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INVARIANT SUBSPACES WITH NO GENERATOR
AND A PROBLEM OF H. HELSON

by Jun-ichi TANAKA (*)

Dedicated to the memory of Henry Helson

Abstract. — In the almost-periodic context, the $H_0^2$-space cannot be generated by one of its elements. Together with a cocycle argument, this implies that there exist all kinds of invariant subspaces without a single generator, from which we answer some questions on invariant subspace theory.

Résumé. — Dans le contexte presque périodique, aucun espace $H_0^2$ ne peut être engendré par un de ses éléments. En tenant compte d’un argument faisant intervenir les cocycles, on peut en déduire qu’il existe de nombreux types de sous-espaces invariants qui ne peuvent pas être engendrés par un seul de leurs éléments; ceci permet de répondre à quelques questions de la théorie des sous-espaces invariants.

1. Introduction

The theory of invariant subspaces has been developed in the context of compact abelian groups with ordered duals, which is a natural generalization of such a theory on the unit circle $\mathbb{T}$. Many classical results extend to these cases, nevertheless, one also meets new difficulties. The purpose of this paper is to resolve a longstanding problem formulated by H. Helson in the 1950s.

Let $\Gamma$ be a countable dense subgroup of the real line $\mathbb{R}$, endowed with the discrete topology. Then the dual group $K$ of $\Gamma$ is a compact abelian group that is metrizable. For $\lambda$ in $\Gamma$, it is customary to denote by $\chi_\lambda$ the character on $K$ defined by $\chi_\lambda(x) = x(\lambda)$. Let $\sigma$ be the normalized Haar
measure on $K$. A function $\phi$ in $L^1(\sigma)$ is analytic if its Fourier coefficients

\begin{equation}
(1.1) \quad a_\lambda(\phi) = \int_K \phi \overline{\lambda} d\sigma
\end{equation}

vanish for all negative $\lambda$ in $\Gamma$. The Hardy space $H^p(\sigma), 1 \leq p \leq \infty$, is defined to be the space of all analytic functions in $L^p(\sigma)$. For technical reasons, it is useful to define $H^0_0(\sigma)$ as the subspace of all $\phi$ in $H^p(\sigma)$ with $a_0(\phi) = 0$. A (weak*-?, if $p = \infty$) closed subspace $\mathcal{M}$ of $L^p(\sigma)$ is invariant if $\mathcal{M}$ contains $\chi_\lambda \mathcal{M}$ for all positive $\lambda$ in $\Gamma$. When the inclusion is strict, $\mathcal{M}$ is said to be simply invariant. Of course, both $H^p(\sigma)$ and $H^0_0(\sigma)$ are simply invariant subspaces of $L^p(\sigma)$. If $\phi$ is in $L^p(\sigma)$, and if $\mathcal{M}[\phi]$ denotes the smallest invariant subspace of $L^p(\sigma)$ containing $\phi$, then $\phi$ is called a single generator of $\mathcal{M}[\phi]$. Recall that a function of modulus one is said to be unitary and an analytic unitary function is called an inner function. We say a function $\phi$ in $H^p(\sigma)$ is outer if it satisfies that

\[ \log | a_0(\phi) | = \int_K \log | \phi | d\sigma > -\infty. \]

Let $1 \leq q \leq p \leq \infty$, and let $\mathcal{M}$ be a simply invariant subspace of $L^p(\sigma)$. It follows from the properties of outer functions that $[\mathcal{M} \cap L^\infty(\sigma)]_q \cap L^p(\sigma) = \mathcal{M}$, where $[\mathcal{M} \cap L^\infty(\sigma)]_q$ is the closure of $\mathcal{M} \cap L^\infty(\sigma)$ in $L^q(\sigma)$ (see [3, Chapter V, Section 6] for details). This fact assures that there is a one-to-one correspondence between the invariant subspaces in $L^p(\sigma)$ and those in $L^q(\sigma)$. Therefore, in dealing with invariant subspaces, we may restrict our attention to the case of $p = 2$, in which Hilbert space theory works well. It follows from Szegö’s theorem that $\phi$ is a single generator of $H^2(\sigma)$ if and only if $\phi$ is outer in $H^2(\sigma)$. However, it has been unknown for a long time whether every simply invariant subspace is singly generated or not. In the literature this has come to be known as the single generator problem (refer to [4, §5.4], [2, Remark, p.158] and [3, p.138 and p.177]). The difficulty seems to center on the case of invariant subspace $H^2_0(\sigma)$. In [6, p.183], it is raised in an equivalent form in connection with stochastic processes.

Our objective in this note is to show a negative answer to this problem in the almost periodic settings:

**Theorem.** — The invariant subspace $H^2_0(\sigma)$ cannot be generated by one of its elements.

To the best of author’s knowledge, $H^2_0(\sigma)$ is the first known example of invariant subspace which cannot be singly generated. On the other hand, by [4, §5.3, Theorem 33], it was shown that every invariant subspace is
generated by two of its elements. In more general setting, we can artificially make $H^0_2$-spaces to have a single generator.

For each $t$ in $\mathbb{R}$, let us denote by $e_t$ the element of $K$ defined by $e_t(\lambda) = e^{i\lambda t}$ for $\lambda$ in $\Gamma$. The map sending $t$ to $e_t$ embeds $\mathbb{R}$ continuously onto a dense subgroup of $K$. Define a one-parameter group $\{T_t\}_{t \in \mathbb{R}}$ of homeomorphisms on $K$ by

$$T_t x = x + e_t, \quad x \in K.$$ (1.2)

Then the pair $(K, \{T_t\}_{t \in \mathbb{R}})$ is a strictly ergodic flow, for which $\sigma$ is the unique invariant probability measure. The flow $(K, \{T_t\}_{t \in \mathbb{R}})$ is called an almost periodic flow, because if $\phi$ is continuous on $K$, then $t \to \phi(x + e_t)$ is a uniformly almost periodic function with exponents in $\Gamma$. Let $H^\infty(dt/\pi(1 + t^2))$ be the space of all boundary functions of bounded analytic functions in the upper half-plane $\mathcal{H}$, and let $H^p(dt/\pi(1 + t^2)), 1 \leq p < \infty$, be the closure of $H^\infty(dt/\pi(1 + t^2))$ in $L^p(dt/\pi(1 + t^2))$. For a function $u(x, t)$ on $K \times \mathbb{R}$, the assertion "$t \to u(x, t)$ for $\sigma$–a.e. $x$ in $K$" is sometimes abbreviated to "almost every $t \to u(x, t)$". Then $\phi$ in $L^p(\sigma)$ lies in $H^p(\sigma)$ if and only if almost every $t \to \phi(x + e_t)$ lies in $H^p(dt/\pi(1 + t^2))$. This fact enables us to define Hardy spaces on every ergodic flow (see the end of the next section).

Let $\mathcal{M}$ be a simply invariant subspace of $L^2(\sigma)$. Set $\mathcal{M}_{\lambda} = \chi_{\lambda}\mathcal{M}$ for each $\lambda$ in $\Gamma$. Define

$$\mathcal{M}_+ = \bigwedge_{\lambda < 0} \mathcal{M}_{\lambda} \quad \text{and} \quad \mathcal{M}_- = \bigvee_{\lambda > 0} \mathcal{M}_{\lambda}.$$ 

Since these spaces are at most one dimension apart, $\mathcal{M}$ coincides with either or both its versions $\mathcal{M}_+$ and $\mathcal{M}_-$. When $\mathcal{M} = \mathcal{M}_+$, $\mathcal{M}$ is said to be normalized. For $\phi$ in $L^2(\sigma)$, the subspace $\mathcal{M}[\phi]$ is simply invariant if and only if

$$\int_{-\infty}^{\infty} \log |\phi(x + e_t)| \frac{dt}{1 + t^2} > -\infty, \quad \sigma - \text{a.e. } x \in K,$$ (1.3)

(see [4, §3.3, Theorem 22]). It is well-known that there is a function $\phi$ in $L^2(\sigma)$ satisfying the inequality (1.3), while $\log |\phi|$ does not belong to $L^1(\sigma)$. Our Theorem asserts that any such function $\phi$ must satisfy $\mathcal{M}[\phi]_+ = \mathcal{M}[\phi]_-.$

A unitary Borel function $A(x, t)$ on $K \times \mathbb{R}$ is said to be a cocycle on $K$ if $A(x, t)$ satisfies the cocycle identity

$$A(x, t + s) = A(x, t) \cdot A(x + e_t, s), \quad (x, s, t) \in K \times \mathbb{R} \times \mathbb{R}.$$ 

We identify two cocycles which differ only on a set of $d\sigma \times dt$–measure zero in $K \times \mathbb{R}$. A one-to-one correspondence is established between normalized
invariant subspaces and cocycles (as discussed in [4, §2.3]). More precisely, let $\mathcal{M}$ be a simply invariant subspace of $L^2(\sigma)$ with cocycle $A(x, t)$. Then a function $\phi$ in $L^2(\sigma)$ lies in $\mathcal{M}_+$ if and only if almost every $t \to A(x, t)\phi(x + e_t)$ lies in $H^2(dt/\pi(1 + t^2))$ (see [4, §3.2]). It is easy to see that $\mathcal{M}_+ \neq \mathcal{M}_-$ if and only if $\mathcal{M}_+ = qH^2(\sigma)$ for some unitary function $q$ on $K$. Then the cocycle of $\mathcal{M}$ has the form $q(x) \cdot q(x + e_t)$, which is called a coboundary. If a cocycle is a coboundary multiplied by $\exp(i\alpha t)$ for some $\alpha$ in $\mathbb{R}$, then such a cocycle is said to be trivial. A trivial cocycle $\exp(i\alpha t)$ is not a coboundary only if $\alpha$ lies in $\mathbb{R} \setminus \Gamma$.

We already know from [5] and [10] that some singly generated subspaces have nontrivial cocycles, but we can strengthen this fact by noting the following:

**Corollary 1.1.** — Let $\mathcal{M}$ be a simply invariant subspace of $L^2(\sigma)$. If the cocycle of $\mathcal{M}$ is trivial, then $\mathcal{M}_-$ has no single generator. In other words, if $\mathcal{M}_-$ is singly generated, then the cocycle of $\mathcal{M}$ is always nontrivial, so that $\mathcal{M}_+ = \mathcal{M}_-.$

A cocycle with values in $\{-1, 1\}$ is called a real cocycle. It follows from [7] that there exist real cocycles which are nontrivial.

**Corollary 1.2.** — Let $\mathcal{M}$ be a simply invariant subspace of $L^2(\sigma)$ with real cocycle. Then $\mathcal{M}_-$ has no single generator.

A cocycle $A(x, t)$ is said to be analytic if almost every $t \to A(x, t)$ lies in $H^\infty(dt/\pi(1 + t^2))$. Then a normalized invariant subspace with analytic cocycle contains always $H^2(\sigma)$. We say that an analytic cocycle $A(x, t)$ is a Blaschke or a singular cocycle, if almost every $t \to A(x, t)$ is an inner function of that type in $H^\infty(dt/\pi(1 + t^2))$. Two cocycles are called cohomologous if one is a coboundary times the other. It is known that every cocycle is cohomologous to a Blaschke cocycle in some restricted class (see [4, §4.6, Theorem 26] and [15]). This fact makes Blaschke cocycles so important for the subject. Using our Theorem, we may answer some questions on analytic cocycles:

**Corollary 1.3.** — In the class of analytic cocycles, the following properties hold:

(a) There is a Blaschke cocycle not being cohomologous to any singular cocycle.

(b) There is a Blaschke cocycle not having exactly the same zeros as any function in $H^2(\sigma)$.
It would be helpful to understand the basic idea behind the proof of our Theorem. On the one hand, we claim that if $\phi$ is a single generator of $H_0^2(\sigma)$, then $\phi$ must have a very special form. Assume that $\Gamma$ is the smallest group determined by the nonzero Fourier coefficients of $\phi$ (see below for details). Similarly, let $\Lambda$ be the smallest group determined by the nonzero coefficients of $|\phi|$. Since $\Lambda$ is a subgroup of $\Gamma$, the dual group of $\Lambda$ is represented as $K/H$, where $H$ is the annihilator of $\Lambda$ in $K$. Let $\tau$ be the normalized Haar measure on $K/H$, and fix an element $\alpha$ in $\Gamma$ with $a_\alpha(\phi) \neq 0$. Then it can be shown that $\chi_\alpha \phi$ lies in $L^2(\tau)$ and generates the simply invariant subspace of $L^2(\tau)$ with trivial cocycle $\exp(i\alpha t)$. We also see that $\alpha$ is independent of $\Lambda$, meaning that $n\alpha$ lies in $\Lambda$ only for $n = 0$ in the integer group $\mathbb{Z}$. This implies that $K$ and $d\sigma$ are respectively identified with $K/H \times \mathbb{T}$ and $d\tau \times d\theta/2\pi$, since $H$ is regarded as $\mathbb{T}$. Thus, for each single generator $\phi$ of $H_0^2(\sigma)$, we derive that $\Gamma \neq \Lambda$. On the other hand, if $H_0^2(\sigma)$ is singly generated, we may construct a generator $\phi$ of $H_0^2(\sigma)$ with the property that $\Gamma = \Lambda$, which contradicts the existence of single generator of $H_0^2(\sigma)$.

In the next section, we establish some notation and elementary facts about invariant subspaces in the almost periodic setting. Using group characters, we develop certain properties of single generators of $H_0^2$-spaces in Section 3. In Section 4, the proof of our Theorem is provided and then Corollaries are proved by using a lemma on cocycles. We conclude the paper with some remarks in Section 5.

We refer the reader to [9], [3, Chapter VII], [4] and [14, Chapter VIII] for further details on analyticity on compact abelian groups. Basic results concerning the Hardy space theory based on uniform algebras can be found in [3, Chapter IV] and [11].

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2. Extension of almost periodic functions

It is easy to show that a function $\phi$ in $H^2(\sigma)$ is outer if and only if $a_0(\phi) \neq 0$ and almost every $t \to \phi(x + e_i)$ is outer in $H^2(dt/\pi(1 + t^2))$. A weak version of this fact stated below is often used in what follows:
LEMMA 2.1. — Let \( \mathcal{M} \) be a simply invariant subspace of \( L^2(\sigma) \) with cocycle \( A(x,t) \). A function \( \phi \) in \( L^2(\sigma) \) generates \( \mathcal{M}_- \) if and only if \( \log |\phi| \) does not lie in \( L^1(\sigma) \) and almost every \( t \to A(x,t)\phi(x+e_t) \) is outer in \( H^2(dt/\pi(1+t^2)) \). In particular, \( H^2_0(\sigma) \) is singly generated by \( \phi \) if and only if \( a_0(\phi) = 0 \) and almost every \( t \to \phi(x+e_t) \) is outer in \( H^2(dt/\pi(1+t^2)) \).

Proof. — Suppose that \( \mathcal{M}[\phi] = \mathcal{M}_- \) for \( \phi \) in \( L^2(\sigma) \). If \( \log |\phi| \) lies in \( L^1(\sigma) \), then there is a unitary function \( q \) on \( K \) such that \( \mathcal{M}[\phi] = qH^2(\sigma) \) by Szegö’s theorem. This implies that \( \mathcal{M}[\phi] \neq \mathcal{M}_- \), so \( \log |\phi| \) cannot lie in \( L^1(\sigma) \). Let \( B(x,t) \) be the analytic cocycle defined by the inner part of \( t \to A(x,t)\phi(x+e_t) \). Let \( \mathcal{M} \) be the invariant subspace with cocycle \( AB(x,t) \).

By [4, §3.2, Theorem 21], we see that \( \mathcal{M}_- \) is contained in \( \mathcal{M}_- \). On the other hand, since almost every \( t \to AB(x,t)\psi(x+e_t) \) lies in \( H^2(dt/\pi(1+t^2)) \) for each \( \psi \) in \( \mathcal{M}[\phi] \), \( \mathcal{M}_+ \) includes \( \mathcal{M}[\phi] \). This shows that \( \mathcal{M}_+ = \mathcal{M}_+ \), so \( B(x,t) \equiv 1 \). Then almost every \( t \to A(x,t)\phi(x+e_t) \) is outer in \( H^2(dt/\pi(1+t^2)) \).

Conversely, suppose that \( \mathcal{M}[\phi] \) is contained strictly in \( \mathcal{M}_- \). Then there is a nonzero function \( q \) in \( \mathcal{M}_- \) such that

\[
\int_K \psi \overline{\phi q} \, d\sigma = 0, \quad \psi \in H^\infty(\sigma).
\]

This shows that \( \phi q \) lies in \( H^1(\sigma) \), so almost every \( t \to \phi q(x+e_t) \) lies in \( H^1(dt/\pi(1+t^2)) \). Notice that \( t \to A(x,t)q(x+e_t) \) is in \( H^2(dt/\pi(1+t^2)) \).

Since

\[
\phi(x+e_t)q(x+e_t) = A(x,t)\phi(x+e_t)\overline{A(x,t)q(x+e_t)},
\]

and since \( t \to A(x,t)\phi(x+e_t) \) is outer in \( H^2(dt/\pi(1+t^2)) \), we see that almost every \( t \to A(x,t)q(x+e_t) \) is also in \( H^2(dt/\pi(1+t^2)) \). This shows that \( t \to A(x,t)q(x+e_t) \) is constant for \( \sigma - a.e. x \) in \( K \), and so is \( t \to |q(x+e_t)| \). It follows from the ergodic theorem that \( |q(x)| \) is constant. We then assume \( q \) is a unitary function on \( K \). Therefore, \( A(x,t) \) is the coboundary \( q(x)\overline{q(x+e_t)} \) and \( \mathcal{M}_- = qH^2_0(\sigma) \). Thus \( q \) does not lie in \( \mathcal{M}_- \), which is a contradiction.

The last part of assertion follows from the fact that the cocycle of \( H^2(\sigma) \) equals 1. Under the assumption that almost every \( t \to \phi(x+e_t) \) is outer in \( H^2(dt/\pi(1+t^2)) \), we see easily \( a_0(\phi) = 0 \) if and only if \( \log |\phi| \) does not lie in \( L^1(\sigma) \). Then \( \mathcal{M}[\phi] = H^2_0(\sigma) \), so the proof is complete.

Let \( L^1(dt) \) be the usual Lebesgue space on \( \mathbb{R} \). Using \( \{T_t\}_{t \in \mathbb{R}} \), one may convolve a function \( \phi \) in \( L^p(\sigma), 1 \leq p < \infty \), with a function \( f \) in \( L^1(dt) \) by
setting

$$(\phi \ast f)(x) = \int_{-\infty}^{\infty} \phi(x + e_t)f(-t) \, dt = \int_{-\infty}^{\infty} \phi(x - e_t)f(t) \, dt,$$

where the integral is a Bochner integral. When $p = \infty$, the convolution

$$(\phi \ast f)$$

is defined in the same way as the weak*-convergent integral. Under

the operation of convolution, $L^p(\sigma)$ becomes an $L^1(dt)$-module such that

$$\|\phi \ast f\|_p \leq \|\phi\|_p \|f\|_1,$$

for $f \in L^1(dt)$. The Fourier transform $\hat{f}$ of $f$ is defined by the formula

$$(2.1) \quad \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} \, dt, \quad \lambda \in \mathbb{R},$$

as usual. We see easily $a_\lambda(\phi \ast f) = a_\lambda(\phi) \hat{f}(\lambda)$, if $\lambda$ is in $\Gamma$. The Poisson

kernel $P_{ir}(t)$ for $\mathcal{H}$ is given by $P_{ir}(t) = r/\pi(t^2 + r^2)$ for an $r > 0$. If $\phi$ is

in $L^1(\sigma)$, then the convolution $\phi \ast P_{ir}$ is considered as the Poisson integral

of $t \rightarrow \phi(x + e_t)$, that is,

$$(\phi \ast P_{ir})(x + e_s) = \int_{-\infty}^{\infty} \phi(x + e_t)P_{ir}(s - t) \, dt.$$

**Lemma 2.2.** Suppose that $H^2_0(\sigma)$ is singly generated. Then we obtain

the following properties:

(a) There is a single generator of $H^2_0(\sigma)$ that is bounded.

(b) If $\phi$ is a bounded generator of $H^2_0(\sigma)$, then so is each of the functions

$\phi \ast P_{ir}$ with $r > 0$ and $\phi^n$ for $n = 1, 2, \cdots$.

**Proof.** Let $\psi$ be a single generator of $H^2_0(\sigma)$. Then there is an outer

function $h$ in $H^2(\sigma)$ such that $|h| = \min(1, |\psi|^{-1})$. From Lemma 2.1, we

deduce that the bounded function $\psi h$ generates $H^2_0(\sigma)$, thus we obtain (a).

To show (b), we observe that $t \rightarrow (\phi \ast P_{ir})(x + e_t)$ as well as $t \rightarrow \phi^n(x + e_t)$
is outer in $H^2(dt/\pi(1+t^2))$ for $\sigma$-a.e. $x$ in $K$. Since $a_0(\phi \ast P_{ir}) = a_0(\phi^n) = 0$, (b) follows from Lemma 2.1 immediately. \qed

We next introduce a local product decomposition of $K$, which is useful

for studying analytic functions on $K$. Fix a positive $\gamma$ in $\Gamma$, and let $K_\gamma$

be the closed subgroup of all $x$ in $K$ such that $\chi_\gamma(x) = 1$. Then $K_\gamma \times [0, 2\pi/\gamma)$
is identified with $K$ via the map $(y,s) \rightarrow y + e_s$. Let $\sigma_1$ be the normalized

Haar measure on $K_\gamma$. Then the probability measure $(\gamma/2\pi)d\sigma_1 \times dt$ on

$K_\gamma \times [0, 2\pi/\gamma)$ is carried by the map to $d\sigma$ on $K$. The one-parameter group

$\{T_t\}_{t \in \mathbb{R}}$ given by (1.2) is represented as

$$T_t(y, s) = (y + [(t + s)\gamma/2\pi]e_{2\pi/\gamma}, t + s - [(t + s)\gamma/2\pi][2\pi/\gamma])$$
on $K \gamma \times [0, 2\pi/\gamma)$, where $[t]$ is the largest integer not exceeding $t$. Define the homeomorphism $T$ on $K \gamma$ by $Ty = y + e_{2\pi/\gamma}$. We denote by $\mathcal{O}(\omega, T)$ the orbit of a point $\omega$ in $(K \gamma, T)$, that is, the set of all $T^n\omega$ for $n$ in $\mathbb{Z}$. Since $\mathcal{O}(\omega, T)$ is dense in $K \gamma$, the discrete flow $(K \gamma, T)$ is also a strictly ergodic flow, on which $\sigma_1$ is the unique invariant probability measure. Since $\Gamma$ is countable, $K \gamma$ is metrizable (see [14, 2.2.6]).

A function $\phi$ on $K$ has the automorphic extension $\phi^\sharp$ to $K \gamma \times \mathbb{R}$ defined by

$$\phi^\sharp(y, t) = \phi(y + [t\gamma/2\pi]e_{2\pi/\gamma}, t - [t\gamma/2\pi]2\pi/\gamma).$$

Since a function $f$ in $H^1(dt/\pi(1 + t^2))$ extends analytically to $\mathcal{H}$ by $f(s + ir) = (f * P_{\gamma})(s)$, we write

$$\phi^\sharp(y, z) = (\phi^\sharp * P_{\gamma})(y, s), \quad z = s + ir \in \mathcal{H},$$

for each $\phi$ in $H^1(\sigma)$. It is clear that $(\phi^\sharp * P_{\gamma})(y, s) = (\phi * P_{\gamma})^\sharp(y, s)$ on $K \gamma \times \mathbb{R}$.

The following is due to a property of Lebesgue sets.

**Lemma 2.3.** — If $E_1$ is a compact subset of $K \gamma$ with $\sigma_1(E_1) > 0$, then there is a closed subset $E$ of $E_1$ with $\sigma_1(E_1) = \sigma_1(E)$ such that $\mathcal{O}(\omega, T) \cap E$ is dense in $E$, for $\sigma_1 - a.e. \omega$ in $K \gamma$.

**Proof.** — Recall that the metric density of $E_1$ is 1 at $\sigma_1 - a.e. \omega$ in $E_1$, meaning that

$$\lim_{\delta \to 0} \frac{\sigma_1(E_1 \cap B(\omega, \delta))}{\sigma_1(B(\omega, \delta))} = 1,$$

where $B(\omega, \delta)$ is the open ball with center $\omega$ and radius $\delta > 0$. Define $E$ to be the closure of the set of points of $E_1$ at which the metric density of $E_1$ is 1. Clearly, we have $\sigma_1(E_1) = \sigma_1(E)$, since $E_1$ is closed. If $\sigma_1(E) = 1$, then $E = K \gamma$. Since $(K \gamma, T)$ is strictly ergodic every orbit $\mathcal{O}(\omega, T)$ is dense in $E$. Assume that $0 < \sigma_1(E) < 1$. It follows from the ergodic theorem that there is a $\sigma_1$-null set $N$ in $K \gamma$ outside which

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} I_E(T^j \omega) = \sigma_1(E),$$

where $I_E$ denotes the characteristic function of $E$. Let $H_\omega$ be the closure of $\mathcal{O}(\omega, T) \cap E$ in $K \gamma$. We claim that if $E \neq H_\omega$, then $\omega$ lies in $N$. Indeed, we see that $\sigma_1(E \setminus H_\omega) > 0$, since the metric density of $E$ does not vanish identically on $E \setminus H_\omega$. Let $p$ be a continuous function on $K \gamma$ such that $0 \leq p \leq 1$, $p \equiv 1$ on $H_\omega$, and $\int_{K \gamma} p d\sigma < \sigma_1(E)$. Since $I_E(T^j \omega) = I_{H_\omega}(T^j \omega)$
for $j$ in $\mathbb{Z}$ and since $(K_{\gamma}, T)$ is strictly ergodic, we have
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} I_E(T^j \omega) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} p(T^j \omega) = \int_{K_{\gamma}} p \, d\sigma_1 < \sigma_1(E)
\]
by [13, §4.2, Proposition 2.8]. Thus $\omega$ has to lie in the null set $N$. 

For each $\phi$ in $H^\infty(\sigma)$, there is a $\sigma_1$-null set of $K_{\gamma}$ outside which $z \to \phi^*(y, z)$ is analytic and uniformly bounded on the upper half plane $\mathcal{H}$. Recall that if a family of analytic functions is uniformly bounded, then it forms a normal family. The next proposition may be regarded as a strengthened form of Lusin’s theorem for analytic functions on $K$, so that it has some interest of its own. Here we denote by $\text{cl}(\mathcal{H})$ the closure of $\mathcal{H}$ in $\mathbb{R}^2$.

**Proposition 2.4.** — Let $\phi$ be a function in $H^\infty(\sigma)$, and let $\epsilon > 0$. Then there is a closed subset $E$ of $K_{\gamma}$ with $\sigma_1(E) > 1 - \epsilon$ having the following properties:

(a) The convolution $(\phi^* \ast P_{ir})(y, t)$ is continuous on $E \times \mathbb{R}$, for a given $r > 0$.

(b) For $\sigma_1$-a.e. $\omega$ in $K_{\gamma}$, the function $(\phi^* \ast P_{ir})(T^j \omega, z)$ on $(\mathcal{O}(\omega, T) \cap E) \times \text{cl}(\mathcal{H})$ extends to $(\phi^* \ast P_{ir})(y, z)$ on $E \times \text{cl}(\mathcal{H})$.

**Proof.** — Since $\phi \ast P_{ir}$ lies in $H^\infty(\sigma)$, Lusin’s theorem asserts that there is a compact subset $F$ of $K$ with $\sigma(F) > 1 - \epsilon^2$ on which $\phi \ast P_{ir}$ is continuous. Considering $F$ as a subset of $K_{\gamma} \times [0, 2\pi/\gamma]$, we choose a compact subset $E$ of $K_{\gamma}$ with $\sigma_1(E) > 1 - \epsilon$ such that $E$ satisfies the property of Lemma 2.3 and
\[
(2.2) \quad \frac{\gamma}{2\pi} \int_0^{2\pi/\gamma} I_F(y, s) \, ds > 1 - \epsilon, \quad y \in E.
\]
In addition, we assume that $z \to (\phi^* \ast P_{ir/2})(y, z)$ is analytic on $\mathcal{H}$ and
\[
| (\phi^* \ast P_{ir/2})(y, z) | \leq \| \phi \|_\infty, \quad y \in E.
\]
Then the family
\[
\mathcal{F} = \{ (\phi^* \ast P_{ir/2})(y, z) ; y \in E \}
\]
forms a normal family on $\mathcal{H}$. Let $\{y_n\}$ be a sequence in $E$ tending to $y$. Since $\mathcal{F}$ is normal, there is a subsequence $\{y_j\}$ of $\{y_n\}$ such that $(\phi^* \ast P_{ir/2})(y_j, z)$ converges uniformly on compact subsets of $\mathcal{H}$ to a bounded analytic function $f(z)$ on $\mathcal{H}$. Let us show that $f(z) = (\phi^* \ast P_{ir/2})(y, z)$. Indeed, we observe by (2.2) that $F \cap \{y\} \times [0, 2\pi/\gamma]$ contains an infinite compact set of the form $\{y\} \times J$. Since
\[
(\phi^* \ast P_{ir})(y, t) = (\phi^* \ast P_{ir/2})(y, t + ir/2) = f(t + ir/2), \quad t \in J,
\]
it follows from the uniqueness principle that $f(z) = (\phi^z \ast P_{ir/2})(y, z)$. This shows that if $(y_n, t_n)$ tends to $(y, t)$, then $(\phi^z \ast P_{ir})(y_n, t_n)$ tends to $(\phi^z \ast P_{ir})(y, t)$. Thus (a) holds. We notice that $(\phi^z \ast P_{ir/2})(y, z)$ is also continuous on $E \times H$.

On the other hand, by Lemma 2.3, $\mathcal{O}(\omega, T) \cap E$ is dense in $E$ for $\sigma_1 - a.e. \omega$ in $K_\gamma$. Since $(\mathcal{O}(\omega, T) \cap E) \times cl(H)$ is dense in $E \times cl(H)$ and since $(\phi^z \ast P_{ir})(y, z)$ is continuous on $E \times cl(H)$, the function $(\phi^z \ast P_{ir})(T^j\omega, t)$ on $(\mathcal{O}(\omega, T) \cap E) \times cl(H)$ extends to $(\phi^z \ast P_{ir})(y, t)$ on $E \times H$. Thus (b) follows immediately. \hfill \Box

We make some remarks on Proposition 2.4. Since $t \to \phi^t(y, t)$ lies in $H^\infty(dt/\pi(1 + t^2))$ for each $y$ in $E$, we see that $(\phi^\ast \ast P_{ir})(y, t + 2\pi/\gamma) = (\phi^\ast \ast P_{ir})(Ty, t)$. Then $E \cup TE \cup \cdots \cup T^nE$ also satisfies the properties (a) and (b) and $\sigma(E \cup TE \cup \cdots \cup T^nE)$ converges to 1, as $n \to \infty$, by the recurrence theorem (see [13, §2.3, Theorem 3.2]). However, to obtain $\phi$ itself, we need a version of Fatou’s theorem as discussed in [12, Theorem II]. Denote by $O(x, \{T_t\}_{t \in \mathbb{R}})$ the orbit of $x$ in $(K, \{T_t\}_{t \in \mathbb{R}})$. With the notation above, when $x = (y, s)$ in $K_\gamma \times [0, 2\pi/\gamma)$, we see that $O(x, \{T_t\}_{t \in \mathbb{R}}) = O(y, T) \times [0, 2\pi/\gamma)$. For $x$ in $K$, we say that $t \to (\phi \ast P_{ir})(x + e_t)$ extends to $\phi \ast P_{ir}$ if, for each $\epsilon > 0$, there is a compact subset $F = F(\epsilon, \phi)$ of $K$ with $\sigma(F) > 1 - \epsilon$ such that $\phi \ast P_{ir}$ is continuous on $F$ and $O(x, \{T_t\}_{t \in \mathbb{R}}) \cap F$ is dense in $F$. The above proof may be modified so as to apply to functions in $H^1(\sigma)$ as well.

The next lemma is an immediate consequence of Proposition 2.4.

**Lemma 2.5.** — Let $\phi$ be a function in $H^\infty(\sigma)$, and let $r > 0$. Then there is an invariant $\sigma$–null set $N = N(\phi)$ in $K$ outside which $t \to (\phi \ast P_{ir})(x + e_t)$ extends to $\phi \ast P_{ir}$.

**Proof.** — For a given $\epsilon > 0$, let $E$ be a closed subset of $K_\gamma$ with $\sigma_1(E) > 1 - \epsilon$ which has the property (a) and (b) of Proposition 2.4. Putting $F = E \times [0, 2\pi/\gamma)$, we regard $F$ as a compact subset of $K$. By (b) of Proposition 2.4, we choose an invariant null set $N' = N'(\phi)$ in $(K_\gamma, T)$ outside which $O(\omega, T) \cap E$ is dense in $E$. If we set $N = N' \times [0, 2\pi/\gamma)$, then the $\sigma$–null set $N$ satisfies the desired property. \hfill \Box

Let $\Omega$ be a compact metric space on which $\mathbb{R}$ acts as a Borel transformation group. This means that there is a one-parameter group $\{U_t\}_{t \in \mathbb{R}}$ of Borel isomorphisms on $\Omega$ such that the map $(\omega, t) \to U_t\omega$ of $\Omega \times \mathbb{R}$ to $\Omega$ is a Borel map. The pair $(\Omega, \{U_t\}_{t \in \mathbb{R}})$ is referred to as a Borel flow. Especially, $(\Omega, \{U_t\}_{t \in \mathbb{R}})$ is called a continuous flow, if $U_t$ is a homeomorphism on $\Omega$ and the map $(\omega, t) \to U_t\omega$ is continuous on $\Omega \times \mathbb{R}$. We often write $\omega + t$ as $U_t\omega$. An immediate application of Lemma 2.5 is the following.

**Corollary.** — Let $\mathcal{C}$ be a $\sigma$–null set in $K$ and $\mathcal{O}$ a measurable subset of $K$ with $\mathcal{O} \cap \mathcal{C}$ a $\sigma$–null set. Then $(\mathcal{O} \cap \mathcal{C}) \cup \mathcal{O}$ is measurable. For $\omega \in \mathcal{C}$, $t \to (\phi \ast P_{ir})(x + e_t)$ extends to $\phi \ast P_{ir}$ on $(\mathcal{O} \cap \mathcal{C}) \cup \mathcal{O}$.

**References.** — [12, Theorem II].
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for the translate $U_t\omega$ of $\omega$ by $t$. Let $\mu$ be an invariant probability measure on $(\Omega, \{U_t\}_{t \in \mathbb{R}})$ which is ergodic, meaning that $\mu(E) = 1$ or 0 for each invariant subset $E$ of $\Omega$. A function $\phi$ in $L^1(\mu)$ is analytic if $t \rightarrow \phi(\omega + t)$ lies in $H^1(dt/\pi(1 + t^2))$ for $\mu$-a.e. $\omega$ in $\Omega$. Then the ergodic Hardy space $H^p(\mu), 1 \leq p \leq \infty$, is defined to be the space of all analytic functions in $L^p(\mu)$. It follows from [11, Theorem I] that $\mu$ is a representing measure for $H^\infty(\mu)$, for which $H^\infty(\mu)$ is a weak*-Dirichlet algebra in $L^\infty(\mu)$. This fundamental result enables us to apply the Hardy space theory based on uniform algebras, and most of the machinery of invariant subspaces on an almost periodic flow $(K, \{T_t\}_{t \in \mathbb{R}})$ can be reconstructed (see [1], [11] and [12] for related topics). As we mentioned earlier, the $H^2_0$-spaces may be singly generated in the situation of ergodic flows other than almost periodic flows (see [16] and §5 (b)).

Let $A(x, t)$ be a cocycle on an almost periodic flow $(K, \{T_t\}_{t \in \mathbb{R}})$ and define the Borel flow $(K \times \mathbb{T}, \{S_t\}_{t \in \mathbb{R}})$ by

$$S_t(x, e^{i\theta}) = (T_t x, A(x, t)e^{i\theta}), \quad (x, e^{i\theta}) \in K \times \mathbb{T},$$

which is called the skew product of $K$ and $\mathbb{T}$ induced by $A(x, t)$. Then $d\sigma \times d\theta/2\pi$ is an invariant probability measure on $K \times \mathbb{T}$. Observe that each function $f$ in $L^2(d\sigma \times d\theta/2\pi)$ is represented as

$$f(x, e^{i\theta}) = \sum_{n=-\infty}^{\infty} \phi_n(x)e^{in\theta},$$

where the coefficients $\phi_n$ are in $L^2(\sigma)$. From this fact, it follows easily that $d\sigma \times d\theta/2\pi$ is ergodic on $(K \times \mathbb{T}, \{S_t\}_{t \in \mathbb{R}})$ if and only if $A(x, t)^n$ is a coboundary only for $n = 0$ (see [4, §6.2] for details).

### 3. Approximation to generators

We now turn to the structure of compact group $K$, under the assumption that $H^2_0(\sigma)$ is singly generated by $\phi$ in $H^2_0(\sigma)$. By multiplying by a suitable outer function, if necessary, we can assume that $\phi$ is a function in $L^\infty(\sigma)$ with $1 \leq \|\phi\|_\infty < \infty$. Furthermore, we also assume that $\Gamma$ is the smallest group containing all $\lambda$ such that $a_\lambda(\phi) \neq 0$, that is, the smallest group over which Fourier series,

$$\phi(x) \sim \sum_{\Gamma \ni \lambda \neq 0} a_\lambda(\phi)\chi_\lambda(x),$$
Let \( \| \) function in \( H \) we obtain a sequence \( g \). Theorem shows that
\[
\| \| \rightarrow 0
\]
is represented on \( \Gamma \), by considering the Taylor series of \( z \to \log z \) at a large positive. This shows that \( \Lambda \) is a subgroup of \( \Gamma \), since
\[
a_\Lambda(\| \|) = \lim_{\epsilon \to +0} a_\Lambda \left( (\| \| + \epsilon)^{1/2} \right)
\]
by (1.1). Since \( \log \| \| \) does not lie in \( L^1(\sigma) \), the generator \( \phi \) cannot be periodic in \( (K, \{ T_t \}_{t \in \mathbb{R}} \). Then \( \Gamma \) as well as \( \Lambda \) is a countable dense subgroup of \( \mathbb{R} \), endowed with discrete topology. Let \( H \) be the annihilator of \( \Lambda \), meaning that \( H \) is the closed subgroup of all \( x \) in \( K \) such that \( \chi_\Lambda(x) = 1 \) for all \( \lambda \) in \( \Lambda \). Then the dual group of \( \Lambda \) is identified with the quotient group \( K/H \) (see [14, 2.1]). We denote by \( \tau \) the normalized Haar measure on \( K/H \). Let \( \tau \) be the canonical homomorphism of \( K \) onto \( K/H \). For each \( x \) in \( K \), we write \( \bar{x} \) for \( x + H \). When a function \( \psi \) on \( K \) is represented as \( \psi = \tilde{\psi} \circ \pi \) for a function \( \tilde{\psi} \) on \( K/H \), we usually identify \( \psi \) with \( \tilde{\psi} \), so that \( \psi(x) = \tilde{\psi}(x) \). Then we say descriptively that \( \psi \) is generated by a function on \( K/H \). If \( 1 \leq p \leq \infty \), then \( L^p(\tau) \) and \( H^p(\tau) \) are subspaces of \( L^p(\sigma) \) and \( H^p(\sigma) \), respectively.

Since almost every \( t \to \phi(x + e_t) \) is outer in \( H^\infty(dt/\pi(1 + t^2)) \) by Lemma 2.1, we see that
\[
-\infty < \log |(\phi * P_{ir})(x)| = (\log |\phi| * P_{ir})(x)
\]
for a given \( r > 0 \). Since \( \log |\phi| \) is not in \( L^1(\sigma) \) and \( \log |\phi| \leq \|\phi\|_\infty \), Fubini’s theorem shows that
\[
\int_K \log |\phi * P_{ir}| d\sigma = \int_K (\log |\phi| * P_{ir}) d\sigma = \int_K \log |\phi| d\sigma = -\infty.
\]
Let \( g = \phi * P_{ir} \). Then Lemma 2.1 shows that \( g \) is also a bounded generator of \( H^2_2(\sigma) \). Since \( \tilde{P}_{ir}(\lambda) = e^{-r|\lambda|} \) by (2.1), we obtain \( a_\Lambda(g) = a_\Lambda(\phi * P_{ir}) = a_\Lambda(\phi)e^{-r|\lambda|} \), hence \( a_\Lambda(\phi) \neq 0 \) if and only if \( a_\Lambda(g) \neq 0 \). Thus the generator \( g \) plays the same role as \( \phi \). For \( n = 1, 2, \ldots \), we then denote by \( \phi_n \) the outer function in \( H^\infty(\tau) \) with \( |\phi_n| = \max(1/n, |\phi|) \). Since \( -\log n \leq \log |\phi_n| \leq \|\phi\|_\infty \), each \( \phi_n^{-1} \) is also an outer function in \( H^\infty(\tau) \). Putting \( g_n = \phi_n * P_{ir} \), we obtain a sequence \( \{ g_n \} \) of outer functions in \( H^\infty(\tau) \) with \( \|g_n\|_\infty \leq \|\phi\|_\infty \). Notice that \( t \to g(x + e_t) \) and \( t \to g_n(x + e_t) \) extend analytically up to \( \{ Re z > -r \} \). Let us look into the relation between \( g \) and \( g_n \). Since
\[
|g_n(x)| = \exp\{(\log |\phi_n| * P_{ir})(x)\},
\]
we obtain

\[(3.1) \quad |g_1(x)| \geq |g_2(x)| \geq \cdots \geq |g_n(x)| \rightarrow |g(x)|, \quad n \rightarrow \infty,
\]

for \( \sigma - a.e. x \) in \( K \). Although \( g \) may not be in \( L^\infty(\tau) \), we observe that

\[|g_n(x)| = |g_n(\bar{x})| \quad \text{and} \quad |g(x)| = |g(\bar{x})|.\]

By (3.1), it is easy to see that almost every \( t \rightarrow |(g/g_n)(x + e_t)| \) converges pointwise to 1 on \( \mathbb{R} \). Let \( G_n^x(t) = g_n(x + e_t) \) and \( G^x(t) = g(x + e_t) \). Let \( N_0 \) be an invariant null set in \( K \) outside which the property of Lemma 2.5 holds simultaneously for all \( \phi_n \). Moreover, for \( x \) in \( K \setminus N_0 \), we may assume \( G_n^x(t) \) and \( G^x(t) \) are outer functions in \( H^\infty(dt/\pi(1 + t^2)) \). Then the family of all analytic extensions \( G_n^x(z) \) of \( G_n^x(t) \) to \( \{Re z > -r\} \) forms a normal family, since \( |G_n^x(z)| \leq ||\phi||_\infty \).

The following lemma is crucial in our proof of the Theorem.

**Lemma 3.1.** — For a bounded generator \( \phi \) of \( H_0^2(\sigma) \), let \( \Lambda \), \( H \) and \( \tau \) be as above. Choose an \( \alpha \) in \( \Gamma \) with \( a_\alpha(\phi) \neq 0 \). Then \( \overline{\alpha}\phi \) is generated by a function on \( K/H \), so lies in \( L^\infty(\tau) \). Consequently, \( \Gamma \) is generated by \( \Lambda \) and \( \alpha \).

**Proof.** — Let \( \{\delta_k\} \) be a decreasing sequence tending to 0. Then there is a sequence \( \{f_k\} \) in \( L^1(dt) \) such that \( \hat{f}_k(\alpha) = 1, \|f_k\|_1 = 1 \) and \( \hat{f}_k = 0 \) outside \( (\alpha - \delta_k, \alpha + \delta_k) \), by modifying the function \( t \rightarrow (1/\pi) \sin^2 t/t^2 \) in \( L^1(dt) \). Since \( a_\lambda(g) = a_\lambda(\phi)e^{-r|\lambda|} \), we see that \( \overline{\alpha}\phi \) lies in \( L^2(\tau) \) if and only if so does \( \overline{\alpha}g \). Thus we may replace \( \phi \) with \( g \) in our argument. Since \( a_\lambda(g * f_k) = a_\lambda(g)\hat{f}_k(\lambda) \), we observe that

\[\|g * f_k - a_\alpha(g)\chi_\alpha\|_2^2 = \sum_{0 < |\lambda| < \delta_k} |a_{\alpha + \lambda}(g)\hat{f}_k(\alpha + \lambda)|^2 \rightarrow 0, \quad k \rightarrow \infty,\]

by the Parseval theorem and that

\[\|g * f_k - a_\alpha(g)(\overline{\alpha}g)\|_2 \leq \|g * f_k - a_\alpha(g)\chi_\alpha\|_2 \|g\|_\infty.\]

From these facts, we conclude that if each \( (g * f_k)g \) lies in \( L^\infty(\tau) \), then so does \( \overline{\alpha}g \). Since the outer function \( \phi_n \) lies in \( L^\infty(\tau) \), so do \( g_n \) and \( g_n * f_k \). Then each \( (g_n * f_k)g_n \) lies in \( L^\infty(\tau) \). Let us show that the sequence \( \{(g_n * f_k)g_n\} \) converges to \( \{(g * f_k)g\} \) in \( L^2(\tau) \), from which we obtain that \( (g * f_k)g \) lies in \( L^\infty(\tau) \). Indeed, in the notation above, if we fix an \( x \) in \( K \setminus N_0 \), there is a subsequence \( \{g_{n_m}\} \) of \( \{g_n\} \) such that \( \{G_{n_m}^x(t)\} \) converges pointwise to \( e^{it}\gamma G^x(t) \) in \( H^\infty(dt/\pi(1 + t^2)) \) with \( 0 \leq \gamma < 2\pi \), where \( \gamma \) depends on \( x \) and \( \{g_{n_m}\} \). This implies that

\[(g_{n_m} * f_k)(x + e_t) \rightarrow e^{-it\gamma}(g * f_k)(x + e_t), \quad m \rightarrow \infty,\]
pointwise in $L^\infty(dt/\pi(1+t^2))$. Note that every subsequence of \( \{g_n\} \) contains such a subsequence \( \{g_m\} \). Since $e^{-i\gamma}e^{i\gamma} = 1$, the sequence \( \{g_n\} \) itself satisfies

$$
(g_n * f_k)g_n(x + e_t) \to (g * f_k)g(x + e_t), \quad n \to \infty,
$$

pointwise in $L^\infty(dt/\pi(1+t^2))$. Since

$$
\| (g_n * f_k)g_n \|_\infty \leq \| g_n \|_\infty^2 \| f_k \|_1 \leq \| \phi \|_\infty^2 \| f_k \|_1,
$$

it follows from the bounded convergence theorem that

$$
\| (g_n * f_k)g_n - (g * f_k)g \|_2 \to 0, \quad n \to \infty,
$$

so that $(g * f_k)g$ lies in $L^\infty(\tau)$. Therefore, $\chi_{\alpha}g$ as well as $\chi_{\alpha}\phi$ is generated by a function on $K/H$. On the other hand, by the property of $\Gamma$, each element in $\Gamma$ has the form $\lambda + n\alpha$ for $\lambda$ in $\Lambda$ and $n$ in $\mathbb{Z}$, thus the proof is complete.

Recall that $K/H$ coincides with the dual group of $\Lambda$. Let $\alpha$ be as in Lemma 3.1 and let $C(\bar{x}, t)$ be the trivial cocycle on $K/H$ defined by $C(\bar{x}, t) = \exp(i\alpha t)$. Since $\alpha$ is positive, $C(\bar{x}, t)$ is an analytic cocycle. We denote by $(K/H \times \mathbb{T}, \{S_t\}_{t \in \mathbb{R}})$ the skew product of $K/H$ and $\mathbb{T}$ induced by $C(\bar{x}, t)$, which is the continuous flow obtained by

$$
S_t(\bar{x}, e^{i\theta}) = (T_t\bar{x}, C(\bar{x}, t)e^{i\theta}), \quad (\bar{x}, e^{i\theta}) \in K/H \times \mathbb{T}.
$$

Then $d\tau \times d\theta/2\pi$ is the invariant probability measure on $K/H \times \mathbb{T}$ (see the end of the preceding section). Let us represent the generator $g$ and all the limits of subsequences of $\{g_n\}$ on $K/H \times \mathbb{T}$, which is the smallest product group with such property. Each function $\psi$ on $K/H$ extends naturally to the one on $K/H \times \mathbb{T}$ by setting $\psi(\bar{x}, e^{i\theta}) = \psi(\bar{x})$. Since $|g|$ and $g_n$ are functions on $K/H$, they belong to $L^\infty(d\tau \times d\theta/2\pi)$.

With the above notation, we fix a $w$ in $K \setminus N_0$. Since $G_n^w(t)$ and $G^w(t)$ are outer functions in $H^2(dt/\pi(1+t^2))$ which extend analytically to $\{Re z > -r\}$, we may assume that $G_n^w(t)$ converges pointwise to $G^w(t)$ on $\mathbb{R}$, by multiplying each $g_n$ by a suitable constant of modulus one. By regarding Lemma 2.5, the functions $G_n^w(t)$ and $G^w(t)$ extend to $g_n$ and $g$, respectively. However, we obtain the following:

**Lemma 3.2.** — For $\sigma - a.e. x$ in $K$, $G_n^x(t)$ never converges pointwise on $\mathbb{R}$. Consequently, we find two subsequences $\{g_m\}$ and $\{g_k\}$ of $\{g_n\}$ such that $G_m^x(t)$ and $G_k^x(t)$ converge to $e^{i\beta}G^x(t)$ and $e^{i\gamma}G^x(t)$ with $0 \leq \beta < \gamma < 2\pi$, respectively.
Proof. — Since $1/n \leq |g_n(x)| \leq \|\phi\|_{\infty}$, each $g_n^{-1}$ is also an outer function in $H^\infty(\sigma)$. This implies that almost every $t \rightarrow (g/g_n)(x + e_t)$ is an outer function in $H^\infty(dt/\pi(1 + t^2))$. Furthermore, since

$$a_0(g/g_n) = \int_K g/g_n \, d\sigma = \int_K g \, d\sigma \int_K g_n^{-1} \, d\sigma = 0,$$

Lemma 2.1 assures that each $g/g_n$ is also a single generator of $H^2_0(\sigma)$. If $K/H$ function in of modulus one on $P$. Since $\beta/2\pi$ is rational, then the order of $Z(\beta)$ is finite. Fix two points $w$ and $x$ in $K \setminus N_0$. We assume by Lemma 3.2 that a subsequence $\{g_k\}$ of $\{g_n\}$ satisfies that $G_k^w(t)$ and $G_k^x(t)$ converge respectively to $e^{ij\beta}G^w(t)$ and $e^{i(j+1)\beta}G^x(t)$ for $j \in \mathbb{Z}$, by multiplying each $g_k$ by a suitable constant of modulus one. Denote by $O(\bar{w})$ the orbit $O(\bar{w}, \{T_t\}_{t \in \mathbb{R}})$ of $\bar{w}$ in $(K/H, \{T_t\}_{t \in \mathbb{R}})$. Then $g$ is determined naturally on $O(\bar{w}) \times Z(\beta)$ and $O(\bar{x}) \times Z(\beta)$ to represent the limits of the subsequence $\{g_k\}$ of $\{g_n\}$ on them. For each $m \in \mathbb{Z}$, we see also that every limit of $\{g_k^m\}$ is represented on these product subsets.

If $\ell$ is a positive integer, then $g^\ell$ as well as $\phi^\ell$ is also a bounded generator of $H^2_0(\sigma)$ by Lemma 2.2. We choose an invariant null set $N(\ell)$ including $N_0$ outside which a subsequence $\{G^x_0(t)^\ell\}$ of $\{G^x_n(t)^\ell\}$ converges to $e^{i\gamma}G^x(t)^\ell$ with $0 < \gamma < 2\pi$. Define the invariant null set $N_1$ by $N_1 = \cup_{\ell=1}^{\infty} N(\ell)$.

When $\ell = m!$, we take again a subsequence $\{G^x_k(t)\}$ of $\{G^x_j(t)\}$ converging to $e^{i\beta(m)}G^x(t)$ with $e^{i\beta(m)} = e^{i\gamma}$. Then the order of $Z(\beta(m))$ is larger than $m$, so $\cup_{m=1}^{\infty} Z(\beta(m))$ is dense in $\mathbb{T}$. Therefore, to represent $g$ and all the limits of subsequences of $\{g_n\}$ on each orbit, the product group $K/H \times \mathbb{T}$ is the smallest one. Let us explain the meaning more precisely. Under the assumption of Lemma 3.1, we put $h_\alpha = \overline{x_\alpha}g$. Then $h_\alpha$ lies in $L^2(\tau)$. Define the group character $\mathcal{P}_\alpha$ of $K/H \times \mathbb{T}$ by the projection $\mathcal{P}_\alpha(\bar{x}, e^{i\theta}) = e^{i\theta}$. Since

$$(h_\alpha \mathcal{P}_\alpha)(S_t(\bar{x}, e^{i\theta})) = h_\alpha(\bar{x} + e_t)C(\bar{x}, t)e^{i\theta} = h_\alpha(\bar{x} + e_t)e^{i\alpha t}e^{i\theta},$$

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the function \( t \to (h_{\alpha}p_{\alpha})(s_{t}(\bar{x}, e^{i\theta})) \) is an outer function in \( H^{\infty}(dt/\pi(1 + t^2)) \) for \( dt \times d\theta/2\pi - \text{a.e.} \) in \( K/H \times \mathbb{T} \). Then the outer function \( G^{x}(t) \) equals \( t \to (h_{\alpha}p_{\alpha})(s_{t}(\bar{x}, e^{i\theta})) \) for some \( \theta \) with \( 0 \leq \theta < 2\pi \). In order to represent consistently all kinds of limits of subsequences \( \{G_{k}^{x}(t)\} \), we require the family of all outer functions \( t \to (h_{\alpha}p_{\alpha})(s_{t}(\bar{x}, e^{i\theta})) \) with \( 0 \leq \theta < 2\pi \).

**Lemma 3.3.** — Let \( \Gamma \) and \( \Lambda \) be as above. Then \( \Lambda \) cannot be equal to \( \Gamma \).

**Proof.** — Let \( \alpha \) be as in Lemma 3.1. Then \( \alpha \) lies in \( \Lambda \) if and only if \( \Lambda = \Gamma \). We suppose, on the contrary, that \( \alpha \) lies in \( \Lambda \). Since \( K/H = K \), let us consider the skew product \( (K \times T, \{s_{t}\}_{t \in \mathbb{R}}) \) of \( K \) and \( T \) induced by the cocycle \( C(x, t) = e^{i\alpha t} \). We use freely the notation above. Since

\[
\mathcal{F}(x, e^{i\theta}) = (\chi_{\alpha}p_{\alpha})(x, e^{i\theta}), \quad (x, e^{i\theta}) \in K \times T,
\]

is an invariant function that is not constant, \( d\sigma \times d\theta/2\pi \) is not an ergodic measure on \( (K \times T, \{s_{t}\}_{t \in \mathbb{R}}) \). Now \( K \) is represented as the local product decomposition \( K_{\alpha} \times [0, 2\pi/\alpha) \), in which \( K_{\alpha} \) is the closed subgroup of all \( x \) in \( K \) such that \( \chi_{\alpha}(x) = 1 \). If we put

\[
\mathcal{G}(x, e^{i\theta}) = h_{\alpha}(x)p_{\alpha}(x, e^{i\theta}), \quad (x, e^{i\theta}) \in K \times T,
\]

then, for each \( x = (y, s) \) in \( K_{\alpha} \times [0, 2\pi/\alpha) \), the equation

\[
(3.2) \quad \mathcal{G}(s_{t}(x, e^{i\theta})) = e^{i(\theta + \alpha t)}h_{\alpha}(x + e_{t}) = e^{i(\theta - \alpha s)}g(x + e_{t})
\]

holds, since \( e^{i(\theta + \alpha t)}\chi_{\alpha}(y + e_{s} + e_{t}) = e^{i(\theta - \alpha s)} \) and \( h_{\alpha} = \chi_{\alpha}g \). By regarding \( T \) as the interval \( [0, 2\pi/\alpha) \), \( K \times T \) is identified with \( K_{\alpha} \times [0, 2\pi/\alpha) \times [0, 2\pi/\alpha) \). Let \( E \) be the subset of \( K \times T \) defined by

\[
E = K_{\alpha} \times \{(s, s) ; 0 \leq s < 2\pi/\alpha\}.
\]

Then \( E \) is a closed invariant set in \( (K \times T, \{s_{t}\}_{t \in \mathbb{R}}) \), for which \( (K, \{T_{t}\}_{t \in \mathbb{R}}) \) is isomorphic to \( (E, \{s_{t}\}_{t \in \mathbb{R}}) \) via the map \( (y, s) \to (y, s, s) \). We see also that the ergodic measure \( d\sigma \) is carried to \( (\alpha/2\pi)d\sigma_{1} \times ds \) on \( E \) by this map, where \( \sigma_{1} \) is the normalized Haar measure on \( K_{\alpha} \). We regard \( g_{n} \), \( g \) and \( h_{\alpha} \) as the functions on \( (K \times T, \{s_{t}\}_{t \in \mathbb{R}}) \). Recall that almost every \( G_{n}^{x}(t) \) and \( G^{x}(t) \) are outer functions in \( H^{\infty}(dt/\pi(1 + t^2)) \).

Let \( x \) be in \( K \backslash N_{1} \) and let \( \{g_{k}\} \) be a subsequence of \( \{g_{n}\} \) such that \( G_{k}^{x}(t) \) converges pointwise to \( t \to e^{i\alpha \beta}e^{i\alpha t}h_{\alpha}(x + e_{t}) \) with \( 0 \leq \beta < 2\pi/\alpha \). Notice that \( t \to e^{i\alpha \beta}e^{i\alpha t}h_{\alpha}(x + e_{t}) \) is an outer function in \( H^{\infty}(dt/\pi(1 + t^2)) \) and that \( |h_{\alpha}(x + e_{t})| = |g(x + e_{t})| \). Let \( x = (y, s) \) in \( K_{\alpha} \times [0, 2\pi/\alpha) \) as above.
Since $x$ may be replaced by any point in the orbit $\mathcal{O}(x)$ of $x$, we consider $x$ as a function of $s$ on $[0, 2\pi/\alpha)$. It follows from (3.2) that
\[
e^{i\alpha}e^{iat}h_\alpha(y + e_s + e_\ell) = e^{i\alpha(\beta - s)}\mathcal{G}(S_\ell(y + e_s, e^{ias})),
\]
\[(s, t) \in [0, 2\pi/\alpha) \times \mathbb{R}.
\]
Putting $t = 0$ and replacing $y$ with $y + e_{[s\alpha/2\pi]}$, if necessary, we observe that
\[
e^{i\alpha(\beta - s)}\mathcal{G}(y + e_s, e^{ias}) = e^{i\alpha}e^{-ias}G_y(s), \quad s \in \mathbb{R}.
\]
This shows that $G^\mu_n(s)$ converges pointwise to $s \rightarrow e^{i\alpha}e^{i\alpha_{(\lambda, n)}}G_y(s)$, which cannot be an outer function in $H^\infty(dt/\pi(1 + t^2))$. Hence any subsequence of $\{G^\mu_n(t)\}$ cannot converge to an outer function in $H^\infty(dt/\pi(1 + t^2))$ for \(\sigma - a.e. x \in K\). Thus we have a contradiction.

In view of Lemma 3.3, we know that there are two possibilities in relation to $\alpha$ and $\Lambda$. Either $n\alpha$ lies in $\Lambda$ only for $n = 0$ or $\ell\alpha$ lies in $\Lambda$ for an integer $\ell \geq 2$. We claim that the latter case cannot occur, meaning that $\alpha$ is independent to $\Lambda$.

**Lemma 3.4.** — Let $\Lambda$, $H$ and $\alpha$ be as above. Then $n\alpha$ lies in $\Lambda$ if and only if $n = 0$ in $\mathbb{Z}$. Consequently, $H$ is isomorphic to $\mathbb{T}$, so that $K$ and $d\sigma$ are identified with $K/H \times \mathbb{T}$ and $d\tau \times d\theta/2\pi$, respectively.

**Proof.** — Suppose that $\ell\alpha$ lies in $\Lambda$ for some $\ell \geq 2$. By Lemma 2.2, $\phi^\ell$ is also a bounded generator of $H^2_\alpha(\sigma)$. It follows from Lemma 3.2 that $\chi_{\ell\alpha}$ and $(\overline{\chi_{\alpha}}\phi)^\ell$ lie in $L^2(\tau)$, so does $\phi^\ell$ itself. Let $\Gamma_\ell$ and $\Lambda_\ell$ be the smallest groups determined by the nonzero Fourier coefficients of $\phi^\ell$ and $|\phi^\ell|$ as above. Then they both are subgroups of $\Lambda$. On the other hand, since
\[
a_\lambda(|\phi|) = \lim_{\epsilon \rightarrow 0^+} a_\lambda \left( (|\phi|^\epsilon + \epsilon)^{1/\epsilon} \right),
\]
each $\lambda$ in $\Lambda$ with $a_\lambda(|\phi|) \neq 0$ lies in $\Lambda_\ell$. This implies that $\Lambda = \Lambda_\ell = \Gamma_\ell$. By replacing $\phi$ with $\phi^\ell$ in Lemma 3.3, this gives a contradiction. Thus $n\alpha$ lies in $\Lambda$ if and only if $n = 0$.

Since $C(\bar{x}, t)$ is a coboundary only for $n = 0$, the measure $d\tau \times d\theta/2\pi$ is ergodic on $(K/H \times \mathbb{T}, \{S_\ell\}_{t \in \mathbb{R}})$. Define the isomorphism of $\Lambda \times \mathbb{Z}$ onto $\Gamma$ by
\[
g(\lambda, n) = \lambda + n\alpha, \quad (\lambda, n) \in \Lambda \times \mathbb{Z}.
\]
Then the conjugate map $g^* \circ g$ is given by $g^* \circ g = (\bar{x}, e^{i\theta})$ on $K$, where $\chi_\alpha(x) = e^{i\theta}$. Indeed, we observe that
\[
\chi_{\lambda}(\bar{x})e^{in\theta} = \langle (\lambda, n), (\bar{x}, e^{i\theta}) \rangle = \chi_{\lambda + n\alpha}(x) = \chi_{\lambda}(\bar{x})\chi_\alpha(x)^n,
\]
for each \((\lambda, n)\) in \(\Lambda \times \mathbb{Z}\). Via the map \(g^*\), \(K\) is identified with \(K/H \times \mathbb{T}\), and \(d\tau \times d\theta / 2\pi\) is carried by the map to \(d\sigma\) on \(K\).

We notice that the annihilator \(H\) of \(\Lambda\) is isomorphic to \(\mathbb{T}\), and \(|g(x)|\) as well as \(|\phi(x)|\) is constant on almost every coset \(\bar{x} = x + H\) in \(K/H\).

4. Contradiction to existence

We may now offer our proof of the main result stated in Section 1.

Proof of the Theorem. — Suppose, on the contrary, that a bounded function \(\phi\) generates \(H_0^2(\sigma)\). Let \(\Gamma\) and \(\Lambda\) be the dense subgroups of \(\mathbb{R}\) defined as in Section 3 with respect to \(\phi\) and \(|\phi|\), respectively. Choose an \(\alpha\) in \(\Gamma\) with \(a_{\alpha}(\phi) \neq 0\). It follows from Lemma 3.4 that \(\alpha\) is independent of \(\Lambda\) and \(\Gamma\) is generated by \(\alpha\) and \(\Lambda\). Let \(0 < \beta < 1\). Since the function

\[
(1 + \beta \chi_{\alpha})^{-1} = \sum_{k=0}^{\infty} (-\beta)^k \chi_{k\alpha}
\]

lies in \(H^\infty(\sigma)\), \((1 + \beta \chi_{\alpha})^2\) is an outer function in \(H^\infty(\sigma)\). Define \(\phi_1 = (1 + \beta \chi_{\alpha})^2 \phi\). In view of Lemma 2.1, \(\phi_1\) is also a bounded generator of \(H_0^2(\sigma)\). As above, let \(\Gamma_1\) and \(\Lambda_1\) be the smallest groups determined by the nonzero Fourier coefficients of \(\phi_1\) and \(|\phi_1|\), respectively. Notice that \(\Gamma_1\) is a subgroup of \(\Gamma\). We claim that the generator \(\phi_1\) cannot satisfy the property of Lemma 3.3. Indeed, since \(|\phi_1| = (1 + \beta^2 + \beta \overline{\chi_{\alpha}} + \beta \chi_{\alpha}) |\phi|\), we obtain by (1.1) that

\[
a_{\lambda}(|\phi_1|) = (1 + \beta^2) a_{\lambda}(|\phi|) + \beta a_{\lambda + \alpha}(|\phi|) + \beta a_{\lambda - \alpha}(|\phi|).
\]

Since \(\alpha\) does not lie in \(\Lambda\), if \(\lambda\) is in \(\Lambda\), then \(a_{\lambda + \alpha}(|\phi|) = a_{\lambda - \alpha}(|\phi|) = 0\). Then we have

\[
a_{\lambda}(|\phi_1|) = (1 + \beta^2) a_{\lambda}(|\phi|) \quad \text{and} \quad a_{\lambda + \alpha}(|\phi_1|) = \beta a_{\lambda}(|\phi|),
\]

for each \(\lambda\) in \(\Lambda\). These facts imply that \(\Lambda_1\) contains \(\Lambda\) and \(\alpha\), so that \(\Gamma = \Lambda_1 = \Gamma_1\), which contradicts Lemma 3.3.

The next proof is of independent interest, because it suggests that our Theorem is regarded essentially as the converse to Corollary 1.1.

Proof of Corollary 1.1. — We consider the case where the cocycle \(C(x, t)\) of \(\mathfrak{M}\) has the form \(C(x, t) = e^{iat}\). Then \(\mathfrak{M}_-\) is the space of all \(\psi\) in \(L^2(\sigma)\) satisfying that

\[
\psi(x) \sim \sum_{\Gamma \ni \lambda > -\alpha} a_{\lambda}(\psi) \chi_{\lambda}(x).
\]
Suppose that $\mathcal{M}_-$ has a generator $\phi$. Then log $|\phi|$ does not lie in $L^1(\sigma)$ and we may assume that $\phi$ is bounded. If $\ell \alpha$ is in $\Gamma$ for a positive integer $\ell$, then the bounded function $(\chi_{x,0^n}(x)\phi)^\ell$ is a single generator of $H^2_0(\sigma)$ by Lemma 2.1, which is contrary to Theorem. We next consider the case that we may assume that $\mathcal{M}_-$ is singly generated. Indeed, by the cocycle identity, $M_\mu(x)$ lies in $L^2(\mu)$ by Lemma 2.1, which contradicts our Theorem.

Proof of Corollary 1.2. — Denote by $C(x,t)$ the real cocycle of $\mathcal{M}$. Suppose that $\mathcal{M}_-$ has a generator $\phi$, for which log $|\phi|$ does not lie in $L^1(\sigma)$. It follows from Lemma 2.1 that almost every $t \rightarrow C(x,t)\phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$. We may assume that $\phi$ is bounded. Since $C(x,t)^2 \equiv 1$, $\phi^2$ is a single generator of $H^2_0(\sigma)$ by Lemma 2.1, which contradicts our Theorem.

By the same way as above, we may show that if $C(x,t)$ takes only finite values, then $\mathcal{M}_-$ cannot be singly generated. Indeed, by the cocycle identity, the set of values of $C(x,t)$ forms a group of order $k$,

$$Z(2\pi/k) = \left\{ e^{i2\pi j/k} ; j = 0, \ldots, k-1 \right\} .$$

Then if $\phi$ generates $\mathcal{M}_-$, then $\phi^k$ is a generator of $H^2_0(\sigma)$.

Let $\mathcal{M}$ be the normalized simply invariant subspace of $L^2(\sigma)$ with cocycle $A(x,t)$. Recall that $\psi$ lies in $\mathcal{M}$ if and only if almost every $t \rightarrow A(x,t)\psi(x+e_t)$ lies in $H^2(dt/\pi(1+t^2))$. Denote by $\mathcal{M}$ the invariant subspace with cocycle $\tilde{A}(x,t)$ (as discussed in [4, §3.2]). To prove Corollary 1.3. we need the following:

**Lemma 4.1.** — Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be as above. If $\mathcal{M}$ is singly generated, then $\tilde{\mathcal{M}}_-$ cannot be singly generated.
Proof. — Since $A(x,t) \cdot \overline{A(x,t)} \equiv 1$, $H^2_0(\sigma)$ is the smallest subspace of $L^2(\sigma)$ containing all $\psi_1\psi_2$ with $\psi_1$ in $\mathcal{M} \cap L^\infty(\sigma)$ and $\psi_2$ in $(\overline{\mathcal{M}})_- \cap L^\infty(\sigma)$ (see [4, §3.2, Theorem 20]). Suppose that $(\overline{\mathcal{M}})_-$ is singly generated. Then Lemma 2.1 shows that there are bounded single generators $\phi_1$ and $\phi_2$ of $\mathcal{M}$ and $(\overline{\mathcal{M}})_-$, respectively. Thus $\phi_1\phi_2$ is a single generator of $H^2_0(\sigma)$, which contradicts our Theorem.

Proof of Corollary 1.3.

(a) Let $\mathcal{M}$ be a simply invariant subspace with nontrivial cocycle $A(x,t)$. It follows from [8] that $\mathcal{M}$ is singly generated if and only if $A(x,t)$ is cohomologous to a singular cocycle. On the other hand, by [4, §4.6, Theorem 26], every cocycle is cohomologous to a Blaschke cocycle. By virtue of Lemma 4.1, we obtain easily a desired Blaschke cocycle.

(b) From Lemma 4.1, we choose a Blaschke cocycle $B(x,t)$ such that the invariant subspace $\mathcal{N}$ having the cocycle $B(x,t)$ is not singly generated. We claim that $B(x,t)$ satisfies the desired property. Suppose, on the contrary, that some function $\psi$ in $H^2(\sigma)$ has exactly the same zeros as $B(x,t)$. By multiplying by a suitable outer function, we assume that $\psi$ is bounded. Then $\psi$ generates the invariant subspace with cocycle $B(x,t)S(x,t)$, where $S(x,t)$ is the singular cocycle determined by the inner part of $t \to B(x,t)\psi(x + e_t)$ in $H^2(dt/\pi(1 + t^2))$. On the other hand, it follows from [8] and Lemma 2.1 that there is a function $h$ in $L^2(\sigma)$ such that almost every $t \to S(x,t)h(x + e_t)$ is outer in $H^2(dt/\pi(1 + t^2))$. Observe that

$$(h\psi)(x + e_t) = B(x,t) \cdot S(x,t)h(x + e_t) \cdot \overline{B(x,t)S(x,t)}\psi(x + e_t).$$

Since the inner part of $t \to (h\psi)(x + e_t)$ is $t \to B(x,t)$, the subspace $\mathcal{N}$ is singly generated by $h\psi$, thus we have a contradiction.

In the proof of (b) above, if the singular cocycle $S(x,t)$ is a coboundary, then $h$ is taken as a unitary function, otherwise $\log |h|$ does not lie in $L^1(\sigma)$.

5. Remarks

Remark A. It is sometimes useful to study the spectral measures associated with invariant subspaces. Let $\mathcal{M}$ be a simply invariant subspace of $L^2(\sigma)$ and put

$$\mathcal{M}_\lambda = \bigwedge_{\lambda \geq \nu} \chi_\nu \mathcal{M}. $$

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for each $\lambda$ in $\mathbb{R}$. Denote by $P_\lambda$ the orthogonal projection of $L^2(\sigma)$ onto $\mathfrak{M}_\lambda$. By the property that
\[
\bigwedge_{-\infty < \lambda < \infty} \mathfrak{M}_\lambda = \{0\} \quad \text{and} \quad \bigvee_{-\infty < \lambda < \infty} \mathfrak{M}_\lambda = L^2(\sigma),
\]
we obtain the continuity of the spectral resolution of identity $\{I - P_\lambda\}_{\lambda \in \mathbb{R}}$ on $L^2(\sigma)$, where $I$ is the identity map on $L^2(\sigma)$. Let $A(x,t)$ be the cocycle of $\mathfrak{M}$. By Stone’s theorem, a unitary group $\{V_t\}_{t \in \mathbb{R}}$ on $L^2(\sigma)$ is defined as
\[
V_t \phi(x) = A(x,t)T_t \phi(x) = -\int_{-\infty}^{\infty} e^{i\lambda t} dP_\lambda \phi(x), \quad \phi \in L^2(\sigma),
\]
where $T_t \phi(x) = \phi(x+e_t)$. For a nonzero function $\phi$ in $L^2(\sigma)$, $-d(P_\lambda \phi, \phi)$ is a finite positive measure on $\mathbb{R}$. On almost periodic flows, by comparing with Lebesgue measure $d\lambda$, the type of such measures is uniquely determined. We then say that each of $\mathfrak{M}, A(x,t)$ and $\{V_t\}_{t \in \mathbb{R}}$ is of absolutely continuous, or singular continuous, or discrete type (as discussed in [4, §2.4]). This fact plays an important role to classify invariant subspaces in this special context. It is easy to observe that $A(x,t)$ and $\overline{A(x,t)}$ have the same spectral type, so the following is an immediate consequence of Lemma 4.1.

**Proposition 5.1.** — There is a simply invariant subspace of $L^2(\sigma)$ of either absolutely continuous or singular continuous type which has no single generator.

Let $w$ be a nonnegative function in $L^2(\sigma)$ satisfying (1.3), while log $w$ does not lie in $L^1(\sigma)$. We know that a cocycle is trivial if and only if it is of discrete type (see [4, §2.4, Theorem 15]). It follows from Corollary 1.1 that the type of $\mathfrak{M}[w]$ has to be continuous. However, we have no idea to decide what kind of continuous spectrum $\mathfrak{M}[w]$ may have.

**Remark B.** Using a suitable cocycle, we may construct a skew product on which the $H^2_0$–space is singly generated. Indeed, let $w$ be a bounded function as above and let $A(x,t)$ be the cocycle of $\mathfrak{M}[w]$. By Lemma 2.1 we see that almost every $t \to A(x,t)w(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$. Denote by $(K \times \mathbb{T}, \{S_t\}_{t \in \mathbb{R}})$ the skew product induced by $A(x,t)$. If $A(x,t)^n, n \geq 1$, is a coboundary $\overline{q(x)}q(x+e_t)$ with unitary function $q$ on $K$, then $qw^n$ is a single generator of $H^2_0(\sigma)$. It then follows from Theorem that $A(x,t)^n$ is a coboundary only for $n = 0$. Hence $d\mu = d\sigma \times d\theta/2\pi$ is an ergodic measure on $(K \times \mathbb{T}, \{S_t\}_{t \in \mathbb{R}})$. If we set
\[
\phi(x,e^{i\theta}) = w(x)e^{i\theta}, \quad (x,e^{i\theta}) \in K \times \mathbb{T},
\]
then $\phi$ is a single generator of $H^2_0(\mu)$, since $\log |\phi|$ does not lie in $L^1(\mu)$ and almost every $t \to \phi(S_t(x,e^{it}))$ is outer in $H^2(dt/\pi(1+t^2))$ (see [16] for another construction).

**Remark C.** We have a bit of information on the distribution of zeros of functions in $H^2(\sigma)$ which are connected with Dirichlet series (refer to [17] for related topics). Let $\{\lambda_n\}$ be a sequence in $\Gamma$ such that

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \lambda, \quad n \to \infty,$$

for some $\lambda$ in $\Gamma$. Define a function $\psi$ in $H^2(\sigma)$ by

$$\psi = \sum_{n=1}^{\infty} a_n \chi_{\lambda_n}$$

with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Observe that almost every $t \to \psi(x+e_t)$ extends to an entire function.

**Proposition 5.2.** — Let $\psi$ be as above and let $\delta > 0$. Then there is a decreasing sequence $\{m_n\}$ with $m_n \to -\infty$ such that the number of zeros of $z \to \psi(x+e_z)$ in the strip

$$S_n = \{ z = t + iu ; m_n > u > m_n - \delta \}$$

is infinite, for $\sigma - a.e. x$ in $K$.

**Proof.** — Putting $\nu_n = \lambda - \lambda_n$, we let $\phi = \sum_{n=1}^{\infty} \overline{a_n} \chi_{\nu_n}$. Since $z \to e^{i\lambda z}$ has no zero, $z \to \psi(x+e_z)$ has zero at $z$ if and only if so does $z \to \phi(x+e_z)$ at $\bar{z}$. For each $r > 0$, $t \to \phi * P_{ir}(x+e_t)$ cannot be an outer function in $H^2(dt/\pi(1+t^2))$, even if $\log |\phi|$ does not lie in $L^1(\sigma)$. Since $\phi$ has no weight at infinity, the inner part of $t \to \phi * P_{ir}(x+e_t)$ derives a Blaschke cocycle being not constant. From this fact, we may choose easily a desired decreasing sequence $\{m_n\}$. □

**BIBLIOGRAPHY**


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