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BI-QUOTIENT MAPS AND CARTESIAN PRODUCTS OF QUOTIENT MAPS

by Ernest MICHAEL (1)

1. Introduction.

In this paper we introduce a new class of maps which seems to have many desirable properties. In particular, it permits us to characterize those quotient maps whose cartesian product with every quotient map is a quotient map. All maps in this paper are assumed to be *continuous* and *onto*.

DEFINITION 1.1. — *A map* $f\colon X \to Y$ *is called bi-quotient* (2) *if, whenever* $y \in Y$ *and* $\mathcal{U}$ *is a covering of* $f^{-1}(y)$ *by open subsets of* X, *then finitely many* $f(U)$, *with* $U \in \mathcal{U}$, *cover some neighborhood of* y *in* Y (3).

As we shall see, all open and all proper (4) maps are bi-quotient, and all bi-quotient maps are quotient maps. These and other general properties of bi-quotient maps are proved in sections 2 and 3. Our principal concern, however, is with product maps.

If $f_\alpha : X_\alpha \to Y_\alpha$ is a map for all $\alpha \in A$, then the *product map* $f = \Pi_\alpha f_\alpha$ from $\Pi_\alpha Y_\alpha$ to $\Pi_\alpha Y_\alpha$ is defined by $f(x) = (f_\alpha(x_\alpha))$. That brings us to out first theorem.

(1) Supported by an N.S.F. contract.

(2) This terminology is explained by Theorem 1.3.

(3) Professor O. Hájek has kindly pointed out that bi-quotient maps are equivalent (in view of our Proposition 2.2) to the limit lifting maps which he defined in [*Comment. Math. Univ. Carolinae* 7 (1966), 319-323], and that our Theorem 1.3 is thus equivalent to Proposition 2 in that paper. It is a pleasure to acknowledge Professor Hájek's priority.

(4) A map $f\colon X \to Y$ is called *proper* (= *perfect* in Russian terminology) if f is closed and if $f^{-1}(y)$ is compact for all $y \in Y$. Example 8.1 shows that not all closed maps are bi-quotient.

THEOREM 1.2. — *Any product (finite or infinite) of bi-quotient maps is a bi-quotient map.*

Theorem 1.2 stands in sharp contrast to the behavior of quotient maps, which are not even preserved by finite products ([4]). In fact, the following theorem shows that the only quotient maps which are well behaved under even the simplest products are bi-quotient maps. We denote the identity map on Z by i_Z.

THEOREM 1.3. — *If $f: X \to Y$ is a map, and if Y is Hausdorff, then the following are equivalent.*

(a) *f is bi-quotient.*

(b) *$f \times i_Z$ is a quotient map for every space Z.*

(c) *$f \times i_Z$ is a quotient map for every paracompact space Z.*

Theorem 1.3 should be compared with Bourbaki's result [4; p. 117, Theorem 1] that a map $f: X \to Y$ is proper if and only if $f \times i_Z$ is a closed map for every space Z. (This is actually Bourbaki's *definition* of a proper map, which he shows, in the above theorem, to be equivalent to our definition in footnote ([3])).

Theorem 1.3 raises the question of what happens if the second factor is not an identity map. Here a useful criterion was provided by J. H. C. Whitehead [17; Lemma 4] in case the *first* factor is an identity map: If Y is locally compact Hausdorff, then $i_Y \times g$ is a quotient map for every quotient map g. In [12; Theorem 2.1], it was shown that local compactness is not only sufficient here, but also necessary. These results, when combined with Theorem 1.3, finaly lead to the following theorem.

THEOREM 1.4. — *If $f: X \to Y$ is a map, and if Y is regular, then the following are equivalent.*

(a) *f is bi-quotient and Y is locally compact.*

(b) *$f \times g$ is a quotient map for every quotient map g.*

([4]) Provided products and quotient maps retain their usual meanings, as they do in this paper. If these meanings are suitably modified, however, then the product of any two quotient maps becomes a quotient map. (See R. Brown [6; Proposition 3.1] and N. Steenrod [15; Theorem 4.4].)

(c) $f \times g$ *is a quotient map for every closed map* g *with paracompact domain and range.*

Theorem 1.4 does not, of course, entirely settle the problem of products of quotient maps, since such a product may be a quotient map even though neither factor satisfies the conditions of Theorem 1.4 (a). The following theorem covers many such cases, and seems to have rather minimal hypotheses. (For instance, Example 8.6 shows that the assumption that X_1 is a k-space [5] cannot be dropped.)

THEOREM 1.5. — *If* $f_i : X_i \to Y_i$ $(i = 1,2)$ *are quotient maps, and if* X_1 *and* $Y_1 \times Y_2$ *are both Hausdorff* k*-spaces, then* $f_1 \times f_2$ *is a quotient map.*

Characterizations and general properties of bi-quotient maps are developed in sections 2 and 3, while Theorems 1.2-1.5 are proved in sections 4-7. Section 8 is denoted to examples.

2. Two Characterizations.

PROPOSITION 2.1. — *If* Y *is Hausdorff, then the following properties of a map* $f : X \to Y$ *are equivalent.*

(a) f *is bi-quotient.*

(b) *If* $y \in Y$ *and* $\mathfrak{U}$ *is an open cover of* X, *then finitely many* $f(U)$, *with* $U \in \mathfrak{U}$, *cover some neighborhood of* y *in* Y.

Proof. — That (a) $\to$ (b) is clear. Let us therefore assume (b) and prove (a). Let $y \in Y$ and $\mathfrak{U}$ a cover of $f^{-1}(y)$ by open subsets of X. Since Y is Hausdorff, there is a cover $\mathfrak{V}$ of $Y - \{y\}$ by open sets whose closures in Y do not contain y. Let $\mathfrak{W} = \mathfrak{U} \cup f^{-1}(\mathfrak{V})$. Then $\mathfrak{W}$ is an open cover of X, so (b) implies that y has a neighborhood $N = A \cup B$, where A is the union of finitely many $f(U)$ with $U \in \mathfrak{U}$ and B is the union of finitely $V \in \mathfrak{V}$. But then $y \notin \overline{B}$, so $A \supset N - \overline{B}$ is also a neighborhood of y in Y. That completes the proof.

The identity map from a finite space with discrete topology

[5] A topological space X is called a *k-space* if a set $A \subset X$ is closed whenever $A \cap K$ is closed in K for every compact $K \subset X$. Locally compact spaces and first-countable spaces are k-spaces.

to the same space with a non-discrete topology shows that Proposition 2.1 may be false if Y is not Hausdorff.

Recall now that a *filter base* $\mathcal{B}$ is a non-empty collection of non-empty sets such that the intersection of two elements of $\mathcal{B}$ always contains an element of $\mathcal{B}$ [4; p. 66]. If $\mathcal{B}$ is a filter base in Y, and if $y \in Y$, then y *adheres* to $\mathcal{B}$ if $y \in \overline{B}$ for every $B \in \mathcal{B}$.

PROPOSITION 2.2. — *The following properties of a map* $f: X \to Y$ *are equivalent.*

(a) f *is bi-quotient.*

(b) *If* $\mathcal{B}$ *is a filter base in* Y, *and if* $y \in Y$ *adheres to* $\mathcal{B}$, *then some* $x \in f^{-1}(y)$ *adheres to* $f^{-1}(\mathcal{B})$.

Proof. — (a) $\to$ (b). If (b) is false, there is a filter base $\mathcal{B}$ in Y, and a $y \in Y$ adherent to $\mathcal{B}$, such that each $x \in f^{-1}(y)$ has an open neighborhood U_x in X which is disjoint from $f^{-1}(B)$ for some $B \in \mathcal{B}$. But then $\{U_x : x \in f^{-1}(y)\}$ covers $f^{-1}(y)$, but no neighborhood of y in Y is covered by finitely many $f(U_x)$.

(b) $\to$ (a). If (a) is false, there is a $y \in Y$, and a cover $\mathcal{U}$ of $f^{-1}(y)$ by open subsets of X, such that no neighborhood of y in Y is the union of finitely many $f(U)$. Let $\mathcal{B}$ consist of all complements in Y of such finite unions. Then y adheres to $\mathcal{B}$, but no $x \in f^{-1}(y)$ adheres to $f^{-1}(\mathcal{B})$. That completes the proof.

Proposition 2.2 should be compared with Bourbaki's result [4; p. 117, Theorem 1] that a map $f: X \to Y$ is proper if and only if, whenever $\mathcal{B}$ is a filter base in X such that $y \in Y$ adheres to $f(\mathcal{B})$, then some $x \in f^{-1}(y)$ adheres to $\mathcal{B}$.

3. General Properties.

Recall that a map $f: X \to Y$ is called a *quotient* map if $V \subset Y$ is open in Y whenever $f^{-1}(V)$ is open in X. This immediately implies that every bi-quotient map is a quotient map. We can, however, do somewhat better. A map $f: X \to Y$ is called *hereditarily quotient* if $f(U)$ is a neighborhood of y

in Y whenever U is a neighborhood of $f^{-1}(y)$ in X. (According to A. Arhangel'skiĭ [2; Theorem 1], this is equivalent to requiring that $f|f^{-1}(S)$ be a quotient map from $f^{-1}(S)$ onto S for every $S \subset Y$, which explains the terminology.) The following result is now immediate.

Proposition 3.1. — *Every bi-quotient map is hereditarily quotient.*

Let us record in passing that closed maps, as well as quotient maps with first-countable range, are also hereditarily quotient.

Before continuing, let us note two other simple properties of bi-quotient maps. First, every bi-quotient map f is *hereditarily bi-quotient*, in the sense that $f|f^{-1}(S)$ is a bi-quotient map from $f^{-1}(S)$ onto S for every $S \subset Y$; this yields another proof of Proposition 3.1. Second, if $f: X \to Y$, and if there is some $X' \subset X$ such that $f(X') = Y$ and $f|X'$ is bi-quotient, then f is bi-quotient.

Proposition 3.2. — *Each of the following conditions implies that a map $f: X \to Y$ is bi-quotient.*

(a) *f is open.*

(b) *f is hereditarily quotient, and $\partial f^{-1}(y)$ is compact for every y in Y (where ∂ denotes boundary).*

(c) *f is proper.*

Proof. — The sufficiency of (a) is clear, (b) requires only routine verification, and (c) is a special case of (b). That completes the proof.

For the next result, we need some more terminology. A map $f: X \to Y$ is called *compact-covering* if every compact subset of Y is the image of some compact subset of X. A space Y is called a *q-space* [11; p. 173] if every $y \in Y$ has a sequence of neighborhoods Q_n such that, if $y_n \in Q_n$ for all n and if $S = \{y_n\}_{n=1}^{\infty}$ is infinite, then S has an accumulation point in Y [6]. Clearly first-countable spaces and locally compact spaces are q-spaces.

[6] A. Arhangel'skiĭ has pointed out in a letter that, in the presence of paracompactness, q-spaces coincide with the spaces of pointwise countable type which he defined in [1].

Proposition 3.3. — *If* Y *is Hausdorff, each of the following conditions implies that a map* $f\colon X \to Y$ *is bi-quotient.*

(a) f *is compact-covering, and* Y *is locally compact.*

(b) X *is paracompact,* f *is closed, and* Y *is a q-space.*

(c) X *is Lindelöf,* f *is quotient, and* Y *is a q-space.*

(d) *Each* $\partial f^{-1}(y)$ *is Lindelöf,* f *is quotient, and* Y *is first-countable.*

Proof. — (a) This follows from Proposition 2.1 and the definition.

(b) These conditions imply [11; Theorem 2.1] that each $\partial f^{-1}(y)$ is compact. Since closed maps are hereditarily quotient, f must be bi-quotient by Proposition 3.2 (b).

(c) By Proposition 2.1, it suffices to show that, if $y \in Y$ and $\mathcal{U}$ is an open cover of X, then finitely many $f(U)$ cover some neighborhood of y in Y. Suppose not. Let $\{U_i\}_{i=1}^{\infty}$ be a countable subcover of $\mathcal{U}$, and let $V_n = \bigcup_{i=1}^{n} U_i$ for all n. Let $\{Q_n\}_{n=1}^{\infty}$ be the sequence of neighborhoods of y guaranteed by the fact that Y is a q-space. By assumption, no $f(V_n)$ contains Q_n, so there is a $y_n \in Q_n - f(V_n)$ for each n. Since the $f(V_n)$ cover Y, the set S of all these y_n must be infinite, so S has an accumulation point, and hence a subset $R \subset S$ which is not closed in Y. But $f^{-1}(R) \cap V_n$ is relatively closed in V_n for all n (since $S \cap f(V_n)$ is finite), so $f^{-1}(R)$ is closed in X. This contradicts the assumption that f is a quotient map.

(d) Let $y \in Y$, and $\mathcal{U}$ a covering of $f^{-1}(y)$ by open subsets of X. Pick $U_1, \ldots, U_n$ in $\mathcal{U}$ to cover $\partial f^{-1}(y)$; if $\partial f^{-1}(y)$ is empty, pick $U_1 \in \mathcal{U}$ to intersect $f^{-1}(y)$. Let $V_n = \bigcup_{i=1}^{n} U_i$, and let $W_n = V_n \cup (\partial f^{-1}(y))$. Then $\{W_n\}$ is an increasing sequence of open subsets of X which covers $f^{-1}(y)$, so some $f(W_k)$ is a neighborhood N of y in Y by a result of A. H. Stone [16; p. 694, Lemma 1]. But $f(V_k) = f(W_k)$, so N is covered by $f(U_1), \ldots, f(U_k)$. That completes the proof.

Our next result shows that bi-quotient maps preserve some topological properties which are not preserved by all hereditarily quotient maps (see Examples 8.1 and 8.4). In view of Proposition 3.3 (c), this result slightly generalizes two lemmas of A. H. Stone [16; p. 695, Lemma 2 and 3].

PROPOSITION 3.4. — *Let* $f: X \to Y$ *be bi-quotient.*

(a) *If* X *is locally compact, so is* Y.

(b) *If* X *has a countable base, so does* Y.

Proof. — (a) For each $x \in X$, let K_x be a compact neighborhood of x in X. Then the interiors of these K_x cover X, so if $y \in Y$, then the union of finitely many $f(K_x)$ is a neighborhood N of y in Y. But this N is compact, so that y has a compact neighborhood.

(b) It is easy to check that, if $\mathcal{B}$ is a base for X, then the interiors of finite unions of sets $f(B)$, with $B \in \mathcal{B}$, from a base for Y. That completes the proof.

COROLLARY 3.5. — *Let* $f: X \to Y$ *be a quotient map, with* Y *Hausdorff, and let* X *have a countable base. Then* Y *has a countable base if and only if* f *is bi-quotient.*

Proof. — This follows from Proposition 3.3 (c) and Proposition 3.4 (b).

We conclude this section with a result on the composition of bi-quotient maps.

PROPOSITION 3.6. — *Let* $f: X \to Y$ *and* $g: Y \to Z$ *be maps.*

(a) *If* f *and* g *are bi-quotient, so is* $f \circ g$.

(b) *If* $f \circ g$ *is bi-quotient, so is* g.

Proof. — Both (a) and (b) can be routinely verified by using the characterization of bi-quotient maps in Proposition 2.2.

4. Proof of Theorem 1.2.

Throughout this section, assume that $f_\alpha : X_\alpha \to Y_\alpha$ is a map for all $\alpha \in A$, that $X = \Pi_\alpha X_\alpha$, $Y = \Pi_\alpha Y_\alpha$, $f: X \to Y$

is defined by $f = \Pi_\alpha f_\alpha$, and $p_\alpha : X \to Y_\alpha$ and $q_\alpha : Y \to Y_\alpha$ are the projections.

LEMMA 4.1. — *If* $U_\alpha \subset X_\alpha$ *for all* α, *then*

$$f\left(\bigcap_\alpha p_\alpha^{-1} U_\alpha\right) = \bigcap_\alpha f p_\alpha^{-1} U_\alpha.$$

Proof. — Observe that

$$f\left(\bigcap_\alpha p_\alpha^{-1} U_\alpha\right) = f(\Pi_\alpha U_\alpha) = \Pi_\alpha f_\alpha U_\alpha = \bigcap_\alpha q_\alpha^{-1} f_\alpha U_\alpha = \bigcap_\alpha f p_\alpha^{-1} U_\alpha.$$

That completes the proof.

THEOREM 1.2. — *If each* f_α *is bi-quotient, then so is* f.

Proof. — We use the characterization of bi-quotient maps given in Proposition 2.2. Let y adhere to $\mathcal{B}$ for some filter base $\mathcal{B}$ in Y. Then there exists an ultrafilter $\mathcal{F}$ which contains $\mathcal{B}$ and converges to y [4; p. 66, Proposition 2; p. 67, Theorem 1; p. 79, Corollary]. Hence $q_\alpha \mathcal{F}$ converges to y_α for all α. Since f_α is bi-quotient, there is an $x(\alpha) \in f_\alpha^{-1}(y_\alpha)$ which adheres to $f_\alpha^{-1} q_\alpha \mathcal{F}$. Let $x \in X$ be the point with $x_\alpha = x(\alpha)$ for all α. Then $x \in f^{-1}(y)$, and it remains to show that x adheres to $f^{-1}(\mathcal{F})$.

If U_α is a neighborhood of x_α in X_α, and if $F \in \mathcal{F}$, then U_α intersects $f_\alpha^{-1} q_\alpha F = p_\alpha f^{-1}(F)$ (since x_α adheres to $f_\alpha^{-1} q_\alpha \mathcal{F}$), so $f p_\alpha^{-1}(U_\alpha)$ intersects F. Since $\mathcal{F}$ is an ultrafilter, $f p_\alpha^{-1}(U_\alpha)$ is an element of $\mathcal{F}$, and hence so is any finite intersection $\bigcap_{i=1}^{n} f p_{\alpha_i}^{-1}(U_{\alpha_i})$, so that these intersections intersect every $F \in \mathcal{F}$. Hence, by Lemma 4.1, $\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})$ intersects $f^{-1}(F)$ for every $F \in \mathcal{F}$, and this means that x adheres to $f^{-1}(\mathcal{F})$ and hence to $f^{-1}(\mathcal{B})$.

5. Proof of Theorem 1.3.

(a) $\to$ (b). If f is bi-quotient, then its product with any bi-quotient map, and hence surely with any identity map i_Z, is bi-quotient by Theorem 1.2, and is thus a quotient map by Proposition 3.1.

(b) $\to$ (c). Obvious.

(c) $\to$ (a). Suppose that f is not bi-quotient, and let us find a paracompact space Z such that $f \times i_Z$ is not a quotient map.

By Proposition 2.1, there is a $y_0 \in Y$ and an open cover $\mathcal{U}$ of X such that no neighborhood of y_0 in Y is the union of finitely many $f(U)$. Let $\mathcal{B}$ be the collection of all complements in Y of such finite unions. Then $y_0 \in \overline{B}$ for all $B \in \mathcal{B}$. We may suppose that each $U \in \mathcal{U}$ intersects $f^{-1}(y_0)$, so that no $B \in \mathcal{B}$ contains y_0.

Let Z be the set Y, topologized as follows: Each point of $Z - \{y_0\}$ is open, and the sets $\{y_0\} \cup B$, with $B \in \mathcal{B}$, form a base for the neighborhoods of y_0 in Z. Then Z is Hausdorff because $\bigcap \mathcal{B} = \varnothing$, and Z is paracompact because it has only one non-isolated point y_0. To prove that $h = f \times i_Z$ is not a quotient map, let

$$S = \{(y, y) \in Y \times Z : y \neq y_0\},$$

and let us show that S is not closed in $Y \times Z$ while $h^{-1}(S)$ is closed in $X \times Z$.

To show that S is not closed in $Y \times Z$, we need only check that $(y_0, y_0) \in \overline{S}$. Let $N = V \times (B \cup \{y_0\})$ be a basic neighborhood of (y_0, y_0) in $Y \times Z$, where V is a neighborhood of y_0 in Y and $B \in \mathcal{B}$. Since $y_0 \in \overline{B}$ but $y_0 \notin B$, there is some $y \in V \cap B$ with $y \neq y_0$ and $(y, y) \in N$. Hence $(y_0, y_0) \in \overline{S}$.

To show that $h^{-1}(S)$ is closed in $X \times Z$, suppose that $(x, y) \notin h^{-1}(S)$, and let us find a neighborhood W of (x, y) in $X \times Z$ which misses $h^{-1}(S)$. If $y \neq y_0$, then $f(x) \neq y$ (since $(x, y) \notin h^{-1}(S)$), and we can take

$$W = f^{-1}(Y - \{y\}) \times \{y\}.$$

If $y = y_0$, pick some $U \in \mathcal{U}$ which contains x, let $B = Y - f(U)$ and let

$$W = U \times (B \cup \{y_0\}).$$

That completes the proof.

6. Proof of Theorem 1.4.

(a) $\to$ (b). Assume that Y is locally compact Hausdorff and $f: X \to Y$ bi-quotient, and let us show that $f \times g$ is a quotient map for every quotient map $g: R \to S$. Observe that

$$f \times g = (i_Y \times g) \circ (f \times i_R).$$

Now $i_Y \times g$ is a quotient map by the result of J. H. C. Whitehead [17; Lemma 4] quoted in the introduction, and $f \times i_R$ is a quotient map by Theorem 1.3. Hence $f \times g$ is also a quotient map.

(b) $\to$ (c). Obvious.

(c) $\to$ (a). Assume that (a) is false, and let us contradict (c). If f is not bi-quotient, then Theorem 1.3 already assures us that $f \times i_Z$ is not a quotient map for some paracompact space Z. If Y is not locally compact, then [12; Theorem 2.1] implies that, for some closed map g with paracompact domain and range, $i_Y \times g$ is not a quotient map. But then $f \times g$ cannot be a quotient map either, and that completes the proof.

7. Proof and applications of Theorem 1.5.

The theorem asserts that, if $f_i: X_i \to Y_i$ $(i = 1,2)$ are quotient maps, and if X_1 and $Y_1 \times Y_2$ are Hausdorff k-spaces, then $f = f_1 \times f_2$ is a quotient map. We begin with two lemmas, each of which asserts the theorem under additional hypotheses.

Lemma 7.1. — *The theorem is true if* $X_1 \times X_2$ *is a Hausdorff k-space.*

Proof. — This follows immediately from a theorem of R. Brown [6; Proposition 3.1].

Lemma 7.2. — *The theorem is true if* X_1 *is locally compact Hausdorff.*

Proof. — In this case, let $g = i_{X_1} \times f_2$ and $h = f_1 \times i_{Y_2}$, and note that $f = h \circ g$. Now g is a quotient map by the theorem of J. H. C. Whitehead which is incorporated in Theorem 1.4 (a) → (b), and h is a quotient map by Lemma 7.1 (since the product of the locally compact space X_1 and the k-space Y_2 is locally compact by D. E. Cohen [7; 3.2]). Hence f is a quotient map, and that proves the lemma.

Let us now prove the theorem in full generality. Since X_1 is a Hausdorff k-space, it is the image, under a quotient map g_1, of a locally compact Hausdorff space X_1' [7]. Let $f_1' = f_1 \circ g_1$, and let $f' = f_1' \times f_2$. Then f' is a quotient map by Lemma 7.2. But $f' = f \circ (g_1 \times i_{X_2})$, so f is also a quotient map. That completes the proof of the theorem.

We conclude this section with three specific cases where the rather general conditions of Theorem 1.5 are satisfied. All spaces will be assumed Hausdorff.

(7.3). X_1 is a k-space and Y_2 is locally compact: This suffices, because quotients of k-spaces are k-spaces, and the product of a locally compact Hausdorff space and a k-space is k-space [7].

(7.4). X_1, Y_1 and Y_2 are first-countable: This is clear (and slightly generalizes a result of R. Brown [5; Corollary 4.10], where X_1, X_2, Y_1 and Y_2 are all assumed first countable).

(7.5). X_1 and X_2, or X_1 and Y_2, are both k_ω-spaces: Here we call a space X a *k_ω-space* if it is the union of countably many compact subsets K_n such that a set $A \subset X$ is closed whenever $A \cap K_n$ is closed in K_n for all n. K. Morita [14] showed that quotients of k_ω-spaces (which he calls space of class $\mathfrak{S}'$) are k_ω-spaces, and it is implicit in a result of J. Milnor [13; Lemma 2.1] that the product of two Hausdorff k_ω-spaces is a k_ω-space. These facts imply our assertion.

8. Examples.

Examples 8.1-8.5 describe various quotient maps $f: X \to Y$ which are not bi-quotient. It follows from Theorem 1.3 that, in each case, $f \times i_Z$ is not a quotient map for some paracompact space Z. In Examples 8.1 and 8.4, we go on to show that this Z can, in fact, be chosen separable metric; for the other examples, Theorem 1.5 implies that Z cannot be chosen metric.

Example 8.6 describes two quotient maps with compact range whose product is not a quotient map. The section concludes with Lemma 8.7, which describes a method of constructing quotient maps f such that $f \times f$ is not a quotient map.

EXAMPLE 8.1. — *A closed map $f: X \to Y$, with X locally compact separable metric, which is not bi-quotient. Moreover, $f \times i_Z$ is not a quotient map for any metric space Z which is not locally compact.*

Proof. — Let X be the disjoint union of countably many copies of the interval [0, 1], let Y be the space obtained by identifying all the 0's in X, and let $f: X \to Y$ be the quotient map.

If Z is any metric space which is not locally compact, then [12; Theorem 4.1] implies that $f \times i_Z$ is not a quotient map.

EXAMPLE 8.2. — *A compact-covering, hereditarily quotient map $f: X \to Y$, with Y locally compact metric and Y separable metric, which is not bi-quotient.*

Proof. — Let Y be any separable metric space which is not locally compact. Let X be the disjoint union of all the compact subsets of Y, and $f: X \to Y$ the obvious map.

EXAMPLE 8.3. — *A hereditarily quotient map $f: X \to Y$, with X locally compact metric and Y compact metric, which is not bi-quotient.*

Proof. — Let Y be a closed interval, let X be the disjoint union of all the convergent sequences in Y, and $f: X \to Y$ the obvious map.

EXAMPLE 8.4. — *A quotient map* $f: X \to Y$, *with* X *locally compact separable metric and with each* $f^{-1}(y)$ *compact, which is not hereditarily quotient. Moreover,* $f \times i_Z$ *is not a quotient map for some separable metric space* Z.

Proof. — Let X be the disjoint union of the open interval $X_1 = (0, 1)$ and the space $X_2 = \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Let Y be the space obtained from X by identifying each $\frac{1}{n} \in X_1$ with $\frac{1}{n} \in X_2$, and let $f: X \to Y$ be the quotient map. (This is [9; Example 1.8]). Here X_2 is a neighborhood of $\{0\} = f^{-1}f(0)$ in X, but $f(X_2)$ is not a neighborhood of $f(0)$ in Y.

Let

$$Z = (0, 1) - \left\{\frac{1}{n} : n = 1, 2, \ldots\right\}.$$

To see that $h = f \times i_Z$ is not quotient, let

$$S = \{(f(x), x) \in Y \times Z : x \in Z, x > 0\}.$$

Then S is not closed in $Y \times Z$, but $h^{-1}(S)$ is closed in $X \times Z$.

EXAMPLE 8.5. — *A quotient map* $f: X \to Y$, *with* X *locally compact Hausdorff and* Y *compact Hausdorff, and with each* $f^{-1}(y)$ *compact, but with* f *not hereditarily quotient.*

Proof. — Let Y be $[0, \Omega]$, the set of ordinals $\leqslant$ the first uncountable ordinal Ω. Let $X_1 = Y - \{\Omega\}$, let X_2 be the set of limit ordinals in Y, let X be the disjoint union of X_1 and X_2, and let $f: X \to Y$ be the obvious map. It is not hard to check that f is a quotient map. However, X_2 is a neighborhood of $\{\Omega\} = f^{-1}(\Omega)$ in X, but $f(X_2)$ is not a neighborhood of Ω in Y.

Remark. — By starting with Y as the space constructed by S. P. Franklin in [10; Example 6.2.], we can slightly improve Example 8.5 so as to make X paracompact. We cannot make X metrizable, however, for that, by a result of A. Arhangel'skiĭ [3; Theorem 2.6], would make Y metrizable, which is impossible by Proposition 3.3 (d).

EXAMPLE 8.6. — *Hereditarily quotient maps* $f_i: X_i \to Y_i$ $(i = 1, 2)$, *with* Y_1 *and* Y_2 *compact metric and with* X_1 *and* X_2 *paracompact, but with* $f_1 \times f_2$ *not quotient.*

Proof. — Let $N = \{1, 2, \dots\}$, and let Y_1 and Y_2 both be the convergent sequence $N \cup \{\omega\}$. Now consider the Stone-Čech compactification βN. Since $\beta N - N$ is a compact Hausdorff space without isolated points, a theorem of E. Hewitt [8; Theorem 47] implies that $\beta N - N$ has two disjoint dense subsets E_1 and E_2. For $i = 1, 2$, let X_i be the disjoint union of all sets $N \cup \{x\} \subset \beta N$, with $x \in E_i$. Since each $N \cup \{x\}$ is a regular Lindelöf space, and hence paracompact, both X_i are also paracompact. Define $f_i: X_i \to Y_i$ by mapping each element of a copy of N to itself, and mapping each $x \in E_i$ to ω.

The maps f_i are clearly continous. Let us check that each f_i is hereditarily quotient. We must show that, if U is a neighborhood of $f_i^{-1}(\omega)$ in X_i, then $f_i(U)$ is a neighborhood of ω in Y_i. Suppose not. Then $Y_i - f_i(U)$ is an infinite set S. Let $\bar{S}$ be the closure of S in βN. Since S is open in N, it is open in βN, and hence so is $\bar{S}$ because βN is extremally disconnected. Let $S^* = \bar{S} \cap (\beta N - N)$. Then S^* is open $\beta N - N$, and it is non-empty because S is infinite. Thus $x \in \bar{S}$ for some $x \in E_i$. But that is impossible, since $U \cap (N \cup \{x\})$ is disjoint from $S = f_i^{-1}(S)$.

To see that $h = f_1 \times f_2$ is not a quotient map, let

$$S = \{(y, y) \in Y_1 \times Y_2 : y \in N\}.$$

Then S is not closed in $Y_1 \times Y_2$, since (ω, ω) is in $\bar{S} - S$. However, $R = h^{-1}(S)$ is closed in $X_1 \times X_2$, because any point in $\bar{R} - R$ would have to be of the form (x, x) with $x \in E_1$ and $x \in E_2$, and that can't happen since E_1 and E_2 are disjoint. That completes the proof.

We conclude this section with a simple lemma, which shows how our examples could be modified to produce quotient maps f such that $f \times f$ is not a quotient map.

LEMMA 8.7. — *Suppose* $f_i: X_i \rightarrow Y_i$ $(i = 1, 2)$ *are quotient maps such that* $f_1 \times f_2$ *is not a quotient map. Let* $X = X_1 + X_2$ *and* $Y = Y_1 + Y_2$ *be the disjoint unions, and define* $f: X \rightarrow Y$ *by* $f|X_i = f_i$. *Then* f *is a quotient map, but* $f \times f$ *is not a quotient map.*

Proof. — That f is a quotient map is clear. Let us show that $f \times f$ is not a quotient map.

Note that $Y_1 \times Y_2$ is a subset of $Y \times Y$, and that $f_1 \times f_2$ is the restriction of $f \times f$ to $(f \times f)^{-1}(Y_1 \times Y_2)$, so that this restriction is not a quotient map. Since $Y_1 \times Y_2$ is closed in $Y \times Y$, and since the restriction of a quotient map to the inverse image of a closed set is always a quotient map, it follows that $f \times f$ cannot be a quotient map. That completes the proof.

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