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## PROJECTIVE INVARIANTS OF AN ORTHOGONAL ENNUPLE IN A FINSLER SPACE

by H. D. PANDE (\*)

### 1. Introduction.

We consider an  $n$ -dimensional Finsler space  $F_n$  with the fundamental metric function  $F(x, \dot{x})$ . This fundamental function is positive homogeneous of the first degree in  $\dot{x}^i$ , it is  $> 0$  for  $\sum (\dot{x}^i)^2 \neq 0$  and the quadratic form  $(\partial^2 F^2 / \partial \dot{x}^i \partial \dot{x}^j) \xi^i \xi^j$  is positive definite in the variables  $\xi^i$ . The metric tensor is given by

$$(1.1) \quad g_{ij}(x, \dot{x}) \stackrel{\text{def}}{=} \frac{1}{2} \partial_i \partial_j F^2(x, \dot{x}) \quad (1), (2)$$

This tensor is symmetric in the indices  $i, j$  and positive homogeneous of degree zero in  $\dot{x}^i$ . The contravariant components  $g^{ij}(x, \dot{x})$  of the metric tensor is determined by

$$(1.2) \quad g^{ij}(x, \dot{x}) g_{jk}(x, \dot{x}) = \delta_k^i \quad \begin{cases} 1 & \text{if } k = i. \\ 0 & \text{if } k \neq i \end{cases}$$

The covariant components of the unit vector along the direction of the element of support  $(x^i, \dot{x}^i)$  are given by

$$(1.3) \quad l_i(x, \dot{x}) = \partial_i F(x, \dot{x}).$$

The covariant derivative of a vector  $X^i(x, \dot{x})$ , depending on the element of support, with respect to  $x^k$  in the sense of

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(1)  $\partial_i = \partial / \partial x^i$  and  $\dot{\partial}_i = \partial / \partial \dot{x}^i$ .

(2) Numbers in brackets refer to the references at the end of the paper.

Cartan is given by

$$(1.4) \quad X^i_{|k}(x, \dot{x}) = (\partial_k X^i) - (\delta_j X^i) G^j_k + X^j \Gamma^*_{jk}{}^i,$$

where

$$(1.5a) \quad G^i_k(x, \dot{x}) \stackrel{\text{def}}{=} \delta_k G^i(x, \dot{x}),$$

$$(1.5b) \quad 2G^i(x, \dot{x}) \stackrel{\text{def}}{=} \gamma^i_{jk}(x, \dot{x}) \dot{x}^j \dot{x}^k,$$

$\gamma^i_{jk}(x, \dot{x})$  being the Christoffel's symbols of second kind [1] and  $\Gamma^*_{jk}{}^i(x, \dot{x})$  are the Cartan's connection coefficients symmetric in their lower indices and homogeneous of degree zero in their directional arguments. We have [1]

$$(1.6) \quad G^i_{jk}(x, \dot{x}) \dot{x}^j = \Gamma^*_{jk}{}^i(x, \dot{x}) \dot{x}^j = G^i_k(x, \dot{x}),$$

where  $G^i_{jk}(x, \dot{x}) \stackrel{\text{def}}{=} \partial_k G^i_j(x, \dot{x})$ .

Let  $\lambda_{(a)}\{a = 1, 2, \dots, n\}$  be the unit tangents of  $n$ -congruences of an orthogonal ennuple. The subscript «  $a$  » in the paranthesis simply distinguishes one congruence from the other. The covariant and contravariant components of  $\lambda_{(a)}$  will respectively be denoted by  $\lambda_{(a)i}$  and  $\lambda^i_{(a)}$ . Since  $n$ -congruences are mutually orthogonal, we have [2]

$$(1.7) \quad g_{ij}(x, \dot{x}) \lambda^i_{(a)} \lambda^j_{(b)} = \delta_{ab},$$

where the Kronecker delta  $\delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$ . We have the Ricci coefficients of rotation, given by [2, 3]

$$(1.8) \quad Y_{abc}(x, \dot{x}) \stackrel{\text{def}}{=} \lambda^i_{(a)|j} \lambda^j_{(b)i} \lambda^l_{(c)},$$

where the symbol  $|$  denotes the covariant derivative with respect to  $x^k$  in the sense of Cartan and

$$(1.9) \quad \mu^i_{(m)}(x, \dot{x}) \stackrel{\text{def}}{=} \sum_h Y_{mhm} \lambda^i_{(h)}.$$

The geometric entities  $\mu^i_{(m)}(x, \dot{x})$  are called the first curvature vector of a curve of congruence in Finsler space [3].

## 2. Projective transformation.

The equation of a geodesic

$$(2.1) \quad \frac{d^2 x^i}{ds^2} + \Gamma^*_{jk}{}^i(x, dx/ds) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$

assumes the following form by the transformation of its parameter  $s$  to  $t$  [4]:

$$(2.2) \quad \dot{x}^i \left\{ \frac{d^2 x^i}{dt^2} + \Gamma_{jk}^*(x, \dot{x}) \dot{x}^j \dot{x}^k \right\} - \dot{x}^i \left\{ \frac{d^2 x^i}{dt^2} + \Gamma_{jk}^{*i}(x, \dot{x}) \dot{x}^j \dot{x}^k \right\} = 0,$$

where

$$(2.3) \quad \Gamma_{jk}^{*i}(x, \dot{x}) = \Gamma_{kj}^{*i}(x, \dot{x}).$$

The equation (2.2) remains unchanged if we replace the Cartan's connection coefficient  $\Gamma_{jk}^{*i}(x, \dot{x})$  by a new symmetric coefficient  $\bar{\Gamma}_{jk}^{*i}(x, \dot{x})$ , given by [6]

$$(2.4) \quad \bar{\Gamma}_{jk}^{*i}(x, \dot{x}) \stackrel{\text{def}}{=} \Gamma_{jk}^{*i}(x, \dot{x}) + 2\delta_{(j}^i p_{k)} + p_{jk} \dot{x}^i,$$

where  $p_k(x, \dot{x})$  is a covariant vector, positively homogeneous of degree zero in its directional arguments and

$$(2.5) \quad p_{jk}(x, \dot{x}) \stackrel{\text{def}}{=} \delta_j p_k(x, \dot{x}).$$

DÉFINITION 2.1. — Let  $F_n$  and  $\bar{F}_n$  be two spaces with fundamental tensor  $g_{ij}(x, \dot{x})$  and  $\bar{g}_{ij}(x, \dot{x})$  at the corresponding points. Then the spaces are said to be in geodesic correspondence if their geodesics are the same and we shall call (2.4) a « projective change » of the Cartan's function  $\Gamma_{jk}^{*i}(x, \dot{x})$ .

Contracting (2.4) with respect to the indices  $i$  and  $j$ , we get

$$(2.6) \quad \bar{\Gamma}_{\gamma k}^{*\gamma}(x, \dot{x}) = \Gamma_{\gamma k}^{*\gamma}(x, \dot{x}) + (n + 1)p_k(x, \dot{x}).$$

Differentiating (2.6) with respect to  $\dot{x}^l$ , we obtain

$$(2.7) \quad \delta_l \bar{\Gamma}_{\gamma k}^{*\gamma}(x, \dot{x}) = \delta_l \Gamma_{\gamma k}^{*\gamma}(x, \dot{x}) + (n + 1)p_{lk}(x, \dot{x}).$$

### 3. Projective invariants.

THEOREM 3.1. — If  $\lambda_{(a)}^i(x)$  and  $\lambda_{(a)i}(x)$  are the contravariant and covariant components of an orthogonal ennuple, then the following geometric entities are invariant under the projective change:

$$(3.1) \quad A_k^i(x, \dot{x}) \stackrel{\text{def}}{=} \lambda_{(a)lk}^i - \frac{1}{n + 1} \lambda_{(a)}^j \left\{ 2 \sum_b \lambda_{(b)m} \delta_{ij} \lambda_{(b)lk}^m \right\},$$

and

$$(3.2) \quad A_k^*(x, \dot{x}) \stackrel{\text{def}}{=} \sum_a \lambda_{(a)i} \lambda_{(a)|k}^i - \Gamma_{\gamma k}^{*\gamma}$$

*Proof.* — If we denote by  $\lambda_{(a)|\bar{k}}^i$  the covariant derivative of  $\lambda_{(a)}^i$  in the sense of Cartan for the connection coefficients  $\bar{\Gamma}_{jk}^{*i}(x, \dot{x})$ , then we have

$$(3.3) \quad \lambda_{(a)|\bar{k}}^i = \delta_k \lambda_{(a)}^i + \lambda_{(a)}^j \bar{\Gamma}_{jk}^{*i}$$

Hence we get in consequence of (2.4)

$$(3.4) \quad \lambda_{(a)|\bar{k}}^i - \lambda_{(a)|k}^i = \lambda_{(a)}^j \{ 2\delta_{(j} p_{k)} + p_{jk} \dot{x}^i \}.$$

Multiplying (3.4) by  $\lambda_{(a)i}$  throughout and summing with respect to  $a$  and using the orthogonality condition (1.7), we obtain

$$(3.5) \quad \sum_a \lambda_{(a)i} (\lambda_{(a)|\bar{k}}^i - \lambda_{(a)|k}^i) = (n + 1) p_k.$$

Eliminating the vector  $p_k(x, \dot{x})$  from equations (3.4) and (3.5), we get

$$(3.6) \quad \lambda_{(a)|\bar{k}}^i - \lambda_{(a)|k}^i = \frac{1}{n + 1} \lambda_{(a)}^j \left[ \delta_j^i \sum_b \lambda_{(b)m} (\lambda_{(b)|\bar{k}}^m - \lambda_{(b)|k}^m) + \delta_k^i \sum_b \lambda_{(b)m} (\lambda_{(b)|\bar{j}}^m - \lambda_{(b)|j}^m) \right].$$

Again, with the help of (2.6), equation (3.5) yields

$$(3.7) \quad \sum_a \lambda_{(a)i} (\lambda_{(a)|\bar{k}}^i - \lambda_{(a)|k}^i) = \bar{\Gamma}_{\gamma k}^{*\gamma} - \Gamma_{\gamma k}^{*\gamma}$$

which gives us (3.2).

**THEOREM 3.2.** — *When  $F_n$  and  $\bar{F}_n$  are in geodesic correspondence, we have the following geometric entities which are invariant under the projective change:*

$$(3.8) \quad C_k^i(x, \dot{x}) \stackrel{\text{def}}{=} \lambda_{(a)|k}^i - \frac{1}{n + 1} \lambda_{(a)}^j \{ 2\delta_{(j}^i \Gamma_{k)\gamma}^{*\gamma} + \dot{x}^i \delta_j \Gamma_{\gamma k}^{*\gamma} \},$$

and

$$(3.9) \quad C_k^{*i}(x, \dot{x}) \stackrel{\text{def}}{=} \lambda_{(a)|k}^i - \frac{1}{n + 1} \lambda_{(a)}^j \left\{ \sum_b 2\lambda_{(b)m} \delta_{(j}^i \lambda_{(b)|k)}^m + \dot{x}^i \delta_j \Gamma_{\gamma k}^{*\gamma} \right\}.$$

*Proof.* — Using equations (2.6), (2.7) and (3.4), we get

$$(3.10) \quad \lambda_{(a)|\bar{k}}^i - \lambda_{(a)|k}^i = \frac{1}{n+1} \lambda_{(a)}^j [\delta_j^i (\bar{\Gamma}_{\gamma k}^{*\gamma} - \Gamma_{\gamma k}^{*\gamma}) + \delta_k^i (\bar{\Gamma}_{\gamma j}^{*\gamma} - \Gamma_{\gamma j}^{*\gamma}) + \dot{x}^i \delta_j^i (\bar{\Gamma}_{\gamma k}^{*\gamma} - \Gamma_{\gamma k}^{*\gamma})],$$

which yields the result (3.8).

Again, eliminating  $p_k(x, \dot{x})$  and  $p_{ik}(x, \dot{x})$  from equations (2.7), (3.4) and (3.5), we obtain

$$(3.11) \quad \lambda_{(a)|\bar{k}}^i - \lambda_{(a)|k}^i = \frac{1}{n+1} \lambda_{(a)}^j \sum_b (b)_m [\delta_j^i (\lambda_{(b)|\bar{k}}^m - \lambda_{(b)|k}^m) + \delta_k^i (\lambda_{(b)|\bar{j}}^m - \lambda_{(b)|j}^m)] + \frac{\dot{x}^i}{n+1} \lambda_{(a)}^j \delta_j^i (\bar{\Gamma}_{\gamma k}^{*\gamma} - \Gamma_{\gamma k}^{*\gamma}),$$

which gives us (3.9).

**THEOREM 3.3.** — *When  $F_n$  and  $\bar{F}_n$  are in geodesic correspondence, we have the following projective invariant geometric entities :*

$$(3.12) \quad S_{abc}(x, \dot{x}) \stackrel{\text{def}}{=} Y_{abc} - \frac{1}{n+1} (\delta_{ab} \lambda_{(c)}^k \Gamma_{\gamma k}^{*\gamma} + \delta_{bc} \lambda_{(a)}^j \Gamma_{\gamma j}^{*\gamma}) - \frac{1}{n+1} \lambda_{(a)}^j \lambda_{(b)i} \lambda_{(c)}^k \dot{x}^i \delta_j^k \Gamma_{\gamma k}^{*\gamma},$$

and

$$(3.13) \quad S_{acd}^*(x, \dot{x}) \stackrel{\text{def}}{=} Y_{acd} - \frac{1}{n+1} \left[ \sum_b (\delta_{ca} Y_{bbd} + \delta_{cd} Y_{bba}) \right] - \frac{1}{n+1} \lambda_{(a)}^j \lambda_{(c)i} \lambda_{(d)}^k \dot{x}^i \delta_j^k \Gamma_{\gamma k}^{*\gamma},$$

and

$$(3.14) \quad S_b(x, \dot{x}) \stackrel{\text{def}}{=} \sum_a Y_{aab} - \lambda_{(b)}^k \Gamma_{\gamma k}^{*\gamma},$$

where  $Y_{abc}$  are Ricci coefficients of rotation.

*Proof.* — Multiplying (3.10) by the product  $\lambda_{(b)i} \lambda_{(c)}^k$  and using the orthogonality relation (1.7), we get

$$(3.15) \quad \bar{Y}_{abc} - \frac{1}{n+1} (\delta_{ab} \lambda_{(c)}^k \bar{\Gamma}_{\gamma k}^{*\gamma} + \delta_{bc} \lambda_{(a)}^j \bar{\Gamma}_{\gamma j}^{*\gamma}) + \lambda_{(a)}^j \lambda_{(b)i} \lambda_{(c)}^k \dot{x}^i \delta_j^k \bar{\Gamma}_{\gamma k}^{*\gamma} = Y_{abc} - \frac{1}{n+1} (\delta_{ab} \lambda_{(c)}^k \Gamma_{\gamma k}^{*\gamma} + \delta_{bc} \lambda_{(a)}^j \Gamma_{\gamma j}^{*\gamma} + \lambda_{(a)}^j \lambda_{(b)i} \lambda_{(c)}^k \dot{x}^i \delta_j^k \Gamma_{\gamma k}^{*\gamma}),$$

where the projectively transformed Ricci coefficients of rotation are given by

$$(3.16) \quad \bar{Y}_{abc} \stackrel{\text{def}}{=} \lambda_{(a)|k}^i \lambda_{(b)i} \lambda_{(c)}^k.$$

Similarly, multiplying (3.11) by the product  $\lambda_{(c)i} \lambda_{(d)}^k$  and using the orthogonal relation (1.7), we obtain (3.13).

Again, multiplying (3.7) by  $\lambda_{(b)}^k$  and making use of (1.7), we get

$$(3.17) \quad \sum_a \bar{Y}_{aab} - \lambda_{(b)}^k \bar{\Gamma}_{\gamma k}^{*\gamma} = \sum_a Y_{aab} - \lambda_{(b)}^k \Gamma_{\gamma k}^{*\gamma},$$

which shows that  $S_b(x, \dot{x})$  are invariant under the projective change.

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