chi$ is a character of K, and the characters of K generate a discrete group. But three vectors in R x Z with rationally independent real parts generate a non-discrete group. Hence there are at most two possible choices for $\alpha_n$ so that $\chi$, with $n = (2p_n - 1)/\alpha_n$, has period K. The result for general periodic $f$ follows from approximation by finite linear combinations of periodic characters.

Now we prove that $\mu$ does not decompose. Let $g_k$ be a positive periodic continuous function with real period $2\alpha_k$, which takes the value 1 at $(2\alpha_k, \beta_k)$ and 0 at $(\alpha_k, \beta_k)$. Then $\mu(g_k) = 1/2$. If $\mu = \mu^+ - \mu^-$ were a decomposition of $\mu$ as a difference of two positive densities, then we would have $\mu^+(g_k) \geqslant 1/2$. Thus $\mu^+(h_k) \geqslant 1/2$ where $h_k$ is any positive periodic function constant on each real line, which is 1 at $(r, \beta_k)$ for all real $r$, and such that $h_k \geqslant g_k$. For any positive integer N, there exist such functions $h_k$, $k = 1, 2, \ldots, N$ such that if $\rho_N = h_1 + h_2 + \cdots + h_N$, then $|\rho_N| \leqslant 1$. Yet $\mu^+(\rho_N) \geqslant N/2$, which makes it impossible for $\mu$ to be a density, and the result is proved.

## 4. Uniform distribution of densities.

Uniformly distributed sequences of measures on compact groups have, in one form or another, attracted interest for many years. We can extend this notion in a natural way to LCA groups by means of densities. Indeed it was partly to facilitate this extension that densities were developed.

However, we will not treat uniform distributions exhaustively in this paper. We will not in general give proofs since we will use standard techniques in uniform distributions in compact groups in conjunction with ideas which have been thoroughly developed in the preceding sections of our paper.

DEFINITION. – *We will say a uniformly bounded sequence of densities* $\{\mu_n\}$ *in an* LCA *group* G *is uniformly distributed if* $\left\{\dfrac{1}{n} \sum\limits_{j=1}^{n} \mu_j\right\}$ *tends weakly to Haar density,* $\lambda$. *That is, for each periodic continuous* $f$ *we have* $\lim\limits_{n} \dfrac{1}{n} \sum\limits_{j=1}^{n} \mu_j(f) = \lambda(f)$.

With this definition the following criterion is immediate.

Weyl Criterion : The uniformly bounded sequence of densities $\{\mu_n\}$ is uniformly distributed if and only if for each non-trivial $\chi \in \pi(G)$ we have $\lim\limits_{n} \dfrac{1}{n} \sum\limits_{j=1}^{n} \mu_j(\chi) = 0$ and $\lim\limits_{n} \dfrac{1}{n} \sum\limits_{j=1}^{n} \mu_j(1) = 1$.

DEFINITION. – *If* $\{\widetilde{\mu}_n\}$ *is a uniformly bounded sequence of measures on the* LCA *group* G *we say* $\{\widetilde{\mu}_n\}$ *is uniformly distributed if* $\{\mu_n\}$, *the sequence of densities induced by* $\{\widetilde{\mu}_n\}$, *is uniformly distributed.*

It is clear that our definition coincides with the standard definition if G is compact. Furthermore it includes most of those extensions to special cases with which we are familiar. In this regard see Rubel [6], where uniformly distributed sequences of elements of G were considered. It is easy to see that the sequence $\{x_\nu\}$ of elements of G is uniformly distributed if and only if the associated sequence $\{\varepsilon_{x_\nu}\}$ of point masses is uniformly distributed.

If a density $\mu$ is induced by a measure $\widetilde{\mu}$ in M(G) then there is clearly a unique measure $\overline{\widetilde{\mu}}$ in M($\overline{G}^p$) which induces the same density in the sense that $\overline{\widetilde{\mu}}(f) = \widetilde{\mu}(f) = \mu(f)$ for $f \in$ P(G). In addition

$$\| \overline{\widetilde{\mu}} \|_{M(\overline{G}^p)} = \| \widetilde{\mu} \|_{M(G)} \geqslant \| \mu \|_{D(G)} \, .$$

Now Haar density $\lambda$, even though not induced by any member of M(G), is induced by Haar measure in M($\overline{G}^p$). Therefore it is natural to attempt to characterize uniform distribution of measures in G in terms of uniform distribution of measures in $\overline{G}^p$. The following theorem is easy to prove by use of the Weyl criterion.

THEOREM 4.1. – *Let* $\pi(G)$ *be a group. Let* $\{\mu_n\}$ *be a sequence of measures in* M(G). *Then, for each n, there is a unique* $\widetilde{\mu}_n$ *in* M($\overline{G}^p$)

*such that the density induced by $\mu_n$ is the density induced by $\{\widetilde{\mu}_n\}$ restricted to the compact quotients of G. Furthermore $\{\mu_n\}$ is uniformly distributed in G if and only if $\{\widetilde{\mu}_n\}$ is uniformly distributed in the compact group $\overline{G}^p$. That is, $\{\mu_n\}$ is uniformly distributed in G*

*if and only if* $\left\{\dfrac{1}{n}\sum\limits_{j=1}^{n}\widetilde{\mu}(j)\right\}$ *tends weakly to Haar measure in $\overline{G}^p$.*

Thus if $\pi(G)$ is not a group we are impelled to look for a compactification $\widetilde{G}$ of G such that if $\{\mu_n\}$ is a sequence of densities induced by members of M(G) and if $\{\widetilde{\mu}_n\}$ is a corresponding sequence of measures on $\widetilde{G}$ then $\{\mu_n\}$ is uniformly distributed if and only if $\{\widetilde{\mu}_n\}$ is uniformly distributed.

However, in our paper with M. Rajagopalan [2], it is shown that in the case where the measures of M(G) are point masses, it is never possible to find such a $\widetilde{G}$ unless $\pi(G)$ is a group. In the same paper those LCA groups G such that $\pi(G)$ is a group are characterized.

We now give an analogue of a classical theorem.

DEFINITION. — *If $\mu$ is a density then $\mu^-$ is the reflected density defined by*

$$\int f(x)\,d\mu^-(x) = \int f(-x)\,d\mu(x).$$

DEFINITION. — *We call a density $\mu$ normalized if it is positive and $\mu(1) = 1$.*

DEFINITION. — *A sequence $\mu$ of densities is uniformly distributed of order two if there is a uniformly distributed sequence $\{v_h\}$, $h = 1, 2, \ldots$ of densities such that for $h = 1, 2, \ldots$*

$$\frac{1}{n}\sum_{k=1}^{n}\mu_{k+h} * \mu_k^-$$

*tends weakly to $v_h$ as n tends to $\infty$.*

For example, it is easy to see that in the group R, the sequence $n^{3/2}$ is uniformly distributed of order two. The next result follows directly from an extension by Cigler [4] of the so-called Fundamental Theorem of Van der Corput.

THEOREM 4.2. – *If $\{\mu_n\}$ is a sequence of normalized densities that is uniformly distributed of order two, then it is uniformly distributed.*

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