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PROBABILITY AND A DIRICHLET PROBLEM FOR MULTIPLY SUPERHARMONIC FUNCTIONS (1)

by John B. WALSH

Introduction.

We shall use the notation $C(A)$ for the set of continuous real-valued functions on the set A. Let D be a bounded domain in R^n and ∂D its boundary. It is classical that under mild smoothness restrictions on ∂D, if $f \in C(\partial D)$ there exists a function h_f which solves the Dirichlet problem, i.e., h_f is harmonic in D, continuous in $\overline{D}$, and equal to f on ∂D. The function h_f can be gotten by the Perron-Wiener-Brelot method, that is h_f is equal to the lower envelope of functions lower semicontinuous in $\overline{D}$ and superharmonic in D which are greater than or equal to f on the boundary. Kakutani [25] was the first to treat the Dirichlet problem probabilistically. He showed that if $f \in C(\partial D)$ and Z^x is Brownian motion from $x \in D$, then $g_f(x) = E\{f(Z^x(\tau))\}$ is a harmonic function of x, where τ is the first time Z^x hits ∂D; the function g_f solves the Dirichlet problem whenever a solution exists.

A deeper treatment of the problem was initiated by Doob ([15], also [17] and [18] where the approach is more general) who observed that Brownian motion has the supermartingale property with respect to the class of superharmonic functions, i.e., if u is superharmonic and does not grow too fast at infinity, $u(Z^x(t))$ is a supermartingale. Thus martingale

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theory could be applied to the Dirichlet problem, and indeed to most of classical potential theory. The probabilistic approach to potential theory is not limited to the classical case. General probabilistic treatments of the Dirichlet problem have been given by Doob ([17] and [18]) and Courrège and Priouret [12], and of potential theory in general by Hunt [23] in his celebrated papers in the Illinois Journal.

In this paper, we wish, not to generalize, but rather to specialize in that we treat the Dirichlet problem not for the harmonic functions but for a proper subset of them, the multiply harmonic functions. A function is multiply harmonic in two sets of variables in R^n if $n = p + q$ for positive integers p, q and f is harmonic in the first p variables and separately is harmonic in the next q. One defines multiple harmonicity for more than two sets of variables similarly. When we speak of the class of multiply harmonic functions we shall always mean the class of functions which are multiply harmonic with respect to a given partition of the variables.

Although we treat a smaller class of functions, we get a much larger class of processes than in the classical case; it turns out that there are processes quite different from Brownian motion which satisfy the supermartingale property with respect to the multiply superharmonic functions. Thus, while previous probabilistic treatments of potential theory deal with a Markov process which is unique apart from its initial distribution and behavior at the boundary, our set-up involves a family of stochastic processes which are not necessarily Markov. The strong Markov property, vital in the classical case, has as a counterpart the dual concepts of continuing and conditioning stochastic processes, which we discuss in section one.

Our approach is based on the following observation. Let T_x be the set of all continuous stochastic processes from x which satisfy the supermartingale property relative to multiply superharmonic functions. If D is a domain in R^n and f is a bounded Baire function on ∂D, and if $U_x \subset T_x$ is a set of processes which are « nice » for small times, then the function Φ_f defined by

$$\Phi_f(x) = \sup_{X \in U_x} E\{f(X(\tau_D))\}$$

is multiply superharmonic in D, where τ_D is the first time X hits ∂D. We call Φ_f a Dirichlet solution; one of the aims of this paper is to justify this.

These Dirichlet solutions are closely connected with the solutions of a Dirichlet problem of a type introduced by H. J. Bremermann ([10]; see also [21], [26], [28]). Bremermann studied the problem for pluriharmonic functions, i.e., functions which are the real parts of functions holomorphic in several complex variables, and for plurisubharmonic functions, but the problem is readily transferred to our case. He used the Perron-Wiener-Brelot method to get solutions for the Dirichlet problem. Bremermann's problem, stated in our terms, is as follows: Let $S \subset \partial D$ and $f \in C(S)$, and let h_f be the lower envelope of functions multiply superharmonic in D, lower semicontinuous in $\overline{D}$ and greater than or equal to f on S. Then h_f is said to solve the Dirichlet problem if $\liminf_{y \to x} h_f(y) = f(x)$, $x \in S$. In the classical theory, such envelopes are harmonic; a fundamental difference between this and the classical case is that the functions h_f are multiply superharmonic but are seldom multiply harmonic. Under mild restrictions on the domain we find that Φ_f solves the above problem and that $h_f = \Phi_f$.

A different approach, using Leja's method of extremal points [29] was used by Gorski [20] and Siciak [35]. This method yields pluisubharmonic solutions which in certain cases turn out to agree with Bremermann's.

The concept of the Šilov boundary enters the problem in a natural way. This boundary is one of the logical boundaries on which to pose the Dirichlet problem; it plays a central role, for example, in Bauer's general treatment of the Dirichlet problem [3]. (The Šilov boundary of a closed set A with respect to a class G of functions on A is defined to be the smallest closed set on which every function in G has its minimum, or sometimes maximum, depending on the class.) In [10], Bremermann showed that in certain domains, his Dirichlet problem is solvable only for functions defined on a certain Šilov boundary; his results have been extended to somewhat more general regions by Gorski [21], Kimura [26], and Kusunoki [20].

A Šilov boundary turns out to be pivotal in our treatment too. We give several characterizations of it, the most interesting from the probabilistic viewpoint being that it is the smallest closed set S with the property that for any fixed $x \in D$ there is a process $X \in T_x$ which hits ∂D in the set S w.p. 1.

1. Conditioned and continued processes.

1.1. Notation.

We will denote Euclidean n-space by R^n, and write $R = R^1$, $R_+ = [0, \infty)$. If a, $b \in R$ we will use the lattice notations $a \wedge b$ and $a \vee b$ for $\inf(a, b)$ and $\sup(a, b)$ respectively. We will also use this notation for σ-fields; if F and G are σ-fields of subsets of a set Ω, then $F \vee G$ is the smallest σ-field containing both F and G. If $F(t)$ is a σ-field for each t in an index set Ω, then $\bigvee_{t \in T} F(t)$ is the smallest σ-field containing all $F(t)$, $t \in T$.

Let Q be a separable complete metric space. The set of all stochastic processes $\{X(t), t \in R_+\}$ with values in Q and right-continuous sample functions will be denoted by $\mathcal{X}_Q$. In situations where the context makes it clear that the parameter set is R_+, we shall write $\{X(t)\}$ rather than $\{X(t), t \in R_+\}$.

Let Ω_{rc} be the space of all right-continuous functions $\omega : R_+ \to Q$. Let $\mathcal{B}(t)$ be the σ-field generated by sets of the form:

$$\{\omega \in \Omega_{rc} : \omega(s) \in A\} \qquad \text{where} \qquad 0 \leqslant s \leqslant t$$

and $A \subset Q$ is Borel.

Define $\mathcal{B} = \bigvee_t \mathcal{B}(t)$. We call $(\Omega_{rc}, \mathcal{B})$ the canonical space and $\{\mathcal{B}(t), t \in R_+\}$ the natural fields. For any probability measure P on $(\Omega_{rc}, \mathcal{B})$ there is a stochastic process $X = \{X(t), t \in R_+\}$ on $(\Omega_{rc}, \mathcal{B}, P)$ defined by

$$X(t, \omega) = \omega(t), \qquad \omega \in \Omega_{rc}.$$

We say that X is canonically defined on $(\Omega_{rc}, \mathcal{B}, P)$. Conversely, if $X \in \mathcal{X}_Q$ is defined on a probability space $(\Omega, \mathcal{F}, P)$, there is a unique measure $\hat{P}$ on $(\Omega_{rc}, \mathcal{B})$ such that the process $\hat{X} = \{\hat{X}(t),\ t \in R_+\}$ defined canonically on $(\Omega_{rc}, \mathcal{B}, \hat{P})$ has the same finite-dimensional joint distributions as X. $\hat{X}$ is called the canonical process of X. The probability $\hat{P}$ on $(\Omega_{rc}, \mathcal{B})$ is defined by: $\hat{P}(\Lambda) = P\{\omega : X(\cdot, \omega) \in \Lambda\}$, $\Lambda \in \mathcal{B}$, where on the right-hand side we are considering $X(\cdot, \omega)$ as an element of Ω_{rc}. If $X_1, X_2 \in \mathcal{X}_Q$, we say they are equivalent, and write $X_1 \sim X_2$, if they have the same canonical process.

1.2. Stopping Times.

If $X = \{X(t)\}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$, a family $\{\mathcal{F}(t),\ t \in R_+\}$ of sub σ-fields of $\mathcal{F}$ is said to be admissible for X if

a) $\mathcal{F}(s) \subset \mathcal{F}(t)$ whenever $s < t$.

b) $X(t)$ is measurable $\mathcal{F}(t)$, $t \in R_+$.

Let $X \in \mathcal{X}_Q$ be defined on the probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}(t)\}$ be a family of σ-fields admissible for X. A measurable function $\tau : \Omega' \to [0, \infty]$, where Ω' is some measurable subset of Ω, is a stopping time for $\{\mathcal{F}(t)\}$ if $t \in R_+$ implies $\{\omega : \tau(\omega) < t\} \in \mathcal{F}(t)$, and is a strict stopping time if $t \in R_+$ implies $\{\omega : \tau(\omega) \leqslant t\} \in \mathcal{F}(t)$.

We find it convenient not to insist that τ be defined everywhere. This is a minor point for if τ is a stopping time which is not defined everywhere it can be extended by setting $\tau = +\infty$ on the set where it was not previously defined. We will always make the convention that if $\tau(\omega)$ is not defined, $X(\tau(\omega), \omega)$ and $X(t + \tau(\omega), \omega)$ are undefined also, but $X(\tau(\omega) \wedge t, \omega) = X(t, \omega)$. (These conventions would automatically hold were we to set $\tau = +\infty$ off its original domain.)

Remark. — If $X \in \mathcal{X}_Q$ and $\{\mathcal{F}(t)\}$ is a family of σ-fields admissible for X, then for any set $\Lambda \in \mathcal{B}(t)$,

$$\{\omega : X(\cdot, \omega) \in \Lambda\} \in \mathcal{F}(t).$$

To see this it is enough to notice that it is clearly true for sets of the form $\{\omega : \omega(s) \in A\}$, where $0 \leqslant s \leqslant t$ and A is a Borel set in Q, and that these sets generate $\mathcal{B}(t)$.

Let $X \in \mathcal{X}_Q$ be defined on $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}(t)\}$ be a family of σ-fields adapted to X. Let $\hat{\tau}$ be a stopping time on Ω_{rc} relative to the natural fields. Then there is a unique random variable τ on Ω defined by

$$\tau(\omega) = \hat{\tau}(X(\cdot,\ \omega))$$

where on the right-hand side we consider $X(\cdot,\ \omega)$ as an element of Ω_{rc}. It follows directly from our previous remark that τ is a stopping time, and is strict if $\hat{\tau}$ is.

Definition. — *Let τ and $\hat{\tau}$ be as above. We say that τ is the natural image of $\hat{\tau}$. Any stopping time which is the natural image of a stopping time on Ω_{rc} is a natural stopping time. A stopping time which is the natural image of a stopping time on Ω_{rc} is a natural stopping time and any stopping time which is the image of a strict stopping time on Ω_{rc} is a natural strict stopping time.*

Definition. — *Let $\hat{\tau}$ be a natural stopping time on Ω_{rc} and let $X_1, X_2 \in \mathcal{X}_Q$. Let τ_1 and τ_2 be the natural images of $\hat{\tau}$ for X_1 and X_2 respectively. We say that X_1 and X_2 are equivalent up to time $\hat{\tau}$, and write $X_1 \sim (\hat{\tau})X_2$, if $\tilde{X}_1 \sim \tilde{X}_2$ where $\tilde{X}_i(t) = X_i(t \wedge \tau_i)$, $i = 1,\ 2$.*

1.3. Conditioned Processes.

The idea of conditioning a stochastic process on a given event has been applied in connection with Markov processes, where the conditioned process often has a simple relation to the original process. This idea is closely connected with the strong Markov property. In probabilistic potential theory the concept of a Markov process conditioned to hit a given point has proved useful [19]. In general, the concept of a conditional process falls under the heading of conditional distribution. In this section we develop one special case.

Let $X \in \mathcal{X}_Q$ be defined on the probability space $(\Omega, \mathcal{F}, P)$ and let τ be a stopping time relative to an admissible set of σ-fields $\{\mathcal{F}_t,\ t \geqslant 0\}$. The idea of the process $\{X(\tau + t),$

$t \geqslant 0\}$ conditioned on $X(\tau)$ will be useful in the following. The definition is analogous to that of a conditional expectation; we define a family of processes rather than a single process. The family so defined is unique only in a certain almost everywhere sense.

DEFINITION. — *The conditional* $X(\tau + \cdot)$ *process, denoted* $\mathcal{E}_\tau(X)$, *is any family* $\{X^z, z \in A\}$ *of right-continuous stochastic processes satisfying.*

E1) $A \subset Q$ *is a Borel set.*

E2) *For any* $\Lambda \in \mathcal{B}$, *the function* $z \to P\{X^z(\cdot) \in \Lambda\}$ *is Borel measurable.*

E3) *For each* $\Lambda \in \mathcal{B}$ *and Borel set* $B \subset Q$, *if* μ_τ *is the distribution of* $X(\tau)$ *in* Q:

$$(1.3.1) \quad P(\{X(\tau + \cdot) \in \Lambda\} \cap \{X(\tau) \in B\}) = \int_{B \cap A} P\{X^z(\cdot) \in \Lambda\}\, d\mu_\tau(z).$$

PROPOSITION 1.3.1. — $\mathcal{E}_\tau(X)$ *always exists. If* $\{X^z, z \in A\}$ *and* $\{Y^z, z \in A'\}$ *are two versions of* $\mathcal{E}_\tau(X)$, *then for* z *not in some set of* μ_τ*-measure zero,* $X^z \sim Y^z$.

Proof. — The requirements for a conditioned process depend only on the joint distribution of τ and this process. Thus, if we define $\tau'(\omega)$ to be equal to $\tau(\omega)$ when the latter is defined, and -1 otherwise, it is enough to consider the pair $(\tau', X(\cdot))$ defined canonically on the space $\Omega' = R \times \Omega_{rc}$, relative to fields $\mathcal{A} \times \mathcal{B}$, where $\mathcal{A}$ is the class of Borel sets of R. The sample functions of X are right-continuous, so $\mathcal{B}$ is generated by $X(t)$ for t rational. It follows easily that $(\Omega', \mathcal{A} \times \mathcal{B})$ is a Lusin space in Blackwell's sense [5], so if $\mathcal{F}'$ is the σ-field generated by $X(\tau)$, P has a conditional distribution P^* relative to $\mathcal{F}'$, that is, a real valued function on $\Omega' \times (\mathcal{A} \times \mathcal{B})$ satisfying:

(*a*) For $A \in \mathcal{A} \times \mathcal{B}$, $P^*(\cdot, A)$ is $\mathcal{F}'$-measurable and equal to $P\{A \mid \mathcal{F}'\}$ w.p.1.

(*b*) For fixed ω, $P^*(\omega, \cdot)$ is a probability distribution on $\mathcal{B}$.

For each z in the range of $X(\tau)$ define a process X^z canonically on Ω_{rc} by:

if $\Lambda \in \mathcal{B}$, $P\{X^z(\cdot) \in \Lambda\} = P^*(\omega, \{\tau' \geqslant 0\} \times \Lambda)$ where ω

is any element of $\{\tau' \geqslant 0, X(\tau') = z\}$. The family of processes so defined then satisfies (E1)-(E3).

The second assertion follows easily from the observation that the field $\mathcal{B}$ is generated by a countable field of sets, hence any two functions satisfying (*a*) and (*b*) must agree off some ω-set of probability zero. Q.E.D.

It is important in the above that we be able to reduce to a canonical space; for an arbitrary space Ω and sub-σ-field $\mathcal{F}$ the conditional distribution P^* may not exist. (See for instance [5], where Blackwell discusses this problem and gives a class of probability spaces for which conditional distributions do exist.)

1.4. Continuation of Stochastic Processes.

Let $X \in \mathcal{X}_Q^v$ and for each point $z \in R^n$, let $X^z \in \mathcal{X}_Q^v$ have the property that $P\{X^z(0) = z\} = 1$. Let τ be a stopping time for X. In this section we consider the problem of the existence of a process $\hat{X}$ answering the description: « $\hat{X}$ is equivalent to X up to time τ. If $\hat{X}(\tau) = z$, then $\hat{X}$ is equivalent to X^z from then on ». This problem has been considered by Courrège and Priouret [14]; similar results for strong Markov processes have been proved by Ikeda, Nagasawa and Watanabe [24].

Definition. — *Let* $A \subset Q$ *be a Borel set. The family* $\{X^z, z \in A\}$ *of continuous processes is coherent if* $\Lambda \in \mathcal{B}$ *implies:*

(1.4.1) *the function* $z \to P\{X^z(\cdot) \in \Lambda\}$ *is Borel measurable.*

One can show in the usual way that it is enough that (1.4.1) holds for all sets in some finitely additive field generating $\mathcal{B}$, and it is even enough that (1.4.1) hold for all sets of the form $\{\omega : \omega(t_i) \in A_i,\ i = 1, \ldots, m\}$ where $t_i \in R_+$ and $A_i \in Q$ are Borel sets.

Note that to say that $\{X^z, z \in A\}$ is coherent is merely to say that the corresponding distributions in the sample space are measurable functions of z.

We can now state the central theorem of this section, which is a consequence of a theorem of Courrege and Priouret.

THEOREM 1.4.1. — *Let* $X \in \mathcal{X}_Q^v$ *and let* τ *be a natural strict stopping time for* X, *finite where defined, which is the natural image of* $\hat{\tau}$ *on* Ω_{rc}. *Let* $A \subset Q$ *be a Borel set such that* $P\{X(\tau) \notin A\} = 0$. *If* $\{X^z, z \in A\}$ *is a coherent family of right continuous processes such that* $P\{X^z(0) = z\} = 1$ *for each* $z \in A$, *then there exists a canonical process* $\hat{X}$ *such that*

a) $\hat{X} \sim (\hat{\tau})X$

b) $\{X^z, z \in A\}$ *is a version of* $\mathcal{E}_{\hat{\tau}}(\hat{X})$

c) $\hat{X}(\cdot \wedge \hat{\tau})$ *and* $\hat{X}(\cdot + \hat{\tau})$ *are conditionally independent given* $\hat{X}(\hat{\tau})$.

Note. — Another way of phrasing (b) is « X^z is equivalent to the $\hat{X}(t + \tau)$ process conditioned on $\hat{X}(\hat{\tau}) = z$ ».

Proof. — The conclusions of the theorem involve only equivalences so there is no loss of generality in assuming X and $\{X^z, z \in A\}$ are defined canonically. In this case $\hat{\tau} = \tau$. In what follows write τ instead of $\hat{\tau}$.

Let P be the distribution of X and P_z the distribution of X^z. By Theorem 1.1.1 of [14], there is a measure $\hat{P}$ on $(\Omega_{rc}, \mathcal{B})$ such that

(*a*) If $\Lambda \in \mathcal{B}(\tau)$, $\hat{P}(\Lambda) = P(\Lambda)$.

If $\theta_\tau : \Omega_{rc} \to \Omega_{rc}$ is the shift operator, i.e.,

$$(\theta_\tau \omega)(\cdot) = \omega(\tau + \cdot),$$

then

(*b*) $\hat{P}\{\theta_\tau^{-1}(\Lambda) | \mathcal{B}(\tau)\} = P_{\hat{X}(\tau)}\{\Lambda\}$ with $\hat{P}$-probability one.

Let $\hat{X}$ be canonically defined on $(\Omega_{rc}, \mathcal{B}, \hat{P})$. Conditional independence of $\hat{X}(\tau \wedge \cdot)$ and $\hat{X}(\tau + \cdot)$ given $\hat{X}(\tau)$ follows from (*b*) as does the fact that $\{X^z, z \in A\}$ is a version of $\mathcal{E}_\tau(X)$.

τ is clearly a natural stopping time for $\hat{X}$. Note that if $\Lambda \in \mathcal{B}$, $\{\hat{X}(\tau \wedge \cdot) \in \Lambda\} \in \mathcal{B}(\tau)$; this is obvious if $\Lambda \in \mathcal{B}(\tau)$, in which case $\hat{X}(\tau \wedge \cdot) \in \Lambda \Longleftrightarrow X(\cdot) \in \Lambda$. If Λ is in the field G generated by $X(\tau + \cdot)$, then if ω and ω' are such that $X(\tau(\omega) + \cdot, \omega) \in \Lambda$ and $\hat{X}(t, \omega) = \hat{X}(t, \omega')$ for $t \leqslant \tau(\omega)$, then $\tau(\omega') = \tau(\omega)$ and $X(\tau(\omega') + \cdot, \omega') \in \Lambda$. Thus by Theorem 1.4 of [13], $\{\hat{X}(\tau + \cdot) \in \Lambda\} \in \mathcal{B}(\tau)$. The conclusion follows since sets of the form $\Lambda_1 \cap \Lambda_2$, where $\Lambda_1 \in \mathcal{B}(\tau)$,

$\Lambda_2 \in G$, generate $\mathscr{B}$ [13]. Thus (a) implies

$$\hat{X}(\cdot \wedge \tau) \sim X(\cdot \wedge \tau), \qquad \text{or} \qquad \hat{X} \sim (\tau)X,$$

which completes the proof.

The necessity for a strict stopping time in Theorem 1.4.1 is illustrated by the following trivial example. Let X be uniform motion to the right on the real line, starting from the origin, and let X^0 be uniform motion to the left. Define a natural stopping time τ for X by $\tau = \inf\{t: X(t) > 0\}$. Then if $\hat{X}$ is the process which is equal to X until time τ, and is X^0 from then on, $\hat{X} = X^0$; then $\hat{X} \not\sim (\tau)X$ for, as $\hat{X}$ is never greater than zero, τ is not defined for $\hat{X}$.

Theorem 1.4.1 leads us to introduce the operator Υ_τ which is in a sense the dual of the operator $\mathscr{E}_\tau$.

Let $X \in \mathscr{X}_Q$ and let τ be a natural stopping time for X. Suppose $A \subset Q$ is a Borel set with the property that $P\{X(\tau) \in A^c\} = 0$, and let $U = \{X^z, z \in A\}$ be a coherent family of $\mathscr{X}_Q$-processes. Then define $\Upsilon_\tau(X; U)$ to be the right-continuous stochastic process satisfying.

1) $\Upsilon_\tau(X; U)$ is canonically defined.
2) $\Upsilon_\tau(X; U) \sim (\tau)X$.
3) U is a version of $\mathscr{E}_\tau(\Upsilon(_\tau X; U))$.

The existence of $\Upsilon_\tau(X; U)$ is guaranteed by Theorem 1.4.1.

It is an interesting though trivial fact that it is not necessarily true that $\Upsilon_\tau(X, \mathscr{E}_\tau(X)) \sim X$ for $X \in \mathscr{X}_Q$, i.e., we may cut a process in two and then glue it back together in the same place and come up with a different process. The reason for this is that the behaviors of the process $\Upsilon_\tau(X, \mathscr{E}_\tau(X))$ before and after time τ are conditionally independent given the process at τ; this may not be true of the original process X.

Let $X_1 \in \mathscr{X}_Q$ and let $U_n = \{X_n^z, z \in A_n\}$, $n = 1, 2, \ldots$ be coherent families of continuous processes with $P\{X_n^z(0) = z\} = 1$. We assume for simplicity that all these processes are canonically defined; if not we can reduce to equivalent canonical processes.

Let $\tau_1, \tau_2, \ldots$ be a sequence of finite natural strict times on Ω_{rc} and suppose that $n \geqslant m$ implies $\tau_n \geqslant \tau_m$ in the sense that whenever $\tau_n(\omega)$ is defined, so is $\tau_m(\omega)$, and $\tau_n(\omega) \geqslant \tau_m(\omega)$.

Define a sequence $\hat{X}_1, \hat{X}_2, \ldots$ of continuous processes, with corresponding probability measures $\hat{P}_1, \hat{P}_2, \ldots$ on Ω_{rc} by induction.

a) $\hat{X}_1 = X_1$

b) If $\hat{X}_j, j \leqslant n$ have been defined such that

$$\hat{P}_n\{\omega : \omega(\tau_n(\omega)) \in A^c_{n+1}\} = 0,$$

then define $\hat{X}_{n+1} = \Upsilon_{\tau_n}(\hat{X}_n, U_{n+1})$.

By definition $\hat{X}_{n+1} \sim (\tau_n)\hat{X}_n$. This induction defines a process for each n. Even more is true, however.

THEOREM 1.4.2. — *Let* X_1, U_n, τ_n *and* $\hat{X}_n, n = 2, 3, \ldots$ *be defined as above. Then there exists a process* $\hat{X}_\infty \in \mathcal{X}_Q$ *with the property that for each* n, $\hat{X}_\infty \sim (\tau_n)\hat{X}_n$.

Proof. — Let $\hat{P}_n$ be the probability measure on $(\Omega_{rc}, \mathcal{B})$ associated with $\hat{X}_n$; let X be the coordinate variable $X(t, \omega) = \omega(t)$ on Ω_{rc} and let $\mathcal{B}_n$ be the σ-field generated by $\{X(t \wedge \tau_n), t \in R_+\}$. $\hat{X}_{n+1} \sim (\tau_n)\hat{X}_n$, hence $\hat{P}_{n+1}$ agrees with $\hat{P}_n$ on $\mathcal{B}_n$, or more generally, if $m > n$, $\hat{P}_m$ agrees with $\hat{P}_n$ on $\mathcal{B}_n$. If Λ is a set of the form:

$$\Lambda = \{X(t_j \wedge \tau_{n_j}) \in A_j, \ j = 1, \ldots, m\}, \tag{1.4.2}$$

A_j Borel sets, then $\Lambda \in \mathcal{B}_n$ for some n.

For each fixed $t_1, \ldots, t_m, \tau_{n_1}, \ldots, \tau_{n_m}$, we can uniquely define a probability measure on the σ-field of all sets of the form (1.4.2) by $P^*(\Lambda) = \hat{P}_n(\Lambda)$, $n \geqslant \max n_j$. For different sets of $t_1, \ldots, t_m, \tau_{n_1}, \ldots, \tau_{n_m}$, these measures are consistent, so by Kolmogorov's extension theorem, there is a measure $\hat{P}_\infty$ on the σ-field $\mathcal{B}_\infty = \bigvee_n \mathcal{B}_n$ generated by all sets of the form (1.4.2); and $\hat{P}_\infty$ is consistent with all the measures $\hat{P}_n$. In particular, $\hat{P}_\infty = \hat{P}_n$ on $\mathcal{B}_n$.

Now let $x_0 \in Q$ and define $\hat{X}_\infty$ on $(\Omega_{rc}, \mathcal{B}, P_\infty)$ by:

$$\begin{aligned} \hat{X}_\infty(t, \omega) &= \lim_{n \to \infty} X(t \wedge \tau_n), && \text{if the limit exists} \\ &= x_0 && \text{otherwise.} \end{aligned}$$

Note that $\hat{X}_\infty$ is not canonically defined but is right continuous. $X(t \wedge \tau_n)$ is $\mathcal{B}_\infty$-measurable for each n, so the set on which the limit of $X(t \wedge \tau_n)$ exists is in $\mathcal{B}_\infty$ for each t, hence $\hat{X}_\infty$ is $\mathcal{B}_\infty$ measurable. For each n, $\hat{X}_\infty(t \wedge \tau_n) = X(t \wedge \tau_n)$. Since $\hat{P}_\infty$ agrees with P_n on $\mathcal{B}_n$, we have $\hat{X}_\infty \sim (\tau_n)\hat{X}_n$.

Q.E.D.

2. Multiply superharmonic functions.

Let $Q = R^n$ and for $k \geqslant 1$ let $\mathfrak{D}$ be a decomposition of Q into the direct product of k subspaces $Q_1, \ldots, Q_k$. k will be called the degree of $\mathfrak{D}$. Let m_i be the dimension of Q_i, $i = 1, \ldots, k$. If $z \in Q$, then there are $z^1, \ldots, z^k$, where $z^i \in Q_i$, such that $z = (z^1, \ldots, z^k)$. We call z^i the i^{th} coordinate of z.

DEFINITION. — *Let $\mathfrak{D}$ be a decomposition of Q of degree k and let $D \subset Q$ be a domain. An extended real valued function f on D is multiply superharmonic relative to $\mathfrak{D}$ if*

(a) $-\infty < f \leqslant +\infty$ and $f \not\equiv +\infty$;

(b) f *is bounded below on each compact* $K \subset D$;

(c) *for each integer* p, $1 \leqslant p \leqslant k$, *if* $\zeta^i \in Q_i$ *are fixed for* $i \neq p$, *then the function* $z^p \to f(\zeta^1, \ldots, z^p, \ldots, \zeta^k)$ *is either superharmonic or identically* $+\infty$ *on each component of* $D \cap \{z : z^i = \zeta^i, i \neq p\}$.

Unless there is danger of ambiguity, we will ordinarily assume the decomposition $\mathfrak{D}$ of Q is fixed and will not indicate it.

If $m_1 = \cdots = m_k = 2$, the class of multiply superharmonic functions contains the class of plurisuperharmonic functions, introduced by Lelong (see [30]). These functions have close connections with functions of several complex variables: if g is a holomorphic function of several complex variables its real part is pluriharmonic and the negative of its absolute value is plurisuperharmonic. On the other hand, the Bergman extended class of pluriharmonic functions in certain domains turns out to be exactly the class of multiply superharmonic functions [4].

The following theorem is due to Avanissian [2].

THEOREM 2.1 (Avanissian). — *If f is multiply superharmonic in D, it is superharmonic in D. Consequently, if f is multiply harmonic, it is also harmonic.*

The corresponding theorem for plurisubharmonic functions was proved by Lelong [30]. Avanissian actually stated Theorem 2.1 for the case where $m_i \geqslant 3$, all i, but the theorem remains valid in our setting.

Note that we allow the possibility that $m_i = 1$ for some or all of $i = 1, \ldots, k$, with the convention that in one dimension, harmonic and superharmonic functions are linear and concave respectively. This case is of some interest, and is a source of easily visualized examples.

For each $i = 1, \ldots, k$, let $B_i(z^i, r_i)$ be the ball in Q_i with center z^i and radius r_i. If $z = (z^1, \ldots, z^k)$ and $\rho = (r_1, \ldots, r_k)$, define the polycylinder $B(z, \rho)$ by:

$$B(z, \rho) = B_1(z^1, r_1) \times \cdots \times B_k(z^k, r_k)$$

$B(z, \rho)$ has a distinguished boundary surface which we shall denote by $B^*(z, \rho)$:

$$B^*(z, \rho) = \partial B_1(z^1, r_1) \times \cdots \times \partial B_k(z^k, r_k).$$

Let λ_i be normalized Lebesgue measure on $\partial B_i(z^i, r_i)$, $i = 1, \ldots, k$, i.e., $\lambda_i(\partial B_i(z^i, r_i)) = 1$, and define the measure $\lambda_{z\rho}$ on $B^*(z, \rho)$ by $\lambda_{z\rho} = \lambda_1 \times \ldots \times \lambda_k$.

The following characterization of multiply superharmonic functions is a trivial consequence of some results in [2].

PROPOSITION 2.2. — *Let f be a lower semicontinuous, extended-real-valued function on a domain $D \subset Q$. If $-\infty < f \leqslant +\infty$ and $f \not\equiv +\infty$, then f is multiply superharmonic iff for each $z \in D$ and each sufficiently small polycylinder $B(z, \rho)$ centered at z,*

$$f(z) \geqslant \int_{B^*(z, \rho)} f(\zeta) \lambda_{z\rho}(d\zeta).$$

We will denote the class of all functions multiply superharmonic in a domain $D \subset Q$ by $S(D)$. A fact that we shall often use is that if $f \in S(D)$, there exists a sequence of functions $\{f_n\}$, each of which is multiply superharmonic and continuous in a given relatively compact subdomain, such that $f_n \uparrow f$. The functions f_n can even be chosen to be infinitely differentiable ([2], p. 142).

3. T_x processes.

3.1. Elementary properties.

A stochastic process $\{X(t), \mathcal{F}_t, t \in R_+\}$, where the σ-fields $\mathcal{F}_t$ are admissible for X, is said to be a local supermartingale if there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that for each n, $\{X(t \wedge \tau_n), \mathcal{F}_{t \wedge \tau_n}, t \in R^+\}$ is a supermartingale. If G is a class of functions defined on some subset of Q, a stochastic process $\{X(t), \mathcal{F}_t, t \in R_+\}$ is said to satisfy the supermartingale property relative to G if $f \in G$ implies $\{f(X(t)), \mathcal{F}_t, t \in R_+\}$ is a local supermartingale.

We change notation slightly: from now on, $\mathcal{X}_Q$ will represent the class of all continuous stochastic processes with values in Q, rather than the class of right-continuous processes.

Definition. — *If* $x \in Q$, T_x *is the class of all processes* $X = \{X(t), \mathcal{F}_t, t \in R_+\}$ *where* $\{\mathcal{F}_t\}$ *is admissible for* $\{X(t)\}$, *satisfying.*

1. $X \in \mathcal{X}_Q$ *and* $P\{X(0) = x\} = 1$.

2. X *satisfies the supermartingale property relative to the class of all functions which are multiply superharmonic in* Q *and finite at* x.

If $f \in S(Q)$ and $X \in T_x$, the stopping times $\{\tau_n\}$ relative to which $f(X(t))$ is a local supermartingale can always be taken to be of the form τ_n = first time X hits ∂D_n, where $\{D_n\}$ is any sequence of bounded domains containing x which increase to the whole space. This is clear if f is continuous; for a general $f \in S(Q)$ it follows from the fact f is the limit of an increasing sequence of continuous multiply superharmonic functions.

Note that we get an equivalent definition if we require only that X have the supermartingale property for continuous multiply superharmonic functions, for $f \in S(Q)$ implies there are continuous $f_n \in S(Q)$ such that $f_n \uparrow f$. Then $f(X(t \wedge \tau_m))$ is the increasing limit of supermartingales $(f_n X(t \wedge \tau_m))$ and therefore must be one as well, providing $E\{f(X(0))\} > \infty$, i.e., $f(x) < \infty$. Further, the class C(Q) of continuous real-valued functions on Q is separable in the topology of uniform convergence on compact sets, so we need require only that X satisfy the supermartingale property relative to any countable dense subset of $C(Q) \cap S(Q)$.

Let $\{Z^x(t),\ t \in R_+\}$ be Brownian motion on Q with initial point x. Let Z_j^{xj} be Brownian motion on Q_j with initial point x^j; then the following processes are easily seen to be in T_x:

a) $\{Z^x(t),\ t \in R_+\}$.

b) $\{X^{(j)}(t),\ t \in R_+\}$ where $X^{(j)}(t) = (z^1, \ldots, Z^{xj}(t), \ldots, z^k)$.

Note that there are non-trivial T_x-processes with the property that for some $f \in S(Q)$, $f(X(t)) \equiv +\infty$. For example, let $Q_1 = Q_2 = R^3$, so $Q = R^6$. For $x = (x^1, x^2)$ define: $f(x) = 1/|x^1| . 1/|x^2|$. Then $f \in S(Q)$ and f is infinite on the set $\{x^1 = 0\} \cup \{x^2 = 0\}$. The process $X^{(1)}(t) = (Z^0(t), 0)$ is in T_x by (*b*) above but $f(X^{(1)}(t)) \equiv +\infty$.

Let $\Pi_i : Q \to Q_i$ be the projection of Q onto Q_i, i.e., $\Pi_i(z^1, \ldots, z^k) = z^i$. If u is superharmonic in Q_i, the function $u \circ \Pi_i$ is multiply superharmonic in Q, so if $X \in T_x$, $u \circ \Pi_i(X(t))$ is a local supermartingale. According to a theorem of Kunita and Watanabe [27], this implies the process $Y_i(t) = \Pi_i(X(t))$ can be obtained from Brownian motion by a random time change. More precisely, Kunita and Watanabe's theorem gives us.

THEOREM 3.1.1. — *Let* $X \in T_x$. *Then there is a Brownian motion process* $\{Z(t)\}$ *on some space* $(\Omega, \mathcal{F}, P)$, *admissible fields* $\{\mathcal{F}_t\}$, *and a family* $\{\tau_t\}$ *of stopping times relative to these fields, such that* $t \to \tau_t(\omega)$ *is right-continuous and increasing for* a.e. ω, *and the process* $\{\Pi_i(X(t))\}$ *is equivalent to the process* $\{Z(\tau_t),\ t \in R_+\}$.

PROPOSITION 3.1.2. — *If* D *is a bounded domain in* Q *and* $x \in D$, $X \in T_x$, *then* $\lim_{t \to \infty} X_D(t)$ *exists* w. p. 1, *where* X_D *is the* X-*process stopped the first time it hits* ∂D. *Consequently,* X_D *either eventually converges to* ∂D *or converges to a point in the interior of* D w. p. 1.

Proof. — Let $l_1, \ldots, l_n$ be a set of linear functions on Q which separates points. Then $\pm l_i \in S(Q)$, $i = 1, \ldots, n$ so $l_i(X_D(t))$ is a martingale. It is bounded since $X_D(t) \subset D$ for all t and l_i is bounded on D. Therefore $\lim_{t \to \infty} l_i(X_D(t))$

exists w. p. 1, for all i. The l_i separate points of D so this implies $X_D(t)$ converges w. p. 1.

3.2. A Martingale Lemma and Corollary.

The following lemma, which is a partial converse to the martingale sampling theorem, seems to be a part of martingale folklore, but as we have not seen it stated explicitly and shall need it, we prove it here. We state it for an increasing set of stopping times indexed by the ordinals rather than for a sequence of times; this slight generalization adds little to the difficulty of the proof and is often convenient, if not necessary, in applications.

We first introduce some notation. Let $\{Z(t)\}$ be a right-continuous real-valued stochastic process and $\{\mathcal{F}(t)\}$ an admissible set of σ-fields. Let $\{\tau_\alpha, \alpha \in I\}$ be an increasing set of everywhere defined stopping times, where I is some countable initial segment of the ordinals, and $\tau_0 = 0$. If β is a limit ordinal, we assume $\tau_\beta = \sup_{\alpha<\beta} \tau_\alpha$. When α is an ordinal not in I, we make the convention that $\tau_\alpha \equiv +\infty$.

For $\alpha \in I$ set $Z_\alpha(t) = Z(t \wedge \tau_\alpha)$ and let

$$\begin{aligned} F_\alpha(t) &= \mathcal{F}^+(t \wedge \tau_\alpha) \quad \text{if } \alpha \text{ is not a limit ordinal,} \\ &= \bigvee_{\beta<\alpha} F_\beta(t) \quad \text{if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Then for $\alpha \in I$, let:

$$\begin{aligned} G_{\alpha+1}(t) &= F_{\alpha+1}(\tau_\alpha + t) \\ &= \{\Lambda : \Lambda \cap \{\tau_\alpha + t < s\} \in F_{\alpha+1}(s),\ s \in R_+\}. \end{aligned}$$

LEMMA 3.2.1. — *With the above notation suppose that for each* $\alpha \in I$, $\{Z_{\alpha+1}(\tau_\alpha + t), G_{\alpha+1}(t), t \in R_+\}$ *is a supermartingale. Then*

(a) *For all finite* α, $\{Z_\alpha(t), F_\alpha(t), t \in R_+\}$ *is a supermartingale.*

(b) *If in addition* $\{Z(t)\}$ *is continuous and bounded below then* $\{Z_\alpha(t)\ F_\alpha(t), t \in R^+\}$ *is a supermartingale for all* $\alpha \in I$.

COROLLARY 3.2.2. — *Under the hypothesis of Lemma* 3.2.1, *if either*

(a) I *is finite*

(b) $I = \{1, 2, \ldots\}$ *and* $\lim_{n\to\infty} \tau_n = \infty$ w. p. 1 *or*

(c) Z *is continuous and bounded below and* $\sup_\alpha \tau_\alpha = \infty$ w. p. 1, *then* $\{Z(t), \mathcal{F}(t), t \in R^+\}$ *is a supermartingale.*

Proof of Lemma 3.2.1. — The proof is by induction on α. The conclusion is automatically true if $\alpha = 0$.

Case I. $\alpha = \beta + 1$, some β. Let $t_1 < t_2$ and $\Lambda \in F_\alpha(t_1)$. Consider

$$(3.2.1)\quad \int_\Lambda Z_\alpha(t_1) = \int_{\Lambda \cap \{\tau_\beta < t_1\}} Z_\alpha(t_1) + \int_{\Lambda \cap \{t_1 \leqslant \tau_\beta < t_2\}} Z_\alpha(t_1) + \int_{\Lambda \cap \{\tau_\beta \geqslant t_2\}} Z_\alpha(t_1) = I_1 + I_2 + I_3.$$

Let $\sigma_i = (t_i - \tau_\beta) \vee 0$, $i = 1, 2$. If $\tau_\beta \leqslant t_i$, then $t_i = \tau_\beta + \sigma_i$. Now $\tau_\beta \leqslant \tau_\alpha$, hence τ_β is a stopping time relative to $\{\mathcal{F}_\alpha^+(t)\}$ ([11] Proposition 21). By Proposition 23.1 of [11], σ_1 and σ_2 are stopping times relative to $\{\mathcal{F}_\alpha(\tau_\beta + s)\} = \{G_\alpha(s)\}$. It is straightforward to verify $\Lambda \cap \{\tau_\beta < t_1\} \in G_\alpha(\sigma_1)$, and since $\{Z_\alpha(\tau_\beta + t), G_\alpha(t), t \in R^+\}$ is a separable supermartingale by hypothesis:

$$(3.2.2)\quad I_1 = \int_{\Lambda \cap \{\tau_\beta < t_1\}} Z_\alpha(\tau_\beta + \sigma_1) \geqslant \int_{\Lambda \cap \{\tau_\beta < t_1\}} Z_\alpha(\tau_\beta + \sigma_2) = \int_{\Lambda \cap \{\tau_\beta < t_1\}} Z_\alpha(t_2)$$

I_2 can be handled similarly by stopping the supermartingale $\{Z_\alpha(\tau_\beta + s), G_\alpha(s), s \in R_+\}$ at the bounded times 0 and σ_2 to get:

$$(3.2.3)\quad I_2 \geqslant \int_{\Lambda \cap \{t_1 \leqslant \tau_\beta < t_2\}} Z_\alpha(\tau_\beta + \sigma_2) = \int_{\Lambda \cap \{t_1 \leqslant \tau_\beta < t_2\}} Z_\alpha(t_2).$$

If $\tau_\beta \geqslant t_2$, $Z_\alpha(t_i) = Z_\beta(t_i)$, $i = 1, 2$. By the induction hypothesis, $Z_\beta(t)$ is a supermartingale, so

$$(3.2.4)\quad I_3 = \int_{\Lambda \cap \{\tau_\beta \geqslant t_2\}} Z_\beta(t_1) \geqslant \int_{\Lambda \cap \{\tau_\beta \geqslant t_2\}} Z_\beta(t_2) = \int_{\Lambda \cap \{\tau_\beta \geqslant t_2\}} Z_\alpha(t_2).$$

The supermartingale inequality follows by adding (3.2.2), (3.2.3) and (3.2.4). This proves (a).

Case II. Let $\alpha = \sup_{\beta < \alpha} \beta$. Assume $\{Z(t), t \in R_+\}$ is bounded below. By the inductive hypothesis $\{Z_\beta(t), F_\beta(t), t \in R^+\}$ is a supermartingale if $\beta < \alpha$. Then the processes $Z^n_\beta(t) = Z_\beta(t) \wedge n$ are bounded supermartingales.

Let $\Lambda \in \mathcal{F}_\alpha(t) = \bigvee_{\beta<\alpha} \mathcal{F}^+_\beta(t)$. The fields $\mathcal{F}^+_\beta(t)$ increase with β so by choosing γ large enough, $\gamma < \alpha$, we can find a set $\Lambda_\gamma \in F_\gamma(t_1)$ such that $P\{\Lambda_\gamma + \Lambda\}$ is as small as desired. Then for $\gamma < \beta < \alpha$:

$$(3.2.5) \qquad \int_{\Lambda_\gamma} Z^n_\beta(t_1) \geqslant \int_{\Lambda_\gamma} Z^n_\beta(t_2) \qquad \text{since} \qquad \Lambda_\gamma \in F_\gamma(t_1) \subset F_\beta(t_1).$$

Z^n_β is bounded and Z is continuous so we can go to the limit under the integral in (3.2.5) as $\beta \uparrow \alpha$. Since $t \wedge \tau_\beta \uparrow t \wedge \tau_\alpha$, $\forall t$, $Z_\beta(t) \to Z_\alpha(t)$ so

$$(3.2.6) \qquad \int_{\Lambda_\gamma} Z^n_\alpha(t_1) \geqslant \int_{\Lambda_\gamma} Z^n_\alpha(t_2).$$

Let $\gamma \uparrow \alpha$ and $P\{\Lambda_\gamma + \Lambda\} \to 0$. By the dominated convergence theorem we can go to the limit:

$$(3.2.7) \qquad \int_{\Lambda} Z^n_\alpha(t_1) \geqslant \int_{\Lambda} Z^n_\alpha(t_2).$$

Now let n go to infinity. $Z^n_\alpha(t) \uparrow Z_\alpha(t)$, and we have: $E\{Z_\alpha(0)\} \geqslant E\{Z^n_\alpha(0)\} \geqslant E\{Z^n_\alpha(t)\}$ for all n, hence $E\{Z_\alpha(t)\} < \infty$. By increasing convergence we can go to the limit in (3.2.7) to get the supermartingale inequality.

Proof of Corollary. — This is trivial from the lemma if I is finite since then I has a largest element, say γ, and by our convention, $Z(t) = Z_{\gamma+1}(t)$, which is a supermartingale.

Suppose $\{Z(t)\}$ is bounded below and let $Z^n(t) = n \wedge Z(t)$, $Z^n_\alpha(t) = n \wedge Z_\alpha(t)$. By the lemma, $\{Z^n_\alpha(t)\}$ is a supermartingale relative to the fields $\{F_\alpha(t)\}$. If $t_1 < t_2$ and $\Lambda \in \mathcal{F}(t_1)$ then $\Lambda \cap \{\tau_\alpha \geqslant t_1\} \in F_\alpha(t_1)$ and

$$(3.2.8) \qquad \int_{\Lambda \cap \{\tau_\alpha \geqslant t_1\}} Z^n(t_1) = \int_{\Lambda \cap \{\tau_\alpha \geqslant t_1\}} Z^n_\alpha(t_1) \geqslant \int_{\Lambda \cap \{\tau_\alpha \geqslant t_1\}} Z^n_\alpha(t_2).$$

As $\tau_\alpha \uparrow \infty$, the set being integrated over increases to Λ and $Z^n_\alpha(t) \to Z^n(t)$. By bounded convergence:

$$(3.2.9) \qquad \int_{\Lambda} Z^n(t_1) \geqslant \int_{\Lambda} Z^n(t_2).$$

Let $n \to \infty$ to get the supermartingale inequality. Q.E.D.

We remark that the assumption in Lemma 3.2.1 and Corollary 3.2.2 that Z be continuous can be weakened; it is enough if Z is only quasileft continuous, i.e., if $\tau_1, \tau_2, \ldots \uparrow \tau$ are stopping times, then $Z(\tau_n) \to Z(\tau)$ w. p. 1. The hypothesis that Z be bounded below was used only in going to the limits in (3.2.5) and (3.2.8) and can be weakened considerably. It is enough, for instance, that the negative part of Z is of class (D), which in turn is satisfied if the negative part of Z is uniformly integrable (see [33] pp. 101-102).

3.3. Local Behavior of T_x Processes.

THEOREM 3.3.1. — *Let* $X = \{X(t), t \in R_+\} \in T_x$ *and let* D *be a bounded domain, with* $x \in D$. *Let* f *be multiply superharmonic in a neighborhood of* $\overline{D}$ *such that* $f(x) < \infty$. *If* X_D *is the* X *process stopped the first time it hits* ∂D, *then* $f(X_D(t))$ *is a supermartingale.*

We first prove the following lemma, which is known to be true without the continuity hypothesis in case $k = 1$ ([6] p. 32), where k is the degree of the decomposition of Q.

LEMMA 3.3.2. — *Let* $D \subset Q$ *be a bounded convex domain and suppose* f *is continuous and multiply superharmonic in some open neighborhood of* $\overline{D}$. *Then there exists a continuous* g, *multiply superharmonic in all of* Q, *such that* $g(x) = f(x)$ *for* $x \in \overline{D}$.

Proof. — Choose a domain D′ such that $\overline{D} \subset D' \subset \overline{D}' \subset$ domain of definition of f. If $x \in \partial D'$, by convexity of D there is a linear function h such that $h(x) < f(x)$ and $h(y) \geqslant \sup_{z \in \overline{D}} f(z)$ for all $y \in \overline{D}$.

By continuity of f and compactness of $\partial D'$, there exists a finite number of linear functions $h_1, \ldots, h_n$ such that if $H = \inf(h_1, \ldots, h_n)$ then 1) $H(x) < f(x)$ for $x \in \partial D'$ and 2) $H(x) \geqslant f(x)$ on $\overline{D}$. Then the function g satisfies the requirements of the proposition, where

$$g(x) = \begin{cases} \min(f(x),\ H(x)) & \text{if} \quad x \in D' \\ H(x) & \text{if} \quad x \in Q - D'. \end{cases}$$

Proof of 3.3.1. — It is enough to prove the supermartingale property for continuous f.

Choose a domain D' such that $\overline{D} \subset D'$ and f is multiply superharmonic in a neighborhood of $\overline{D}'$. If we could extend f to be multiply superharmonic in Q, the proof would be immediate. However if D' is not convex, we may not be able to do this. Therefore we use Lemma 3.3.2 to extend f from sufficiently small neighborhoods to the whole plane.

We can cover $\overline{D}$ with a countable number of open balls S_i such that $\overline{S}_i \subset D'$. Suppose $x \in S_i$ and let τ_1 be the first time X_D hits ∂S_1, if X_D hits ∂S_1, and undefined otherwise. If $\tau_1(\omega) < \infty$, then for some i (which may depend on ω) $X_D(\tau_i) \in S_i$. Define $i_1(\omega)$ to be the smallest index i such that $X_D(\tau_1(\omega), \omega) \in S_i$. Then let $\tau_2(\omega)$ be the first time after $\tau_1(\omega)$ that X_D hits $\partial S_{i_1}(\omega)$, etc.

If the limit of $\tau_1, \tau_2, \ldots$ is finite with positive probability, let $\tau_\omega = \lim \tau_n$, and continue with $\tau_{\omega+1}, \ldots$ through the countable ordinals. If $\tau_\alpha(\omega)$ is not defined for some α (and therefore for all $\beta > \alpha$) define $\tau_\alpha(\omega) = \infty$ with the convention that $X_D(\infty) = \lim_{t \to \infty} X_D(t)$, which exists for a.e. ω by Proposition 3.1.2.

For an ordinal α, let $i_\alpha(\omega)$ be the smallest i such that $X_D(\tau_\alpha(\omega), \omega) \in S_i$. For each j, S_j is convex and f is continuous and superharmonic in a neighborhood of $\overline{S}_j$. By Lemma 3.3.2 there is a continuous $g_j \in S(Q)$ such that $g_j = f$ on $\overline{S}_j$.

Since $X \in T_x$, $g_j(X_D(t))$ is a local supermartingale. Let $X_D^\alpha = X_D(t \wedge \tau_\alpha)$. Then note $g_j(X_D^{\alpha+1}(\tau_\alpha + t))$ is a supermartingale since it is obtained from the continuous local supermartingale $g_j(X_D(t))$ by optional stopping. Let $t_2 > t_1$ and let $\Lambda \in \mathcal{F}^+(\tau_\alpha + t_1)$.

$$\int_\Lambda f(X_D^{\alpha+1}(\tau_\alpha + t_1)) = \sum_j \int_{\Lambda \cap \{i_\alpha = j\}} g_j(X_D^{\alpha+1}(\tau_\alpha + t_1)).$$

But $\{i_\alpha = j\} \in \mathcal{F}^+(\tau_\alpha + t_1)$ so this is

$$\geqslant \sum_j \int_{\Lambda \cap \{i_\alpha = j\}} g_j(X_D^{\alpha+1}(\tau_\alpha + t_2)) = \int_\Lambda f(X_D^{\alpha+1}(\tau_\alpha + t_2)).$$

Thus $f(X_D^{\alpha+1}(\tau_\alpha + t))$ is a supermartingale. It is bounded and $\sup_\alpha \tau_\alpha(\omega) = \infty$, so by Corollary 3.2.2, $f(X_D(t))$ is a supermartingale. Q.E.D.

PROPOSITION 3.3.3. — *Let* $D \subset Q$ *be a bounded domain and* $x \in D$. *If* f *is multiply superharmonic in a neighborhood of* $\overline{D}$, *and* $X \in T_x$, *then* $f(X(t \wedge \tau_D))$ *is a right-continuous function of* t *with probability one, and is, a fortiori, a separable supermartingale.*

Proof. — There exists a sequence $\{f_n\}$ of functions which are continuous and multiply superharmonic in some neighborhood of $\overline{D}$, such that $f_n \uparrow f$ in that neighborhood. For each n, $f_n(X(t \wedge \tau_D))$ is a continuous supermartingale, so by a result of Meyer ([33], Thm T16, p. 99), the limit must be right-continuous.

In the case $k = 1$, it is known that the paths of $f(X(t \wedge \tau_D))$ are continuous w.p. 1. We conjecture this is true for $k > 1$ as well.

3.4. Conditioning and Continuing T_x Processes.

In sections 1.3 and 1.4 we introduced the operators $\mathcal{E}_\tau$ and Υ_τ. In this section we consider these operators applied to T_x processes. The following two theorems can be thought of as the T_x process analogues of the strong Markov property.

THEOREM 3.4.1. — *Let* $X \in T_x$ *and let* τ *be a natural stopping time with the property that* $X(t \wedge \tau)$ *is bounded for all* t. *Let* $\{X^z, z \in A\}$ *be a version of* $\mathcal{E}_\tau(X)$. *Then for* a.e. $(\mu_\tau)z$, $X^z \in T_z$, *where* μ_τ *is the distribution of* $X(\tau)$.

Remark. — It follows that there is a version of $\mathcal{E}_\tau(X)$ which satisfies $X^z \in T_z$ for every z in Q.

Proof. — $X^z \in \mathcal{X}_Q$ by definition. It is easily shown that $P\{X^z(0) = z\} = 1$ for a.e. $(\mu_\tau)z$. We must verify the supermartingale property. We can and will assume that the processes X^z are canonically defined relative to the natural fields $\mathcal{B}(t)$.

Let D be a bounded region containing x such that $X(\cdot \wedge \tau)$ never leaves D. Let $t_1 < t_2$ and $\Lambda \in \mathcal{B}_D(t_1)$. Suppose there exists a set $A \subset Q$, such that $\mu_\tau(A) > 0$ and :

$$(3.4.1) \qquad z \in A \Longrightarrow \int_\Lambda f(X^z_D(t_2)) > \int_\Lambda f(X^z_D(t_1)).$$

Let $\hat{\Lambda} = \{\omega : \omega(\tau(\omega) + \cdot) \in \Lambda\}$. $\hat{\Lambda} \in \mathcal{B}$ whenever Λ is.

Since τ and $t+\tau$, $t \in R_+$, are stopping times for the continuous supermartingale $\{f(X_D(t)),\ t \in R_+\}$, the process $\{f(X_D(t+\tau)),\ t \in R_+\}$ is also a supermartingale. Thus:

(3.4.2)
$$\int_{\hat{\Lambda} \cap \{X_D(\tau) \in A\}} f(X_D(\tau + t_1)) \geqslant \int_{\hat{\Lambda} \cap \{X_D(\tau) \in A\}} f(X_D(\tau + t_2)).$$

But the left-hand side is:

$$= \int_A \Big[\int_\Lambda f(X_D^z(t_1))\Big]\, d\mu_\tau(z).$$

By (3.4.1) this is:

$$< \int_A \Big[\int_\Lambda f(X_D^z(t_2))\Big]\, d\mu_\tau(z) = \int_{\hat{\Lambda} \cap \{X_D(\tau) \in A\}} f(X_D(\tau + t_2))$$

which contradicts (3.4.2). Therefore $\mu_\tau(A) = 0$.

Thus for fixed f, D, t_1, t_2 and $\Lambda \in \mathcal{B}_D(t_1)$, there is a set $A(D, f, t_1, t_2, \Lambda)$ of μ_τ-measure zero such that if $z \notin A(D, f, t_1, t_2, \lambda)$, then

$$\text{(3.4.3)} \qquad \int_\Lambda [f(X_D^z(t_1) - f(X_D^z(t_2))] \geqslant 0.$$

$\mathcal{B}_D(t_1)$ is generated by a countable finitely additive field G, and (3.4.3) holds for all $\Lambda \in G$ if z is not in some null set $A(D, f, t_1, t_2)$. But if (3.4.3) holds for every set in G, it must hold for every set in $\mathcal{B}_D(t_1)$. Next, outside of some null set $A(D, f)$, (3.4.3) holds for all rational t_1 and t_2 such that $t_1 < t_2$, and hence for all values of t_1 and t_2 by continuity. If $D_1, D_2, \ldots \uparrow Q$ is an increasing sequence of regions with the property that $X(\cdot \wedge \tau)$ never leaves D_1, then for z not in a null set $A(f) = \bigcup_n A(D_n, f)$, X^z satisfies the supermartingale property relative to the function f. Finally if z is not in some null set A, X^z satisfies the supermartingale inequality simultaneously for all functions in some countable set which is dense in $C(Q) \cap S(Q)$, and therefore X^z must be in T_x.

THEOREM 3.4.2. — *Let* $X \in T_x$ *and let* $U = \{X^z,\ z \in A\}$ *be a coherent family such that for each* z, $X^z \in T_z$. *Let* τ *be a natural strict stopping time for* X, *and suppose* $P\{X(t) \notin A\} = 0$.

Then $\Upsilon_\tau(X;\ U) \in T_x$ *(where* Υ_τ *is the operator introduced in* 1.4).

Proof. — Let $\hat{X} = \Upsilon_\tau(X;\ U)$. This is canonical by definition, and we may assume X is canonically defined as well.

Conditions (T1) and (T2) are clearly satisfied, so we must verify the supermartingale condition (T3). Let D be a bounded domain containing x and let τ_D be the first time X hits ∂D if X ever hits ∂D, and $+\infty$ otherwise. Set $\hat{X}_D(t) = \hat{X}(t \wedge \tau_D)$. Let $\tau_1 = \tau \wedge \tau_D$ and let f be a continuous multiply superharmonic function. $\hat{X} \sim (\tau)X$ implies $\hat{X} \sim (\tau_1)X$. This together with the fact that $X \in T_x$ implies $f(\hat{X}_D(t \wedge \tau_1))$ is a supermartingale.

We claim that the post-τ_1 process is a supermartingale relative to the fields $\mathcal{B}_D(t) = \mathcal{B}(t \wedge \tau_D)$.

Let $t_1 < t_2$ and consider the σ-field $\mathcal{F}(t_1)$ generated by $\hat{X}_D(\tau_1 + s)$, $0 \leqslant s \leqslant t_1$. Let $\Lambda \in \mathcal{F}(t_1)$ and let

$$\hat{\Lambda} = \{\omega : \omega(\tau(\omega) + \cdot) \in \Lambda\}.$$

Then $\hat{\Lambda} \in \mathcal{B}(t_1)$. Since the family U is a version of $\mathcal{E}_\tau(X)$ we see :

$$\int_{\Lambda \cap \{\tau_D > \tau\}} f(\hat{X}_D(\tau_1 + t_1)) = \int_Q \left[\int_{\hat{\Lambda}} f(X_D^z(t_1))\right] d\mu_\tau(z).$$

Now $X_D^z \in T_z$ so $f(X_D^z(t))$ is a supermartingale, and $\Lambda \in \mathcal{B}(t_1)$ so the right hand side is :

$$\geqslant \int_A \left[\int_{\hat{\Lambda}} f(X_D^z(t_2))\right] d\mu_\tau(z) = \int_{\Lambda \cup \{\tau_D > \tau\}} f(\hat{X}_D(\tau_1 + t_2)).$$

Since $f(\hat{X}_D(\tau_1 + t))$ is constant on the set $\{\tau_D \leqslant \tau\}$, this give us the supermartingale inequality. This does not quite prove our claims, as we have only shown that $f(\hat{X}_D(\tau_1 + t))$ is a supermartingale relative to the fields $\mathcal{F}(t)$. However, $\hat{X}(s)$ for $s \geqslant \tau_1$ and $\hat{X}(s)$ for $s \leqslant \tau_1$ are defined to be conditionally independent given $\hat{X}(\tau)$, so that this remains a supermartingale relative to the larger fields $\mathcal{B}(\tau) \vee \mathcal{F}(t)$. By a theorem of Courrège and Priouret ([13]) $\mathcal{B}(\tau) \vee \mathcal{F}(t) = \mathcal{B}(\tau + t)$, which establishes our claim.

By Corollary 3.2.2, $\{f(\hat{X}_D(t)),\ \mathcal{B}(t),\ t \in R_+\}$ is a supermartingale. Q.E.D.

4. U_x processes and the the Dirichlet problem.

4.1. U_x processes.

Let $D \subset Q$ be a domain. Let $\overline{D}$ be a metrizable compactification of D; that is, $\overline{D}$ is a compact, metrizable topological space in which D is a dense subspace. All topological notions in this section will be understood to be relative to $\overline{D}$ unless it is specifically stated otherwise. Thus the boundary of D is $\overline{D} - D$, which we will denote by ∂D.

In the classical case, where $k = 1$ and the notions of harmonic and multiply harmonic functions coincide, one ordinarily takes ∂D to be the geometrical boundary of D or the geometrical boundary of D together with the Alexandroff point at infinity if D is unbounded. One may wish to use other compactifications, however, such as those corresponding to the Feller or Kuramochi boundary; or the Martin boundary, which is the most natural boundary from the standpoint of the Dirichlet problem and its probabilistic treatment.

If $f \in S(D)$ is bounded below we denote the greatest lower semicontinuous extension of f to $\overline{D}$ by $\overline{f}$; that is $\overline{f} = f$ on D and if $x \in \partial D$, $f(x) = \liminf\limits_{\substack{y \in D \\ y \to x}} f(y)$. Let

$$S(\overline{D}) = \{\overline{f} : f \in S(D),\ f \text{ bounded below}\}.$$

A sample path originating at a point of D is said to converge to ∂D if it eventually leaves any compact subset of D.

We will put two restrictions on the domain D and its compactification, the first one minor and the second somewhat more stringent. We require that

H1) for $x \in D$ there is a process $x \in T_x$ which converges to the boundary of D;

H2) if $x, y \in \partial D$, there is a function $f \in S(\overline{D})$ and sets N and N', where N is a neighborhood of either x or y and N' is a neighborhood of the remaining point, such that $f > 1$ on N, $f < 1$ on N'. We say f strongly separates x and y.

Note that since $\overline{D}$ has a countable base for its topology, if (H2) holds there must be a countable subset of $S(\overline{D})$ which strongly separates points of ∂D.

By a theorem of Lévy, if the dimension of Q is three or greater, Brownian motion will eventually leave any compact subset of Q. If the dimension of Q is two and the degree of decomposition of Q is two, it is easy to find examples of T_x-processes (not Brownian motion) with the same property. Thus the only cases excluded by (H1) are $Q = D = R$ and the case in which $\dim Q = 2$, $k = 1$ and the complement of D in Q has logarithmic capacity zero; these two cases are of little interest in the present context.

Let $D_1 \subset D_2 \subset \dots$ be a sequence of relatively compact subdomains of D and for any process X let

$$\tau_i(\omega) = \inf \{t : X(t, \omega) \in D_i^c\},$$

undefined if there is no such t. Then let

$$\tau_D(\omega) = \sup_i \tau_i(\omega) \qquad \text{if} \qquad X(t, \omega)$$

converges to D, undefined otherwise.

The following is a standard lemma.

LEMMA 4.1.1. — *Let* $h \in S(D)$ *be bounded below on* D. *Let* $x \in D$ *and* $X \in T_x$, *and suppose* X *converges to* ∂D w. p. 1. *Then* $\lim_{t \uparrow \tau_D} h(X(t)) = h_\infty$ *exists* w. p. 1. *and*

$$h(x) \geqslant E\{h_\infty\}.$$

Proof. — We may assume $h \geqslant 0$. Suppose $h(x) < \infty$. (The case $h(x) = \infty$ can be handled by considering $h \wedge n$ and letting $n \uparrow \infty$.) The process

$$\begin{aligned} Y(t) &= h(X(t)) \qquad &\text{if} \qquad t < \tau_D \\ &= 0 \qquad &\text{if} \qquad t \geqslant \tau_D \end{aligned}$$

is a supermartingale. By Proposition 3.3.3 Y is right-continuous, and hence separable. Thus Y has left limits w. p. 1., so $\lim_{t \uparrow \tau_D} Y(t) = h_\infty$ exists. If $\tau_1, \tau_2, \dots$ are the stopping times defined in the preceding paragraph, by the optional sampling theorem $h(x), Y(\tau_1), Y(\tau_2), \dots$ is a positive supermartingale. Thus :

$$h(x) \geqslant \lim_{n \to \infty} E\{Y(\tau_n)\} \geqslant E\{\lim_{n \to \infty} Y(\tau_n)\} = E\{h_\infty\} \qquad \text{Q.E.D.}$$

PROPOSITION 4.1.2. — *Under* (H1) *and* (H2), *if* $x \in \mathrm{D}$ *and* $\mathrm{X} \in \mathrm{T}_x$ *and if* Ω' *is the* ω*-set on which* $\mathrm{X}(\cdot, \omega)$ *converges to* $\partial\mathrm{D}$, *then for* a.e. $\omega \in \Omega'$, $\lim\limits_{t \Uparrow \tau_{\mathrm{D}}(\omega)} \mathrm{X}(t, \omega)$ *exists.*

Proof. — Let W be the countable subset of $\mathrm{S}(\overline{\mathrm{D}})$ which separates boundary points. If $f \in \mathrm{W}$ we may assume f is bounded by taking $f \wedge 2$ if necessary. By Lemma 4.1.1, if ω is not in some exceptional null-set Γ, for all $f \in \mathrm{W}$ $f(\mathrm{X}(t, \omega))$ has a limit as $t \uparrow \tau_{\mathrm{D}}$ if $\omega \in \Omega'$, or as $t \uparrow \infty$ if $\omega \notin \Omega'$. If $\omega \in \Omega' - \Gamma$, there must be at least one point of $\partial\mathrm{D}$ in the closure of $\{\mathrm{X}(t, \omega), 0 \leqslant t < \tau_{\mathrm{D}}\}$. But there can be at most one point of $\partial\mathrm{D}$ in this closure, for if x and y were two such points, take $f \in \mathrm{W}$ to be a function which separates x and y. Then there must be infinitely many times s, t arbitrarily close to τ_{D} for which $f(\mathrm{X}(s, \omega)) > 1 + \varepsilon$ and $f(\mathrm{X}(t, \omega)) < 1$, which contradicts the convergence of $f(\mathrm{X}(t, \omega))$ as $t \uparrow \tau_{\mathrm{D}}$.

Remark. — A corollary to the above proof is that if $\mathrm{X} \in \mathrm{T}_x$, for a.e. ω there can be at most one point of $\partial\mathrm{D}$ in the closure of the set $\{\mathrm{X}(t, \omega), t < \tau_{\mathrm{D}}\}$, where if $\tau_{\mathrm{D}}(\omega)$ is not defined, « $t < \tau_{\mathrm{D}}$ » is taken to mean « $\forall t \geqslant 0$ ».

DEFINITION. — *Let* $\mathrm{C} \subset \partial\mathrm{D}$ *be a Borel set. If* $x \in \mathrm{D}$, $\mathrm{U}_x(\mathrm{C})$ *is the class of processes* X *satisfying:*

(U1) $\mathrm{X} \in \mathrm{T}_x$,

(U2) X *is Brownian motion until it first leaves some neighborhood of* x,

(U3) $\mathrm{P}\{\mathrm{X}(\tau_{\mathrm{D}}) \in \mathrm{C}\} = 1$.

When $\mathrm{C} = \partial\mathrm{D}$, we will write U_x for $\mathrm{U}_x(\partial\mathrm{D})$. The significant property of U_x processes is (U2) which assures us that they are « nice » in a neighborhood of their origin.

DEFINITION. — *A set* $\mathrm{C} \subset \partial\mathrm{D}$ *is a* U*-boundary for* D *if* C *is closed and if for every* $x \in \mathrm{D}$, $\mathrm{U}_x(\mathrm{C}) \neq 0$. *Under* (H1), $\partial\mathrm{D}$ *is always a* U*-boundary.*

Before proceeding to the Dirichlet problem we need some facts concerning coherent families of stochastic processes.

Suppose $\mathrm{C} \subset \partial\mathrm{D}$, $z \in \mathrm{D}$ and $\mathrm{X} \in \mathrm{U}_z$; and let N be a neighborhood of z with closure in D. Then if $\mathrm{V} = \{\mathrm{X}^x, x \in \partial\mathrm{N}\}$ is a coherent family of processes such that $\mathrm{X}^x \in \mathrm{T}_x$

and $P\{X^x(\tau_D) \in C\} = 1$, the process $\Upsilon_{\tau_N}(X;\ V)$ is in $U_z(C)$. This is an immediate consequence of Theorem 3.4.2.

LEMMA 4.1.3. — *Let* V_1 *and* V_2 *be coherent families of canonically defined processes, where* $V_i = \{X_i^z, z \in A_i\}$, $i = 1, 2$. *Let* τ *be a natural strict time on* Ω_c *such that for* $z \in A_1$, $P\{X_1^z(\tau) \notin A_2\} = 0$. *Then the family:*

$$V_3 = \{\Upsilon_\tau(X_1^z;\ V_2),\quad z \in A_1\}$$

is coherent.

Note. — The hypothesis that the processes be canonically defined is merely a device to define τ for all processes, and can be dispensed with at the expense of a slightly more cumbersome statement.

Proof. — Let P_i^z be the distribution of X_i^z, $i = 1, 2$, and let ${}_xP_1^z$ and ${}_\infty P_1^z$ be the distributions of P_1^z conditioned on « $X_1^z(\tau) = x$ » and « τ not defined » respectively.

If $\Lambda_1, \Lambda_2 \in \mathcal{B}$, let

$$\hat{\Lambda}_1 = \{X(\cdot \wedge \tau) \in \Lambda_1\} \qquad \text{and} \qquad \hat{\Lambda}_2 = \{X(\cdot + \tau) \in \Lambda_2\}.$$

If $\hat{P}_z$ is the distribution of $\Upsilon_\tau(X_1^z, V_2)$ on Ω_c:

$$\hat{P}_z(\hat{\Lambda}_1 \cap \hat{\Lambda}_2) = \int P_2^x(\Lambda_2)\,{}_xP_1^z(\hat{\Lambda}_1)\, d\mu_\tau(x) + [1 - \mu_\tau^z(Q)]\,{}_\infty P^z\{\hat{\Lambda}_1 \cap \hat{\Lambda}_2\}.$$

We must show $z \to \hat{P}_z(\hat{\Lambda}_1 \cap \hat{\Lambda}_2)$ is Borel measurable. The second term is clearly measurable. Let $d\nu_z = {}_xP_1^z(\Lambda_1)\, d\mu_\tau^z(x)$ and let G be the class of functions $f\colon Q \to R$ such that $z \to \int f(x)\, d\nu_z(x)$ is measurable. G contains indicator functions of Borel sets since if $A \subset Q$ is Borel:

$$\int I_A(x)\, d\nu_z(x) = \int_A {}_xP_1^z(\Lambda_1)\, d\mu_\tau^z(x) = P_1^z(\Lambda_1 \cap \{X_1^z(\tau) \in A\}).$$

This is measurable in z since V_1 is coherent. G is clearly linear and closed under increasing convergence, so G contains all Baire functions. In particular, $x \to P_2^x(\Lambda_2) \in G$.

Q.E.D.

We can now prove an important fact about the existence of coherent families of U_x processes. The hypothesis (U2) plays a fundamental role in the proof; it is not known whether the corresponding theorem is true for T_x processes.

THEOREM 4.1.4. — *Let* A *be a Borel subset of* ∂D *and* f *a Baire function on* A. *Let* $K \subset D$ *be Borel and* $\varphi \in C(K)$. *If there is a family* $\{X^z, z \in K\}$ *such that for each* $z \in K$:

a) $X^z \in U_z(A)$

and

b) $E\{f(X^z(\tau_D))\} > \varphi(z)$,

then there is a coherent family satisfying the same conditions.

Proof. — Let $z \in K$; $X^z \in U_z(A)$, so X^z is Brownian motion until it first leaves some open ball B with $\overline{B} \subset D$.

Let $\{{}_xX, x \in \partial B\}$ be a version of $\mathcal{E}_{\tau_B}(X^z)$. For $y \in B$, let μ_y be harmonic measure on ∂B relative to y. By Theorem 3.4.1 ${}_xX \in T_x$ for a.e. $(\mu_z)x \in \partial B$ and further

$$1 = P\{X^z(\tau_D) \in A\} = \int_{\partial B} P\{{}_xX(\tau_D) \in A\}\, d\mu_z(x),$$

so $P\{{}_xX(\tau_D) \in A\} = 1$ for a.e. $(\mu_z)x$.

Let Z^y be Brownian motion from y and consider $\hat{X}^y_z = \Upsilon_{\tau_B}(Z^y, \mathcal{E}_{\tau_B}(X^z))$. (Intuitively this is the process which is Brownian motion until it hits ∂B and is ${}_xX$ from then on if $X^y(\tau_B) = x$.) $\hat{X}^y_z \in U_y(A)$, and, if we let $g(x) = E\{{}_xX(\tau_D)\}$,

$$\begin{aligned} E\{f(\hat{X}^y_z(\tau_D))\} &= E\{E\{f(\hat{X}^y_z(\tau_D)) \mid \hat{X}^y_z(\tau_B)\}\} \\ &= \int_{\partial B} g(x)\, d\mu_y(x) = h(y), \end{aligned}$$

where this defines $h(y)$.

Now $h(z) > \varphi(z) > -\infty$. Thus h is either $\equiv +\infty$ or harmonic, and therefore continuous, in B. φ is continuous so in either case there is an open neighborhood N_z of z on which $h > \varphi$.

The family $V_z = \{\hat{X}^y_z, y \in N_z\}$ is coherent by Lemma 4.1.3. For each $z \in K$ we can find such a neighborhood N_z and family $V_z = \{\hat{X}^y_z, y \in N_z\}$. Since $\overline{D}$ has a countable base there is a countable set $\{z_i\}$ such that $K \subset \bigcup_i N_{z_i}$, and we can clearly find sequence $\{M_i\}$ of disjoint Borel sets such that $M_i \subset N_{z_i}$ and $K = \bigcup_i N_i$. Then the family $\{\hat{X}^y, y \in K\}$ given by $\hat{X}^y = \hat{X}^y_{z_i}$ if $y \in M_i$ satisfies the requirements of the theorem. Q.E.D.

4.2. The Dirichlet Problem.

We follow Bremermann in allowing multiply superharmonic, as well as multiply harmonic solutions. We consider the problem on certain subsets of the boundary rather than on the whole boundary. This is somewhat analogous to the case of the Dirichlet problem for certain parabolic differential equations (see [16]) where large portions of the boundary may be irregular for the Dirichlet problem and therefore can be ignored.

Given a boundary function on one of these sets, we define a multiply superharmonic function — we prejudice the issue somewhat by calling this function a « Dirichlet solution » — which is connected with the boundary function in a natural way. We then have two problems : first to study the function and in particular its boundary behavior, and second to study the class of boundary sets for which such an approach is possible. We address ourselves to the first problem in this and the next section, and to the second problem in section 6.

Let C be a U-boundary and f a Baire function on C. We say $f \in \mathrm{I(C)}$ if for each $x \in \mathrm{D}$ there is an $x \in \mathrm{U}_x(\mathrm{C})$ such that $-\infty < \mathrm{E}\{f(\mathrm{X}(\tau_\mathrm{D}))\} \leqslant +\infty$. In particular, f is in I(C) if it is bounded below.

Definition. — *Let $f \in \mathrm{I(C)}$. The Dirichlet solution Φ_f^C of f is:*

$$\Phi_f^\mathrm{C}(x) = \sup_{\mathrm{X} \in \mathrm{U}_x(\mathrm{C})} \mathrm{E}\{f(\mathrm{X}(\tau_\mathrm{D}))\}.$$

If f is defined on some larger set than C, then Φ_f^C will mean the Dirichlet solution of the restriction of f to C.

Note that we have defined Φ_f^C relative to the classes $\mathrm{U}_x(\mathrm{C})$, not relative to the apparently more natural classes $\mathrm{T}_x(\mathrm{C})$ of T_x processes which hit C with probability one. The reason is that Φ_f^C is multiply superharmonic whereas the corresponding fact for the function $\tilde{\Phi}_f^\mathrm{C}$ defined using $\mathrm{T}_x(\mathrm{C})$ depends strongly on the nature of the boundary and boundary function f. In fact, Φ_f^C is multiply superharmonic iff $\tilde{\Phi}_f^\mathrm{C} = \Phi_f^\mathrm{C}$.

Roughly speaking, $\tilde{\Phi}_f^\mathrm{C}$ corresponds to the lower envelope of all multiply superharmonic functions g satisfying $\liminf_{y \to x} g(y) \geqslant f(x)$ for $x \in \mathrm{C}$, while Φ_f^{C} corresponds to that lower envelope regularized to be lower semicontinuous.

THEOREM 4.2.1. — *Let* $\mathrm{D} \subset \mathrm{Q}$ *be a domain and* C *a U-boundary. If* $f \in \mathrm{I}(\mathrm{C})$, *then* Φ_f^{C} *is either multiply superharmonic or identically* $+\infty$.

This is a result of the following two lemmas and Proposition 2.2.

LEMMA 4.2.2. — Φ_f^{C} *is lower semicontinuous.*

Proof. — Let $\mathrm{X}^z \in \mathrm{U}_z(\mathrm{C})$. X^z is Brownian motion until it leaves some ball B relatively compact in D. Consider $\mathcal{E}_{\tau_\mathrm{B}}(\mathrm{X}) = \{\mathrm{X}^x,\ x \in \partial\mathrm{B}\}$ and let $g(x) = \mathrm{E}\{f(\mathrm{X}^x(\tau_\mathrm{D}))\}$. Now if Z^y is Brownian motion from $y \in \mathrm{B}$, let $\hat{\mathrm{X}}^y = \Upsilon_{\tau_\mathrm{B}}(\mathrm{Z}^y; \mathcal{E}_{\tau_\mathrm{B}}(\mathrm{X}^z))$. Then $\hat{\mathrm{X}}^y \in \mathrm{U}_y(\mathrm{C})$ and

$$\mathrm{E}\{f(\hat{\mathrm{X}}^y(\tau_\mathrm{D}))\} = \mathrm{E}\{\mathrm{E}\{f(\hat{\mathrm{X}}^y(\tau_\mathrm{D}))|\hat{\mathrm{X}}^y(\tau_\mathrm{B})\}\}.$$

The distribution of $\hat{\mathrm{X}}_y(\tau_\mathrm{B})$ is $\mathrm{u}_y =$ harmonic measure on $\partial\mathrm{B}$ relative to z so this is just $\int_{\partial\mathrm{B}} g(x)\ \mu_y(x) = h(y)$. $h(y)$ is either identically $+\infty$ or else harmonic, and therefore continuous, in B. By definition $\Phi_f^{\mathrm{C}}(y) \geqslant h(y)$, $y \in \mathrm{B}$. If $a < \Phi_f^{\mathrm{C}}(z)$, we can choose X to make $h(z) > a$, hence Φ_f^{C} must be lower semicontinuous at z. Q.E.D.

For the notation in the following lemma, see Section 2.

LEMMA 4.2.3. — *Let* $\mathrm{B} = \mathrm{B}(z;\ \rho)$ *be a polycylinder such that* $\overline{\mathrm{B}} \subset \mathrm{D}$ *and let* B^* *be its distinguished boundary. If* $\lambda_{z\rho}$ *is the uniform measure on* B^*, *then for any* $f \in \mathrm{I}(\mathrm{C})$,

$$\Phi_f^{\mathrm{C}}(z) \geqslant \int_{\mathrm{B}^*} \Phi_f^{\mathrm{C}}(x)\lambda_{z\rho}\,(dx).$$

Proof. — There are T_z processes which hit B^* before leaving D. One such process X^* can be described as follows:

Let $z = (z^1, \ldots, z^k)$ and $\mathrm{B} = \mathrm{B}_1 \times \cdots \times \mathrm{B}_k$, where $\mathrm{B}_j \subset \mathrm{Q}_j$ is a ball centered at z^j. Let Z^j be Brownian motion from z^j which is stopped the first time it hits $\partial\mathrm{B}_j$. Then let $\mathrm{X}^* = (\mathrm{Z}^1, \ldots, \mathrm{Z}^k)$. It is easily seen that $\mathrm{X}^* \in \mathrm{T}_z$. Further, if N is a neighborhood of z such that $\overline{N} \subset B$, X is Brownian motion until it leaves N. Finally, since for each j, Z^j hits ∂B_j and is stopped there in finite time, X hits

$$B^* = \partial B_1 \times \cdots \times \partial B_k$$

in a finite time.

Let $\{\varphi_n\}$ be a sequence of continuous functions increasing to Φ_f^C. For fixed n and $x \in B^*$, there is $X_n^x \in U_x(C)$ such that $E\{f(X_n^x(\tau_D))\} > \varphi_n(x)$. By Theorem 4.1.4, there is a coherent family $V_n = \{X^x, x \in B^*\}$ satisfying the same conditions. Let $\hat{X}_n^z = \Upsilon_{\tau_B}(X^*; V_n)$. $\hat{X}_n^z \in U_z(C)$ and

$$\begin{aligned}\Phi_f^C(z) \geqslant E\{f(\hat{X}_n^z(\tau_D))\} &= E\{E\{f(\hat{X}_n^z(\tau_D)) | \hat{X}^n(\tau_B)\}\} \\ &= \int E\{f(X_n^x(\tau_D))\}\, d\lambda\,(x) \geqslant \int_{B^*} \varphi_n\, d\lambda\end{aligned}$$

where λ is the uniform measure on B^*. Since n was arbitrary, $\Phi_f^C(z) \geqslant \int_{B^*} \Phi_f^C\, d\lambda$. Q.E.D.

Proposition 4.2.4. — *Let* $D \subset Q$ *be a domain and* C *a* U-*boundary for* D. *If* $f, f_1, f_2, \ldots$ *are such that* $f_i \in I(C)$, $i = 1, 2, \ldots$ *and* $f_n \uparrow f$, *then* $\Phi_{f_n}^C \uparrow \Phi_f^C$.

Proof. — Clearly $\Phi_{f_n}^C \leqslant \Phi_f^C$ for all n. If $a < \Phi_f^C(x)$, there is $X \in U_x(C)$ such that $E\{f(X(\tau_D))\} > a$. Then

$$\Phi_{f_n}^C(x) \geqslant E\{f_n(X(\tau_D))\} \uparrow E\{f(X(\tau_D))\} > a. \qquad \text{Q.E.D.}$$

Proposition 4.2.5. — (Minimality Property). *Let* $D \subset Q$ *be a bounded domain and* C *a* U-*boundary for* D. *Suppose* $f \in I(C)$. *If* h *is multiply superharmonic and bounded below in* D *and satisfies* $\liminf_{x \to z} h(x) \geqslant f(z)$, $z \in C$, *then* $h \geqslant \Phi_f^C$.

Note. — The theorem remains true if C is not closed.

Proof. — Let $x \in D$, $X \in U_x(C)$. By Lemma 4.1.1, $\lim_{t \uparrow \tau_D} h(X(t, \omega)) = h_\infty(\omega)$ exists and $E_\infty\{h\} \leqslant h(x)$. But $X(t) \to X(\tau_D) \in C$ as $t \uparrow \tau_D$ and for $z \in C$, $\liminf_{y \to z} h(y) \geqslant f(z)$, so $E\{h_\infty\} \geqslant E\{f(X(\tau_D))\}$. Thus

$$h(x) \geqslant \sup_{X \in U_x(C)} E\{f(X(\tau_D))\} = \Phi_f^C(x). \qquad \text{Q.E.D.}$$

4.3. Regularity of boundary points.

It is customary to characterize regularity properties of boundary points in terms of barriers, but, in keeping with our probabilistic approach, we will define them in terms of the functions Φ_f. We will show that under suitable restrictions on the domain, these probabilistic definitions coincide with the usual definitions.

DEFINITION. — *Let* C *be a* U-*boundary for the domain* $D \subset Q$ *and let* $x_0 \in C$. x_0 *is* C-*semiregular if for each neighborhood* N *of* x_0, *if* $g = I_{C-N}$, $\liminf_{x \to x_0} \Phi_g^C(x) = 0$, *and is strongly* C-*regular if* $\lim_{x \to x_0} \Phi_g^C(x) = 0$ *for each neighborhood* N *of* x_0.

THEOREM 4.3.1. — *Suppose* $\overline{D}$ *satisfies* (H1) *and* (H2). *Then for any* U-*boundary* C *there exists at least one* C-*semiregular point.*

Proof. — We first show that if D has no semiregular points there exist T_x processes which fail to converge, and then show this violates the separation hypothesis.

If $x \in \partial D$ is not C-semiregular there are neighborhoods N'_x and N''_x of x, such that, if g_x is the indicator function of $\overline{N}{}^c_x$:

(*a*) $\Phi_{g_x}^C > \varepsilon_x$ on N''_x.

If the boundary of D has no semiregular points we can find such N'_x, N''_x and ε_x for each $x \in \partial D$. Since ∂D is compact, there is a set $\{x_1, \ldots, x_n\}$ such that $\partial D \subset \bigcup_1^n N''_{x_i}$.

Let $N_1 = N_{x_1}$, $N_2 = N''_{x_2} - N''_{x_1}$, $\ldots$, $N_n = N''_{x_n} - \bigcup_1^{n-1} N''_{x_i}$. Then the N_i are disjoint and cover ∂D. Write N'_i instead of N'_{x_i} and let $\varepsilon = \min_i \varepsilon_{x_i}$.

By (a), the definition of Φ^C, and Theorem 4.1.4, there is a coherent family $V = \{X^z, z \in D \cap \bigcup_1^n N_i\}$ such that $X^z \in U_z(C)$ and

(*b*) $z \in D \cap N_i \Longrightarrow P\{X^z(\tau_D) \notin \overline{N'_i}\} > \varepsilon$.

We construct a sequence of processes by induction. Let

$\zeta \in D$, $X_1 \in U_\zeta$ and let A_1 be a neighborhood of ∂D not containing ζ, with $\bar{A}_1 \subset \bigcup_1^n N_i$. Suppose we have defined a sequence $X_1, \ldots, X_m$ of U_ζ-processes and a decreasing sequence $A_1, \ldots, A_m$ of neighborhoods of ∂D such that for $j = 1, \ldots, m-1$;

(*c*) $X_{j+1} = \Upsilon_{\tau_{A_j}}(X_j, V)$.

(*d*) There is an exceptional ω-set of probability less than $\frac{1}{j}$ outside of which

$$P\{X_j(\tau_{A_j}) \in N_i \quad \text{and} \quad X(\tau_{A_{j+1}}) \in N_i' | X_j(\tau_{A_j})\} < 1 - \varepsilon.$$

Now choose a neighborhood A_{m+1} of ∂D such that $A_{m+1} \subset A_m$ and such that (*d*) holds for $j = m$. This we can do since by (*b*) and (*c*) the conditional probability that $X_m(\tau_{A_j}) \in N_i$ and $X_m(\tau_D) \in \bar{N}_i'$ given $X_m(\tau_{A_m})$ is less than $1 - \varepsilon$, so the probability is greater than ε that $X_m(t)$ will eventually leave N_i'. Thus by making A_{m+1} small enough we can assure (*d*). Then let

$$X_{m+1} = \Upsilon_{\tau_{A_{m+1}}}(X_m, V).$$

By Theorem 1.4.2 there is a process $\hat{X} \in T_\zeta$ such that $\hat{X} \sim (\tau_{A_j})X_j$, $j = 1, 2, \ldots$ Notice that since $X_j \in U_\zeta(C)$, X_j (and hence $\hat{X}$) hits ∂A_j w. p. 1. We will show that for arbitrarily large n the sequence $\hat{X}(\tau_{A_j})$, $j = n, n+1, \ldots$ moves from some N_i to $(N_i')^c$.

Choose n such that $\frac{1}{n} < \varepsilon$ and let $i_j(\omega)$ be that index i for which $\hat{X}(\omega, \tau_{A_j}(\omega)) \in N_i$. Consider

$$P\{\hat{X}(\tau_{A_{j+1}}) \in N_{i_j}, j = n, n+1, \ldots\}.$$

By (*d*) and the fact that the behavior of $\hat{X}$ before and after τ_{A_j} are conditionally independent given $X(\tau_{A_j})$ this is

$$\leqslant \prod_n^\infty \left(1 + \frac{1}{j} - \varepsilon\right) = 0.$$

The proof is now readily completed. By there mark following Proposition 4.1.2, w. p. 1 there can be at most one point of ∂D in the closure of $\{\hat{X}(t), t < \tau_D\}$. But w. p. 1 for some i, $X(\tau_{A_j}) \in N_i$ for infinitely many values of j and $X(\tau_{A_k}) \in (N_i')^c$ for infinitely many values of k, so there must be at least two points of ∂D in the closure, which is a contradiction.

Q.E.D.

The following proposition is connected to Theorem 4.3.1 primarily through its proof. We will state it now for future reference.

PROPOSITION 4.3.2. — *If* C *is a* U-*boundary and* $x \in C$ *is not* C-*semiregular, there is a neighborhood* N *of* x *such that* $C - N$ *is a* U-*boundary.*

The proof involves a simpler version of the above construction, so we only outline it here. One finds a neighborhood N of x such that for some $\varepsilon > 0$, $\Phi^{\partial D}_{I_{N^c}} > \varepsilon$ on N, and a coherent family $V = \{X^z, z \in N\}$ of U(C) processes with the property that $P\{X^z(\tau_D) \in \overline{N}\} < 1 - \varepsilon$. If $z \in D$, one defines by induction a sequence of $U_z(C)$ processes and a decreasing sequence $A_1, A_2, \ldots$ of neighborhoods of $\overline{N} \cap \partial D$, such that $X_{j+1} = \Upsilon_{\tau_{A_j}}(X_j, V)$ and such that $P\{X_{j+1} \text{ hits } \partial A_{j+1}\} < \varepsilon^n$. Then the process $\hat{X}$ satisfying $\hat{X} \sim (\tau_{A_j})X_j$ for all j is in $U_x(C)$ but does nothit $\partial D \cap N$, and hence is in $U_x(C - N)$.

In the classical theory, the regularity properties of a boundary point are local properties. In our case in particular we would like the definition of semiregularity to be independant of the U-boundary. In order for this to be true it is necessary to introduce slight further restrictions on the region. We will say a region is normal if its boundary points can be strongly separated from closed sets. More precisely, a point $x \in \overline{D}$ is said to be normal in $\overline{D}$ if for any closed set $K \subset \overline{D}$ not containing x there is a function f, positive and multiply superharmonic on $\overline{D}$ satisfying $f(x) > 1$ and $f \leqslant 1$ on K. A region D is normal if all points of ∂D are normal in $\overline{D}$. In analogy with accepted topological terminology we should call such regions « regular », but this word is already sorely overused in this paper.

If D is bounded and $\overline{D}$ is the closure of D in Q, D is normal, for in this case the separating functions can be taken to be lower envelopes of finitely many positive linear functions.

LEMMA 4.3.3. — *Let* $D \subset Q$ *be a domain and let* x *be a normal point in* ∂D, *and* N *a neighborhood of* x. *Then there exists a neighborhood* N' *of* x *such that* $y \in D - N$ *and* $X \in T_y$ *imply that*

$$P\{X(\tau_D) \in N'\} < 1 - \varepsilon.$$

Proof. — By normality of x let h be a positive multiply superharmonic function such that for some $\varepsilon > 0$, $h \leqslant 1 - \varepsilon$ on $D - N$ and $\liminf_{y \to x} h(y) > 1$. Let N' be a neighborhood of x such that $N' \cap D = \{y : h(y) > 1\}$, and let $h' = h \wedge 1$. If $y \in D - N$, $X \in T_y$, by Lemma 4.1.1:

$$1 - \varepsilon \geqslant h'(y) \geqslant E\{\lim_{t \uparrow \tau_D} h'(X(t))\} \geqslant P\{X(\tau_D) \in N'\} \qquad \text{Q.E.D.}$$

THEOREM 4.3.4. — *Let* $D \subset Q$ *be a normal region and let* C_1 *and* C_2 *be U-boundaries. Suppose* x *is* C_1*-semiregular* (C_1*-regular*). *Then* $x \in C_2$ *and* x *is* C_2*-semiregular* (C_1*-regular*).

Proof. — Let N be a neighborhood of x and choose $N' \subset N$ in such a way that for some $m > 0$, if $g' = I_{\partial D - N'}$, $\Phi_{g'}^{C_1}(y) \geqslant m$ if $y \in D - N$, which we can do by Lemma 4.3.3. Then, if $g = I_{\partial D - N}$, we have

$$\liminf_{y \to z} \frac{1}{m} \Phi_{g'}^{C_1}(y) \geqslant g(z) \qquad \text{if} \qquad z \in \partial D.$$

Therefore, by the minimality property,

$$\Phi_g^{C_2} \leqslant \frac{1}{m} \Phi_{g'}^{C_1}.$$

In particular, since x is C_1-semiregular, (C_1-regular) this implies that the limit inferior (limit) of $\Phi_g^{C_2}$ at x is zero, hence x must be C_2-semiregular (C_2-regular). Q.E.D.

Since strongly C-regular and C-semiregular points are independent of the U-boundary C we shall call them strongly regular and semiregular points respectively from now on. Notice that if C is a U-boundary, C contains the set of all semiregular points.

There is an important class of boundary points not included among the strongly regular and semiregular points.

Definition. — *Let* C *be a* U*-boundary and suppose* $x \in C$. x *is weakly* C*-regular if for any neighborhood* N *of* x, $\lim_{y \to x} \Phi^{C}_{I_N}(y) = 1$.

Equivalently, x is weakly C-regular iff for any neighborhood N of x,

$$\sup_{X \in U_{y}(c)} P\{X(\tau_D) \in N\} \to 1 \qquad \text{as} \qquad y \to x_0.$$

Unlike the strong and semiregular points, the set of weakly C-regular points depends upon the U-boundary C. When $C = \partial D$, we say a weakly C-regular point is weakly regular. We reserve « regular » for points regular for the classical Dirichlet problem.

a) Strong regularity implies semiregularity, regularity and weak C-regularity for any U-boundary C.

b) If $C_1 \subset C_2$ are U-boundaries, then weak C_1-regularity implies weak C_2-regularity.

c) Regularity implies weak regularity, but regularity does not imply weak C-regularity for all U-boundaries C.

The above implications are all obvious, but none of the reverse implications are true. Semiregularity does not in general imply weak regularity.

Proposition 4.3.5. — $x \in \partial D$ *is semiregular iff there exists a sequence* $x_1, x_2, \ldots, \in D$ *such that* $x_n \to x$ *and for any neighborhood* N *of* x,

$$(*) \qquad \sup_{X \in U_{x_n}} P\{X(\tau_D) \notin N\} \to 0 \qquad \text{as} \qquad n \to \infty.$$

Proof. — This is slightly stronger than the definition but follows directly from it. Let $N_1, N_2, \ldots \downarrow \{x\}$ be a sequence of neighborhoods of x. For each n we can find $X_n \in B_n$ such that $\sup_{X \in U_{x_n}} P\{X(\tau_D) \in B\} < \frac{1}{n}\cdot$ Then $x_n \to x$ and (*) is satisfied for this sequence. The converse is obvious.

Q.E.D.

Corollary 4.3.6. — *Let* D^{**} *be the set of all semiregular points of* ∂D. *Then* D^{**} *is closed.*

The regularity properties defined above are local properties, i.e., if $x \in \partial D$ is strongly regular, semiregular or weakly regular, and N is a neighborhood of x, then x is a strongly regular, semiregular or weakly regular point of $N \cap D$ respectively, and conversely. We will postpone the proof of this until Section 4.5.

4.4. Boundary behavior of the Dirichlet solution.

THEOREM 4.4.1. — *Let $D \subset Q$ be a normal domain and C a U-boundary for D. Let $f \in I(C)$ be bounded and continuous at x_0. Then*

a) *If x_0 is weakly C-regular and f is bounded below,*

$$(*) \qquad \liminf_{x \to x_0} \Phi_f^C(x) \geqslant f(x_0).$$

b) *If x_0 is semiregular and f is bounded above,*

$$(**) \qquad \liminf_{x \to x_0} \Phi_f^C(x) \leqslant f(x_0).$$

Hence if x_0 is both weakly C-regular and semiregular and f is bounded there is equality in (**).

c) *If x_0 is strongly regular and f is bounded,*

$$(***) \qquad \lim_{x \to x_0} \Phi_f^C(x) = f(x_0).$$

d) *x_0 is weakly C-regular, semiregular or strongly regular iff (*), (**), or (***) respectively holds for all $f \in C(C)$.*

Proof. — Let

$$m = \inf\{f(x) : x \in C\} \qquad \text{and} \qquad M = \sup\{f(x) : x \in C\}.$$

(We may have $m = -\infty$ and/or $M = +\infty$.) f is continuous at x_0, so for $\varepsilon > 0$ there is a neighborhood N of x_0 such that $|f(x) - f(y)| < \varepsilon$ for $x, y \in N$. Let $h = I_N$ and $g = I_{C-N}$. Then, whenever the terms are well-defined we have:

$$(4.4.2) \quad [m + f(x_0)]h - m - \varepsilon \leqslant f \leqslant [M - f(x_0)]g + f(x_0) + \varepsilon$$

so that:

$$(4.4.3) \quad [m + f(x_0)]\Phi_h^C - m - \varepsilon \leqslant \Phi_f^C \leqslant [M - f(x_0)]\Phi_g^C + f(x_0) + \varepsilon.$$

a) f is bounded below so the left-hand inequalities in (4.4.2) and (4.4.3) make sense. x_0 is weakly regular, hence $\lim_{x \to x_0} \Phi_g^C(x) = 1$. By (4.4.3), $\liminf_{x \to x_0} \Phi_f^C(x) \geqslant f(x_0) - \varepsilon$. ε is arbitrary, which proves (*a*).

b) f is bounded above so $M < \infty$ and the right-hand inequalities of (4.4.2) and (4.4.3) make sense. x_0 is semiregular so $\liminf_{x \to x_0} \Phi_h^C(x) = 0$, hence by (4.4.3) $\liminf_{x \to x_0} \Phi_f^C \leqslant f(x_0) + \varepsilon$. ε is arbitrary, which proves (*b*).

c) f is bounded so both M and m are finite, hence (4.4.2) and (4.4.3) make sense. Strong regularity implies weak C-regularity so

$$\lim_{x \to x_0} \Phi_h^C(x) = 1 \qquad \text{and} \qquad \lim_{x \to x_0} \Phi_g^C(x) = 0$$

so (4.4.3) becomes :

$$f(x_0) - \varepsilon \leqslant \liminf_{x \to x_0} \Phi_f^C(x) \leqslant \limsup_{x \to x_0} \Phi_f^C(x) \leqslant f(x_0) + \varepsilon.$$

d) If x_0 is not weakly C-regular, there is a neighborhood N of x_0 such that $\liminf_{x \to x_0} \Phi_{I_N}^C(x) < 1$. Choose a continuous f on C satisfying $f(x_0) = 1$ and $f \leqslant I_N$. Then

$$\liminf_{x \to x_0} \Phi_f^C(x) \leqslant \liminf_{x \to x_0} \Phi_{I_N}^C(x) < f(x_0).$$

If x_0 is not semiregular (strongly regular) there is a neighborhood N of x_0 such that if $g = I_{\partial D - N}$ then $\liminf_{x \to x_0} \Phi_g^{\partial D}(x) > 0$ ($\limsup_{x \to x_0} \Phi_g^{\partial D}(x) > 0$). Choose a continuous f on ∂D such that $f(x_0) = 0$ and $f \geqslant g$. Then $\Phi_f^{\partial D} \geqslant \Phi_g^{\partial D}$, so $\liminf_{x \to x_0} \Phi_f^{\partial D}(x) > f(x_0)$ ($\limsup_{x \to x_0} \Phi_f^{\partial D}(x) > f(x_0)$). Q.E.D.

4.5. Regularity and barriers.

DEFINITION. — *Let* N *be a neighborhood of* $x_0 \in \partial D$. *A function* $f \in S(N \cap D)$ *is a barrier at* x_0 *if*

a) $f \geqslant 0$.

b) $\liminf_{x \to x_0} f(x) = 0$.

c) *If* N′ *is any neighborhood of* x_0, f *is bounded away from zero on* $N \cap D - N'$.

We can characterize strongly regular and semiregular points in terms of barriers. One can show as usual that if f is a barrier at x_0, it can be extended to a barrier on all of D.

PROPOSITION 4.5.1. — *Let* $D \subset Q$ *be a normal domain, and let* $x \in \partial D$. Then :

a) x *is semiregular iff there is a barrier at* x.

b) x *is strongly regular iff there is a barrier at* x *which has a limit there.*

Proof. — We will just prove (a). (b) is similar. Define $f \in C(\partial D)$ such that $f(x) = 0$ and x is the unique minimum of f. By Theorem 4.4.1 (b), $\liminf_{y \to x} \Phi_f^{\partial D}(y) = 0$ and by Lemma 4.3.3 $\Phi_f^{\partial D}$ is bounded away from zero outside any neighborhood of x.

Conversely, suppose g is a barrier at x, and let $f(z) = \liminf_{x \to z} g(x)$ for $z \in \partial D$. Then $f(x) = 0$ and f is bounded away from zero outside of any neighborhood of x. Let N be a neighborhood of x. There is a constant n such that $nf \geqslant I_{\partial D - N} = h$. By the minimality property, $ng \geqslant \Phi_h^{\partial D}$ But $\liminf_{y \to x} g(y) = 0$, so $\liminf_{y \to x} \Phi_h^{\partial D}(y) = 0$. Q.E.D.

Using this and the fact that barriers are defined locally we have :

COROLLARY 4.5.2. — *Let* $D \subset Q$ *be a normal domain. If* x *is a semiregular (strongly regular) boundary point of* D *and* N *is a neighborhood of* x, *then* x *is a semiregular (strongly regular) boundary point of* $N \cap D$ *and conversely.*

One can show similarly that weak regularity is a local property, but as we shall have no use for this we omit it.

4.6. Examples.

If $D \subset Q$ is a domain, let D^{**} be the set of semiregular points of ∂D.

4.6.1. D = unit ball in Q. Then all boundary points are strongly regular, hence $D^{**} = \partial D$. To see this, just note that since D is strictly convex, there is a linear function which takes on its unique minimum in $\overline{D}$ at a given boundary point x. This function is continuous and multiply harmonic, so x is strongly regular.

4.6.2. $D = B(x;\ \rho)$, a polycylinder (see Section 2.) Then $D^{**} = B^*$, the distinguished boundary, and all points of $B^\star$ are strongly regular.

4.6.3. Let $Q = R \times R$ and let D be the unit disc minus a sector of central angle less than 180°, where the sector is contained in the left half-plane. A function is multiply superharmonic in D if it is concave in each coordinate. Then $D^{**} = \partial D$, and all boundary points except (0,0) are strongly regular. (0,0) is semiregular, being in the closure of strongly regular points, but is not strongly regular. To see this last, let N be any small neighborhood of (0,0) and define f on ∂D by: $f(x) = I_{\partial D - N}(x),\ x \in \partial D$.

Consider the behavior of $\Phi_f^{\partial D}(z)$ as $z \to (0,0)$ along the positive real axis. From any point $(x,0) \in D$ the process $X(t) = (Z^x(t),\ 0)$, where Z^x is 1-dimensional Brownian motion from x, is in T_x and will hit ∂D at either $(x, \sqrt{1 - x^2})$ or $(x, -\sqrt{1 - x^2})$. At both these points $f = 1$, so $\Phi_f(x,\ 0) = 1$, and hence $\limsup_{z \to (0,0)} \Phi_f(z) = 1 > 0$.

4.6.4 Let $k = 1$ and $Q = R^3$. In this case the notions of harmonic and multiply harmonic functions coincide, and the concepts of regular, strongly regular, and weakly regular boundary points are identical. Let D be a region with a Lebesgue thorn, and let x be the tip of that thorn. As is will-known, x is semiregular but not regular — and therefore is semiregular but not weakly regular.

5. Behavior in the interior.

5.1. Continuity of $\Phi_f^{\partial D}$.

In comparing our multiply superharmonic Dirichlet solutions with classical Dirichlet solutions one is led to remark that while the classical solutions « smooth out » the boundary function in that the Dirichlet solution of a badly discontinuous boundary function is always infinitely differentiable in the interior, the multiply superharmonic solutions retain much of the boundary function's roughness. The multiply superharmonic solution can be $+\infty$ at interior points, for instance, without being identically $+\infty$.

In this section we will give sufficient conditions for continuity of a Dirichlet solution, and give examples to show the Dirichlet solution of an arbitrary boundary function may not be continuous, even if the function is continuous.

As in the previous section, D is a domain in Q and $\overline{D}$ is a metrizable compactification of D in which D is an everywhere-dense subspace. All topological notions are relative to $\overline{D}$ unless it is specifically stated otherwise. If $x, y \in \overline{D}$, $\|x - y\|$ will denote their distance in the $\overline{D}$ metric. If $A \subset Q$ and $x \in Q$, we will use $x + A$ to denote the set $\{z : z = x + y,\ y \in A\}$. If f is multiply superharmonic on D, $\overline{f}$ will mean the greatest lower semicontinuous extension of f to $\overline{D}$.

THEOREM 5.1.1. — *Let* C *be a* U-*boundary for* D, *and let* $g \in I(C)$. *If* $\overline{\Phi}_g^C$ *is continuous at all points of* ∂D *and is greater than or equal to* g *on* C, *then* $\overline{\Phi}_g^C$ *is continuous on* $\overline{D}$.

Proof. — Since Φ_g^C is multiply superharmonic, hence lower semicontinuous on D, it is enough to show it is upper semicontinuous as well.

By Lemma 4.1.1 and the fact that $\overline{\Phi}_g \geqslant g$ on C, if $x \in D$ and $X \in T_x$ is such that it hits C w. p. 1, then

a) $\Phi_g^C(x) \geqslant E\{g(X(\tau_D))\}$.

By continuity of Φ_g^C on ∂D and Theorem 4.1.4, given $\varepsilon > 0$ we can find a $d > 0$, a compact set $K \subset D$, and a coherent family $V = \{X^s,\ z \in D - K\}$ such that

b) If $z, w \in \overline{D} - K$ and $\|z - w\| < d$, then

$$|\overline{\Phi}_g^C(z) - \overline{\Phi}_g^C(w)| < \frac{\varepsilon}{3}.$$

c) $X^z \in U_z(C)$, $z \in D - K$.

d) $E\{g(X^z(\tau_D))\} \geqslant \Phi_g^C(z) - \frac{\varepsilon}{3}$, $z \in D - K$.

Choose a further compact subset K' of D such that K is contained in the interior of K'. Using the fact that on D the metric $\|\ \ \|$ is equivalent to the Euclidean metric, which is homogeneous, we can see that there is a number $d' > 0$

such that for any $x \in K$, $y \in D$ and $z \in \partial K'$, that $\|x - y\| < d'$ implies that both $z + x - y \in D - K$ and $\|(z + x - y) - z\| < d$.

Now let x be any point of D; we can assume without loss of generality that $x \in K$. Choose y such that $\|y - x\| < d'$ and choose $X^y \in U_y(C)$ such that

$$e)\quad E\{g(X^y(\tau_D))\} \geqslant \Phi_g^C(y) - \frac{\varepsilon}{3}.$$

Let $X^x = X^y + x - y$. Then $X^y \in T_y$. Let $\hat{\tau}$ be the first time X^x hits the set $x - y + \partial K'$. Then

$$X^x(\hat{\tau}) = x - y + X^y(\tau_{K'});$$

This is in $D - K$ by assumption.

Now let $\hat{X}^x = \Upsilon_{\hat{\tau}}(X^x;\ V)$. Then $\hat{X}^x \in U_x(C)$ and, by (e)

$$\Phi_g^C(x) \geqslant E\{g(\hat{X}^x(\tau_D))\} \geqslant E\{\Phi_g^C(\hat{X}^x(\hat{\tau}))\} - \frac{\varepsilon}{3}.$$

By (b) and (a) applied to the family $\mathcal{E}_{\tau_{K'}}(X^y)$ we have:

$$\geqslant E\{\Phi_g^C(X^y(\tau_{K'}))\} - \frac{2\varepsilon}{3} \geqslant E\{g(X^y(\tau_D))\} - \frac{2\varepsilon}{3}.$$

By (e) this is

$$\geqslant \Phi_g^C(y) - \varepsilon.$$

Since y was any element satisfying $\|x - y\| < d'$, this shows Φ_g^C is upper semicontinuous at x. Q.E.D.

Using Theorem 4.4.1 (c) we immediately have:

Corollary 5.1.2. — *Let* D *be a normal region. If every boundary point of* ∂D *is strongly regular and* g *is continuous on* ∂D, $\overline{\Phi}_g^{\partial D}$ *is continuous on* $\overline{D}$.

5.2. Examples.

Let $Q = \mathbf{C} \times \mathbf{C}$ where $\mathbf{C}$ is the complex plane, and let v be superharmonic in $\mathbf{C}$. Define u by: $u(z_1, z_2) = v(z_1)$, $z_1, z_2 \in \mathbf{C}$. Then u is multiply superharmonic in Q, and is even plurisuperharmonic. Let D be a bounded domain in Q, and $\overline{D}$ its closure in Q. We now show that $\Phi_u^{\partial D} = u$ in D.

u is bounded below in $\overline{D}$ and satisfies $\liminf\limits_{x \to x_0} u(x) \geqslant u(x_0)$,

$x_0 \in \partial D$, so by the minimality property, $\Phi_u^{\partial D} \leqslant u$ in D. Conversely, for $z \in D$ and $a < u(z)$ there is a neighborhood $N \subset D$ of z such that $u > a$ on $\bar{N}$.

Consider the process $X \in U_z$ which is Brownian motion on Q from z until it hits ∂N, and is the process $(z_1, Z^{z_2}(t))$ from then on if it hits ∂N at (z_1, z_2), where Z^{z_2} is plane Brownian motion from z_2. (X can be defined rigorously using Υ_{τ}.) Then $u(X(\tau_D)) > a$ w. p. 1, hence $\Phi_u^{\partial D}(z) > a$; thus $\Phi_u^{\partial D} = u$ in D.

We can use this to get examples of discontinuous Dirichlet solutions and of lower envelopes of plurisuperharmonic functions of the kind studied by Bremermann and others (see Bremermann [10], Gorski [21], Kimura [26] and Kusonoki [28]). Observe that if f is plurisuperharmonic and therefore multiply superharmonic and $\liminf_{x \to x_0} f \geqslant u(x_0)$ for $x_0 \in \partial D$, then the minimality property implies $u = \Phi_u^{\partial D} \leqslant f$. u itself is plurisuperharmonic, and therefore equal to the lower envelope of functions plurisuperharmonic in D with limit inferior greater or equal to φ on ∂D.

(*a*) Let D be the unit ball and $u(z_1, z_2) = -\log|z_1|$. The Dirichlet solution of u is infinite at interior points of D without being $\equiv +\infty$. This is a counter example to a conjecture of Bremermann (Bremermann [10] 9.2).

(*b*) Let φ be a superharmonic function with infinities dense in the plane, and let $u(z_1, z_2) = \varphi(z_1)$. Then $\Phi_u^{\partial D}$ has a set of infinites dense in D.

(*c*) Let u be as above and set $\hat{u} = u \wedge M$ for a constant $M > u(0, 0)$. Then $\Phi_{\hat{u}}^{\partial D}$ has discontinuities on a set of positive Lebesgue measure in D, although the boundary function is bounded and lower semicontinuous and the boundary is strongly regular. This also shows that envelopes of Bremermann's type need not be continuous, even if they are bounded. This has been an open question (see Siciak [35]).

It is somewhat more complicated to find a region D and continuous boundary function g such that $\Phi_g^{\partial D}$ is not continuous. This can be done as follows.

Let B be the open unit ball in R^3 and let A be the unit ball in R^3 with a Lebesgue thorn; suppose the tip of this thorn is the origin. Let $Q = Q_1 \times Q_2$ where $Q_1 = R$, $Q_2 = R^3$. Denote points of Q by $z = (z^1, z^2)$, $z^i \in Q_i$. Let $D \subset Q$ be the region:

$$D = (I_1 \times A) \cup (I_{1/2} \times B), \qquad \text{where for} \qquad r > 0,$$

I_r is the interval $(-r, r)$. Let g be the function

$$g(z) = 2(1 - |z^1|)(1 - |z^2|).$$

We claim $\Phi_g^{\partial D}$ is not continuous; in fact $\limsup_{z \to 0} \Phi_g^{\partial D}(z) = 1$ while $\Phi_g^{\partial D}(0) < 1$.

To see the former, let z^2 be an interior point of $B - A$ and let $z_0 = (0, z^2) \in D$. By considering U_{z_0} processes which are four dimensional Brownian motion until leaving some small neighborhood of z_0, and are of the form $(Z(t), \zeta^2)$ from then on, where $Z(t)$ is linear Brownian motion, we see there are U_{z_0} processes which hit ∂D in arbitrarily small neighborhoods of $\left(-\frac{1}{2}, z^2\right)$ and $\left(\frac{1}{2}, z^2\right)$. Thus $\Phi_g^{\partial D}(z_0) \geqslant 1 - |z^2|$. Since $|g| \leqslant 1$ on ∂D, this implies $\limsup_{z \to 0} \Phi_g^{\partial D}(z) = 1$.

Now if Y is a process on Q_2, let τ' be the natural stopping time $\tau' = \inf \{t > 0 : Y(t) \in \partial A\}$, and undefined if there is no such t. Since the tip of the Lebesgue thorn is irregular for A, if $Z(t)$ is Brownian motion from 0, $\tau' > 0$ w. p. 1 for Z, so there is a neighborhood N^2 of 0 such that $P\{Z(\tau') \in N_2\} = \alpha > 1$. Now consider any $X \in U_0$. X can hit ∂D in the set $Q_1 \times N^2$ only if $(\Pi_2 X)(\tau') \in N^2$, where Π_2 is the projection of Q onto Q_2. By Theorem 3.1.1, $\Pi_2 X$ is a time change of Brownian motion, hence $P\{(\Pi_2 X)(\tau') \in N^2\} \leqslant \alpha < 1$. Since g is bounded away from one outside of $Q_1 \times N$, this implies $\Phi_g^{\partial D}(0) < 1$.

One can use a device of Kusunoki [28] to modify this example and get a region $\hat{D}$ in which $\Phi_g^{\hat{D}*}$ is not continuous. One does this by covering the boundary of D with a countable number of balls $\{B_n\}$ which are small enough so that the set $\hat{D} = D \cup \left(\bigcup_1^\infty B_n\right)$ is a domain and $\Phi_g^{\partial \hat{D}}$ is approximately equal to $\Phi_g^{\partial D}$ near the origin, and so is not continuous. Now any point in $\partial \hat{D}$ which is a boundary point of exactly one of the B_n will be strongly regular, for the definition of strong regularity depends only on the local behavior of $\partial\hat{D}$. Since we can choose the balls B_n in such a way that $\partial\hat{D}$ is the closure of these points, $\partial\hat{D} = \hat{D}^*$.

6. Minimal U-boundaries and the Šilov boundary.

The concept of the Šilov boundary with respect to a class of functions was introduced in connection with algebras of continuous functions and has been extended to cover semi-groups of continuous and semi-continuous functions by Arens and Singer [1], and by Siciak [34].

Certain Šilov boundaries turn out to be very natural sets on which to specify the boundary function of the Dirichlet problem; they play a fundamental role in Bauer's theory (Bauer [3]). Bremermann [10] introduced them to the extended Dirichlet problem for plurisubharmonic functions.

As before, $\overline{D}$ is a metrizable compactification of D; all topological notions in the following are relative to $\overline{D}$. Let $D \subset Q$ be a bounded domain and let $S(\overline{D})$ be the set of extended real-valued functions f defined on $\overline{D}$ which are multiply superharmonic and bounded below on D, and satisfy $f(x) = \liminf_{y \to x} f(y)$ if $x \in \partial D$.

Definition. *If* $D \subset Q$ *is a domain, the Šilov boundary* D^* *of* D *with respect to* $S(\overline{D})$ *is any set* $K \subset \overline{D}$ *satisfying:*

(S1) K *is closed and if* $f \in S(\overline{D})$, *then* f *takes on its minimum on* K.

(S2) *If* C *satisfies* (S1), *then* $C \supset K$.

The existence of D^* for a bounded domain D can be immediately deduced from a theorem of Siciak [34]; we shall get its existence for arbitrary normal regions as a by-product of the following theorem on U-boundaries.

Theorem 6.1.1. — *Let* $D \subset Q$ *be a normal domain. There exists a smallest* U-*boundary* D^{**}, *and* D^{**} *is the set of semiregular points of* ∂D.

Proof. — The proof is in two parts: First, we use Zorn's lemma to show that a minimal U-boundary exists, then we show that this minimal boundary is unique by identifying it with the set of semiregular points.

I. Let $\{C_\alpha\}_{\alpha\in I}$ be a linearly ordered (by set inclusion) set of U-boundaries. We claim that $C = \bigcap_{\alpha\in I} C_\alpha$ is a U-boundary. C is closed since each C_α is. Q has a countable base, so there is a countable subset $C_{\alpha_1} \supset C_{\alpha_2} \supset \cdots$ such that

$$\bigcap_{i=1}^{\infty} C_{\alpha_i} = \bigcap_{\alpha\in I} C_{\alpha_i}.$$

Let $\{N_n\}$ be a decreasing sequence of open sets such that N_n is a hoodneighbor of C_{α_n} and $\bigcap_{i=1}^{\infty} \overline{N}_i = C$.

Let $x \in D$. For some n, $x \in D - \overline{N}_n$. By renumbering if necessary, assume $n = 1$. We will show that there exists $\hat{X} \in U_x$ such that $P\{\hat{X}(\tau_D) \in V\} = 1$.

Let $X_1 \in U_x(C_{\alpha_1})$. Then with probability one, X_1 hits ∂D. We define $\hat{X}_1, \hat{X}_2, \ldots$ by induction as follows: $\hat{X}_1 = X_1$. Suppose $\hat{X}_n$ has been defined such that $\hat{X}_n \in U_x(C_{\alpha_n})$, so that $P\{\hat{X}_n$ hits ∂N_n before $\partial D\} = 1$.

Since $C_{\alpha_{n+1}}$ is a U-boundary, there exists a coherent family $V = \{X^y, y \in D\}$ such that for each y, $X^y \in U_y(C_{\alpha_{n+1}})$. Then let $\hat{X}_{n+1} = \Upsilon_{\tau_n}(\hat{X}_n, U_n)$, where τ_n is the first time $\hat{X}$ hits ∂N_n. Then $\hat{X}_{n+1} \in U_x(C_{\alpha_{n+1}})$ and $\hat{X}_{n+1} \sim (\tau_n)\hat{X}_n$. Using Theorem 1.4.2, let $\hat{X}$ be the process which is equivalent to each $\hat{X}_n$ until time τ_n. Then $\hat{X} \in U_x(C)$, hence $U_x(C) \neq \varphi$; C must be a U-boundary. By Zorn's Lemma, there exists a minimal U-boundary.

Uniqueness is now immediate. All U boundaries contain D^{**}, the set of semiregular points. This set is closed. If C is a minimal U-boundary such that $C \neq D^*$, let $x \in C - D^{**}$. Then x is not semiregular, hence by Proposition 4.3.2, there is a neighborhood N of x such that $C - N$ is a U-boundary, contradicting the minimality of C. Q.E.D.

Corollary 6.1.2. — *Suppose* D *is a normal domain. Then the Šilov boundary* D^* *of* D *with respect to* $S(\overline{D})$ *exists and is* D^{**}.

Proof. — Let $v \in \hat{S}(D)$ and suppose that $v \geqslant 0$ on D^{**}. D^{**} is a U-boundary by 6.1.1 so by the minimality property, $v \geqslant \Phi_0^{D^{**}} = 0$ in D, hence $v \geqslant 0$ in $\overline{D}$; moreover, D^{**} is closed, so it satisfies (S1). Conversely, if $x \in D^{**}$, x is semiregular, so there is a barrier at x. Therefore any set satisfying (S1) contains D^{**}. Q.E.D.

We now have the following characterizations of the Šilov boundary.

THEOREM 6.1.3. — *Suppose* D *is a normal domain, and* D^* *its Šilov boundary. Then*

(a) $x \in D^*$ *iff* x *is semiregular*

(b) $x \in D^*$ *iff there exists a barrier at* x

(c) D^* *is the smallest* U-*boundary.*

We remark that it is now clear that a set $C \subset \partial D$ is a U-boundary iff it is closed and contains D^*.

7. Other Dirichlet problems.

7.1. Bremermann's Dirichlet problem.

Let $D \subset Q$ be a domain and let $\overline{D}$ be a metrizable compactification of D. Suppose D is a normal domain and let K be a closed subset of ∂D. Recall that functions of $S(\overline{D})$ are defined on $\overline{D}$. If $f \in C(K)$, let $V_k(f) = \{u : u \in \hat{S}(D),\ u \geqslant f$ on $K\}$. Let $u_f = \inf\{u : u \in V_K(f)\}$.

When is it true that:

(DPI) For each $f \in C(K)$, u_f satisfies

1. $u_f \in \hat{S}(D)$

2. $u_f = f$ on K.

This is the analogue for multiply superharmonic functions of a Dirichlet problem for plurisubharmonic functions investigated by Bremermann [10]. We will call it Bremermann's Dirichlet problem, and will say that K is resolutive for Bremermann's Dirichlet problem if (DPI) obtains. Bremermann showed in his case that if D was pseudo-convex then K is resolutive iff it is a certain Šilov boundary.

The following theorem contains the analogous result for multiply superharmonic functions in arbitrary normal domains and shows that under mild restrictions on the boundary the multiply superharmonic Dirichlet solution solves Bremermann's Dirichlet problem.

If $D \supset Q$ is a normal domain with Šilov boundary D^*, we will write Φ_f^* and U_f^* instead of $\Phi_f^{D^*}$ and $U_x(D^*)$ respectively. We will speak of weak* regular points instead of weakly D^* regular points.

THEOREM 7.1.1. — *Let $D \subset Q$ be a normal domain, D^* its Šilov boundary, and $K \subset \partial D$ a closed set. Then*

(a) *A necessary condition that K be resolutive for Bremermann's Dirichlet problem is that $K = D^*$.*

(b) *A sufficient condition that D^* be resolutive for Bremermann's Dirichlet problem is that all points of D^* be weak* regular.*

(c) *Let $f \in C(D^*)$, and let $\overline{\Phi}_f^*$ be the lower semicontinuous extension of Φ_f^* to $\overline{D}$. Then a necessary and sufficient condition that $\overline{\Phi}_f^*$ solves Bremermann's Dirichlet problem for all $f \in C(D^*)$ is that all points of D^* be weak* regular.*

Proof. — (*a*) If $D^* - K \neq \varphi$, let $x_0 \in D^* - K$. K is closed so there is a neighborhood N of x_0 such that $\overline{N} \cap K = \varphi$. By 6.1.5 (b) there is a function $v \in \hat{S}(D)$ such that $v \geqslant 0$ on $D - N$ and $v(x_0) < 0$. Let $f = 0$ on K and consider $v_K(f)$. For all constants $M > 0$, $Mv \in V_K(f)$. If $x \in D$ is such that $v(x) < 0$, then, by letting $M \to \infty$ we see $\inf \{u(x); u \in V_K(f)\} = -\infty$, so Bremermann's Dirichlet problem is not solvable.

Suppose now $D^* \subset K$ but $D^* \neq K$, and let $x_0 \in K - D$. If $f \in C(K)$ satisfies $f = 0$ on D^*, $f(x_0) < 0$, then $u \in V_K(f)$ implies $u \geqslant 0$ on D^*, hence $u \geqslant 0$ on D. Thus $u_f \geqslant 0$; in particular, $u_f(x_0) \geqslant 0 > f(x_0)$. Therefore K is not resolutive.

(c) Points of D^* are semiregular by Theorem 6.1.3, so by Theorem 4.4.1 (b) and (d), if $x_0 \in D^*$: x_0 is weak* regular iff $\liminf_{x \to x_0} \Phi_f^*(x) = f(x_0)$, all $f \in C(D^*)$, and this is true iff $\Phi_f^* \in V_{D^*}(f)$, $f \in C(D^*)$. On the other hand, if $u \in V_{D^*}$ then

$\liminf_{x \to x_0} u(x) \geqslant f(x_0)$, $x_0 \in D^*$. Thus $\inf_{x \in D} u(x) \geqslant \inf_{x \in D^*} f(x) > -\infty$ so by the minimality property $u \geqslant \Phi_f^*$. Therefore $\Phi_f^* = u_f$.

Part (b) is now immediate from (c). Q.E.D.

7.2. Multiply harmonic Dirichlet solutions.

Φ_f^k is not in general multiply harmonic, and the question naturally arises as to when Φ_f^k is multiply harmonic for all $f \in C(K)$. It is, for instance, when D is a product region and K its distinguished boundary (we will treat this case in the next section) or when D is some subregion of a product region D′ such that all semiregular points of D are contained in the distinguished boundary of D′, but these appear to be nearly the only cases. It is never true, for instance, if D is non-void and $K = \partial D$.

We keep the notation of 7.1. If for each $f \in C(K)$, u_f is multiply harmonic and (DPI) obtains, we say K is h-resolutive. The question of h-resolutivity is closely connected with a general Dirichlet problem formulated by Bauer [3]; as we have stated it, it does not quite fall under Bauer's theory, for the Šilov boundary D^H of D with respect to the class of functions bounded and multiply harmonic in D, which is the boundary relevant to Bauer's formulation, may be strictly smaller than D^*. Bremermann [8], [9], and Kusunoki [28] have given examples showing that the Šilov boundaries for pluriharmonic and plurisuperharmonic functions need not be identical; using a result of Avanissian we can give an example in the present case (the construction follows that of Kusunoki). Let $Q = Q_1 + Q_2$ where $\dim Q_i \geqslant 3$, $i = 1, 2$, and let A and B be bounded domains in Q with $\bar{A} \subset B$. Let $C = B - \bar{A}$. Points of C^* are characterized by the local behavior of ∂C, so we can choose A to make $C^* \cap \partial A \neq \varphi$. If h is harmonic in C, by a theorem of Avanissian [2], it can be extended to be multiply harmonic in B; and therefore must have its infimum in C as a limiting value at some point of ∂B — hence $C^H = B^H$; in particular $C^H \cap \partial A = \varphi$ so C^H is strictly smaller than C^*.

In spite of these differences, the following two results of Bauer [3] carry over; we state them in our setting.

(a) K is h-resolutive only if $K = D^*$.

(b) A necessary condition for D^* to be h-resolutive is that for each $x \in D^*$ there is a multiply harmonic barrier which has a limit there.

It is an immediate consequence of Lemma 4.2.5 that if u_f is multiply harmonic in D, continuous on $D \cup K$, where K is a U-boundary, and equal to f on K, that $u_f = \Phi_f^K$.

THEOREM 7.2.1. — *Consider the following conditions.*

(a) Φ_f^* *is an additive function of* f *for* $f \in C(D^*)$.

(b) *For all* $x \in D$ *the distribution of* $X(\tau_D)$ *is independent of* X *for* $X \in U_x^*$.

(c) D^* *is h-resolutive.*

Then (a) and (b) are equivalent, and if D^* is weakly regular, (a), (b), and (c) are equivalent.

Proof. — Obviously $(b) \Longrightarrow (a)$. If Φ_f^* is an additive function of f, for fixed $x \in D$, $\Phi_f^*(x)$ is a linear functional of norm one on $C(D^*)$, hence there is a Borel measure μ on D^* such that

$$\Phi_f^*(x) = \int_{D^*} f(z)\mu\,(dz).$$

Suppose that for some $X \in U_x^*$ the distribution of $X(\tau_D)$ is not equal to μ. Then for some function $g \in C(D^*)$ we have either $E\{g(X(\tau_D))\} > \Phi_g^*(x)$ or $E\{-g(X(\tau_D))\} > \Phi_{-g}^*(x)$. By the definition of Φ_g^* and Φ_{-g}^*, this is impossible. Thus $(b) \Longrightarrow (a)$.

To show $(a) \Longrightarrow (c)$, let $f \in C(D^*)$. Then $\Phi_f^* + \Phi_{-f}^* = 0$ or $\Phi_f^* = -\Phi_{-f}^*$, so Φ_f^* is both multiply super- and subharmonic, hence multiply harmonic. D^* is weakly regular so if $x \in D^*$ we have both

$$\liminf_{z \to x} \Phi_f^*(z) \geqslant f(x) \qquad \text{and} \qquad \liminf_{z \to x} \Phi_{-f}^*(z) \geqslant -f(x).$$

Since $\Phi_{-f}^* = -\Phi_f^*$, this implies $f(x) = \lim_{z \to x} \Phi_f^*(z)$.

Finally, if (c) holds and $x \in D$, let X_1 and X_2 be in U_x^*. Let μ_1 and μ_2 be the distributions of $X_1(\tau_D)$ and $X_2(\tau_D)$ respectively. Then for each $f \in C(D^*)$

$$\Phi_f^*(x) = u_f(x) = \int f(z)\mu_1\,(dz) = \int f(z)\mu_2\,(dz)$$

where the first equality follows from the remarks preceding this theorem and the other equalities are applications of the minimality property. This being true for all $f \in C(D^*)$ implies $\mu_1 = \mu_2$. Q.E.D.

8. Product regions.

8.1. Dirichlet solutions.

If $Q = C^k$, and $Q_i = C$, $i = 1, \ldots, k$, and D is a polycylinder, the Dirichlet problem with boundary values specified on the distinguished boundary of D becomes a special case of one discussed by Bergman [4], for then the Bergman extended class coincides with the class of multiply harmonic functions. In this section we consider the Dirichlet problem for a more general product domain. Product domains are of special interest as they provide the primary examples of domains in which Dirichlet solutions are multiply harmonic.

The most useful compactification of such a domain is derived from the Martin boundary. For simplicity of notation we will assume the degree of decomposition k of Q is two; extensions to $k > 2$ will all be trivial. Let $Q = Q_1 \times Q_2$, where $\dim Q_i \geqslant 2$, $i = 1, 2$, and let D_i be a domain in Q_i which has a positive boundary. Let ∂D_i be its Martin boundary and set $\overline{D}_i = D_i \cup \partial D_i$, with the Martin topology, which is metrizable ([32] and [7]).

We will use the following facts about the Martin boundary ([18], [32]).

(8.1.1) If $z \in D_i$, there is a harmonic measure $\mu_i(z, d\zeta)$ on ∂D; if f is a bounded Borel function on ∂D, then $g(z) = \int \mu_i(z, d\zeta)\, f(\zeta)$ is harmonic.

(8.1.2) If $z \in D_i$ and Z^z is Brownian motion from z, $Z^z(t)$ converges to a point of the Martin boundary as $t \uparrow \tau_D$; the distribution of this point is $\mu_i(z, d\zeta)$.

Let $D = D_1 \times D_2$ and $\overline{D} = \overline{D}_1 \times \overline{D}_2$. The distinguished boundary of D is $\delta D = \partial D_1 \times \partial D_2$. All topological concepts in the following are relative to $\overline{D}$ unless specifically stated otherwise.

Given any point $z = (\xi, \eta) \in D$, we can easily construct processes in $U_z(\delta D)$. One way of doing this is as follows. Choose domains $D_i^1 \subset D_i^2 \subset \cdots \uparrow D_i$, where $\overline{D}_i^n \subset D_i^{n+1}$, $i = 1, 2$ and where $\xi \in D_1^1$, $\eta \in D_2^1$. Let Z_1 and Z_2 be independent Brownian motions on Q_1 and Q_2 from ξ and η respectively. Let τ_i^n be the first time Z_i hits ∂D_i^n, $i = 1, 2$ and set $\tau_i^0 = 0$. For each $i = 1, 2$, we define a second process $\hat{Z}_i$ by

$$\hat{Z}_i(t, \omega) = Z_i(t - n, \omega) \quad \text{if} \quad \tau_i^n(\omega) \leqslant t < \tau_i^{n+1}, \; n = 0, 1, \ldots$$

and $\hat{Z}_i(\cdot, \omega)$ is constant from $\tau_i^n(\omega) + n - 1$ to $\tau_i^n(\omega) + n$; that is, $\hat{Z}_i$ is stationary for one unit of time after hitting any ∂D_i^n for the first time, and then proceeds as Brownian motion.

Then let $X(t, \omega) = (\hat{Z}_1(t, \omega), \hat{Z}_2(t, \omega))$. This is the desired process; it converges to δD as $t \to \infty$ since each $\hat{Z}_i(t)$ converges to the boundary of D_i as $t \to \infty$; the distribution of $X(\tau_D)$ is then $\mu(z, \cdot) = \mu_1(\xi, \cdot) \times \mu_2(\eta, \cdot)$ independent of the regions D_i^n. We will call $\mu(z, \cdot)$ m-harmonic measure relative to z.

A boundary point x_0 of ∂D_i for which there exists a superharmonic barrier which has the limit zero at z_0 is called regular. A point $z = (\xi, \eta)$ of δD is called m-regular if both ξ and η are regular. The set of all non-regular points of ∂D_i is negligible; that is, there is a positive superharmonic function u_i which has the limit $+\infty$ at every non-regular point of ∂D_i [7]. It is no restriction to assume u_i is finite in D; for if x is a point at which $u_i(x) = \infty$, we can replace u_i in a small ball centered at x by the Dirichlet solution of the restriction of u_i to the boundary of the ball. The resulting function is superharmonic and has the same boundary limits as u_i. Thus there exists a finite positive multiply function u in D which has the limit $+\infty$ at all non-m-regular points of δD, namely the function defined by

$$u(z^1, z^2) = u_1(z^1) + u_2(z^2) \qquad \text{where} \qquad z^i \in D_i, \quad i = 1, 2.$$

PROPOSITION 8.1. — (a) *Let* $x \in D$ *and* $X \in U_x(\delta D)$. *Then if* G *is the set of all non-m-regular points of* δD, $P\{X(\tau_D) \in G\} = 0$.

(b) $X(t)$ *has the limit* $X(\tau_D)$ w. p. 1 *as* $t \uparrow \tau_D$.

Proof. — Let u be the function of the preceding paragraph and F the set on which it has infinite limits. Then $G \subset F$ and if f is the indicator function of F, by the minimality property, for any $\varepsilon > 0$:

$$P\{X(\tau_D) \in G\} \leqslant \Phi_f^{\delta D}(x) \leqslant \varepsilon u(x),$$

which implies (a).

To prove (b), let $x = (\xi, \eta)$ and consider the processes $\Pi_i X$, $i = 1, 2$, which are the projections of X on Q_i, $i = 1, 2$. These processes are time changes of Brownian motion by Theorem 3.1.1, hence by the fact that Brownian motion has limits at ∂D_i, $i = 1, 2$, $\Pi_i X$ has the limit $\Pi_i X(\tau_D)$, $i = 1, 2$; this implies (b).

Let f be a bounded Borel function on δD. Define a function Ψ_f on D by

$$\Psi_f(z) = \int_{\delta D} f(\zeta) u(z, d\zeta).$$

Ψ_f is easily seen to be multiply harmonic; if f is of the form $f(z) = f_1(z_1) f_2(z_2)$ this follows from (8.1.1), and the class of functions for which Ψ_f is multiply harmonic is linear and closed under bounded monotone convergence.

Proposition 8.2. — *Let $f \in C(\delta D)$. Then Ψ_f has the boundary limit f at all m-regular points.*

Proof. — If $z = (z^1, z^2)$ is m-finely regular, z^1 and z^2 are finely regular in ∂D_1 and ∂D_2 respectively, hence for each $i = 1, 2$, given $\varepsilon > 0$ and a neighborhood N_i of z^i there exists a fine neighborhood F_i such that for $\zeta \in F_i$, $\mu_i(\zeta, N_i) > 1 - \varepsilon/2$. Thus if $x \in D$ is a point such that $x \in F_1 \times F_2$, $\mu(x, N_1 \times N_2) > 1 - \varepsilon$. The conclusion now follows from the continuity of f and the fact that sets of the form $F_1 \times F_2$ form a base the topology at z. Q.E.D.

The following corollary will be strengthened later:

Corollary 8.3. — *Let $f \in C(\delta D)$, $x \in D$ and $X \in U_x(\delta D)$. Then* w. p. 1, $\lim_{t \uparrow \tau_D} \Psi_f(X(t)) = f(X(\tau_D))$.

Theorem 8.4. — *If $X \in U_x(\delta D)$, the distribution of $X(\tau_D)$ is m-harmonic measure relative to x.*

Proof. — For each $f \in C(\delta D)$, Ψ_f is multiply harmonic and bounded, hence by Lemma 4.1.1,

$$\int f(\zeta)\mu(x, d\zeta) = \Psi_f(x) = E\{\lim_{t \uparrow \tau_D} \Psi_f(X(t))\}.$$

By Corollary 8.3 this is

$$= E\{f(X(\tau_D))\} = \int f(\zeta)\nu\,(d\zeta)$$

where ν is the distribution of $X(\tau_D)$. This being true for all $f \in C(\delta D)$ implies $\nu = u(z, \cdot)$. Q.E.D.

Theorem 8.5. — *The Šilov boundary of* D *with respect to* $S(\overline{D})$ *is equal to the set of semiregular points of* δD, *which is in turn equal to the support of the m-harmonic measure* $u(z, \cdot)$, *where* z *is an arbitrary point of* D.

Proof. — This would be an immediate consequence of Theorem 6.1.1 if we had verified the separation hypothesis; since we have not, we prove it directly.

By Theorem 8.4, it is clear that the set of all semiregular points is contained in the support of m-harmonic measure; on the other hand, m-regularity implies semiregularity, hence the semiregular points are dense in the support of harmonic measure. Both sets are closed, so they must be equal.

If K is any closed set on which all functions in $S(\overline{D})$ take on their minimum, K must contain all m-regular points, for these have barriers, and so contains the support of m-harmonic measure. But $f \in S(\overline{D})$ and $f \geqslant 0$ on the support of harmonic measure implies (by the minimality property and Theorem 8.4)

$$f \geqslant \Phi_0^{\delta D} = 0 \qquad \text{on} \qquad D. \qquad\qquad \text{Q.E.D.}$$

An immediate consequence of this corollary is that if f is a bounded Borel function on δD, integrable with respect to m-harmonic measure for one (and therefore all) $z \in D$, that $\Phi_f^* = \Phi_f^{\delta D} = \Psi_f$. This last equality shows:

Theorem 8.6. — Φ_f^* *is multiply harmonic.*

The following theorem is now an immediate consequence of Theorem 8.4 and the characterization of regular points of ∂D_1 and ∂D_2 in terms of harmonic measure.

THEOREM 8.7. — *Let* $z = (\xi, \eta)$ *be a point of* δD. *Then*

(a) z *is strongly regular iff* ξ *and* η *are both regular in* ∂D_1 *and* ∂D_2.

(b) z *is semiregular iff both* ξ *and* η *are semiregular in* ∂D_1 *and* ∂D_2.

(c) *Weak* regularity and strong regularity are equivalent; a point is weakly regular (that is, weakly* ∂D*-regular) iff at least one of* ξ, η, *is regular in the corresponding* ∂D_i.

Since Φ_f^* is multiply harmonic instead of only multiply superharmonic, many of the techniques of the ordinary Dirichlet problem can be adapted. The following generalization of Corollary 8.3 is due to Doob in case $k = 1$ [15]; the proof we give is just an adaptation of his.

THEOREM 8.8 — *Let* f *be a Baire function on* δD *and suppose* Φ_f^* *exists and is finite. Then for any* $z \in D$ *and* $X \in U_x^*$, Φ_f^* *has the limit* f *on almost every* X*-path to the boundary.*

Proof. — Extend the function Φ_f^* to $D \cup \delta D$ by defining it to be equal to f on δD. Let $\hat{X}$ be the X process stopped on the boundary, i.e., $\hat{X}(t) = X(t \wedge \tau_D)$. Let V be the class of functions f such that

a) the family $\{\Phi_f^*(\hat{X}(t)),\ 0 \leqslant t < \infty\}$ is uniformly integrable and

b) $\lim\limits_{t \to \infty} \Phi_f^*(\hat{X}(t)) = f(\hat{X}(\tau_D))$ w. p. 1.

i) V contains the continuous functions by Corollary 8.3.

ii) V is linear since Φ_f^* is linear in f.

iii) Let $f_n \in V, n = 1, 2, \ldots$ and let $f_n \uparrow f$. Suppose $\Phi_f^* < \infty$ in D. We claim $f \in V$.

We suppose $\hat{X}$ is canonically defined relative to the fields $\mathcal{B}(t)$, and $\mathcal{B} = \bigvee\limits_{t \in R^+} \mathcal{B}(t)$. $\Phi_{f_n}^*$ is multiply harmonic so for each n, $\{\Phi_{f_n}^*(\hat{X}(t)), \mathcal{B}(t), t \in R^+\}$ is a martingale which converges to $f_n(\hat{X}(\tau_D))$ with probability one. It is uniformly integrable since $f \in V$, so

$$\Phi_{f_n}^*(\hat{X}(t)) = E\{f_n(\hat{X}(\tau_D)) | \mathcal{B}(t)\} \quad \text{w. p. 1.}$$

As $n \to \infty$, $f_n \uparrow f$ and $\Phi^*_{f_n} \uparrow \Phi^*_f$ so

$$\Phi^*_f(\hat{X}(t)) = E\{f(\hat{X}(\tau_D)) \mid \mathcal{B}(t))\}.$$

This implies f satisfies (a). We must show that Φ^*_f has continuous sample paths, which we shall do by showing that w. p. 1., $\Phi^*_{f_n}(\hat{X}(t))$ converges uniformly in t to $\Phi^*_f(\hat{X}(t))$.

For fixed m and n, $n > m$, $\Phi^*_{f_n}(\hat{X}(t)) - \Phi^*_{f_m}(\hat{X}(t))$ is a positive martingale, so by the maximal inequality:

$$\begin{aligned} P\{\sup_{0 \leqslant t \leqslant \infty} [\Phi^*_{f_n}(\hat{X}(t)) - \Phi^*_{f_m}(\hat{X}(t))] \geqslant \lambda\} \\ &\leqslant \frac{1}{\lambda} E\{\Phi^*_{f_n}(\hat{X}(\tau_D)) - \Phi^*_{f_m}(\hat{X}(\tau_D))\} \\ &= \frac{1}{\lambda} [\Phi^*_{f_n}(z) - \Phi^*_{f_m}(z)]. \end{aligned}$$

Since the r.h.s. goes to zero as $m, n \to \infty$, this implies uniform convergence w. p. 1, which was to be shown. Thus $f \in V$. But this implies that V contains all Baire functions whose Dirichlet solutions exist. Q.E.D.

8.2. Plurisuperharmonic Dirichlet solutions.

We have discussed only the multiply superharmonic functions in this paper, but the same methods can be applied with little or no change to other subclasses of the superharmonic functions. In particular, many of our results carry over directly to the classically important case of plurisuperharmonic functions.

Let $Q = \mathbf{C}^n$ and define $\hat{T}_x$ to be the set of continuous processes which have the supermartingale property relative to the class of plurisuperharmonic functions. The class $\hat{U}_x(C)$ is defined by substituting $\hat{T}_x$ for T_x in the definition of $U_x(C)$. The classes $\hat{T}_x$ and $\hat{U}_x(C)$ enjoy substantially the same properties as their counterparts T_x and $U_x(C)$; in particular the analogues of Theorems 3.3.1, 3.4.1, 3.4.2 and 4.1.4 remain valid, as do Propositions 3.3.3 and 4.1.2; the proofs go through with little change other than to substitute « plurisuperharmonic » for « multiply superharmonic » throughout. The plurisuperharmonic Dirichlet solution $\hat{\Phi}_f^C$ is defined as in 4.2 by substituting $\hat{U}_x(C)$ for $U_x(C)$. The proof that $\hat{\Phi}_f^C$ is plurisuperharmonic is superficially different from the proof of Theorem 4.2.1 in that plurisuperharmonic functions satisfy a different average property than multiply harmonic functions [30]; in verifying this property one makes use of the fact that if $Z(t)$ is Brownian motion from the origin of $\mathbf{C}$, and $x, y \in \mathbf{C}^n$, that $x + yZ(t) \in \hat{T}_x$.

Once the analogous characterizations of the regularity properties of boundary points have been made, the main results and proofs of Sections 4 through 7 carry over with the usual changes, that is, by substitution of « plurisuperharmonic » for « multiply superharmonic » and replacing T_x, $U_x(C)$, and Φ_f^C by $\hat{T}_x$, $\hat{U}_x(C)$ and $\hat{\Phi}_f^C$ respectively. Of these, the results numbered 4.2.5, 4.4.1, 4.5.1, 5.1.1, 5.1.2, 6.1.3 and 7.1.1 are perhaps of the most interest. Theorem 7.1.1 gives an identification of lower envelopes which leads to new results for the latter. Theorem 4.4.1 for instance is only a slight extension of known results, while Theorems 5.1.1 and 5.1.2 yield results on the continuity of lower envelopes and the non-probabilistic part of Theorem 6.1.3 gives a characterization of the Šilov boundary, which are new even when the region is bounded and has a smooth boundary. The results in Section 8 depend on specific properties of multiply harmonic functions and so don't carry over to the pluriharmonic case.

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