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ON VERLINDE SHEAVES AND STRANGE DUALITY
OVER ELLIPTIC NOETHER-LEFSCHETZ DIVISORS

by Alina MARIAN & Dragos OPREA

ABSTRACT. — We extend results on generic strange duality for $K3$ surfaces by showing that the proposed isomorphism holds over an entire Noether-Lefschetz divisor in the moduli space of quasipolarized $K3$s. We interpret the statement globally as an isomorphism of sheaves over this divisor, and also describe the global construction over the space of polarized $K3s$.

RéSUMÉ. — On établit l’isomorphisme de dualité étrange pour toutes les surfaces $K3$ constituant un diviseur de Noether-Lefschetz dans l’espace de modules de surfaces $K3$ quasipolarisées. On interprète le résultat d’une manière globale, comme un isomorphisme de faisceaux à travers ce diviseur, et on décrit aussi la construction globale sur l’espace de modules des surfaces $K3$s polarisées.

1. Introduction

1.1. Setup

For a fixed polarized complex $K3$ surface $(X, H)$, let $v, w \in H^*(X, \mathbb{Z})$ be two primitive elements which are orthogonal in the sense that

$$\int_X v \cup w = 0.$$ 

Consider the moduli space $\mathcal{M}_v$ of Gieseker $H$-stable sheaves $E$ on $X$ of Mukai vector $v$:

$$\text{ch}(E) \sqrt{\text{Tod}(X)} = v.$$ 

The Mukai vector $w$ induces a determinant line bundle

$$\Theta_w \to \mathcal{M}_v,$$

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constructed in [10][12]. Specifically, if a universal family $E \to \mathcal{M}_v \times X$ is available, we set

$$\Theta_w = \det R\pi_!(E \otimes^L q^* F)^{-1},$$

for a complex $F \to X$ of Mukai vector $w$. Similarly we obtain the line bundle $\Theta_v \to \mathcal{M}_w$.

If $c_1(v \cdot w) \cdot H > 0$, as explained in [17], the set

$$\Theta = \{(E, F) : H^0(E \otimes^L F) \neq 0\} \hookrightarrow \mathcal{M}_v \times \mathcal{M}_w$$

is the zero locus of a section of the line bundle

$$\Theta_w \boxtimes \Theta_v \to \mathcal{M}_v \times \mathcal{M}_w,$$

and induces a map

$$(1.1) \quad D : H^0(\mathcal{M}_v, \Theta_w) \to H^0(\mathcal{M}_w, \Theta_v).$$

According to Le Potier’s strange duality conjecture [11], D is expected to be an isomorphism.

### 1.2. Results

In [14] we established the conjecture for generic surfaces $(X, H)$ in the moduli space $\mathcal{K}_\ell$ of primitively quasipolarized $K3$ surfaces of degree $2\ell$, and for many pairs of Mukai vectors $(v, w)$ which satisfy

$$c_1(v) = c_1(w) = H.$$ 

The proof involves degeneration to the locus of elliptic $K3$ surfaces with section and irreducible at worst nodal fibers.

In the present paper, we study the problem for elliptic $K3$s with arbitrary singular fibers. In other words, we consider the entire Noether–Lefschetz divisor

$$\mathcal{P}_1 \hookrightarrow \mathcal{K}_\ell$$

consisting of pairs $(X, H)$ of elliptically fibered $K3$s which are quasipolarized by means of a numerical section $H$. We show

**Theorem 1.1.** — For any surface $(X, H)$ in $\mathcal{P}_1$, fix two orthogonal Mukai vectors $v$ and $w$ of ranks $r, s \geq 3$ with

$$c_1(v) = c_1(w) = H,$$

and satisfying further

$$\langle v, v \rangle + \langle w, w \rangle \geq 2(r + s)^2.$$ 

Then the duality morphism $D$ is an isomorphism.
In Section 2 we record basic properties of the Noether-Lefschetz divisor $P_1$. In Section 3, we prove the theorem above. In Section 4, the duality is stated globally as an isomorphism of sheaves, the Verlinde sheaves, over the entire divisor $P_1$. The Verlinde sheaves are also constructed more generally over the locus $K_\ell \cap K_\ell$ of polarized $K3$ s. It would be interesting to extend this construction to $K_\ell$ in a suitable manner.

2. The Noether-Lefschetz divisor $P_1$

Let $(X, H) \to K_\ell$ be the moduli stack of quasipolarized $K3$ surfaces $(X, H)$ of degree $H^2 = 2\ell$ with $\ell \neq 1$.

We consider the Noether-Lefschetz loci of quasipolarized elliptically fibered $K3$ surfaces in $K_\ell$. Specifically, for each $k > 0$, we denote by $P_k$ the Noether-Lefschetz stack parametrizing triples $(X, H, F)$ consisting of quasipolarized $K3$ s of degree $2\ell$, and divisor classes $F$ over $X$ satisfying

$$F^2 = 0, \quad F \cdot H = k.$$ 

We claim that

$$P_1 \hookrightarrow K_\ell$$

is a substack of $K_\ell$ parametrizing exactly the quasipolarized $K3$s which can be elliptically fibered with section, and with the quasipolarization a numerical section. This is expressed by the lemma below. The statement is standard, but a reference seemed difficult to find.

**Lemma 2.1.** — Let $(X, H)$ be a quasipolarized $K3$ surface of degree $2\ell$ with $\ell \neq 1$, and let $F$ be a divisor class on $X$ satisfying

$$F^2 = 0, \quad F \cdot H = 1.$$ 

Then

(i) $F$ is effective and $O(F)$ is globally generated;

(ii) the induced map $\pi : X \to \mathbb{P}^1$ is an elliptic fibration with section $\sigma$, having $F$ as the fiber class;

(iii) the quasipolarization equals $H = \sigma + (\ell + 1)F$;

(iv) the class $F$ satisfying the two numerical assumptions above is unique.

**Proof.** — Note first that $\chi(O(F)) = 2$. Since $-F \cdot H = -1$, and $H$ is nef, $-F$ cannot be effective, so

$$h^2(O(F)) = h^0(O(-F)) = 0$$

and $h^0(O(F)) \geq \chi(O(F)) = 2$. 

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Thus $F$ is effective.

We treat separately the two possibilities that $\mathcal{O}(F)$ be nef or not. First, if $\mathcal{O}(F)$ is nef, by the theorem of Piatetski-Shapiro and Shafarevich [15] there exists an elliptic fibration

$$
\pi : X \to \mathbb{P}^1
$$

such that $F = mf$, where $f$ is the class of a fiber. In fact,

$$
F \cdot H = 1 \implies m = 1, \quad F = f, \quad H \cdot f = 1.
$$

We next show that the fibration has a section. It is easy to check that the class

$$
\Sigma = H - (\ell + 1)f
$$

has self-intersection $-2$. Since $\chi(\mathcal{O}(\Sigma)) = 1$, $\Sigma$ is either effective or anti-effective. In fact, $\Sigma$ is effective, since $\Sigma \cdot H > 0$. Let $C$ be a curve in the linear series $\mathcal{O}(\Sigma)$. Now, for any component $R$ of a fiber we have $R \cdot f = 0$ by Zariski’s lemma, cf. III.8.2 [1]. Since $C \cdot f = 1$, $C$ must have a component which intersects each fiber with multiplicity 1. The other components of $C$ must be supported on components of the fibers. The transversal component gives a section $\sigma$ of the elliptic fibration $\pi$.

We now argue that $H = \sigma + (\ell + 1)f$. From the above discussion, we already know that

$$
H = \sigma + mf + \sum m_i R_i
$$

where $R_i$ are components of fibers and $m = \ell + 1$. In fact, by absorbing other fiber classes into the constant $m$, we may assume $R_i$ are supported on fibers with two components or more. We have the following possibilities:

(i) fibers of type $I_n$, consisting in a polygon of rational curves $C_1, \ldots, C_n$;
(ii) fibers of type $III$, consisting of 2 rational curves $C_1, C_2$ meeting tangentially;
(iii) fibers of type $IV$ consisting of 3 concurrent rational curves $C_1, C_2, C_3$;
(iv) fibers of type $I_n^\ast$ which can be written as

$$
C_1 + C_2 + C_3 + C_4 + 2(D_1 + \ldots + D_n)
$$

where

$$
C_1 \cdot D_1 = C_2 \cdot D_1 = C_3 \cdot D_n = C_4 \cdot D_n = 1
$$

and $D_i \cdot D_{i+1} = 1$ for $1 \leq i \leq n - 1$;
(v) fibers of type $II^\ast, III^\ast, IV^\ast$ corresponding to the graphs $E_6, E_7, E_8$. 

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Consider a fiber of type (i) and its contribution $\sum m_i C_i$ to the divisor $H$. We claim this contribution is a multiple of the fiber. Indeed, label the components so that $C_1$ intersects the section $\sigma$. Since $H \cdot C_i \geq 0$ for all $i$, we obtain the inequalities

$$-2m_1 + m_2 + m_n \geq -1, \ -2m_2 + m_1 + m_3 \geq 0, \ldots, -2m_n + m_1 + m_{n-1} \geq 0.$$ 

If $-2m_1 + m_2 + m_n \geq 0$, then after adding the above inequalities, we conclude that we must have equality throughout. Thus $m_1 = \ldots = m_n = m$ which shows that $\sum m_i C_i = mf$ as claimed. The case $-2m_1 + m_2 + m_n = -1$ is impossible. Indeed, since

$$\sum_{k \neq 1} (-2m_k + m_{k-1} + m_{k+1}) = -(-2m_1 + m_2 + m_n) = 1$$

we conclude that for some index $k_0$

$$-2m_k + m_{k-1} + m_{k+1} = \begin{cases} 1 & \text{if } k = k_0 \\ 0 & \text{if } k \neq 1, k_0. \end{cases}$$

This system is easily seen not to have any solutions. The remaining fiber types (ii)-(v) are entirely similar, and we will not verify them explicitly. In all cases, we find that $\sum m_i C_i$ must contribute a multiple of the fiber, hence

$$H = \sigma + mf$$

for some integer $m$. In fact, $m = \ell + 1$ by computing $H^2 = 2\ell$. This completes the proof when $\mathcal{O}(F)$ is nef.

We assume now that $\mathcal{O}(F)$ is not nef and we will reach a contradiction. Then there exists an irreducible curve $\Gamma_1$ such that

$$F \cdot \Gamma_1 < 0.$$ 

The curve $\Gamma_1$ is a component of an effective curve of class $F$ and furthermore $\Gamma_1^2 < 0$. Thus $\Gamma_1$ is a smooth rational curve on $X$. Let $H'$ be an ample class, and set $F_0 = F$. The reflection of $F$ along $\Gamma_1$ then yields an effective class, cf. proof of Theorem 2.2 in [16]:

$$F_1 = F_0 + (F_0 \cdot \Gamma_1) \Gamma_1$$

which has the property that

$$F_1^2 = F_0^2 = 0, \ F_1 \cdot H' < F_0 \cdot H'.$$
If $F_1$ is not nef, then we continue the process reflecting along a smooth rational curve $\Gamma_2$. The process will eventually stop since $F_i \cdot H'$ is a decreasing sequence of non-negative integers. At the end, we find a nef line bundle $\mathcal{O}(F_k)$ of zero self-intersection, where

$$F_k = F + (F_0 \cdot \Gamma_1)\Gamma_1 + (F_1 \cdot \Gamma_2)\Gamma_2 + \ldots + (F_{k-1} \cdot \Gamma_k)\Gamma_k.$$  

Therefore $F_k = mf$, where $m \geq 0$ by nefness. In particular,

$$F = mf + \sum n_i \Gamma_i$$

where $n_i = -F_{i-1} \cdot \Gamma_i > 0$. Using $F \cdot H = 1$ we conclude

$$m(H \cdot f) + \sum n_i(H \cdot \Gamma_i) = 1.$$ 

Since $H$ is nef, the intersection numbers above are nonnegative. If $H \cdot f = 0$, since $H^2 > 0$, by the Hodge index theorem we find $f^2 \leq 0$. Since equality occurs, $f$ must be numerically trivial which is not the case since it intersects $H'$ nontrivially. Therefore

$$H \cdot f = 1, \ m = 1, \ H \cdot \Gamma_i = 0 \text{ for all } i.$$ 

The argument given in the nef case then shows that the elliptic fibration $\pi$ has a section $\sigma$, and

$$H = \sigma + (\ell + 1)f.$$ 

We conclude

$$H \cdot \Gamma_i = \sigma \cdot \Gamma_i + (\ell + 1)f \cdot \Gamma_i = 0.$$ 

Thus either $\sigma \cdot \Gamma_i \leq 0$ or $f \cdot \Gamma_i \leq 0$. This means $\Gamma_i$ is contained in $\sigma$ or in the fiber $f$. The first case cannot occur since then

$$\Gamma_i = \sigma \text{ and } \sigma \cdot \Gamma_i + (\ell + 1)f \cdot \Gamma_i = 0 \text{ shows } \ell = 1$$

which is not allowed. Thus $\Gamma_i$ is a component of the fiber of $f$. However, in this case $f \cdot \Gamma_i = 0$ by Zariski’s lemma. Since

$$F = f + \sum n_i \Gamma_i$$

has zero self intersection, we find

$$\left(\sum n_i \Gamma_i\right)^2 = 0,$$

where $\Gamma_i$ are components of the fiber. This yields $\sum n_i \Gamma_i = nf$ for some integer $n$, again by Zariski’s lemma. Thus $F = (n+1)f$, and since $F \cdot H = 1$ then $F$ is the fiber class.

Finally, we establish the uniqueness of $F$ as claimed in (iv). If $F'$ is another class with

$$F'^2 = 0, \ F' \cdot H = 1$$
then we can write
\[ F' = a\sigma + R \]
where \( R \) is supported on components of fibers. We have \( R \cdot f = 0 \) and
\[ F' \cdot H = (a\sigma + R) \cdot (\sigma + (\ell + 1)f) = 1 \implies R \cdot \sigma = 1 - a(\ell - 1). \]
In addition
\[ F'^2 = 0 \implies -2a^2 + 2a(R \cdot \sigma) + R^2 = 0. \]
This yields
\[ R^2 = -2a + 2a^2(\ell + 1). \]
By Zariski’s lemma, \( R^2 \leq 0 \), which implies \( a = 0 \). Furthermore, we obtain \( R^2 = 0 \), showing that \( R = mf \), again by Zariski’s lemma. Moreover, \( R \cdot \sigma = 1 \) hence \( m = 1 \). Therefore \( F' = f \), proving uniqueness. \( \square \)

3. Strange duality along \( \mathcal{P}_1 \)

We now show Theorem 1.1 of the Introduction. For \((X, H) \in \mathcal{P}_1\), we consider the orthogonal Mukai vectors
\[ v = r + H + a[\text{pt}], \quad w = s + H + b[\text{pt}] \quad (3.1) \]
with \( r, s \geq 3 \), satisfying further
\[ \langle v, v \rangle + \langle w, w \rangle \geq 2(r + s)^2. \quad (3.2) \]
We form the moduli spaces of stable sheaves \( \mathcal{M}_v \) and \( \mathcal{M}_w \) together with the corresponding theta line bundles. Stability of the sheaves in \( \mathcal{M}_v \) and \( \mathcal{M}_w \) is with respect to a polarization which is suitable in the sense of Friedman. For such polarizations, and sheaves of fiber degree 1, stability on the surface is equivalent to stability of the restriction to a generic fiber, cf. Theorem 5, Chapter 6 of [6]. \(^{(1)}\) Both moduli spaces are smooth and projective.

Under these conditions, in [14], the strange duality map
\[ D : H^0(\mathcal{M}_v, \Theta_w) \to H^0(\mathcal{M}_w, \Theta_v) \]
was proven to be an isomorphism over the open sublocus of \( \mathcal{P}_1 \) consisting of surfaces with Picard rank 2.

\(^{(1)}\) As shown in the appendix of [14], this choice of polarization is in fact irrelevant under the stronger assumptions that
\[ \langle v, v \rangle \geq 2(r - 1)(r^2 + 1), \quad \langle w, w \rangle \geq 2(s - 1)(s^2 + 1). \]
Indeed, in this case, the different moduli spaces are birational in codimension 1.
We now assume that $X$ has Picard rank larger than 2. The elliptic fibration has finitely many reducible fibers. Fourier-Mukai functors were studied in this setting in \[8\]. Specifically, let

$$\pi : X \to \mathbb{P}^1$$

be any quasipolarized elliptically fibered $K3$ surface with section class $\sigma$ and fiber class $f$. Consider the product $Y = X \times_{\mathbb{P}^1} X$ with projections $p$ and $q$ to the two factors, and let

$$\Delta \subset X \times_{\mathbb{P}^1} X$$

be the diagonal. The $\pi$-relative Fourier-Mukai functor

$$S : D(X) \longrightarrow D(X)$$

with kernel

$$P = I_\Delta \otimes \mathcal{O}(p^* \sigma + q^* \sigma)$$

is an equivalence of bounded derived categories of coherent sheaves by Proposition 2.16 of \[8\]. As $(X, H)$ is in $\mathcal{P}_1$, by Lemma 2.1

$$c_1(v) = c_1(w) = \sigma + (\ell + 1)f.$$ 

Along the lines of \[3\], we shall prove shortly that the Fourier-Mukai transform $S$ induces a birational morphism, regular in codimension 1, between the moduli spaces $M_v$ and $M_w$ on the one hand, and the Hilbert schemes of $d_v$ respectively $d_w$ points on $X$ on the other:

$$\Psi_v : M_v \longrightarrow X^{[d_v]}, \quad \Psi_w : M_w \longrightarrow X^{[d_w]}.$$ 

Assuming this for the moment, we explain how to complete the proof of Theorem 1.1, much as in \[14\]. We determine first the exact numerics of the transformation $S$ by a cohomological Fourier-Mukai calculation. Let $V \in D(X)$ be any complex of rank $r$, Euler characteristic $\chi$, and first Chern class

$$c_1(V) = k\sigma + mf,$$

for integers $k$ and $m$. Recalling $p$ and $q$ are the projections from $Y = X \times_{\mathbb{P}^1} X$, we have

$$\det S(V) = \det Rq_*(\mathcal{P} \otimes p^* V) = \det Rq_*(I_\Delta \otimes p^* V(\sigma) \otimes q^* \mathcal{O}(\sigma))$$

$$= \det Rq_*(I_\Delta \otimes p^* V(\sigma)) \otimes \mathcal{O}(\sigma)^{\chi(V|_F)}$$

$$= \det Rq_*(p^* V(\sigma)) \otimes \det Rq_*(\mathcal{O}_\Delta \otimes p^* V(\sigma))^{-1} \otimes \mathcal{O}(k\sigma)$$

$$= \det Rq_*(p^* V(\sigma)) \otimes \det V(\sigma)^{-1} \otimes \mathcal{O}(k\sigma)$$

$$= \det Rq_*(p^* V(\sigma)) \otimes \mathcal{O}(-r\sigma - mf).$$
To calculate the first term, it is more convenient to work on the product 
\[ j : Y \hookrightarrow X \times X. \]
Let \( \bar{p}, \bar{q} \) denote the two projections from \( X \times X \), and let \( \text{pr} = \pi \times \pi : X \times X \to \mathbb{P}^1 \times \mathbb{P}^1 \). Observing that 
\[ j_\ast \mathcal{O}_Y = \text{pr}^\ast (\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} - \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1)) \]
\[ = \mathcal{O}_{X \times X} - \bar{p}^\ast \mathcal{O}(-f) \otimes \bar{q}^\ast \mathcal{O}(-f), \]
we calculate 
\[ \det Rq_\ast (p^\ast V(\sigma)) = \det R\bar{q}_\ast (\bar{p}^\ast V(\sigma) \otimes j_\ast \mathcal{O}_Y) \]
\[ = \det R\bar{q}_\ast (\bar{p}^\ast V(\sigma)) \otimes \det R\bar{q}_\ast (\bar{p}^\ast V(\sigma) \otimes \bar{p}^\ast \mathcal{O}(-f) \otimes \bar{q}^\ast \mathcal{O}(-f))^{-1} \]
\[ = \det(R\bar{q}_\ast (\bar{p}^\ast V(\sigma - f)) \otimes \mathcal{O}(-f))^{-1} \]
\[ = \mathcal{O}(-f)^{-\chi(V(\sigma - f))} = \mathcal{O}((\chi - 2r + m - 3k)f). \]
To summarize, we obtained 
\[ \det S(V) = \mathcal{O}(-r \sigma + (\chi - 2r - 3k)f). \]

Now let \( E \) and \( F \) be stable sheaves whose Mukai vectors \( v \) and \( w \) are given by (3.1). By the preceding calculation 
\[ \det S(E^\vee) = \mathcal{O}(-r \sigma + (a - r + 3)f), \]
\[ \det S(F) = \mathcal{O}(-s \sigma + (b - s - 3)f). \]
Assuming the birational isomorphism with the Hilbert scheme, for generic \( E \) and \( F \) we therefore have that 
\[ S(E^\vee) = I_Z \otimes \mathcal{O}(r \sigma - (a - r + 3)f)[-1], \]
\[ S(F) = I_W^\vee \otimes \mathcal{O}(-s \sigma + (b - s - 3)f), \]
where \( Z \) and \( W \) are zero dimensional subschemes of lengths \( d_v \) and \( d_w \) respectively. In fact, we will only explain the first equality below; the second can be deduced from the first by Grothendieck duality as in Proposition 2 of [14].

We finally calculate 
\[ H^0(E \otimes^L F) = \text{Hom}_{D(X)}(E^\vee, F) = \text{Hom}_{D(X)}(S(E^\vee), S(F)) \]
\[ = \text{Ext}^1(I_Z \otimes L, I_W^\vee) = \text{Ext}^1(I_W^\vee, I_Z \otimes L)^\vee \]
\[ = H^1(I_W \otimes^L I_Z \otimes L)^\vee. \]
On the third line, using (3.3) and (3.4), we have set

\[ L = \mathcal{O} ( (r + s)\sigma + (r + s - a - b) f ) . \]

The orthogonality condition

\[ H^2 = -rb - sa \]

for the Mukai vectors \( v \) and \( w \) together with the bound (3.2) on the dimensions \( d_v \) and \( d_w \) ensure that \(-a - b > r + s\), so the line bundle \( L \) is big and nef, without higher cohomology on \( X \).

Thus, under the birational map

\[ \Psi_v \times \Psi_w : \mathcal{M}_v \times \mathcal{M}_w \to X^{[d_v]} \times X^{[d_w]} \]

the two theta divisors

\[ \Theta = \{ (E, F) : H^0 (E \otimes^L F) \neq 0 \} \subset \mathcal{M}_v \times \mathcal{M}_w , \]

and

\[ \theta_L = \{ (I_Z, I_W) : H^0 (I_Z \otimes^L I_W \otimes L) \neq 0 \} \subset X^{[d_v]} \times X^{[d_w]} \]

coincide. The line bundles \( \Theta_w, \Theta_v \) on the two higher-rank moduli spaces and \( L^{[d_v]}, L^{[d_w]} \) on the two Hilbert schemes are also identified. As explained in Section 3 of [13], for line bundles \( L \) without higher cohomology, \( \theta_L \) is known to induce an isomorphism

(3.5) \[ H^0 (X^{[d_v]}, L^{[d_v]})^\vee \to H^0 (X^{[d_w]}, L^{[d_w]}). \]

Therefore, under the identifications above, \( \Theta \) also induces the isomorphism of equation (1.1):

\[ D : H^0 (\mathcal{M}_v, \Theta_w)^\vee \to H^0 (\mathcal{M}_w, \Theta_v). \]

We turn now to the proof that \( \Psi_v \) is an isomorphism in codimension 1, which was given for a surface \( \pi : X \to \mathbb{P}^1 \) with irreducible fibers in [2], [14]. We thus take up the case when the fibration has at least one reducible fiber. We shall explain that the inverse

\[ \Psi_v^{-1} : X^{[d_v]} \to \mathcal{M}_v \]

is a regular embedding defined on a subscheme \( U \subset X^{[d_v]} \) with \( \text{codim}(X^{[d_v]} \setminus U) \geq 2 \). The same is then true about \( \Psi_v \) on \( \mathcal{M}_v \). Indeed, if this were not the case, as the two moduli spaces are holomorphic symplectic, \( \Psi_v \) would at least admit by [9], Section 2.2, an extension \( \overline{\Psi}_v \) to a regular embedding defined away from codimension 2 on \( \mathcal{M}_v \). Thus \( \overline{\Psi}_v \) would extend over a divisorial locus \( D \subset \mathcal{M}_v \) where the original map \( \Psi_v \) is assumed undefined. But then

\[ \overline{\Psi}_v(D) \subset X^{[d_v]} \setminus U, \]
a contradiction as the latter has codimension 2 in \( X^{[d_s]} \).

We are thus left to analyze the domain of \( \Psi_v^{-1} \). The inverse is a Fourier-Mukai transform whose kernel is a complex \( Q[1] \) over \( X \times_{\mathbb{P}^1} X \). We write \( T \) for the Fourier-Mukai transform with kernel \( Q \) so that
\[
S \circ T = [-1], \quad T \circ S = [-1].
\]

We claim that for generic \( Z \), the sheaf
\[
M = I_Z \otimes \mathcal{O}(r\sigma - (a - r + 3)f)
\]
is \( WIT_0 \) for the kernel \( Q \). Its transform is then a stable torsion free sheaf in \( \mathfrak{M}_v \), cf. Section 7 of [3]. To prove the claim, we adapt arguments of [3], as follows. On general grounds, cf. Lemma 6.1 in [3], there is a short exact sequence
\[
0 \to A \to M \to B \to 0
\]
where \( A \) is \( T-WIT_0 \), while \( B \) is \( T-WIT_1 \). We prove that \( B = 0 \), following Lemma 6.4 in [3]. Assuming otherwise, we have \( T(B) \neq 0 \), and therefore there exists \( x \in X \) and a non-zero morphism
\[
T^1(B) \to \mathbb{C}_x.
\]
Note however that
\[
\mathbb{C}_x = T^1(I_x(o)),
\]
where \( I_x \) is the ideal sheaf of the point \( x \) in its fiber, and \( o \) denotes the intersection of the fiber through \( x \) with the section. In fact, \( I_x(o) = S^0(\mathbb{C}_x) \), by Lemma 6.3.7 of [4]. By Parseval, we now obtain a non-zero morphism
\[
M \to B \to I_x(o).
\]
This morphism must factor through the restriction of \( M \) to the fiber \( C \) through \( x \), yielding a non-zero map
\[
I_Z|_C \otimes \mathcal{O}(ro) \to I_x(o).
\]
Thus it suffices to show
\[
\text{Hom}_C(I_Z|_C \otimes \mathcal{O}((r - 1)o), I_x) = 0.
\]
We prove this is the case for \( r \geq 3 \) and subschemes \( Z \) such that
\begin{enumerate}
\item \( Z \) intersects any smooth fiber in at most two points;
\item \( Z \) intersects any singular fiber in at most one point which is not a node or a cusp (if the fiber is irreducible) or does not lie on at least two irreducible components.
\end{enumerate}
This locus has complement of codimension 2 in the Hilbert scheme of $X$. When $C$ is a smooth fiber, $\zeta = Z \cap C$ has length at most equal to 2, by (i). Then

$$I_Z|_C = I_{\zeta/C} \oplus T$$

where $T$ is a torsion sheaf supported at $\zeta$. This can be seen by restricting the ideal sequence of $Z$ to the curve $C$. In fact, the same statement also holds when $C$ is singular, as $Z$ is subject to (ii). When $C$ is smooth, it suffices therefore to prove

$$\text{Hom}_C(I_{\zeta/C}((r-1)o), I_x) = 0 \iff H^0(O_C(-(r-1)o + \zeta - x)) = 0.$$ 

Since for $r \geq 3$ the degree is negative, the conclusion follows. When $C$ is a singular fiber, the scheme $\zeta = Z \cap C$ has length 1. We show

$$\text{Hom}_C(I_{\zeta/C}((r-1)o), O_C) = 0$$

which gives $\text{Hom}_C(I_{\zeta/C}((r-1)o), I_x) = 0$.

Indeed, by duality, this is the same as proving

$$H^1(I_{\zeta/C}((r-1)o))) = 0.$$ 

Here we used that the dualizing sheaf of $C$ is trivial. Assume first $\zeta \neq o$. From the exact sequence

$$0 \to I_{\zeta/C}(o) \to I_{\zeta/C}((r-1)o) \to C^{r-2} \to 0$$

we see it suffices to show

$$H^1(I_{\zeta/C}(o)) = 0.$$ 

Next, from the exact sequence

$$0 \to O(-o) \to O \to C_o \to 0$$

we conclude

$$H^0(O(-o)) = 0, \ H^1(O(-o)) = C \implies H^0(O(o)) = C, \ H^1(O(o)) = 0.$$ 

The exact sequence

$$0 \to I_{\zeta/C}(o) \to O_C(o) \to C_\zeta \to 0$$

and the fact that

$$H^0(O_C(o)) \to C_\zeta$$

is an isomorphism for $\zeta \neq o$ yield $H^1(I_{\zeta/C}(o)) = 0$, as claimed. The vanishing of higher cohomology also holds for $\zeta = o$ since $H^1(O((r-2)o)) = 0$. This completes the proof.
4. The Verlinde sheaves

We will reinterpret Theorem 1.1 as giving an isomorphism of sheaves defined over the divisor $P_1$ in the moduli space of quasipolarized $K3$s.

4.1. Construction

For a fixed integer $n$, we may consider over $K_{\ell}$ the relative Hilbert scheme of $n$ points

$$\pi : \mathcal{X}^{[n]} \to K_{\ell},$$

viewed as the relative moduli stack of rank 1 torsion free sheaves of trivial determinant and second Chern number $-n$.

More generally, to consider spaces of higher rank sheaves as the $K3$ surface varies in moduli, we restrict attention to the open substack

$$K_{\ell}^o \hookrightarrow K_{\ell}$$

where the line bundle $\mathcal{H}$ over the universal surface

$$\pi : \mathcal{X} \to K_{\ell}$$

is ample. We construct

$$M[v] \to K_{\ell}^o,$$

the moduli space of $\mathcal{H}$-semistable sheaves with rank $r$, determinant $d\mathcal{H}$ and Euler characteristic $a - r$ over the fibers of $\pi : \mathcal{X}^o \to K_{\ell}^o$.

The construction of the theta bundles over $M[v]$ is subtler. To start, let

$$\pi : \mathcal{X}^o_1 \to K_{\ell,1}^o$$

be the universal family over the moduli stack $K_{\ell,1}^o$ of polarized $K3$s with a marked point. It has a canonical section

$$\sigma : K_{\ell,1}^o \to \mathcal{X}^o_1.$$
\[ \beta = b - s - eH^2. \]

We further denote as
\[ \pi_v : M[v]_1 \rightarrow \mathcal{K}^0_{\ell,1} \]
the relative moduli space of stable sheaves of type \( v \) over the fibers of \( \pi : \mathcal{X}^0_{1} \rightarrow \mathcal{K}^0_{\ell,1} \). The class \( \mathcal{W} \) induces standardly a determinant line bundle
\[ \Theta_w \rightarrow M[v]_1, \]
via descent from
\[ Q \rightarrow M[v]_1, \]
where \( Q \) is an open subscheme of a suitable quot scheme. Explicitly, over \( Q \), we have
\[ \Theta_w = \det R p_!(E \otimes q^\ast \mathcal{W})^{-1} \]
for the universal quotient sheaf \( E \rightarrow \mathcal{Q} \times \mathcal{K}^0_{\ell,1} \). The fiber of the forgetful map
\[ M[v]_1 \rightarrow M[v] \]
over a point \( (X, H, E \rightarrow X) \in M[v] \) is the surface \( X \). To describe the restriction of \( \Theta_w \) to this fiber, we let \( \Delta \subset X \times X \) be the diagonal and denote by \( p, q \) the projections from \( X \times X \) to the two factors. Then
\[ \Theta_w|_X = \det Rp_*(q^\ast E \otimes ((s - e)\mathcal{O} \oplus q^\ast(eH) \oplus \beta \mathcal{O}_{\Delta}))^{-1} = \det E^{-\beta} = H^{-\beta d}. \]

We conclude that the product line bundle
\[ (4.1) \]
\[ \Theta_w \otimes \pi_v^* \mathcal{H}^{\beta d} \text{ on } M[v]_1 \]
restricts trivially to the fibers of the map
\[ M[v]_1 \rightarrow M[v] \]
forgetting the marking. By the seesaw lemma, the product (4.1) is in fact the pullback to \( M[v]_1 \) of a line bundle \( \Theta_w \rightarrow M[v] \):
\[ \Theta_w \otimes \pi_v^* \mathcal{H}^{\beta d} = \text{pr}^\ast \Theta_w. \]

While the determinant line bundle \( \Theta_w \) is uniquely defined for a fixed \( K3 \) surface, over the relative moduli space \( M[v] \), \( \Theta_w \) depends on choice of \( \mathcal{H} \), and therefore can be canonically defined only up to tensoring by line bundles pulled back from \( \mathcal{K}^0_{\ell} \).

\textbf{Remark 4.1.} — The same construction gives the theta line bundle on the relative moduli space \( SU_g(r) \rightarrow M_g \) of semistable rank \( r \) bundles with trivial determinant over smooth curves of genus \( g \). They are naturally defined on the basechanged moduli space
\[ SU_{g,1}(r) = SU_g(r) \times_{M_g} M_{g,1} \rightarrow M_{g,1}, \]
relative to the $K$-theory class
\[ \mathcal{O} + (g - 1)\mathcal{O}_\sigma \]
on the universal curve $\mathcal{C} \to M_{g,1}$, and are then seen to be pulled back under the forgetful map
\[ SU_{g,1}(r) \to SU_g(r). \]
Pushing forward the $k$-tensor powers of the theta line bundles to $M_g$, we obtain the Verlinde bundles
\[ \mathcal{V}_{r,k} \to M_g. \]
Their first Chern classes remain unknown in general.

4.2. Global strange duality

Over $K^0_\ell$ we define now the Verlinde complexes
\begin{align*}
(4.2) \quad \mathbf{W} &= \mathbf{R}\pi_v^*\Theta_w, \quad \mathbf{V} = \mathbf{R}\pi_w^*\Theta_v.
\end{align*}
Consider the fiber product
\[ \pi : M[v] \times_{K^0_\ell} M[w] \to K^0_\ell, \]
endowed with the canonical Brill-Noether locus,
\begin{align*}
(4.3) \quad \Theta &= \{(X,H,E,F) \text{ so that } \mathbb{H}^0(X,E \otimes L F) \neq 0 \} \subset M[v] \times_{K^0_\ell} M[w].
\end{align*}
One expects $\Theta$ to be a divisor. This was established in [14] when $v$ and $w$ satisfy
\[ c_1(v) = c_1(w) = H. \]
The corresponding line bundle, also denoted for simplicity as $\Theta$, is in any case always defined on the product space, and splits by the seesaw lemma as
\begin{align*}
(4.4) \quad \Theta &\simeq \Theta_w \boxtimes \Theta_v.
\end{align*}
The above equation is correct up to a line bundle twist
\[ \mathcal{T} \to K^0_\ell \]
which will be found explicitly below, and which for now we absorb into any one of the theta bundles. The two line bundles $\Theta_w$ and $\Theta_v$ are ambiguous up to reverse twistings by a line bundle from $K^0_\ell$,
\[ (\Theta_v, \Theta_w) \sim (\Theta_v \otimes \pi_w^*\mathcal{L}, \Theta_w \otimes \pi_v^*\mathcal{L}^{-1}), \text{ for } \mathcal{L} \in \text{Pic} K^0_\ell, \]
while $\Theta$ is canonical. Pushing forward the canonical theta line bundle via $\pi$, we get
\begin{equation}
R_{\pi*}\Theta \simeq W \otimes^L V,
\end{equation}
and the above ambiguity carries over to the Verlinde complexes $W$ and $V$. The divisor (4.3) then induces a morphism
$$D : W^V \to V.$$In [14], also having assumed that
$$\chi(v), \chi(w) \leq 0,$$we showed that over a Zariski open subset of $K^\circ_{[\ell]}$, the higher cohomology sheaves vanish while $\mathcal{H}^0(D)$ induces an isomorphism between the zeroth cohomology sheaves.

**Remark 4.2.** — Even though not necessary for our argument, let us determine the twist $T \to \mathcal{K}^\circ_{[\ell]}$ in the decomposition
\begin{equation}
\Theta = \Theta^w \boxtimes \Theta^v \otimes \text{pr}^* T
\end{equation}
over $M[v] \times K^\circ_{[\ell]} M[w]$, where $\text{pr}$ is the projection to $K^\circ_{[\ell]}$. Above, we absorbed this twist into the Verlinde complexes, for the ease of exposition.

First, we may pass to the moduli stack $M[v]$ and $M[w]$ of all sheaves over $X$, without changing the above equations. We let
$$\mathcal{E} \to M[v]_1 \times K^\circ_{[\ell]} M^\circ_{[\ell]} X^1_1, \quad \mathcal{F} \to M[w]_1 \times K^\circ_{[\ell]} M^\circ_{[\ell]} X^1_1$$be the universal families of sheaves, and further set, on the same product spaces,
$$\overline{\mathcal{E}} = \mathcal{E} - \text{pr}^* \mathcal{V}, \quad \overline{\mathcal{F}} = \mathcal{F} - \text{pr}^* \mathcal{W}.$$Considering now the triple product
$$M[v]_1 \times K^\circ_{[\ell]} M[w]_1 \times K^\circ_{[\ell]} M^\circ_{[\ell]} X^1_1,$$we calculate
$$\Theta \otimes \Theta^{-1}_v \otimes \Theta^{-1}_w$$as the pushforward
\[(\det R_{p_{12*}} (p_{13*} \mathcal{E} \otimes^L p_{23}^* \mathcal{F} - p_{13*} \mathcal{E} \otimes^L L p_{23}^* \mathcal{F} - p_{23}^* \mathcal{V} - p_{23}^* \mathcal{F} \otimes L p_{23}^* \mathcal{V}))^{-1} \otimes \text{pr}^* \mathcal{H}^{-d\beta - e\alpha} = (\det R_{p_{12*}} (p_{13*} \overline{\mathcal{E}} \otimes^L p_{23}^* \overline{\mathcal{F}} - p_{13*} (\mathcal{V} \otimes^L \mathcal{W})))^{-1} \otimes \text{pr}^* \mathcal{H}^{-d\beta - e\alpha},\]
where $\mathcal{H} \to \mathcal{K}^\circ_{[\ell]}$ is viewed on $M[v]_1 \times K^\circ_{[\ell]} M[w]_1$ via pullback by the natural projection
$$\text{pr} : M[v]_1 \times K^\circ_{[\ell]} M[w]_1 \to K^\circ_{[\ell]}.$$
We apply Grothendieck-Riemann-Roch to compute
\[ \text{ch} \, R^{p_{12}}_* \left( p_{13}^* \mathcal{E} \otimes L \, p_{23}^* \mathcal{F} \right). \]

By construction, \( \text{ch} \, \mathcal{E} \) and \( \text{ch} \, \mathcal{F} \) restrict trivially over the fibers of
\[ p_{12} : \mathcal{M}[v]_1 \times \mathcal{K}_{\ell,1} \mathcal{M}[w]_1 \times \mathcal{K}_{\ell,1} \mathcal{X}_1^\circ \rightarrow \mathcal{M}[v]_1 \times \mathcal{K}_{\ell,1} \mathcal{M}[w]_1. \]
The Chern character of the pushforward above is thus supported in codimension 2 or higher, and therefore gives
\[ \det \, R^{p_{12}}_* \left( p_{13}^* \mathcal{E} \otimes L \, p_{23}^* \mathcal{F} \right) = \mathcal{O}. \]

Recalling the morphism \( \pi : \mathcal{X}_1^\circ \rightarrow \mathcal{K}_\ell \) which describes the universal surface, we find that
\[ \Theta \otimes \Theta_v^{-1} \otimes \Theta_w^{-1} = \det \, R^{p_{12}}_* \left[ p_3^* (\mathcal{V} \otimes \mathcal{W}) \right] \otimes \text{pr}^* \mathcal{H}^{-d \beta - e \alpha} \]
\[ = \text{pr}^* \left( \det \, R_{\pi_*} (\mathcal{V} \otimes \mathcal{L} \mathcal{W}) \otimes \mathcal{H}^{-d \beta - e \alpha} \right) \]
\[ = \text{pr}^* \left( \det \, R_{\pi_*} \left[ ((r - d) \mathcal{O} + d \mathcal{H} + \alpha \mathcal{O}_s) \otimes \mathcal{L} \left( (s - e) \mathcal{O} + e \mathcal{H} + \beta \mathcal{O}_s \right) \right] \otimes \mathcal{H}^{-d \beta - e \alpha} \right) \]
\[ = \text{pr}^* \left( \lambda^{-(r-d)(s-e)} \otimes \left( \det \, \pi_* \mathcal{H} \right)^{e(r-d) + d(s-e)} \otimes \left( \det \, \pi_* \mathcal{H}^2 \right)^{de} \right). \]

Here, we wrote
\[ \lambda = \left( \det \, R_{\pi_*} \mathcal{O}_X \right)^{-1} \rightarrow \mathcal{K}_\ell \]
for the Hodge bundle. This yields the following

**Proposition 4.3.** — The twist \( \mathcal{T} \) defined by equation (4.6) is given by
\[ \mathcal{T} = \lambda^{-(r-d)(s-e)} \otimes \left( \det \, \pi_* \mathcal{H} \right)^{e(r-d) + d(s-e)} \otimes \left( \det \, \pi_* \mathcal{H}^2 \right)^{de}. \]

**4.3. Extensions of the Verlinde sheaves and desiderata**

We now turn our attention to the locus of elliptic \( K3 \) with section, where the Verlinde sheaves and the isomorphism \( \mathcal{D} \) can be extended from
\[ \mathcal{P}^0_1 = \mathcal{P}_1 \cap \mathcal{K}_\ell \]
to all of \( \mathcal{P}_1 \) by the results of Section 3, as we now explain.

The universal data over \( \mathcal{P}_1 \) consists of the triple
\[ (\mathcal{X}, \mathcal{H}, \mathcal{F}) \rightarrow \mathcal{P}_1, \]
where \( \mathcal{F} \) denotes the universal fiber class of the elliptic fibration. We consider the line bundle
\[ \mathcal{L} = \mathcal{H}^{r+s} \otimes \mathcal{O} (\mathcal{F})^{-(r+s) \ell - a - b}, \]
which restricts over each \((X, H, F)\) to

\[ L = \mathcal{O}((r + s)\sigma + (r + s - a - b)f). \]

In the product of Hilbert schemes we have the universal theta divisor

\[ \theta = \{(X, Z, W) : H^0(X, I_Z \otimes L \otimes \mathcal{L}|_X) \neq 0\} \subset \mathcal{X}^{[d_v]} \times_{\mathcal{P}_1} \mathcal{X}^{[d_w]} . \]

To write the corresponding line bundle, we denote by

\[ Z \subset \mathcal{X}^{[d_v]} \times_{\mathcal{K}_\ell} \mathcal{X}, \quad \mathcal{W} \subset \mathcal{X}^{[d_w]} \times_{\mathcal{K}_\ell} \mathcal{X}, \]

the universal subschemes, and set standardly

\[ \mathcal{L}^{[d_v]} = \det \mathcal{R}p_*(\mathcal{O}_Z \otimes q^* \mathcal{L}), \quad \mathcal{L}^{[d_w]} = \det \mathcal{R}p_*(\mathcal{O}_W \otimes q^* \mathcal{L}) . \]

From the product

\[ \mathcal{X}^{[d_v]} \times_{\mathcal{K}_\ell} \mathcal{X}^{[d_w]} \times_{\mathcal{K}_\ell} \mathcal{X}, \]

we calculate

\[ \theta = \det \left( \mathcal{R}p_{12*} \left( p_{13}^* \mathcal{I}_Z \otimes \mathcal{L} \otimes p_{23}^* \mathcal{I}_W \otimes p_3^* \mathcal{L} \right) \right)^{-1} \]

\[ = \det \left( \mathcal{R}p_{12*} \left( p_{13}^* (\mathcal{O} - \mathcal{O}_Z) \otimes \mathcal{L} \otimes p_{23}^* (\mathcal{O} - \mathcal{O}_W) \otimes p_3^* \mathcal{L} \right) \right)^{-1} \]

\[ = \mathcal{L}^{[d_v]} \boxtimes \mathcal{L}^{[d_w]} \otimes \pi^*(\det \pi_* \mathcal{L})^{-1} \otimes \det \mathcal{R}p_{12*} \left( p_{13}^* \mathcal{O}_Z \otimes \mathcal{L} \otimes p_{23}^* \mathcal{O}_W \otimes p_3^* \mathcal{L} \right) \]

\[ = \mathcal{L}^{[d_v]} \boxtimes \mathcal{L}^{[d_w]} \otimes \pi^*(\det \pi_* \mathcal{L})^{-1} . \]

On the third line, the last bundle is the determinant of a complex of sheaves supported on the codimension 2 locus of intersecting subschemes in \( \mathcal{X}^{[d_v]} \times_{\mathcal{K}_\ell} \mathcal{X}^{[d_w]} \) – thus it is trivial. Lemma 5.1 of [5] implies that

\[ \pi_* \mathcal{L}^{[d_v]} = \Lambda^{[d_v]} \pi_* \mathcal{L}, \quad \pi_* \mathcal{L}^{[d_w]} = \Lambda^{[d_w]} \pi_* \mathcal{L} . \]

The higher direct images of the line bundles \( \mathcal{L}^{[d_v]}, \mathcal{L}^{[d_w]} \) vanish by Theorem 5.2.1 of [18]. We therefore finally have

\[ \pi_* \theta \simeq \Lambda^{d_v} (\pi_* \mathcal{L}) \otimes \Lambda^{d_w} (\pi_* \mathcal{L}) \otimes (\det \pi_* \mathcal{L})^{-1} \cong \mathcal{W}' \otimes \mathcal{V}' . \]

We set

\[ \mathcal{W}' = \pi_* \mathcal{L}^{[d_v]}, \quad \mathcal{V}' = \pi_* \mathcal{L}^{[d_w]} \otimes (\det \pi_* \mathcal{L})^\vee . \]

As before these sheaves are only defined up to reverse twistings by a line bundle from \( \mathcal{P}_1 \). The divisor \( \theta \) induces the duality isomorphism

\[ D' : \mathcal{W}'^\vee \to \mathcal{V}' \]

over \( \mathcal{P}_1 \), which is a global version of (3.5).

Section 3 shows that the universal relative Fourier-Mukai transform induces a birational map

\[ \mathcal{X}^{[d_v]} \times_{\mathcal{P}_1} \mathcal{X}^{[d_w]} \dashrightarrow M[v] \times_{\mathcal{P}_1} M[w] \]
regular in codimension 1 over each fiber, such that the divisors $\theta$ and $\Theta$ are precisely matched. Because of regularity in codimension 1, the pushforward sheaves $\pi_* \theta$ and $R^0 \pi_* \Theta$ coincide. Therefore

$$W' \otimes V' \cong H^0(W) \otimes H^0(V)$$

over $P_1^\circ$. We can furthermore align the line bundle twists inherent in the definition of $W, V, W', V'$ so that

$$H^0(D) = D'$$

over this locus. We thus extended the Verlinde sheaves from $P_1^\circ \hookrightarrow P_1$.

The resolution of the following query will however be of much greater interest.

**Question 1.** — Is it possible to extend $W, V$ from $K_\ell^\circ \hookrightarrow K_\ell$ in such a fashion that

$$c_1(W) = -c_1(V)?$$

Combined with the results of [14], this would establish the strange duality conjecture over the entire locus where there is no higher cohomology, since the Baily-Borel compactification of $K_\ell$ has one dimensional boundary. It would be interesting to investigate whether $D$ is in fact a quasi-isomorphism between the complexes $W^\vee$ and $V$.

Regarding the canonical line bundle $\Theta$, it is also natural to wonder

**Question 2.** — Is the Chern character $\text{ch}(R\pi_* \Theta)$ in the ring generated by the Hodge class $\lambda = -c_1(R^2 \pi_* \mathcal{O}_{X^\circ})$ studied in [7]?

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