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SEMIBOUNDED UNITARY REPRESENTATIONS OF DOUBLE EXTENSIONS OF HILBERT–LOOP GROUPS

by K. H. NEEB (*)

Abstract. — A unitary representation π of a, possibly infinite dimensional, Lie group G is called semibounded if the corresponding operators $i\mathtt{d}\pi(x)$ from the derived representation are uniformly bounded from above on some non-empty open subset of the Lie algebra $\mathfrak{g}$ of G. We classify all irreducible semibounded representations of the groups $\widehat{\mathcal{L}}_\phi(K)$ which are double extensions of the twisted loop group $\mathcal{L}_\phi(K)$, where K is a simple Hilbert–Lie group (in the sense that the scalar product on its Lie algebra is invariant) and ϕ is a finite order automorphism of K which leads to one of the 7 irreducible locally affine root systems with their canonical $\mathbb{Z}$-grading. To achieve this goal, we extend the method of holomorphic induction to certain classes of Fréchet–Lie groups and prove an infinitesimal characterization of analytic operator-valued positive definite functions on Fréchet–BCH–Lie groups.

This is the first paper dealing with global aspects of Lie groups whose Lie algebra is an infinite rank analog of an affine Kac–Moody algebra. That positive energy representations are semibounded is a new insight, even for loops in compact Lie groups.

Résumé. — Une représentation unitaire π d'un groupe de Lie G est dite semiborné, si les opérateurs $i\mathtt{d}\pi(x)$ de la représentation derivée sont semi-bornés uniformément sur une partie ouverte de l'algèbre de Lie $\mathfrak{g}$ de G. Nous déterminons toutes les représentations irréductibles semi-bornées des groupes $\widehat{\mathcal{L}}_\phi(K)$ qui sont extensions doubles du groupe $\mathcal{L}_\phi(K)$, où K est un groupe de Lie hilbertien et ϕ est une automorphisme de K d'ordre fini qui mène à l'un des 7 systèmes de racines affines irréductibles localement finis. Pour atteindre cet objectif, nous étendons la méthode d'induction holomorphe aux certaines classes de groupes de Lie-Fréchet.

Il s'agit du premier papier traitant des aspects globaux des groupes de Lie dont l'algèbre de Lie est une algèbre de Kac–Moody à rang infini.

Keywords: infinite dimensional Lie group, unitary representation, semibounded representation, Hilbert–Lie algebra, Hilbert–Lie group, Kac–Moody group, loop group, double extension, positive definite function.
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Introduction

This paper is part of a project concerned with a systematic approach to unitary representations of infinite-dimensional Lie groups in terms of semiboundedness conditions on spectra in the derived representation ([42]). For the derived representation to carry significant information, we have to impose a suitable smoothness condition: Let G be a Lie group with Lie algebra $\mathfrak{g}$ and exponential function $\exp\colon \mathfrak{g} \to G$. A unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ is said to be *smooth* if the subspace $\mathcal{H}^\infty \subseteq \mathcal{H}$ of smooth vectors is dense. This is automatic for continuous representations of finite-dimensional Lie groups, but not for Banach–Lie groups ([6]). For any smooth unitary representation, the *derived representation*

$$\mathtt{d}\pi\colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty), \quad \mathtt{d}\pi(x)v := \frac{d}{dt}\Big|_{t=0} \pi(\exp tx)v$$

carries significant information in the sense that the closure of the operator $\mathtt{d}\pi(x)$ coincides with the infinitesimal generator of the unitary one-parameter group $\pi(\exp tx)$. We call $(\pi, \mathcal{H})$ *semibounded* if the function

$$s_\pi\colon \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, \quad s_\pi(x) := \sup\big(\mathrm{Spec}(i\mathtt{d}\pi(x))\big)$$

is bounded on a neighborhood of some point $x_0 \in \mathfrak{g}$. Then the set W_π of all such points x_0 is an open $\mathrm{Ad}(G)$-invariant convex cone. We say that π is *bounded* if $W_\pi = \mathfrak{g}$. All finite-dimensional continuous unitary representations are bounded and most of the unitary representations appearing in physics are semibounded or satisfy similar spectral conditions (*cf.* [57], [58], [56], [9], [52], [10], [34], [50], [14], [4], [42]).

For finite-dimensional Lie groups, the irreducible semibounded representations are precisely the unitary highest weight representations and one has unique direct integral decompositions into irreducible ones [38, X.3/4, XI.6]. Since the traditional tools to obtain classification results for representations of infinite-dimensional groups, such as the group algebra, invariant integration and character theory break down for infinite-dimensional groups, it is important to specify large and interesting classes of representations that, on the one hand side contain the representations showing up in applications, and on the other hand, lead to settings where complete classification results can be obtained. Semibounded representations form such a class.

For many interesting classes of groups such as the Virasoro group and affine Kac–Moody groups (double extensions of loop groups with compact target groups), the irreducible highest weight representations are semibounded by Theorem 6.1 below, but to prove the converse is more difficult and requires a thorough understanding of invariant cones in the corresponding Lie algebras as well as of convexity properties of coadjoint orbits ([42, Sect. 8]). A unifying framework in which semibounded representations can be constructed from bounded representations is provided by the method of holomorphic induction. This scheme has been developed in [46] for Banach–Lie groups, and in Appendix B below we explain under which assumptions it still works for Fréchet–Lie groups. All these assumptions are satisfied by groups of smooth loops (Subsection 5.2).

In this paper we study a class of Lie groups whose Lie algebras are natural infinite rank generalizations of affine Kac–Moody groups [26]. On the Lie algebra level, the theory of unitary highest weight modules, together with some classification results, was developed in [43]. Here we develop the corresponding global picture which fits perfectly into the scheme of semibounded representations. It is amazing that in this context, where neither root bases nor positive systems and dominance of weights is available, one can obtain classification results like Theorem 0.1 below whose formulation formally ressembles the corresponding results for compact Lie algebras and Kac–Moody algebras. Together with the automorphism groups of symmetric domains whose semibounded representations were classified in [45], the groups we study here are among the rare examples where one has complete classification results in a setting where the natural $\mathbb{Z}$-grading on the Lie algebra level has infinite-dimensional grading spaces. One of our guiding motivations in this project was that, compared to the classical loop groups and the Virasoro group ([52], [34], [50]), we are dealing here with representations in Hilbert spaces with $\mathbb{Z}$-gradings by infinite-dimensional subspaces, and it is an important problem to develop a better understanding for such situations. Since $\mathbb{Z}$-gradings correspond to $\mathbb{T}$-actions, it is clear that loop groups with infinite-dimensional targets are the prototypical examples of groups with such representations. Our present results also constitute another step towards a more systematic understanding of groups with a smooth one-parameter group of automorphisms for which irreducible positive energy representations exist; see [68] for a rather complete theory for Heisenberg groups, resp., the canonical commutation relations in this context.

The closest infinite-dimensional relatives of compact Lie algebras are *Hilbert–Lie algebras*. These are real Lie algebras which are Hilbert spaces on which the adjoint group acts by isometries (*cf.* [21, Def. 6.3]). We call a Lie group K whose Lie algebra $\mathfrak{k} = \mathbf{L}(K)$ is a Hilbert–Lie algebra a

Hilbert–Lie group.[(1)] The finite-dimensional Hilbert–Lie algebras are the compact Lie algebras. The main goal of this paper is the classification of all semibounded unitary representations of groups which are double extensions of loop groups with values in a Hilbert–Lie group.

For a Hilbert–Lie group K, we write $\mathrm{Aut}(K)$ for the group of all Lie group automorphisms acting by isometries with respect to the scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{k}$. For an automorphism $\phi \in \mathrm{Aut}(K)$ of order N,

$$\mathcal{L}_\phi(K) := \Big\{ f \in C^\infty(\mathbb{R}, K) \colon (\forall t \in \mathbb{R})\ f\Big(t + \frac{2\pi}{N}\Big) = \phi^{-1}(f(t)) \Big\}$$

is called the corresponding *twisted loop group*. It is a Fréchet–Lie group with Lie algebra

$$\mathcal{L}_\phi(\mathfrak{k}) := \Big\{ \xi \in C^\infty(\mathbb{R}, \mathfrak{k}) \colon (\forall t \in \mathbb{R})\ \xi\Big(t + \frac{2\pi}{N}\Big) = \mathbf{L}(\phi)^{-1}(\xi(t)) \Big\}$$

([48, App. A]). The subgroup $\mathcal{L}_\phi(K) \subseteq C^\infty(\mathbb{R}, K)$ is translation invariant, so that we obtain for each $T \in \mathbb{R}$ an automorphism of $\mathcal{L}_\phi(K)$ by

$$\alpha_T(f)(t) := f(t + T) \quad \text{with} \quad \mathbf{L}(\alpha_T)(\xi)(t) := \xi(t + T). \tag{0.1}$$

Our assumption $\phi^N = \mathrm{id}$ implies that $\alpha_{2\pi} = \mathrm{id}$, which leads to a smooth action of the circle group $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$ on $\mathcal{L}_\phi(K)$. The Fréchet–Lie algebra $\mathcal{L}_\phi(\mathfrak{k})$ carries the positive definite form

$$\langle \xi, \eta \rangle := \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(t), \eta(t) \rangle \, dt,$$

which is invariant under the $\mathbb{R}$-action (0.1). Therefore the derivation $D\xi := \xi'$ is skew-symmetric and thus

$$\omega(\xi, \eta) := \langle D\xi, \eta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi'(t), \eta(t) \rangle \, dt$$

defines a continuous Lie algebra cocycle on $\mathcal{L}_\phi(\mathfrak{k})$ (*cf.* [48, Sect. 3.2]). Let

$$\widetilde{\mathcal{L}}_\phi(\mathfrak{k}) := \mathbb{R} \oplus_\omega \mathcal{L}_\phi(\mathfrak{k})$$

denote the corresponding central extension and observe that ω is D-invariant, so that we obtain the double extension

$$\mathfrak{g} := \widehat{\mathcal{L}}_\phi(\mathfrak{k}) := (\mathbb{R} \oplus_\omega \mathcal{L}_\phi(\mathfrak{k})) \rtimes_D \mathbb{R}.$$

Here we extend D to $\widetilde{\mathcal{L}}_\phi(\mathfrak{k})$ by $D(z, \xi) := (0, \xi')$ to obtain the Lie bracket on $\mathfrak{g}$:

$$[(z_1, \xi_1, t_1), (z_2, \xi_2, t_2)] := (\langle \xi_1', \xi_2 \rangle, t_1\xi_2' - t_2\xi_1' + [\xi_1, \xi_2], 0).$$

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(1) In the literature one also finds a weaker concept of a "Hilbert–Lie group," namely Lie groups whose Lie algebra is a Hilbert space, but no compatibility between the Lie bracket and the scalar product is required.

To formulate our main result, we start with a 1-connected simple Hilbert–Lie group K, *i.e.,* $\mathfrak{k}$ contains no proper closed ideal. Corresponding twisted loop groups $\mathcal{L}_\phi(K)$ and $\mathcal{L}_\psi(K)$ are isomorphic if ϕ and ψ define the same conjugacy class in the group $\pi_0(\mathrm{Aut}(K))$ of connected components of the Lie group $\mathrm{Aut}(K)$. Since every infinite-dimensional simple Hilbert–Lie algebra is isomorphic to the algebra $\mathfrak{u}_2(\mathcal{H})$ of skew-hermitian Hilbert–Schmidt operators on an infinite-dimensional real, complex or quaternionic Hilbert space, it is possible to determine $\mathrm{Aut}(\mathfrak{k})$ explicitly (Theorem 1.15). This in turn leads to a complete classification of the corresponding twisted loop groups. We thus obtain four classes of loop algebras: the untwisted loop algebras $\mathcal{L}(\mathfrak{u}_2(\mathcal{H}))$, where $\mathcal{H}$ is an infinite-dimensional real, complex or quaternionic Hilbert space, and a twisted type $\mathcal{L}_\phi(\mathfrak{u}_2(\mathcal{H}))$, where $\mathcal{H}$ is a complex Hilbert space and $\phi(x) = \sigma x \sigma$ holds for an antilinear isometric involution $\sigma\colon \mathcal{H} \to \mathcal{H}$ (this corresponds to complex conjugation of the corresponding matrices). Our main result is the classification of the semibounded unitary representations of the 1-connected Lie groups $G := \widehat{\mathcal{L}}_\phi(K)$ corresponding to the respective double extensions $\mathfrak{g} = \widehat{\mathcal{L}}_\phi(\mathfrak{k})$.

To describe this classification, we choose a maximal abelian subspace

$$\mathfrak{t} \subseteq \mathfrak{k}^\phi := \{x \in \mathfrak{k}\colon\ \mathbf{L}(\phi)x = x\}.$$

Then $\mathfrak{t}_\mathfrak{g} := \mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}$ is maximal abelian in $\mathfrak{g} = \widehat{\mathcal{L}}_\phi(\mathfrak{k})$. We write $T_G := \exp(\mathfrak{t}_\mathfrak{g}) \subseteq G$ for the corresponding subgroup and identify its character group $\widehat{T}_G$ with a subgroup of $i\mathfrak{t}_\mathfrak{g}'$, where $'$ denotes the topological dual space. In the following, we assume that the corresponding $\mathbb{Z}$-graded root system $\Delta_\mathfrak{g} = \Delta(\mathfrak{g}, \mathfrak{t}_\mathfrak{g})$ is one of the seven irreducible locally affine reduced root systems of infinite rank $A_J^{(1)}, B_J^{(1)}, C_J^{(1)}, D_J^{(1)}, B_J^{(2)}, C_J^{(2)}$ or $BC_J^{(2)}$ (*cf.* Definition 2.6).

Theorem 0.1. — *Irreducible semibounded representations π_λ of $G = \widehat{\mathcal{L}}_\phi(K)$ are characterized by their $\mathfrak{t}_\mathfrak{g}$-weight set*

$$\mathcal{P}_\lambda = \mathrm{conv}(\widehat{\mathcal{W}}\lambda) \cap (\lambda + \widehat{\mathcal{Q}}) \subseteq i\mathfrak{t}_\mathfrak{g}' \quad \textit{with} \quad \mathrm{Ext}(\mathrm{conv}(\mathcal{P}_\lambda)) = \widehat{\mathcal{W}}\lambda,$$

where $\widehat{\mathcal{W}}$ is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{t}_\mathfrak{g})$ and $\widehat{\mathcal{Q}} \subseteq i\mathfrak{t}_\mathfrak{g}'$ the corresponding root group. The set of occurring extremal weights λ is $\pm\mathcal{P}^+$, where

$$\mathcal{P}^+ := \{\mu \in \widehat{T}_G\colon\ \inf(\widehat{\mathcal{W}}\mu)(d) > -\infty\} \quad \textit{for} \quad d := (0, 0, -i) \in \mathfrak{g}.$$

Let $\mathcal{P}_d^+ \subseteq \mathcal{P}^+$ denote the set of those elements μ for which

$$\mu(d) = \min(\widehat{\mathcal{W}}\mu)(d).$$

For $\mu_c := \mu(i,0,0)$, the elements $\mu \in \mathcal{P}^+$ contained in $\mathcal{P}_d^+$ are characterized by:

$$\mu_c \geqslant 0, \quad |\mu(\check{\alpha})| \leqslant \frac{2\mu_c}{(\alpha,\alpha)}, \quad |\mu(\check{\beta})| \leqslant \frac{4\mu_c}{(\beta,\beta)} \quad \text{for } (\alpha,1),(\beta,0) \in \Delta_{\mathfrak{g}},$$

where $\check{\alpha} \in i\mathfrak{t}$ is the associated coroot. The parameter space of the equivalence classes of semibounded representations is given by

$$\pm\mathcal{P}^+/\widehat{\mathcal{W}} \cong \pm\mathcal{P}_d^+/\mathcal{W},$$

where $\mathcal{W} \subseteq \widehat{\mathcal{W}}$ is the Weyl group of the pair $(\mathfrak{k}^\phi, \mathfrak{t})$.

In all cases we obtain an explicit description of the set $\mathcal{P}_d^+/\mathcal{W}$ of $\mathcal{W}$ orbits in the set $\mathcal{P}_d^+$ of d-minimal integral weights which is based on a characterization of the d-minimal weights (Theorem 4.4) and the quite elementary classification of $\mathcal{W}$-orbits (Proposition 5.8). The remarkable observation that the intersection of a $\widehat{\mathcal{W}}$-orbit with the set $\mathcal{P}_d^+$ coincides with a $\mathcal{W}$-orbit is drawn from preliminary work on convex hulls of Weyl group orbits ([20]).

Structure of the paper. We start in Section 1 with the introduction of the simple Hilbert–Lie algebras and their root decompositions which leads to the four locally finite root systems A_J, B_J, C_J and D_J (*cf.* [29]). Our first main result is the determination of the full automorphism groups of the simple Hilbert–Lie algebras (Theorem 1.15). In Section 2 we introduce the double extensions $\mathfrak{g} = \widehat{\mathcal{L}}_\phi(\mathfrak{k})$ for the twisted loop algebras $\mathcal{L}_\phi(\mathfrak{k})$, where we restrict our attention to those automorphisms ϕ for which the corresponding root systems $\Delta_{\mathfrak{g}}$ are, as $\mathbb{Z}$-graded root systems, equal to one of the seven locally affine root systems $A_J^{(1)}, B_J^{(1)}, C_J^{(1)}, D_J^{(1)}, B_J^{(2)}, C_J^{(2)}$ or $BC_J^{(2)}$ (*cf.* Definition 2.6). In Section 3 we mount to the global level by showing that, for every 1-connected simple Hilbert–Lie group K, there exists a 1-connected Fréchet–Lie group $\widehat{\mathcal{L}}_\phi(K)$ which is a central $\mathbb{T}$-extension $\widetilde{\mathcal{L}}_\phi(K) \rtimes_\alpha \mathbb{R}$ of $\mathcal{L}_\phi(K) \rtimes_\alpha \mathbb{R}$. Section 4 focuses on the action of the Weyl group $\widehat{\mathcal{W}}$ on the integral weights. Here the main result is the explicit classification of the d-minimal weights in Theorem 4.4. After these preparations, we attack our goal of classifying the irreducible semibounded representations of $G = \widehat{\mathcal{L}}_\phi(K)$. The first major step is Theorem 5.2, asserting that for a semibounded representation $(\pi, \mathcal{H})$, the operator $\mathtt{d}\pi(d)$ is either bounded from below (positive energy representations) or from above. Up to passing to the dual representation, we may therefore assume that we are in the first case. Then the minimal spectral value of $\mathtt{d}\pi(d)$ turns out to be an eigenvalue and the group $Z_G(d)$ acts on the corresponding eigenspace, which leads to a bounded representation (ρ, V) of this group. To proceed further, we rely on some general results concerning holomorphic induction. This framework has been developed for Banach–Lie groups in [46] and in Appendix C we briefly explain how it can be carried over to certain Fréchet–Lie groups, containing in particular groups such as $\widehat{\mathcal{L}}_\phi(K)$. This permits us to conclude that the representation (ρ, V) is irreducible and that it determines $(\pi, \mathcal{H})$ uniquely. Since an explicit classification of the bounded irreducible representations of the groups $Z_G(d)_0$ is available from [37, 45] (Theorem 5.9) in terms of $\mathcal{W}$-orbits of extremal weights, it remains to characterize those weights λ for which the corresponding representation $(\rho_\lambda, V_\lambda)$ corresponds to a unitary representation of G. This is achieved in Theorem 5.10, asserting that this is equivalent to λ being d-minimal, and the final step consists in showing that the irreducible G-representation $(\pi_\lambda, \mathcal{H}_\lambda)$ corresponding to a d-minimal weight is actually semibounded (Theorem 6.1). Compared to related arguments in other contexts (*cf.* [37, 38, 42]), the argument we give here is rather direct and does not require any convexity results on projections of coadjoint orbits, such as [3, 27]. This brings us full circle and completes the proof of Theorem 0.1.

For untwisted loop groups $\widehat{\mathcal{L}}(K)$ and compact groups K, the corresponding class of representations is well-known from the context of affine Kac–Moody algebras (*cf.* [26], [52]). In this context one thus obtains the class of positive energy representations ($\mathtt{d}\pi(d)$ bounded from below), but this requirement is too weak for infinite-dimensional K. We therefore work with the semiboundedness condition which has the additional advantage that it is invariant under twisting with arbitrary automorphisms. Compared with the classical situation where K is finite-dimensional, we thus obtain the new insight that every positive energy representation is actually semibounded. In various respects our techniques are simpler than the ones used in the classical case to prove the existence of the unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ for a d-minimal weight λ (*cf.* [52], [18], [62]). Instead of using ad hoc operator estimates for the corresponding Lie algebra representation, we combine the technique of holomorphic induction and some general results on analytic positive definite functions (*cf.* Appendix B) to see that the d-minimality of λ, which is already known to lead to a unitary Lie algebra module on the algebraic level ([43]), to integrate to an analytic representation of the Lie group $\widehat{\mathcal{L}}_\phi(K)$. This is done by using the following new characterization: An operator-valued function $\phi\colon V \to B(\mathcal{K})$, $\mathcal{K}$ a Hilbert space, V an identity neighborhood of any Fréchet–BCH–Lie group

G, is positive definite in an identity neighborhood if and only if the corresponding linear map $U(\mathfrak{g}_{\mathbb{C}}) \to B(\mathcal{K})$ obtained by derivatives in $\mathbf{1}$ is positive definite (Theorem B.6).

Although it does not appear on the surface of our arguments, it is crucial that we deal with the Fréchet–Lie group $\widehat{\mathcal{L}}_\phi(K)$ through its Banach analog $\widehat{\mathcal{L}}_\phi^H(K)$ constructed similarly from H^1-maps instead of smooth ones. This is a topological group which is a Banach manifold and a semidirect product $\widetilde{\mathcal{L}}_\phi^H(K) \rtimes_\alpha \mathbb{R}$, where the factor on the left is a Banach–Lie group but the translation action of $\mathbb{R}$ is not smooth. As a byproduct, our techniques imply that the representations π_λ extend to continuous representations of $\widehat{\mathcal{L}}_\phi^H(K)$ which are analytic on $\widetilde{\mathcal{L}}_\phi(K)$ in the sense that the space of analytic vectors is dense (Remark 5.11). For the convenience of the reader, we collect some basic information on groups of H^1-maps in Appendix A, including the existence of the central Lie group extension $\widetilde{\mathcal{L}}_\phi^H(K)$.

In view of the classification of semibounded irreducible representations in terms of extremal weights, it is natural to ask why we first pass to the double extension of the group $\mathcal{L}_\phi(K)$ to study unitary representations. Without the double extension, the representation theory of loop groups is much less interesting: From Theorem 5.15 it follows that all semibounded unitary representations of the central extension $\widetilde{\mathcal{L}}_\phi(\mathfrak{k})$ are trivial on the center and factor through bounded representations of $\mathcal{L}(\mathfrak{k})$, which in turn are finite-dimensional and tensor products of evaluation representations (see [47] for the case of Banach–Lie algebras of maps and [24] for groups of smooth maps). We also show in Theorem 5.16 that all semibounded representations of $\mathcal{L}_\phi(K) \rtimes_\alpha \mathbb{R}$ are trivial on $\mathcal{L}_\phi(K)$. These two results clearly demonstrate that the double extension of $\mathcal{L}_\phi(\mathfrak{k})$ is crucial to get hold of the interesting class of semibounded representations.

To put our results into perspective, it is instructive to recall that if X is a compact space and K a semisimple compact Lie group, then all irreducible bounded unitary representations of $C(X, K)_0$ are finite tensor products of evaluation representations, hence in particular finite-dimensional ([47], [24]). Other irreducible representations $(\pi, \mathcal{H})$ of the loop groups $\mathcal{L}(K) \rtimes_\alpha \mathbb{R}$ (twisted loop modules) constructed by Chari and Pressley in [11] have the property that the spectrum of $\mathtt{d}\pi(d)$ is unbounded from below and above and their restrictions to $\widetilde{\mathcal{L}}(K)$ are not irreducible. For any, not necessary compact, Lie group K, the group $C(X, K)$ has unitary representations obtained as finite tensor products of evaluation representations. However, for some non-compact groups, such as $K = \widetilde{\mathrm{SU}}_{1,n}(\mathbb{C})$, one even has "continuous" tensor product representations which are irreducible (*cf.* [22], [7], [12], [64, 15]). In the algebraic context of loops group, these representations also appear in [23] which contains a classification of various types of unitary representations generalizing highest weight representations. In addition to these representations which actually extend to groups of measurable maps, there exist irreducible representations of mapping groups defined most naturally on groups of Sobolev H^1-maps, the so-called energy representations (*cf.* [1], [2]) and certain variants of positive energy representations of gauge groups of tori (*cf.* [63]). The problem to classify all smooth (projective) irreducible unitary representations of gauge groups is still wide open, although the classification of their central extensions by Janssens and Wockel ([25]) is a major step towards this goal. Bounded (projective) unitary representations of gauge groups are classified in [24], and the present paper contributes to this program by the construction of irreducible unitary representations of gauge groups for infinite-dimensional structure groups.

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Notation

We collect some basic notational conventions used below. We write $\mathbb{N} = \{1, 2, \ldots\}$ for the natural numbers.

Hilbert spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ are mostly denoted $\mathcal{H}$. We write $B(\mathcal{H})$ for the algebra of all bounded operators on $\mathcal{H}$, $B_2(\mathcal{H})$ for the ideal of *Hilbert-Schmidt operators*, $B_1(\mathcal{H})$ for the ideal of *trace class operators* and $K(\mathcal{H})$ for the ideal of *compact operators*. Accordingly we write

$$\mathrm{U}_2(\mathcal{H}) := \mathrm{U}(\mathcal{H}) \cap (\mathbf{1} + B_2(\mathcal{H}))$$

for the Hilbert–Lie group of unitary operators u for which $u - \mathbf{1}$ is Hilbert–Schmidt.

Let G be a Lie group (modeled on a locally convex space) and unit element $\mathbf{1}$. Then we write $\mathfrak{g} = \mathbf{L}(G)$ for its Lie algebra, which is identified with the tangent space $T_{\mathbf{1}}(G)$. The Lie bracket is obtained by identification with the Lie algebra of left invariant vector fields. A smooth map $\exp_G \colon \mathfrak{g} \to G$ is called an *exponential function* if each curve $\gamma_x(t) := \exp_G(tx)$ is a one-parameter group with $\gamma_x'(0) = x$. Not every infinite-dimensional Lie group has an exponential function ([41, Ex. II.5.5]), but exponential functions are unique whenever they exist, and this is in particular the case for all Banach–Lie groups.

1. Hilbert–Lie groups

In this section we briefly introduce the class of Hilbert–Lie algebras, the closest infinite-dimensional relatives of compact Lie algebras.

1.1. Hilbert–Lie algebras

DEFINITION 1.1. — (a) *A Hilbert–Lie algebra* $\mathfrak{k}$ *is a real Lie algebra endowed with the structure of a real Hilbert space such that the scalar product is invariant under the adjoint action, i.e.,*

$$\langle [x,y],z\rangle = \langle x,[y,z]\rangle \quad \text{for } x,y,z \in \mathfrak{k}.$$

From the Closed Graph Theorem and the Uniform Boundedness Principle one derives that the bracket $\mathfrak{k}\times\mathfrak{k}\to\mathfrak{k}$ *is continuous with respect to the norm topology on* $\mathfrak{k}$ *(cf.* [54, p. 70]*). A Hilbert–Lie algebra* $\mathfrak{k}$ *is called simple if* $\{0\}$ *and* $\mathfrak{k}$ *are the only closed ideals.*

Example 1.2. — (a) A finite-dimensional Lie algebra $\mathfrak{k}$ carries the structure of a Hilbert–Lie algebra if and only if it is compact.

(b) For any Hilbert space $\mathcal{H}$ over $\mathbb{K}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, the Lie algebra

$$\mathfrak{u}_2(\mathcal{H}) := \{x \in B_2(\mathcal{H}) : x^* = -x\}$$

is a Hilbert–Lie algebra with respect to the scalar product $\langle x,y\rangle := \mathrm{tr}_{\mathbb{R}}(xy^*) = -\,\mathrm{tr}_{\mathbb{R}}(xy)$. It is simple if $\dim\mathcal{H}=\infty$.

THEOREM 1.3 (Schue). — *Each Hilbert–Lie algebra* $\mathfrak{k}$ *is an orthogonal direct sum* $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \widehat{\bigoplus}_{j\in J}\mathfrak{k}_j$*, where each* $\mathfrak{k}_j$ *is a simple ideal. Each simple infinite-dimensional Hilbert–Lie algebra is isomorphic to* $\mathfrak{u}_2(\mathcal{H})$ *for an infinite-dimensional real, complex or quaternionic Hilbert space* $\mathcal{H}$*.*

Proof. — The orthogonal decomposition into center and simple ideals follows from [54, 1.2, Th. 1]. The classification of the simple Hilbert algebras $\mathfrak{k}$ follows immediately from the classification of the complex L^*-algebras because $\mathfrak{k}_{\mathbb{C}}$ is a complex L^*-algebra. For the separable case, the classification was obtained in [54, 3.7, Th. 3] under the assumption of the existence of a root decomposition whose existence was shown in [55]. The classification was extended to the non-separable case in [13], [49] and [59, Thm. 19.28]. □

DEFINITION 1.4. — *If $\mathcal{H}$ is a real Hilbert space, then we also write*

$$\mathrm{O}(\mathcal{H}) := \mathrm{U}(\mathcal{H}), \quad \mathrm{O}_2(\mathcal{H}) := \mathrm{U}_2(\mathcal{H}), \ \mathfrak{o}(\mathcal{H}) := \mathfrak{u}(\mathcal{H}), \quad \textit{and} \ \ \mathfrak{o}_2(\mathcal{H}) := \mathfrak{u}_2(\mathcal{H}).$$

For a quaternionic Hilbert space $\mathcal{H}$, we write

$$\mathrm{Sp}(\mathcal{H}) := \mathrm{U}(\mathcal{H}), \ \mathrm{Sp}_2(\mathcal{H}) := \mathrm{U}_2(\mathcal{H}), \ \mathfrak{sp}(\mathcal{H}) := \mathfrak{u}(\mathcal{H}) \ \ \textit{and} \ \ \mathfrak{sp}_2(\mathcal{H}) := \mathfrak{u}_2(\mathcal{H}).$$

THEOREM 1.5 (*cf.* [40], Sect. II.4). — *For an infinite-dimensional Hilbert space $\mathcal{H}$, over $\mathbb{R}$, $\mathbb{C}$, resp., $\mathbb{H}$, the homotopy groups of* $\mathrm{O}_2(\mathcal{H})$, $\mathrm{U}_2(\mathcal{H})$, *resp.,* $\mathrm{Sp}_2(\mathcal{H})$ *are given by:*

	$\mathrm{O}_2(\mathcal{H})$	$\mathrm{U}_2(\mathcal{H})$	$\mathrm{Sp}_2(\mathcal{H})$
π_0	$\mathbb{Z}/2$	$\{\mathbf{1}\}$	$\{\mathbf{1}\}$
π_1	$\mathbb{Z}/2$	$\mathbb{Z}$	$\{\mathbf{1}\}$
π_2	$\{\mathbf{1}\}$	$\{\mathbf{1}\}$	$\{\mathbf{1}\}$
π_3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

Remark 1.6. — If $\mathcal{H}$ is infinite-dimensional, Schur's Lemma implies that the center of the groups $\mathrm{O}_2(\mathcal{H})$, $\mathrm{U}_2(\mathcal{H})$ and $\mathrm{Sp}_2(\mathcal{H})$ is trivial. Therefore their fundamental group is isomorphic to the center of the simply connected covering group, so that

$$Z(\widetilde{\mathrm{O}}_2(\mathcal{H})_0) \cong \pi_1(\mathrm{O}_2(\mathcal{H})) \cong \mathbb{Z}/2 \quad \text{and} \quad Z(\widetilde{\mathrm{U}}_2(\mathcal{H})) \cong \pi_1(\mathrm{U}_2(\mathcal{H})) \cong \mathbb{Z}.$$

1.2. Root decomposition

Our parametrization of irreducible semibounded representations is based on weights w.r.t. a maximal abelian subalgebra. In this subsection we recall some basics on roots and root space decompositions.

DEFINITION 1.7. — (a) *Let $\mathfrak{g}$ be a real topological Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ be its complexification. If $\sigma\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}, z = x + iy \mapsto \overline{z} = x - iy$, denotes the complex conjugation with respect to $\mathfrak{g}$, we write $x^* := -\sigma(x)$ for $x \in \mathfrak{g}_{\mathbb{C}}$, so that $\mathfrak{g} = \{x \in \mathfrak{g}_{\mathbb{C}} \colon x^* = -x\}$. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a maximal abelian subalgebra and $\mathfrak{t}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ be its complexification. For a linear functional $\alpha \in \mathfrak{t}_{\mathbb{C}}'$ (the space of $\mathbb{C}$-valued continuous linear functionals on $\mathfrak{t}$ which is identified with the space of $\mathbb{C}$-linear continuous functionals on $\mathfrak{t}_{\mathbb{C}}$),*

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} = \{x \in \mathfrak{g}_{\mathbb{C}} \colon (\forall h \in \mathfrak{t}_{\mathbb{C}}) \ [h, x] = \alpha(h)x\}$$

is called the corresponding root space, and

$$\Delta := \Delta(\mathfrak{g}, \mathfrak{t}) := \{\alpha \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\} \colon \mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}\}$$

is the root system of the pair $(\mathfrak{g}, \mathfrak{t})$*. We then have* $\mathfrak{g}_{\mathbb{C}}^0 = \mathfrak{t}_{\mathbb{C}}$ *and* $[\mathfrak{g}_{\mathbb{C}}^\alpha, \mathfrak{g}_{\mathbb{C}}^\beta] \subseteq \mathfrak{g}_{\mathbb{C}}^{\alpha+\beta}$*, hence in particular* $[\mathfrak{g}_{\mathbb{C}}^\alpha, \mathfrak{g}_{\mathbb{C}}^{-\alpha}] \subseteq \mathfrak{t}_{\mathbb{C}}$*.*

(b) *If* $\mathfrak{g}$ *is the Lie algebra of a group* G *with an exponential function, then we call* $\mathfrak{t}$ *elliptic if the subgroup* $e^{\mathrm{ad}\,\mathfrak{t}} = \mathrm{Ad}(\exp \mathfrak{t}) \subseteq \mathrm{Aut}(\mathfrak{g})$ *is equicontinuous. We then have*

(I1) $\alpha(\mathfrak{t}) \subseteq i\mathbb{R}$ *for* $\alpha \in \Delta$*, and therefore*
(I2) $\sigma(\mathfrak{g}_{\mathbb{C}}^\alpha) = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$ *for* $\alpha \in \Delta$*.*

Lemma 1.8. — *Suppose that* $\mathfrak{t} \subseteq \mathfrak{g}$ *is elliptic. For* $0 \neq x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$*, the subalgebra* $\mathfrak{g}_{\mathbb{C}}(x_\alpha) := \mathrm{span}_{\mathbb{C}}\{x_\alpha, x_\alpha^*, [x_\alpha, x_\alpha^*]\}$ *is* σ*-invariant and of one of the following types:*

(A) *The abelian type:* $[x_\alpha, x_\alpha^*] = 0$*, i.e.,* $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ *is two-dimensional abelian.*
(N) *The nilpotent type:* $[x_\alpha, x_\alpha^*] \neq 0$ *and* $\alpha([x_\alpha, x_\alpha^*]) = 0$*, i.e.,* $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ *is a three-dimensional Heisenberg algebra.*
(S) *The simple type:* $\alpha([x_\alpha, x_\alpha^*]) \neq 0$*, i.e.,* $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cong \mathfrak{sl}_2(\mathbb{C})$*. In this case we distinguish two cases:*
  (CS) $\alpha([x_\alpha, x_\alpha^*]) > 0$*, i.e.,* $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cap \mathfrak{g} \cong \mathfrak{su}_2(\mathbb{C})$*, and*
  (NS) $\alpha([x_\alpha, x_\alpha^*]) < 0$*, i.e.,* $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cap \mathfrak{g} \cong \mathfrak{su}_{1,1}(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{R})$*.*

Proof (*cf.* [42], App. C). — First we note that, in view of $x_\alpha^* \in \mathfrak{g}_{\mathbb{C}}^{-\alpha}$, [37, Lemma I.2] applies, and we see that $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ is of one of the three types (A), (N) and (S). We note that $\alpha([x_\alpha, x_\alpha^*]) \in \mathbb{R}$ because of (I2) and $[x_\alpha, x_\alpha^*] \in i\mathfrak{t}$. Now it is easy to check that $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cap \mathfrak{g}$ is of type (CS), resp., (NS), according to the sign of this number. □

Definition 1.9. — (a) *Assume that* $\mathfrak{g}_{\mathbb{C}}^\alpha = \mathbb{C}x_\alpha$ *is one-dimensional and that* $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ *is of type (S). Then there exists a unique element* $\check{\alpha} \in \mathfrak{t}_{\mathbb{C}} \cap [\mathfrak{g}_{\mathbb{C}}^\alpha, \mathfrak{g}_{\mathbb{C}}^{-\alpha}]$ *with* $\alpha(\check{\alpha}) = 2$*. It is called the coroot of* α*. The root* $\alpha \in \Delta$ *is said to be compact if, for* $0 \neq x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$*, we have* $\alpha([x_\alpha, x_\alpha^*]) > 0$ *and non-compact otherwise. We write* Δ_c *for the set of compact roots. With the notation* $\mathbb{R}_+ := [0, \infty[$*, Lemma 1.8 implies that*

$$\check{\alpha} \in \mathbb{R}_+[x_\alpha, x_\alpha^*] \quad \text{for } \alpha \in \Delta_c. \tag{1.1}$$

(b) *The Weyl group* $\mathcal{W} = \mathcal{W}(\mathfrak{g}, \mathfrak{t}) \subseteq \mathrm{GL}(\mathfrak{t}_{\mathbb{C}})$ *is the subgroup generated by all reflections*

$$r_\alpha(x) := x - \alpha(x)\check{\alpha} \quad \text{for compact roots} \quad \alpha \in \Delta_c. \tag{1.2}$$

It acts on the dual space $\mathfrak{t}_{\mathbb{C}}^*$ *by the dual maps* $r_\alpha^*(\beta) := \beta - \beta(\check{\alpha})\alpha$*.*

(c) *A linear functional* $\lambda \in i\mathfrak{t}'$ *is said to be an integral weight if* $\lambda(\check{\alpha}) \in \mathbb{Z}$ *holds for every compact root* $\alpha \in \Delta_c$*. We write* $\mathcal{P} = \mathcal{P}(\mathfrak{g}, \mathfrak{t}) \subseteq i\mathfrak{t}'$ *for the group of all integral weights.*

Let $\mathfrak{k}$ be a Hilbert–Lie algebra and $\mathfrak{t} \subseteq \mathfrak{k}$ be a maximal abelian subalgebra. According to [55], $\mathfrak{t}_{\mathbb{C}} \subseteq \mathfrak{k}_{\mathbb{C}}$ defines a root space decomposition

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \widehat{\bigoplus_{\alpha \in \Delta}} \mathfrak{k}_{\mathbb{C}}^{\alpha}$$

which is a Hilbert space direct sum with respect to the hermitian extension of the scalar product to $\mathfrak{k}_{\mathbb{C}}$. We now describe the relevant root data for the three types of simple Hilbert algebras $\mathfrak{u}_2(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

Example 1.10 (cf. [45], Ex. C.4 (Root data of unitary Lie algebras)). — Let $\mathcal{H}$ be a complex Hilbert space with orthonormal basis $(e_j)_{j\in J}$ and $\mathfrak{t} \subseteq \mathfrak{k} := \mathfrak{u}_2(\mathcal{H})$ be the subalgebra of all diagonal operators with respect to the e_j. Then $\mathfrak{t}$ is elliptic and maximal abelian, $\mathfrak{t}_{\mathbb{C}} \cong \ell^2(J, \mathbb{C})$. The set of roots of $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{gl}_2(\mathcal{H})$ with respect to $\mathfrak{t}_{\mathbb{C}}$ is given by the root system

$$\Delta = \{\varepsilon_j - \varepsilon_k \colon j \neq k \in J\} =: A_J.$$

Here the operator $E_{jk}e_m := \delta_{km}e_j$ is a $\mathfrak{t}_{\mathbb{C}}$-eigenvector in $\mathfrak{gl}_2(\mathcal{H})$ generating the corresponding eigenspace and $\varepsilon_j(\operatorname{diag}(h_k)_{k\in J}) = h_j$. From $E_{jk}^* = E_{kj}$ it follows that

$$(\varepsilon_j - \varepsilon_k)\check{} = E_{jj} - E_{kk} = [E_{jk}, E_{kj}] = [E_{jk}, E_{jk}^*],$$

so that $\Delta = \Delta_c$, *i.e.*, all roots are compact.

The Weyl group $\mathcal{W}$ is isomorphic to the group $S_{(J)}$ of finite permutations of J, acting in the canonical way on $\mathfrak{t}_{\mathbb{C}} \cong \ell^2(J, \mathbb{C})$. It is generated by the reflections $r_{jk} := r_{\varepsilon_j - \varepsilon_k}$ corresponding to the transpositions of $j \neq k \in J$. The Weyl group acts transitively on the set of roots and, in particular, all roots have the same length 2 w.r.t. the scalar product induced by $\langle x, y\rangle = \operatorname{tr}(xy^*)$ on the dual space.

Remark 1.11. — (a) In many situations it is convenient to describe real Hilbert spaces as pairs $(\mathcal{H}, \sigma)$, where $\mathcal{H}$ is a complex Hilbert space and $\sigma \colon \mathcal{H} \to \mathcal{H}$ is a *conjugation*, *i.e.*, an antilinear isometry with $\sigma^2 = \mathrm{id}_{\mathcal{H}}$. Then we write $A^\top := \sigma A^* \sigma$, which corresponds to the transposition of matrices with respect to any ONB contained in $\mathcal{H}^\sigma$.

(b) A quaternionic Hilbert space $\mathcal{H}$ can be considered as a complex Hilbert space $\mathcal{H}^{\mathbb{C}}$ (the underlying complex Hilbert space), endowed with an *anticonjugation* σ, *i.e.*, σ is an antilinear isometry with $\sigma^2 = -\mathbf{1}$.

(c) That all conjugations and anticonjugations on a complex Hilbert space are conjugate under the unitary group $\mathrm{U}(\mathcal{H})$ has been shown in [5] by describing them in terms of orthonormal bases.

Example 1.12 (cf. [45], Ex. C.5 (Root data of symplectic Lie algebras)). For a complex Hilbert space $\mathcal{H}$ with a conjugation σ, we consider the quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}} := \mathcal{H}^2$, where the quaternionic structure is defined by the anticonjugation $\widetilde{\sigma}(v,w) := (\sigma w, -\sigma v)$. Then $\mathfrak{k} := \mathfrak{sp}_2(\mathcal{H}_{\mathbb{H}}) = \{x \in \mathfrak{u}_2(\mathcal{H}^2)\colon \widetilde{\sigma}x = x\widetilde{\sigma}\}$ and

$$\mathfrak{sp}_2(\mathcal{H}_{\mathbb{H}})_{\mathbb{C}} = \left\{\begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} \in B_2(\mathcal{H}^2)\colon B^\top = B, C^\top = C\right\}.$$

Let $(e_j)_{j\in J}$ be an orthonormal basis of $\mathcal{H}$ with $\sigma(e_j) = e_j$ for every j, and $\mathfrak{t} \subseteq \mathfrak{k}$ be the subalgebra of all diagonal operators with respect to the basis elements $(e_j, 0)$ and $(0, e_k)$ of $\mathcal{H}^2$. Then $\mathfrak{t}$ is elliptic and maximal abelian in $\mathfrak{k}$. Moreover, $\mathfrak{t}_{\mathbb{C}} \cong \ell^2(J,\mathbb{C})$ consists of diagonal operators of the form $h = \mathrm{diag}((h_j), (-h_j))$, and the set of roots of $\mathfrak{k}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ is given by

$$\Delta = \{\pm 2\varepsilon_j, \pm(\varepsilon_j \pm \varepsilon_k)\colon j \neq k, j,k \in J\} =: C_J,$$

where $\varepsilon_j(h) = h_j$. If we write $E_j = \begin{pmatrix} E_{jj} & 0 \\ 0 & -E_{jj} \end{pmatrix} \in \mathfrak{t}_{\mathbb{C}}$ for the element defined by $\varepsilon_k(E_j) = \delta_{jk}$, then the coroots are given by

$$(\varepsilon_j \pm \varepsilon_k)\check{} = E_j \pm E_k \quad \text{for } j \neq k \quad \text{and} \quad (2\varepsilon_j)\check{} = E_j. \tag{1.3}$$

Again, all roots are compact, and the Weyl group $\mathcal{W}$ is isomorphic to the group $\{\pm 1\}^{(J)} \rtimes S_{(J)}$, where $\{\pm 1\}^{(J)}$ is the group of finite sign changes on $\ell^2(J,\mathbb{R})$. In fact, the reflection $r_{\varepsilon_j - \varepsilon_k}$ acts as a transposition and the reflection $r_{2\varepsilon_j}$ changes the sign of the jth component. The Weyl group has two orbits in C_J, the short roots form a root system of type D_J and the second orbit is the set $\{\pm 2\varepsilon_j \colon j \in J\}$ of long roots.

Example 1.13 (cf. [45], Ex. C.6 (Root data of orthogonal Lie algebras)). Let $\mathcal{H}_{\mathbb{R}}$ be an infinite-dimensional real Hilbert space and $\mathfrak{k} := \mathfrak{o}_2(\mathcal{H}_{\mathbb{R}})$ be the corresponding simple Hilbert–Lie algebra. Let $\mathfrak{t} \subseteq \mathfrak{k}$ be maximal abelian. The fact that $\mathfrak{t}$ is maximal abelian implies that the common kernel $\ker \mathfrak{t}$ is at most one-dimensional. Since $\mathfrak{t}$ consists of compact skew-symmetric operators, the Lie algebra $\mathfrak{t}_{\mathbb{C}}$ is diagonalizable on the complexification $\mathcal{H} := (\mathcal{H}_{\mathbb{R}})_{\mathbb{C}}$. We conclude that, on the space $(\ker \mathfrak{t})^\perp$, we have an orthogonal complex structure I commuting with $\mathfrak{t}$ and there exists an orthonormal subset $(e_j)_{j\in J}$ of $(\ker \mathfrak{t})^\perp$ such that $\{e_j, Ie_j \colon j \in J\}$ is an orthonormal basis of $(\ker \mathfrak{t})^\perp$ and all the planes $\mathbb{R}e_j + \mathbb{R}Ie_j$ are $\mathfrak{t}$-invariant. If $\mathcal{H}_{\mathbb{R}}^{\mathfrak{t}}$ is non-zero, we write e_{j_0} for a unit vector in this space. For $j \in J$ put

$$f_j := \frac{1}{\sqrt{2}}(e_j - iIe_j) \quad \text{and} \quad f_{-j} := \frac{1}{\sqrt{2}}(e_j + iIe_j).$$

If $\ker \mathfrak{t} \neq \{0\}$, then we also put $f_{j_0} := e_{j_0}$. Then the f_j form an orthonormal basis of $\mathcal{H}$ consisting of $\mathfrak{t}$-eigenvectors.

We conclude that $\mathfrak{t}_{\mathbb{C}}$ is precisely the set of all those elements in $\mathfrak{k}_{\mathbb{C}} = \mathfrak{o}_2(\mathcal{H}_{\mathbb{R}})_{\mathbb{C}}$ which are diagonal with respect to the ONB consisting of the f_j. This implies that $\mathfrak{t}_{\mathbb{C}} \cong \ell^2(J,\mathbb{C})$, where $x \in \mathfrak{t}_{\mathbb{C}}$ corresponds to the element $(x_j)_{j\in J} \in \ell^2(J,\mathbb{C})$ defined by $xf_j = x_j f_j$, $j \in J$. Writing $\varepsilon_j(x) := x_j$, we see that $\{\pm\varepsilon_j : j \in J\}$, together with ε_{j_0} if $\ker \mathfrak{t} \neq \{0\}$, is the set of $\mathfrak{t}_{\mathbb{C}}$-weights of $\mathcal{H}$. Accordingly, the set of roots of $\mathfrak{k}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ is given by

$$\Delta = \{\pm\varepsilon_j \pm \varepsilon_k \colon j \neq k, j,k \in J\} =: D_J \quad \text{if } \ker \mathfrak{t} = \{0\},$$

and

$$\Delta = \{\pm\varepsilon_j \pm \varepsilon_k \colon j \neq k, j,k \in J\} \cup \{\pm\varepsilon_j \colon j \in J\} =: B_J \quad \text{otherwise.}$$

As in Example 1.12, all roots are compact. For B_J we obtain the same Weyl group $\{\pm 1\}^{(J)} \rtimes S_{(J)}$ as for C_J. For D_J the reflection $r_{\varepsilon_j+\varepsilon_k}$ changes the sign of the j- and the k-component, so that the Weyl group $\mathcal{W}$ is isomorphic to the group $\{\pm 1\}^{(J)}_{\text{even}} \rtimes S_{(J)}$, where $\{\pm 1\}^{(J)}_{\text{even}}$ is the group of finite even sign changes. For D_J the Weyl group acts transitively on the set of roots and all roots have the same length. For B_J we have two $\mathcal{W}$-orbits, the roots $\pm\varepsilon_j$ are short and the roots $\pm\varepsilon_j \pm \varepsilon_k$, $j \neq k$, are long.

Remark 1.14. — In a simple Hilbert–Lie algebra, two maximal abelian subalgebras are conjugate under the full automorphism group if and only if the corresponding root systems are isomorphic (see [59, Prop. 19.24, Rem. 19.25] and [5]). Up to conjugacy by automorphisms, the classification of locally finite root systems thus implies that we have only four types of pairs $(\mathfrak{k}, \mathfrak{t})$ and that they correspond to the root systems A_J, B_J, C_J and D_J.

For a real Hilbert space $\mathcal{H}$, the Hilbert–Lie algebra $\mathfrak{o}_2(\mathcal{H})$ contains two conjugacy classes of maximal abelian subalgebras $\mathfrak{t}$, distinguished by $\dim \mathcal{H}^{\mathfrak{t}} \in \{0, 1\}$. For the classification purposes in this paper, we only need one maximal abelian subalgebra to set up the parametrization of the equivalence classes of unitary representations (*cf.* Theorem 0.1). Passing to a different conjugacy class of maximal abelian subalgebras leads to a different parameter space for the same class of representations.

1.3. Automorphism groups

For a complex Hilbert space $\mathcal{H}$, we write $\mathrm{AU}(\mathcal{H})$ for the group of unitary or antiunitary isometries of $\mathcal{H}$ and

$$\mathrm{PAU}(\mathcal{H}) := \mathrm{AU}(\mathcal{H})/\mathbb{T}\mathbf{1} \cong \mathrm{PU}(\mathcal{H}) \rtimes \{\mathrm{id}, \sigma\},$$

where σ is an anticonjugation of $\mathcal{H}$.

Theorem 1.15. — *The automorphism groups of the simple infinite-dimensional Hilbert algebras are given by*

$$\mathrm{Aut}(\mathfrak{u}_2(\mathcal{H})) \cong \mathrm{PAU}(\mathcal{H})$$

for a complex Hilbert space, and for the real and quaternionic case we have

$$\mathrm{Aut}(\mathfrak{o}_2(\mathcal{H})) \cong \mathrm{O}(\mathcal{H})/\{\pm\mathbf{1}\} \quad \textit{and} \quad \mathrm{Aut}(\mathfrak{sp}_2(\mathcal{H})) \cong \mathrm{Sp}(\mathcal{H})/\{\pm\mathbf{1}\}.$$

Proof. — We know from Schue's Theorem 1.3 that any simple infinite-dimensional Hilbert–Lie algebra $\mathfrak{k}$ is isomorphic to $\mathfrak{u}_2(\mathcal{H})$ for some infinite-dimensional Hilbert space $\mathcal{H} \cong \ell^2(J, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a maximal abelian subalgebra and Δ be the corresponding root system. As we have seen in Examples 1.10, 1.12 and 1.13, it is of type A_J (for $\mathbb{K} = \mathbb{C}$), C_J (for $\mathbb{K} = \mathbb{H}$) or B_J, D_J (for $\mathbb{K} = \mathbb{R}$).

For any $\phi \in \mathrm{Aut}(\mathfrak{k})$, the subspace $\phi(\mathfrak{t}) \subseteq \mathfrak{k}$ is also maximal abelian with isomorphic root system. It follows from [5, Thm. 2] (and its proof) that for real, complex and quaternionic Hilbert spaces, the group $\mathrm{U}(\mathcal{H})$ acts transitive on the set of all maximal abelian subalgebras of $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$ whose root system is of a given type (see also [59, Thm. 19.24]). This implies the existence of $\psi \in \mathrm{U}(\mathcal{H})$ for which the corresponding automorphism $c_\psi(x) := \psi x \psi^{-1}$ satisfies $c_\psi(\mathfrak{t}) = \phi(\mathfrak{t})$. Then $c_\psi^{-1} \circ \phi$ fixes $\mathfrak{t}$, hence induces an automorphism of the root system Δ.

For each root system, the automorphism group is known from [60, Props. 5.1-5.4]:

$$\mathrm{Aut}(A_J) \cong S_J \times \{\pm \mathrm{id}\}, \quad \mathrm{Aut}(B_J) \cong \mathrm{Aut}(C_J) \cong \mathrm{Aut}(D_J) \cong (\mathbb{Z}/2)^J \rtimes S_J.$$

From this description it easily follows that, for Δ of type B_J, C_J or D_J, each automorphism of the root system is implemented by conjugation with an element of the corresponding full group $\mathrm{U}(\mathcal{H})$. For A_J, the elements of S_J are obtained by conjugation with a unitary operator permuting the elements $(e_j)_{j\in J}$ of an orthonormal basis, and $-\mathrm{id}$ is obtained by $\phi(x) = \sigma x \sigma$, where σ is a conjugation fixing each e_j, $j \in J$.

For the realization of $\mathfrak{sp}_2(\mathcal{H}_\mathbb{H})$ as in Example 1.12, we obtain the elements of S_J by conjugation with unitary operators of the form $\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$, where

$U \in \mathrm{U}(\mathcal{H})$ permutes the elements of the ONB $(e_j)_{j\in J}$. To implement the sign changes corresponding to the element $\chi \in \{\pm 1\}^J$ taking the value -1 on the subset $M \subseteq J$ and 1 elsewhere, we consider the projection P on $\mathcal{H}$ with

$$Pe_j = \begin{cases} e_j & \text{for } j \in M \\ 0 & \text{for } j \in J \setminus M. \end{cases}$$

Then conjugation with $u := \begin{pmatrix} \mathbf{1} - P & P \\ -P & \mathbf{1} - P \end{pmatrix} \in \mathrm{Sp}(\mathcal{H}_{\mathbb{H}})$ induces on $\Delta = B_J$ the automorphism corresponding to χ.

For the realization of $\mathfrak{o}_2(\mathcal{H}_{\mathbb{R}})$ as in Example 1.13, we obtain the elements $\pi \in S_J$ by conjugation with orthogonal operators U satisfying $Ue_j = e_{\pi(j)}$ for $j \in J$ and commuting with the complex structure I. To implement the sign changes represented by $\chi \in \{\pm 1\}^J$ as above, we conjugate with $U \in \mathrm{O}(\mathcal{H}_{\mathbb{R}})$ satisfying $Ue_j = e_j$ for $j \in J$ and

$$UIe_j = \begin{cases} -Ie_j & \text{for } j \in M \\ Ie_j & \text{for } j \in J \setminus M. \end{cases}$$

This reduces the problem to show that every automorphism is given by conjugation with a unitary (or antiunitary operator in the complex case) to the special case where it preserves $\mathfrak{t}$ and induces the trivial automorphism on Δ. In view of [60, Lemma 6.2(b)], any such automorphism ϕ satisfies $\phi(x_\alpha) = \chi(\alpha)x_\alpha$ for $x_\alpha \in \mathfrak{k}_{\mathbb{C}}^\alpha$ and a homomorphism $\chi\colon \mathcal{Q} := \langle\Delta\rangle_{\mathrm{grp}} \to \mathbb{C}^\times$. As ϕ is supposed to be isometric, we have $\mathrm{im}(\chi) \subseteq \mathbb{T}$. Conversely, the orthogonality of the root decomposition of $\mathfrak{k}_{\mathbb{C}}$ implies that every homomorphism $\chi\colon \mathcal{Q} \to \mathbb{T}$ occurs. We now show that any such automorphism is given by conjugation with an element of $U \in \mathrm{U}(\mathcal{H})$. For $\Delta = A_J$ we pick an element $j_0 \in J$ and put

$$u_{j_0} := 1 \quad \text{and} \quad u_j := \chi(\varepsilon_j - \varepsilon_{j_0}) \quad \text{for } j \neq j_0$$

to find the required element $U = \mathrm{diag}((u_j)) \in \mathrm{U}(\mathcal{H})$. For C_J we first extend χ to the $\mathbb{Z}$-span of $\{\varepsilon_j \colon j \in J\}$ ($\mathbb{T}$ is divisible) and put $U := \mathrm{diag}(\chi(\varepsilon_j), \chi(-\varepsilon_j))$. For B_J we use the same element $U \in \mathrm{U}((\mathcal{H}_{\mathbb{R}})_{\mathbb{C}})$, and for D_J we first extend χ to the $\mathbb{Z}$-span of B_J and proceed with the diagonal operator U with $u_{j_0} = 1$ and $u_{\pm j} = \chi(\varepsilon_j)^{\pm 1}$ for $j \in J$ (*cf.* [60, Sect. 6] for a similar argument in the algebraic context).

Finally, we note that, if $g \in \mathrm{U}(\mathcal{H})$ induces the trivial automorphism $c_g = \mathrm{id}$ on $\mathfrak{u}_2(\mathcal{H})$, then $g \in \mathbb{T}\mathbf{1}$ in the complex case and $g \in \{\pm\mathbf{1}\}$ in the real and quaternionic case. We thus obtain $\mathrm{Aut}(\mathfrak{o}_2(\mathcal{H})) \cong \mathrm{O}(\mathcal{H})/\{\pm\mathbf{1}\}$, $\mathrm{Aut}(\mathfrak{sp}_2(\mathcal{H})) \cong \mathrm{Sp}(\mathcal{H})/\{\pm\mathbf{1}\}$, and $\mathrm{Aut}(\mathfrak{u}_2(\mathcal{H})) \cong \mathrm{PAU}(\mathcal{H})$. □

Corollary 1.16. — *The automorphism groups of* $\mathfrak{o}_2(\mathcal{H})$ *and* $\mathfrak{sp}_2(\mathcal{H})$ *are connected, whereas the automorphism group of* $\mathfrak{u}_2(\mathcal{H})$ *(*$\mathcal{H}$ *a complex Hilbert space), has two connected components.*

Proof. — It only remains to recall that the groups $\mathrm{O}(\mathcal{H})$, $\mathrm{Sp}(\mathcal{H})$ and $\mathrm{U}(\mathcal{H})$ are connected for any infinite-dimensional real, quaternionic, resp., complex Hilbert space (*cf.* [40, Thm. II.6]). □

Remark 1.17. — The preceding corollary implies in particular that for $\mathfrak{k} = \mathfrak{o}_2(\mathcal{H}), \mathfrak{sp}_2(\mathcal{H})$, each automorphism of $\mathfrak{k}$ acts trivially on the homotopy groups $\pi_j(K), j \in \mathbb{N}$, for any connected Lie group K with Lie algebra $\mathfrak{k}$.

Proposition 1.18. — *Let* $\mathcal{H}$ *be a complex Hilbert space and* σ *be a conjugation of* $\mathcal{H}$*. For* $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$ *and the automorphism* $\phi(x) = \sigma x \sigma$ *of* $\mathrm{U}_2(\mathcal{H})$*, we then have*

$$\pi_{2k-1}(\phi) = (-1)^k \,\mathrm{id} \quad \text{for } k \in \mathbb{N}.$$

For any connected Lie group K *with the Lie algebra* $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$*,* $\pi_1(K)$ *either is trivial or isomorphic to* $\mathbb{Z}$*, there exists a* $\phi_K \in \mathrm{Aut}(K)$ *with* $\mathbf{L}(\phi_K) = \phi$*, and this automorphism satisfies* $\pi_1(\phi_K) = -\,\mathrm{id}$*.*

Proof. — We pick an orthonormal basis $(e_j)_{j\in J}$ in $\mathcal{H}$ fixed pointwise by σ, and represent operators accordingly as matrices. Then the involution ϕ is given by component-wise conjugation $\phi(x_{ij}) = (\overline{x_{ij}})$. Using the approximation techniques described in [40, Thm. II.14, Cor. II.15], it suffices to study the action of ϕ on the subgroup $\mathrm{U}_n(\mathbb{C})$, fixing all but n basis vectors. Therefore it follows from [28, Prop. 19] that $\pi_{2k-1}(\phi) = (-1)^k \,\mathrm{id}$.

If K is a connected Lie group with $\mathbf{L}(K) \cong \mathfrak{u}_2(\mathcal{H})$ which is not simply connected, then it is a quotient of $\widetilde{\mathrm{U}}_2(\mathcal{H})$ by an infinite cyclic group because $Z(\widetilde{\mathrm{U}}_2(\mathcal{H})) \cong \mathbb{Z}$ by Remark 1.6. The automorphism ϕ of $\mathfrak{u}_2(\mathcal{H})$ induces an automorphism $\widetilde{\phi}_K$ of $\widetilde{\mathrm{U}}_2(\mathcal{H})$, and this automorphism preserves the center. In view of $\mathrm{Aut}(\mathbb{Z}) = \{\pm \,\mathrm{id}_{\mathbb{Z}}\}$, it also preserves all subgroups of the center. We conclude that it also induces an automorphism ϕ_K on K. Since $\widetilde{\phi}_K$ acts on $Z(\widetilde{\mathrm{U}}_2(\mathcal{H})) \cong \pi_1(\mathrm{U}_2(\mathcal{H}))$ by inversion, we obtain $\pi_1(\phi_K) = -\,\mathrm{id}$. □

Combining Remark 1.17 with the preceding proposition, we obtain in particular

Corollary 1.19. — *If* K *is a Hilbert–Lie group for which* $\mathfrak{k}$ *is simple and* $\phi \in \mathrm{Aut}(K)$*, then* $\pi_3(\phi) = \mathrm{id}$*.*

2. Double extensions of twisted loop algebras

In this section we introduce the double extensions $\mathfrak{g} = \widehat{\mathcal{L}}_\phi(\mathfrak{k})$ for the twisted loop algebras $\mathcal{L}_\phi(\mathfrak{k})$, where we restrict our attention to those automorphisms ϕ for which the corresponding root systems $\Delta_\mathfrak{g}$ are, as $\mathbb{Z}$-graded root systems, equal to one of the seven locally affine root systems $A_J^{(1)}, B_J^{(1)}, C_J^{(1)}, D_J^{(1)}, B_J^{(2)}, C_J^{(2)}$ or $BC_J^{(2)}$ (*cf.* Definition 2.6).

2.1. Root decomposition of double extensions

Definition 2.1. — *A quadratic (topological) Lie algebra is a pair $(\mathfrak{g}, \kappa)$, consisting of a topological Lie algebra $\mathfrak{g}$ and a non-degenerate invariant symmetric continuous bilinear form κ on $\mathfrak{g}$. Suppose that $\mathfrak{t} \subseteq \mathfrak{g}$ is maximal abelian and elliptic and that $\sum_\alpha \mathfrak{g}_\mathbb{C}^\alpha$ is dense in $\mathfrak{g}_\mathbb{C}$. We extend κ to a hermitian form on $\mathfrak{g}_\mathbb{C}$ which is also denoted κ. Then the root spaces satisfy $\kappa(\mathfrak{g}_\mathbb{C}^\alpha, \mathfrak{g}_\mathbb{C}^\beta) = \{0\}$ for $\alpha \neq \beta$ (cf. Definition 1.7), and in particular, κ is non-degenerate on $\mathfrak{t}_\mathbb{C} = \mathfrak{g}_\mathbb{C}^0$ and all the root spaces $\mathfrak{g}_\mathbb{C}^\alpha$. We thus obtain an injective antilinear map $\flat \colon \mathfrak{t}_\mathbb{C} \to \mathfrak{t}_\mathbb{C}', h \mapsto h^\flat, h^\flat(x) := \kappa(x, h)$, where $\mathfrak{t}_\mathbb{C}'$ denotes the space of continuous linear functionals on $\mathfrak{t}_\mathbb{C}$. For $\alpha \in \mathfrak{t}_\mathbb{C}^\flat := \flat(\mathfrak{t}_\mathbb{C})$ we put $\alpha^\sharp := \flat^{-1}(\alpha)$ and define a hermitian form on on $\mathfrak{t}_\mathbb{C}^\flat$ by*

$$(\alpha, \beta) := \kappa(\beta^\sharp, \alpha^\sharp) = \alpha(\beta^\sharp) = \overline{\beta(\alpha^\sharp)}. \tag{2.1}$$

For $h \in \mathfrak{t}_\mathbb{C}$ and $x_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha$ we then have

$$\alpha(h)\kappa(x_\alpha, x_\alpha) = \kappa([h, x_\alpha], x_\alpha) = \kappa(h, [x_\alpha, x_\alpha^*]). \tag{2.2}$$

Since κ is non-degenerate on $\mathfrak{g}_\mathbb{C}^\alpha$, we may choose x_α such that $\kappa(x_\alpha, x_\alpha) \neq 0$. Then the non-degeneracy of κ on $\mathfrak{t}_\mathbb{C}$ leads to $\alpha \in \mathfrak{t}_\mathbb{C}^\flat$ and

$$[x_\alpha, x_\alpha^*] = \kappa(x_\alpha, x_\alpha)\alpha^\sharp. \tag{2.3}$$

This shows that $\Delta \subseteq \mathfrak{t}_\mathbb{C}^\flat$, so that (α, β) is defined for $\alpha, \beta \in \Delta$ by (2.1).

Remark 2.2. — If α is compact and $\check{\alpha} = [x_\alpha, x_\alpha^*]$ (*cf.* Definition 1.9), then (2.2) and $\alpha(\check{\alpha}) = 2$ imply $\kappa(\check{\alpha}, \check{\alpha}) = 2\kappa(x_\alpha, x_\alpha)$, which leads for $\beta \in \mathfrak{t}_\mathbb{C}^\flat$ to

$$\alpha^\sharp = \frac{2\check{\alpha}}{\kappa(\check{\alpha}, \check{\alpha})}, \quad (\alpha, \alpha) = \frac{4}{\kappa(\check{\alpha}, \check{\alpha})} \quad \text{and} \quad (\beta, \alpha) = \frac{2\beta(\check{\alpha})}{\kappa(\check{\alpha}, \check{\alpha})}. \tag{2.4}$$

Definition 2.3 (Double extensions). — *Let* $(\mathfrak{g},\kappa)$ *be a real quadratic Fréchet–Lie algebra and* $D \in \operatorname{der}(\mathfrak{g},\kappa)$ *be a derivation which is skew-symmetric with respect to* κ. *Then* $\omega_D(x,y) := \kappa(Dx,y)$ *defines a continuous* 2*-cocycle on* $\mathfrak{g}$, *and* D *extends to a derivation* $\widetilde{D}(z,x) := (0,Dx)$ *of the corresponding central extension* $\mathbb{R}\oplus_{\omega_D}\mathfrak{g}$. *The Lie algebra*

$$\widehat{\mathfrak{g}} := \mathfrak{g}(\kappa,D) := (\mathbb{R}\oplus_{\omega_D}\mathfrak{g}) \rtimes_{\widetilde{D}} \mathbb{R}$$

with the Lie bracket

$$[(z,x,t),(z',x',t')] = (\kappa(Dx,x'),[x,x'] + tDx' - t'Dx, 0)$$

is called the corresponding double extension. It carries a non-degenerate invariant symmetric bilinear form

$$\widehat{\kappa}((z,x,t),(z',x',t)) = \kappa(x,x') + zt' + z't,$$

so that $(\widehat{\mathfrak{g}},\widehat{\kappa})$ *also is a quadratic Fréchet–Lie algebra (cf.* [33]*). In terms of the hermitian extension of* κ *to* $\mathfrak{g}_{\mathbb{C}}$, *the Lie bracket on* $\widehat{\mathfrak{g}}_{\mathbb{C}}$ *is given by*

$$(2.5)\qquad [(z,x,t),(z',x',t')] = (\kappa(Dx,\overline{x'}),[x,x'] + tDx' - t'Dx, 0).$$

Example 2.4. — Let $(\mathfrak{k},\langle\cdot,\cdot\rangle)$ be a Hilbert–Lie algebra and let $\mathfrak{t} \subseteq \mathfrak{k}$ be a maximal abelian subalgebra, so that $\mathfrak{k}$ has a root decomposition with respect to $\mathfrak{t}$ (*cf.* [55]) and all roots in $\Delta(\mathfrak{k},\mathfrak{t})$ are compact. We consider the corresponding loop algebra $\mathcal{L}(\mathfrak{k})$ of 2π-periodic smooth functions $\mathbb{R}\to\mathfrak{k}$. Then

$$\langle\xi,\eta\rangle := \frac{1}{2\pi}\int_0^{2\pi}\langle\xi(t),\eta(t)\rangle\,dt$$

defines a non-degenerate invariant symmetric bilinear form on $\mathcal{L}(\mathfrak{k})$. We use the same notation for the unique hermitian extensions of $\langle\cdot,\cdot\rangle$ to $\mathfrak{k}_{\mathbb{C}}$ and to $\mathcal{L}(\mathfrak{k})_{\mathbb{C}}$. Further, $D\xi := \xi'$ is a $\langle\cdot,\cdot\rangle$-skew symmetric derivation on $\mathcal{L}(\mathfrak{k})$, so that we may form the associated double extension

$$\mathfrak{g} := \widehat{\mathcal{L}}(\mathfrak{k}) := (\mathbb{R}\oplus_{\omega_D}\mathcal{L}(\mathfrak{k})) \rtimes_{\widetilde{D}} \mathbb{R},$$

where $\omega_D(\xi,\eta) = \langle\xi',\eta\rangle$ and $\widetilde{D}(z,\xi) := (0,\xi')$ is the canonical extension of D to the central extension $\mathbb{R}\oplus_{\omega_D}\mathcal{L}(\mathfrak{k})$ (*cf.* Definition 2.3). This Lie algebra is called the *affinization of the quadratic Lie algebra* $(\mathfrak{k},\langle\cdot,\cdot\rangle)$. Now

$$\kappa((z_1,\xi_1,t_1),(z_2,\xi_2,t_2)) := z_1t_2 + z_2t_1 + \langle\xi_1,\xi_2\rangle$$

is a continuous invariant Lorentzian symmetric bilinear form on $\mathfrak{g}$ and $\mathfrak{t}_{\mathfrak{g}} := \mathbb{R}\oplus\mathfrak{t}\oplus\mathbb{R}$ is maximal abelian and elliptic in $\mathfrak{g}$ (*cf.* Definition 1.7(b)). In the following we identify $\mathfrak{t}$ with the subspace $\{0\}\times\mathfrak{t}\times\{0\}$ of $\mathfrak{t}_{\mathfrak{g}}$. We put

$$(2.6)\qquad c := (i,0,0) \in i\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}},\quad d := (0,0,-i)\quad\text{and}\quad e_n(t) := e^{int}.$$

Then c is central and the eigenvalue of $\operatorname{ad} d$ on $e_n \otimes \mathfrak{k}_{\mathbb{C}}$ is n. It is now easy to verify that the set $\Delta_{\mathfrak{g}}$ of roots of $(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})$ can be identified with the set

$$(\Delta(\mathfrak{k},\mathfrak{t}) \times \mathbb{Z}) \cup (\{0\} \times (\mathbb{Z} \setminus \{0\})) \subseteq i\mathfrak{t}' \times \mathbb{R},$$
$$\text{where} \quad (\alpha, n)(z,h,t) := (0,\alpha,n)(z,h,t) = \alpha(h) + itn.$$

The roots $(0,n)$, $0 \neq n \in \mathbb{Z}$, corresponding to the root spaces $e_n \otimes \mathfrak{t}_{\mathbb{C}}$, $n \neq 0$, are of nilpotent type and $(\Delta_{\mathfrak{g}})_c = \Delta(\mathfrak{k},\mathfrak{t}) \times \mathbb{Z}$ is the set of compact roots.

For $\alpha \in \Delta(\mathfrak{k},\mathfrak{t})$ pick $x_\alpha \in \mathfrak{k}_{\mathbb{C}}^\alpha$ with $[x_\alpha, x_\alpha^*] = \check{\alpha}$, so that we obtain with equations (2.3) and (2.4) the relation

$$\langle x_\alpha, x_\alpha \rangle = \frac{\langle \check{\alpha}, \check{\alpha} \rangle}{2} = \frac{2}{(\alpha,\alpha)}, \tag{2.7}$$

which leads with (2.5) to

$$\langle D(e_n \otimes x_\alpha), \overline{e_{-n} \otimes x_\alpha^*} \rangle = \langle i n e_n \otimes x_\alpha, -e_n \otimes x_\alpha \rangle = -in \langle x_\alpha, x_\alpha \rangle = -\frac{2in}{(\alpha,\alpha)}.$$

For the corresponding root vectors $x_{(\alpha,n)} = e_n \otimes x_\alpha \in \mathfrak{g}_{\mathbb{C}}^{(\alpha,n)}$, we thus obtain with (2.5)

$$[e_n \otimes x_\alpha, (e_n \otimes x_\alpha)^*] = [e_n \otimes x_\alpha, e_{-n} \otimes x_\alpha^*] = \Big(-\frac{2in}{(\alpha,\alpha)}, \check{\alpha}, 0 \Big).$$

Since, by definition, (α, n) takes the value 2 on this element, it follows that

$$(\alpha,n)\check{} = \Big(-\frac{2in}{(\alpha,\alpha)}, \check{\alpha} \Big) = \check{\alpha} - \frac{2n}{(\alpha,\alpha)} c = \check{\alpha} - \frac{n\|\check{\alpha}\|^2}{2} c. \tag{2.8}$$

For a linear functional $\lambda \in i\mathfrak{t}_{\mathfrak{g}}' \cong i\mathbb{R} \times i\mathfrak{t}' \times i\mathbb{R}$ and $\lambda_c := \lambda(c)$, we conclude that λ is an integral weight if and only if

$$\lambda\big((\alpha,n)\check{}\big) = \lambda(\check{\alpha}) - \frac{2n}{(\alpha,\alpha)} \lambda_c \in \mathbb{Z} \quad \text{for} \quad 0 \neq n \in \mathbb{Z},\ \alpha \in \Delta(\mathfrak{k},\mathfrak{t})$$

(*cf.* Definition 1.9(c)). This means that

$$\mathcal{P}(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}}) = \Big\{ \lambda \in i\mathfrak{t}_{\mathfrak{g}}' \colon \lambda|_{\mathfrak{t}} \in \mathcal{P}(\mathfrak{k},\mathfrak{t}),\ (\forall \alpha \in \Delta(\mathfrak{k},\mathfrak{t}))\ \lambda_c \in \frac{(\alpha,\alpha)}{2}\mathbb{Z} \Big\}. \tag{2.9}$$

Example 2.5. — In addition to the setting of the preceding example, we assume that $\mathfrak{k}$ is semisimple and let $\phi \in \operatorname{Aut}(\mathfrak{k})$ be an automorphism of order N. Then

$$\mathcal{L}_\phi(\mathfrak{k}) := \Big\{ \xi \in C^\infty(\mathbb{R}, \mathfrak{k}) \colon (\forall t \in \mathbb{R})\ \xi\Big(t + \frac{2\pi}{N}\Big) = \phi^{-1}(\xi(t)) \Big\}$$

is a closed Lie subalgebra of $\mathcal{L}(\mathfrak{k})$. Accordingly, we obtain a Lie subalgebra

$$\mathfrak{g} := \widehat{\mathcal{L}}_\phi(\mathfrak{k}) := (\mathbb{R} \oplus_{\omega_D} \mathcal{L}_\phi(\mathfrak{k})) \rtimes_{\widetilde{D}} \mathbb{R} \subseteq \widehat{\mathcal{L}}(\mathfrak{k}),$$

called the *ϕ-twisted affinization of* $(\mathfrak{k}, \langle \cdot, \cdot \rangle)$.

Let $\mathfrak{t} \subseteq \mathfrak{k}^\phi$ be a maximal abelian subalgebra, so that $\mathfrak{t}_\mathfrak{k} := \mathfrak{z}_\mathfrak{k}(\mathfrak{t})$ is maximal abelian in $\mathfrak{k}$ by Lemma D.2. Then $\mathfrak{t}_\mathfrak{g} = \mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}$ is maximal abelian in $\mathcal{L}_\phi(\mathfrak{k})$ and $\Delta_\mathfrak{g} := \Delta(\mathfrak{g}, \mathfrak{t}_\mathfrak{g})$ can be identified with the set of pairs (α, n), where

$$(\alpha, n)(z, h, t) := (0, \alpha, n)(z, h, t) = \alpha(h) + itn, \quad n \in \mathbb{Z},\ \alpha \in \Delta_n,$$

where $\Delta_n \subseteq i\mathfrak{t}'$ is the set of $\mathfrak{t}$-weights in $\mathfrak{k}_\mathbb{C}^n = \{x \in \mathfrak{k}_\mathbb{C} \colon \phi^{-1}(x) = e^{2\pi i n/N} x\}$. For $(\alpha, n) \neq (0, 0)$, the corresponding root space is

$$\mathfrak{g}_\mathbb{C}^{(\alpha,n)} = e_n \otimes \mathfrak{k}_\mathbb{C}^{(\alpha,n)} = e_n \otimes (\mathfrak{k}_\mathbb{C}^\alpha \cap \mathfrak{k}_\mathbb{C}^n), \quad \text{where} \quad e_n(t) = e^{int}.$$

The discussion in Appendix D implies that

$$(\Delta_\mathfrak{g})_c = \{(\alpha, n) \colon 0 \neq \alpha \in \Delta_n, n \in \mathbb{Z}\}.$$

This leads to the N-fold layer structure

$$(\Delta_\mathfrak{g})_c = \bigcup_{n=0}^{N-1} \Delta_n^\times \times (n + N\mathbb{Z}), \quad \text{where} \quad \Delta_n^\times := \Delta_n \setminus \{0\}.$$

For $n \in \mathbb{Z}$ and $x \in \mathfrak{k}_\mathbb{C}^{(\alpha,n)}$ with $[x, x^*] = \check{\alpha}$ (*cf.* Appendix D), the element $e_n \otimes x \in \mathfrak{g}_\mathbb{C}^{(\alpha,n)}$ satisfies $(e_n \otimes x)^* = e_{-n} \otimes x^*$, which leads to the coroot

$$[e_n \otimes x, (e_n \otimes x)^*] = (\alpha, n)\check{} = \Big(-\frac{2in}{(\alpha, \alpha)}, \check{\alpha}, 0\Big) = \check{\alpha} - \frac{2n}{(\alpha, \alpha)} c.$$

As $(\alpha, n) \in \Delta_\mathfrak{g}$ implies $(\alpha, n + kN) \in \Delta_\mathfrak{g}$ for every $k \in \mathbb{Z}$, we obtain the following description of the integral weights

(2.10)
$$\mathcal{P}(\mathfrak{g}, \mathfrak{t}_\mathfrak{g}) = \Big\{\lambda \in i\mathfrak{t}_\mathfrak{g}' \colon (\forall \alpha \in \Delta_n^\times, n \in \mathbb{Z})\ \lambda_c \in \frac{(\alpha, \alpha)}{2N}\mathbb{Z}, \lambda(\check{\alpha}) \in \mathbb{Z} + \frac{2n}{(\alpha, \alpha)}\lambda_c\Big\}.$$

Suppose that $\mathfrak{k}$ is simple. Then Lemma D.3 implies that $(\Delta_\mathfrak{g})_c$ does not decompose into two mutually orthogonal proper subsets, so that

$$\widehat{\mathcal{L}}_\phi(\mathfrak{k})_\mathbb{C}^{\mathrm{alg}} := \mathbb{C}c + \mathbb{C}d + \operatorname{span}_\mathbb{C} \check{\Delta}_\mathfrak{g} + \sum_{(\alpha,n) \in \Delta_\mathfrak{g}} \mathfrak{g}_\mathbb{C}^{(\alpha,n)}$$

is a locally extended affine Lie algebra in the sense of [43, Def. 1.2] (see also [35]). Since only the roots of the form $(0, n)$, $n \in \mathbb{Z}$, are isotropic, [43, Prop. 2.5] further implies that $(\Delta_\mathfrak{g})_c$ is a locally affine root system.

Definition 2.6. — *Instead of going into the axiomatics of locally affine root systems developed in* [67], *we only recall that a locally affine root system is in particular a subset Δ of a vector space V endowed with a positive semidefinite form. For a reduced locally finite root system Δ in the euclidean space V (such as A_J, B_J, C_J or D_J from Examples 1.10, 1.12, 1.13), we put $\Delta^{(1)} := \Delta \times \mathbb{Z} \subseteq V \times \mathbb{R}$, where the scalar product on $V \times \mathbb{R}$*

is defined by $((\alpha,t),(\alpha',t')) := (\alpha,\alpha')$*. According to Yoshii's classification* ([67, Cor. 13])*, there exist* 7 *isomorphy classes of irreducible reduced locally affine root system of infinite rank: the four untwisted reduced root systems* $A_J^{(1)}, B_J^{(1)}, C_J^{(1)}, D_J^{(1)}$*, and, for* $BC_J := B_J \cup C_J$*, the three twisted root systems*

$$\begin{aligned} B_J^{(2)} &:= \big(B_J \times 2\mathbb{Z}\big) \cup \big(\{\pm\varepsilon_j : j \in J\} \times (2\mathbb{Z}+1)\big), \\ C_J^{(2)} &:= (C_J \times 2\mathbb{Z}) \cup \big(D_J \times (2\mathbb{Z}+1)\big) \\ (BC_J)^{(2)} &:= (B_J \times 2\mathbb{Z}) \cup \big(BC_J \times (2\mathbb{Z}+1)\big). \end{aligned}$$

Remark 2.7. — Let $q\colon V \times \mathbb{R} \to V, (v,t) \mapsto v$ denote the projection. Then the root systems $\Delta^{(2)}$ satisfy $q(\Delta^{(2)}) = \Delta$, but $\Delta_0 := \{\alpha \in \Delta\colon (\alpha,0) \in \Delta^{(2)}\}$ may be smaller, as the example BC_J shows. However, in all 7 cases the subsystem $\Delta_0 \subseteq \Delta$ contains enough elements to obtain all generating reflections of the Weyl group of Δ. Hence the subgroup $\mathcal{W}_0 := \langle r_{(\alpha,0)} \colon \alpha \in \Delta_0\rangle \subseteq \widehat{\mathcal{W}}$ is isomorphic to the Weyl group $\mathcal{W}$ of Δ.

2.2. Three involutive automorphisms

In this subsection we introduce three involutive automorphisms whose significance lies in the fact that these automorphism lead to the three twisted affine root systems of infinite rank $B_J^{(2)}$, $C_J^{(2)}$ and $BC_J^{(2)}$. In the following we call these three automorphism *standard*.

Example 2.8. — Consider $\mathfrak{k} = \mathfrak{o}_2(\mathcal{H})$, where $\mathcal{H}$ is a real infinite-dimensional Hilbert space. We consider the automorphism $\phi(x) := gxg^{-1}$, where g is the orthogonal reflection in the hyperplane $v_0^\perp$ for some unit vector $v_0 \in \mathcal{H}$.

Then $\mathfrak{k}^\phi \cong \mathfrak{o}_2(v_0^\perp) \cong \mathfrak{o}_2(\mathcal{H})$ is a simple Hilbert–Lie algebra. Pick

$$\mathfrak{t} \subseteq \mathfrak{k}^\phi = \{x \in \mathfrak{k}\colon xv_0 = 0\}$$

maximal abelian with $\dim(\ker(\mathfrak{t}) \cap v_0^\perp) = 1$. Then $\mathfrak{t}_\mathfrak{k} = \mathfrak{z}_\mathfrak{k}(\mathfrak{t})$ is maximal abelian (Lemma D.2) with $\ker(\mathfrak{t}_\mathfrak{k}) = \{0\}$ and $\mathfrak{t}$ is a hyperplane in $\mathfrak{t}_\mathfrak{k}$. Hence the root system $\Delta(\mathfrak{k},\mathfrak{t}_\mathfrak{k})$ is of type D_J, and $\Delta(\mathfrak{k}^\phi,\mathfrak{t})$ is of type B_{J_0}, where $J_0 = J \setminus \{j_0\}$ for some $j_0 \in J$ (*cf.* Example 1.13). It follows from [43, Thm. 5.7(ii)] that the set of compact roots of $\widehat{\mathcal{L}}_\phi(\mathfrak{k})$ is of type $B_J^{(2)}$ (*cf.* Example 2.5).

Example 2.9. — To obtain a root system of type $C_J^{(2)}$, we start with $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$ for a complex Hilbert space $\mathcal{H}$. We write $\mathcal{H}$ as $\mathcal{H}_0 \oplus \mathcal{H}_0$ for a complex Hilbert space $\mathcal{H}_0$ endowed with a conjugation σ_0 and extend this conjugation by $\sigma(x,y) := (\sigma_0(x), \sigma_0(y))$ to a conjugation of $\mathcal{H}$. Then

$$\phi(g) = S(g^\top)^{-1}S^{-1} = S\sigma g\sigma S^{-1} \quad \text{for} \quad S = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

defines an involutive automorphism of $\mathrm{U}_2(\mathcal{H})$. Let $(e_j)_{j\in J}$ be an ONB of $\mathcal{H}_0$, so that $\{e_j, Se_j : j \in J\}$ is an ONB of $\mathcal{H}$. Let $\mathfrak{t} \subseteq \mathfrak{k}^\phi$ be those elements which are diagonal with respect to this ONB. Then $\mathfrak{t}_\mathfrak{k} = \mathfrak{z}_\mathfrak{k}(\mathfrak{t})$ consists of all elements in $\mathfrak{u}_2(\mathcal{H})$ which are diagonal with respect to this ONB, and from [43, Thm. 5.7(ii)] we know that the set of compact roots of $\widehat{\mathcal{L}}_\phi(\mathfrak{k})$ is of type $C_J^{(2)}$. As $\widetilde{\sigma} := S\sigma = \sigma S$ is an anticonjugation of $\mathcal{H}$, it defines a quaternionic structure $\mathcal{H}_\mathbb{H} = (\mathcal{H}, \widetilde{\sigma})$. In this sense we have

$$U_2(\mathcal{H})^\phi = \{g \in \mathrm{U}_2(\mathcal{H}) : \widetilde{\sigma}g\widetilde{\sigma}^{-1} = g\} \cong \mathrm{Sp}_2(\mathcal{H}_\mathbb{H}).$$

Example 2.10. — To obtain a root system of type $BC_J^{(2)}$, we consider $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$ for a complex Hilbert space $\mathcal{H}$. We write $\mathcal{H}$ as $\mathcal{H}_0 \oplus \mathbb{C} \oplus \mathcal{H}_0$ for a complex Hilbert space $\mathcal{H}_0$ endowed with a conjugation σ_0 and extend σ_0 by $\sigma(x,y,z) := (\sigma_0(x), \overline{y}, \sigma_0(z))$ to a conjugation of $\mathcal{H}$. We consider the automorphism

$$\phi(g) = S(g^{-1})^\top S^{-1} \quad \text{for} \ \ S = \begin{pmatrix} 0 & 0 & \mathbf{1} \\ 0 & 1 & 0 \\ \mathbf{1} & 0 & 0 \end{pmatrix}, g \in \mathrm{U}_2(\mathcal{H}).$$

As S commutes with σ, $\tau := S\sigma$ is a conjugation on $\mathcal{H}$ with $\phi(g) = \tau g \tau$, so that

$$\mathrm{U}_2(\mathcal{H})^\phi = \{x \in \mathrm{U}_2(\mathcal{H}) : S(g^{-1})^\top S^{-1} = g\} = \{x \in \mathrm{U}_2(\mathcal{H}) : \tau g \tau = g\} \cong \mathrm{O}_2(\mathcal{H}^\tau).$$

In particular, $\mathfrak{k}^\phi \cong \mathfrak{o}_2(\mathcal{H})^\tau$ is a simple Hilbert–Lie algebra.

Pick an ONB $(e_j)_{j\in J}$ of $\mathcal{H}_0$, so that the elements $(e_j, 0, 0), (0, 0, e_j)$, $j \in J$, together with $(0, 1, 0)$, form an ONB of $\mathcal{H}$. Let $\mathfrak{t} \subseteq \mathfrak{k}^\phi$ be those elements which are diagonal with respect to this ONB. Then $\mathfrak{t}_\mathfrak{k} = \mathfrak{z}_\mathfrak{k}(\mathfrak{t})$ consists of all elements in $\mathfrak{u}_2(\mathcal{H})$ which are diagonal with respect to this ONB, and from [43, Thm. 5.7(ii)] we know that the set of compact roots of $\widehat{\mathcal{L}}_\phi(\mathfrak{k})$ is of type $BC_J^{(2)}$.

Remark 2.11. — In Examples 2.9 and 2.10 the operator S is contained in the identity component of $\mathrm{U}(\mathcal{H})$ (Theorem 1.15), so that in both cases, the involution ϕ is homotopic to the automorphism $x \mapsto -x^\top = \sigma x \sigma$ of $\mathfrak{u}_2(\mathcal{H})$.

We now record some topological properties of the subgroup K^ϕ for the standard involutions.

Lemma 2.12. — *If K is simply connected, then K^ϕ is connected for each of the standard involutions. It is 1-connected in all cases except the $BC_J^{(2)}$-case, in which $\pi_1(K^\phi) \cong \mathbb{Z}/2$.*

Proof. — (a) First we consider $K := \mathrm{O}_2(\mathcal{H})_0$ and $\phi(k) = gkg^{-1}$, where $g \in \mathrm{O}(\mathcal{H})$ is the reflection in a hyperplane $v_0^\perp$ (Example 2.8). We show that the group $\widetilde{K}^{\widetilde{\phi}}$ of fixed points of the lifted automorphism $\widetilde{\phi}$ of the 2-fold covering group $q_K \colon \widetilde{K} \to K$ is connected.

Let $\mathrm{O}_2(\mathcal{H})_\pm$ denote the two connected components of $\mathrm{O}_2(\mathcal{H})$, where $\mathbf{1} \in \mathrm{O}_2(\mathcal{H})_+$ (Theorem 1.5). The group $\mathrm{O}_2(\mathcal{H})^\phi \cong \{\pm 1\} \times \mathrm{O}_2(v_0^\perp)$ has four connected components, two of which lie in $\mathrm{O}_2(\mathcal{H})_0$, so that

$$K^\phi \cong (\{1\} \times \mathrm{O}_2(v_0^\perp)_+) \times (\{-1\} \times \mathrm{O}_2(v_0^\perp)_-)$$

has two connected components and $(K^\phi)_0 \cong \mathrm{O}_2(v_0^\perp)_+$. We conclude that $\pi_1(K^\phi) \cong \mathbb{Z}/2$ (Theorem 1.5) and that the inclusion $j \colon (K^\phi)_0 \to K$ induces an isomorphism of fundamental groups, so that $\widetilde{K}^{\widetilde{\phi}}$ has a simply connected identity component which is the universal covering group of K_0^ϕ and contains $\ker(q_K)$. In particular, $q_K^{-1}(K_0^\phi)$ is connected. To see that $\widetilde{K}^{\widetilde{\phi}}$ is connected, it therefore suffices to show that q_K maps it into K_0^ϕ, which is equivalent to $(-1, r) \notin q_K(\widetilde{K}^{\widetilde{\phi}})$ for any reflection $r \colon v_0^\perp \to v_0^\perp$ in a hyperplane $w^\perp$ of $v_0^\perp$. Let $V := \mathrm{span}_{\mathbb{R}}\{v_0, w\} \cong \mathbb{R}^2$. Then the inclusion $j \colon \mathbb{T} \cong \mathrm{SO}(V) \to \mathrm{O}_2(\mathcal{H})$ induces a surjection $\mathbb{Z} \cong \pi_1(\mathbb{T}) \to \pi_1(\mathrm{O}_2(\mathcal{H}))$ ([40]), so that it lifts to an inclusion $\widetilde{j} \colon \mathrm{Spin}(V) \to \widetilde{K}$, where $\mathrm{Spin}(V)$ denotes the unique 2-fold covering of $\mathrm{SO}(V)$. In $\mathrm{Spin}(V)$ the inverse image of $-\mathrm{id}_V$ consists of two elements of order 4 which are exchanged by $\widetilde{\phi}$. Therefore $(-\mathbf{1}, r)$ does not lift to a $\widetilde{\phi}$-fixed element. This proves the connectedness of $\widetilde{K}^{\widetilde{\phi}}$.

(b) For $K = \mathrm{U}_2(\mathcal{H})$ and $\phi(g) = \widetilde{\sigma} g \widetilde{\sigma}^{-1}$ as in Example 2.9, the subgroup $K^\phi \cong \mathrm{Sp}(\mathcal{H}_{\mathbb{H}})$ is 1-connected by Theorem 1.5. Since $(\widetilde{K}^\phi)_0$ is a covering of K^ϕ, this group is also simply connected and isomorphic to K^ϕ. For the universal covering $q_K \colon \widetilde{K} \to K$ we thus obtain

$$q_K^{-1}(K^\phi) \cong \ker q_K \times K^\phi \cong \mathbb{Z} \times K^\phi,$$

and $\widetilde{\phi}$ acts by $-\mathrm{id}$ on $\pi_1(K)$ (Proposition 1.18). Therefore $\widetilde{K}^{\widetilde{\phi}} \cong K^\phi$ is connected and hence 1-connected.

(c) For $K = \mathrm{U}_2(\mathcal{H})$ and $\phi(g) = \tau g \tau$ as in Example 2.10, the group $K^\phi \cong \mathrm{O}_2(\mathcal{H}^\tau)$ has 2 connected components and satisfies $\pi_1(K^\phi) \cong \mathbb{Z}/2$ (Theorem 1.5). Since the inclusions $\mathrm{SO}_n(\mathbb{R}) \to \mathrm{U}_n(\mathbb{C})$ induces the trivial homomorphism $\pi_1(\mathrm{SO}_n(\mathbb{R})) \cong \mathbb{Z}/2 \to \pi_1(\mathrm{U}_n(\mathbb{C})) \cong \mathbb{Z}$ for each $n > 2$, the same holds for the inclusion $(K^\phi)_0 \hookrightarrow K$ (*cf.* [40]). In particular, this inclusion lifts to a homomorphism $(K^\phi)_0 \to \widetilde{K}$, so that $\pi_1(\widetilde{K}^{\widetilde{\phi}}) \cong \mathbb{Z}/2$.

To see that $\widetilde{K}^{\widetilde{\phi}}$ is connected, it suffices to show that, for any reflection $r \in \mathrm{O}_2(\mathcal{H}^\tau)$ in a hyperplane $v_0^\perp \subseteq \mathcal{H}^\tau$, there is no $\widetilde{\phi}$-invariant inverse image under the covering map $q_K\colon \widetilde{K} \to K$. Since the homotopy groups of $\mathrm{U}_2(\mathcal{H})$ are obtained from the direct limit $\varinjlim \mathrm{U}_n(\mathbb{C})$ (*cf.* [40]), it suffices to prove the corresponding assertion for the case where $n := \dim \mathcal{H} < \infty$. Then $\widetilde{\mathrm{U}}_n(\mathbb{C}) \cong \mathrm{SU}_n(\mathbb{C}) \rtimes \mathbb{R}$ with $\widetilde{\phi}(g,t) = (\overline{g}, -t)$, so that $\widetilde{\mathrm{U}}_n(\mathbb{C})^{\widetilde{\phi}} \cong \mathrm{SU}_n(\mathbb{C})^\phi = \mathrm{SO}_n(\mathbb{R})$ is connected. This implies that $\widetilde{K}^{\widetilde{\phi}}$ is connected in the general case. □

2.3. The adjoint action of twisted loop groups

We claim that the following formula describes the adjoint action of $\mathcal{L}_\phi(K)$ on $\mathfrak{g} = \widehat{\mathcal{L}}_\phi(\mathfrak{k})$:

$$\mathrm{Ad}_{\mathfrak{g}}(g)(z,\xi,t) = \Big(z + \langle \delta^l(g), \xi\rangle - \frac{t}{2}\|\delta^r(g)\|^2, \mathrm{Ad}(g)\xi - t\delta^r(g), t\Big) \tag{2.11}$$

where $\delta^r(g) = g'g^{-1} \in \mathcal{L}_\phi(\mathfrak{k})$ denotes the *right logarithmic derivative* and $\delta^l(g) = g^{-1}g' \in \mathcal{L}_\phi(\mathfrak{k})$ denotes the *left logarithmic derivative* (*cf.* [52, Prop. 4.9.4] for the case where K is compact and a different choice of sign on the center). In fact, modulo the center, the formula $g.(\xi,t) = (\mathrm{Ad}(g)\xi - t\delta^r(g), t)$ defines a smooth action of $\mathcal{L}_\phi(K)$ on $\mathcal{L}_\phi(\mathfrak{k}) \rtimes \mathbb{R}$, whose derived action is given by

$$\eta.(\xi,t) = ([\eta,\xi] - tD\eta, 0) = [(\eta,0),(\xi,t)]$$

which implies that it is the adjoint action of the group $\mathcal{L}_\phi(K)$. The formula for the central component is now obtained from the invariance of the Lorentzian form on $\widehat{\mathcal{L}}_\phi(\mathfrak{k})$ under the adjoint action.

Remark 2.13. — Since we need it below, we take a closer look at the affine action of $\mathcal{L}_\phi(K)$ on $\mathcal{L}_\phi(\mathfrak{k})$ by

$$g * \xi := \mathrm{Ad}(g)\xi - \delta^r(g) = \mathrm{Ad}(g)\xi - g'g^{-1}.$$

To understand its orbits, for a smooth curve $\xi\colon \mathbb{R} \to \mathfrak{k}$, let $\gamma_\xi\colon \mathbb{R} \to K$ be the unique smooth curve with $\gamma_\xi(0) = \mathbf{1}$ and $\delta^l(\gamma_\xi) = \xi$. We consider the smooth holonomy maps

$$\mathrm{Hol}_t\colon \mathcal{L}_\phi(\mathfrak{k}) \to K, \quad \mathrm{Hol}_t(\xi) := \gamma_\xi(t), \quad t \in \mathbb{R}.$$

If $\xi \in \mathcal{L}_\phi(\mathfrak{k})$, then

$$\delta^l(\gamma_\xi)_{t+2\pi/N} = \xi\Big(t + \frac{2\pi}{N}\Big) = \mathbf{L}(\phi)^{-1}(\xi(t)) = \delta^l(\phi^{-1} \circ \gamma_\xi)_t$$

implies that

$$\gamma_\xi\Big(t + \frac{2\pi}{N}\Big) = \mathrm{Hol}_{2\pi/N}(\xi)\phi^{-1}(\gamma_\xi(t)). \tag{2.12}$$

From

$$\delta^l(\gamma_\xi g^{-1}) = \delta^l(g^{-1}) + \mathrm{Ad}(g)\xi = g * \xi, \tag{2.13}$$

we derive the following equivariance property

$$\mathrm{Hol}_s(g * \xi) = g(0)\,\mathrm{Hol}_s(\xi)g(s)^{-1}.$$

For $s = 2\pi/N$, we obtain in particular

$$\mathrm{Hol}_{2\pi/N}(g * \xi) = g(0)\,\mathrm{Hol}_{2\pi/N}(\xi)\phi^{-1}(g(0)^{-1}). \tag{2.14}$$

Proposition 2.14. — *The map $\mathrm{Hol}_{2\pi/N} \colon \mathcal{L}_\phi(\mathfrak{k}) \to K$ is equivariant with respect to the action of $\mathcal{L}_\phi(K)$ on K for which g acts by $c^\phi_{g(0)}$, where $c^\phi_k(k') := kk'\phi^{-1}(k^{-1})$ is the ϕ^{-1}-twisted conjugation map. The fibers of $\mathrm{Hol}_{2\pi/N}$ coincide with the orbits of the subgroup*

$$\mathcal{L}_\phi(K)_* := \{g \in \mathcal{L}_\phi(K) \colon g(0) = \mathbf{1}\}$$

and the image of the $\mathcal{L}_\phi(K)$-orbits are the ϕ^{-1}-twisted conjugacy classes in K.

Proof. — The asserted equivariance is (2.14). In particular, $\mathrm{Hol}_{2\pi/N}$ is constant on the orbits of the subgroup $\mathcal{L}_\phi(K)_*$. If $\mathrm{Hol}_{2\pi/N}(\xi_1) = \mathrm{Hol}_{2\pi/N}(\xi_2)$, then (2.12) implies that the smooth curve $g := \gamma_{\xi_2}^{-1}\gamma_{\xi_1} \colon \mathbb{R} \to K$ is contained in $\mathcal{L}_\phi(K)_*$. It satisfies $\gamma_{\xi_2} = \gamma_{\xi_1} g^{-1}$, so that (2.13) implies $g * \xi_1 = \xi_2$. This completes the proof. □

3. Double extensions of twisted loop groups

In the preceding section we have introduced the double extension $\widehat{\mathcal{L}}_\phi(\mathfrak{k})$ of the Fréchet–Lie algebra $\mathcal{L}_\phi(\mathfrak{k})$ of smooth ϕ-twisted loops. Since its construction involves a central extension, it is not obvious that this extension is the Lie algebra of a Lie group. In this section we show that such a group always exists for a simple infinite-dimensional Hilbert–Lie algebra $\mathfrak{k}$, in particular, we obtain a corresponding 1-connected Lie group which we denote $\widehat{\mathcal{L}}_\phi(K)$.

3.1. The central extension for smooth loops

Remark 3.1. — Let K be a connected Lie group for which $\mathfrak{k}$ is a simple Hilbert–Lie algebra and write $\mathrm{ev}_0 \colon \mathcal{L}_\phi(K) \to K, f \mapsto f(0)$ for the evaluation map. In [48, Prop. 3.5, Rem. 3.6], we have seen that the long exact homotopy sequence of the Lie group extension

$$\mathbf{1} \to \mathcal{L}_\phi(K)_* := \ker(\mathrm{ev}_0) \to \mathcal{L}_\phi(K) \xrightarrow{\quad \mathrm{ev}_0 \quad} K \to \mathbf{1}$$

provides crucial information on the homotopy groups of $\mathcal{L}_\phi(K)$. For each $j \geqslant 1$, we obtain a short exact sequence

$$\mathbf{1} \to \pi_j(K)_\phi := \pi_j(K)/\operatorname{im}(\pi_j(\phi) - \mathrm{id}) \hookrightarrow \pi_{j-1}(\mathcal{L}_\phi(K)) \twoheadrightarrow \pi_{j-1}(K)^\phi \to \mathbf{1}.$$

As K is connected, $\pi_2(K)$ vanishes (Theorem 1.5) and $\pi_3(\phi) = \mathrm{id}$ (Corollary 1.19), we obtain in particular

$$\pi_0(\mathcal{L}_\phi(K)) \cong \pi_1(K)_\phi,\ \pi_1(\mathcal{L}_\phi(K)) \cong \pi_1(K)^\phi \text{ and } \pi_2(\mathcal{L}_\phi(K)) \cong \pi_3(K) \cong \mathbb{Z}.$$

If K is 1-connected, these relations imply that $\mathcal{L}_\phi(K)$ is also 1-connected.

Definition 3.2. — *In the following we shall identify the Lie algebra* $\mathbf{L}(\mathbb{T})$ *of the circle group* $\mathbb{T} \subseteq \mathbb{C}^\times$ *with* $i\mathbb{R}$, *so that the exponential function is given by* $\exp_{\mathbb{T}}(it) = e^{it}$ *with* $\ker(\exp_{\mathbb{T}}) = 2\pi i\mathbb{Z}$.

The following theorem generalizes the corresponding result for compact target groups which can be found in [52, Sect. 4.4] for the untwisted case.

Definition 3.3. — *We say that the scalar product* $\langle \cdot, \cdot \rangle$ *on the simple Hilbert–Lie algebra* $\mathfrak{k}$ *is normalized if* $\|\check{\alpha}\|^2 = 2$ *holds for the coroots of all long roots*[(2)] $\alpha \in \Delta(\mathfrak{k}, \mathfrak{t}_\mathfrak{k})$, *where* $\mathfrak{t}_\mathfrak{k} \subseteq \mathfrak{k}$ *is maximal abelian. In view of* (2.4), *this is equivalent to* $(\alpha, \alpha) = 2$ *for all long roots.*

Theorem 3.4. — *Let K be a 1-connected simple Hilbert–Lie group and suppose that the scalar product on $\mathfrak{k}$ is normalized. Then the central Lie algebra extension $\widetilde{\mathcal{L}}_\phi(\mathfrak{k})$ defined by the cocycle*

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi'(t), \eta(t) \rangle \, dt$$

integrates to a 2-connected central Lie group extension

$$\mathbf{1} \to Z \to \widetilde{\mathcal{L}}_\phi(K) \to \mathcal{L}_\phi(K) \to \mathbf{1},$$

where $Z = \exp(\mathbb{R}ic)$ *and* $\ker(\exp|_{\mathbb{R}ic}) = 2\pi N\mathbb{Z}ic$.

(2) At most three root lengths occur, the long roots are those of maximal length.

Proof. — Let $\omega^l \in \Omega^2(\mathcal{L}_\phi(K),\mathbb{R})$ denote the left invariant 2-form corresponding to the 2-cocycle ω. We derive from Remark 3.1 that $\mathcal{L}_\phi(K)$ is 1-connected because K is 1-connected. Therefore we can use [39, Thm. 7.9] to see that the required simply connected Lie group extension $\widetilde{\mathcal{L}}_\phi(K)$ exists if and only if the range of the period homomorphism

$$\mathrm{per}_\omega \colon \pi_2(\mathcal{L}_\phi(K)) \to \mathbb{R}, \quad [\sigma] \mapsto \int_{\mathbb{S}^2} \sigma^*\omega^l$$

is discrete and in this case $Z \cong \mathbb{R}/\operatorname{im}(\mathrm{per}_\omega)$.

To use the results from [48], where the circle is identified with $\mathbb{R}/\mathbb{Z}$, we have to translate them to our context where $\mathbb{S}^1 \cong \mathbb{R}/\frac{2\pi}{N}\mathbb{Z}$. So let

$$\mathcal{L}^1_\phi(K) := \big\{f \in C^\infty(\mathbb{R},K) \colon (\forall t\in\mathbb{R})\ f(t+1) = \phi^{-1}(f(t))\big\}$$

and observe that

$$\Phi \colon \mathcal{L}_\phi(K) \to \mathcal{L}^1_\phi(K), \quad \Phi(\xi)(t) := \xi(2\pi t/N)$$

is an isomorphism of Fréchet–Lie groups. For the cocycle $\omega^1(\xi,\eta) := \int_0^1 \langle \xi'(t),\eta(t)\rangle\,dt$ on $\mathcal{L}^1_\phi(\mathfrak{k})$ we then obtain

$$\begin{aligned}\big(\mathbf{L}(\Phi)^*\omega^1\big)(\xi,\eta) &= \int_0^1 \langle \xi'(2\pi t/N)\frac{2\pi}{N}, \eta(2\pi t/N)\rangle\,dt \\ &= \int_0^{2\pi/N} \langle \xi'(t),\eta(t)\rangle\,dt = \frac{2\pi}{N}\omega(\xi,\eta).\end{aligned}$$

This implies that $\operatorname{im}(\mathrm{per}_\omega) = \frac{N}{2\pi}\operatorname{im}(\mathrm{per}_{\omega^1})$.

According to [48, Lemma 3.10], the period homomorphism per_{ω^1} of the restriction of ω^1 to the ideal $\mathcal{L}^1_\phi(\mathfrak{k})_*$ coincides with the homomorphism

$$\frac{1}{2}\,\mathrm{per}_{C_\mathfrak{k}} \colon \pi_2(\mathcal{L}^1_\phi(K)_*) \cong \pi_3(K) \cong \mathbb{Z} \to \mathbb{R},$$

where $C_\mathfrak{k}(x,y,z) := \langle [x,y],z\rangle$ is the 3-cocycle defined by the scalar product on $\mathfrak{k}$ and

$$\mathrm{per}_{C_\mathfrak{k}} \colon \pi_3(K) \to \mathbb{R}, \quad [\sigma] \mapsto \int_{\mathbb{S}^3} \sigma^* C^l_\mathfrak{k}$$

is the period homomorphism of the corresponding closed 3-form $C^l_\mathfrak{k}$ on K. This map can be evaluated quite explicitly as follows. Let $\alpha \in \Delta(\mathfrak{k},\mathfrak{t}_\mathfrak{k})$ be a long root and $\mathfrak{k}(\alpha) \subseteq \mathfrak{k}$ be the corresponding $\mathfrak{su}_2$-subalgebra and $\check{\alpha} \in i\mathfrak{t}$ be the corresponding coroot. The associated homomorphism $\gamma_\alpha \colon \mathrm{SU}_2(\mathbb{C}) \to K$ induces an isometric embedding $\mathbf{L}(\gamma_\alpha) \colon \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{k}$ with respect to the normalized scalar products. Hence [48, Ex. 3.11] implies that

$$\frac{1}{2}\,\mathrm{per}_{C_\mathfrak{k}}([\gamma_\alpha]) = \frac{1}{2}\frac{\|\check{\alpha}\|^2}{2}8\pi^2 = 4\pi^2.$$

Since $\pi_2(K)$ vanishes (Remark 3.1), [48, Thm. 3.12] thus leads to

$$\operatorname{im}(\operatorname{per}_\omega) = \frac{N}{2\pi}\operatorname{im}(\operatorname{per}_{\omega^1}) = 2\pi N\mathbb{Z}.$$

As $\pi_2(\mathcal{L}_\phi(K)_*) \cong \mathbb{Z}$, the non-zero homomorphism $\operatorname{per}_\omega$ is injective. Since $\mathcal{L}_\phi(K)$ is 1-connected by Remark 3.1, [39, Rem. 5.12(b)] now implies that, for $Z := \mathbb{R}/\operatorname{im}(\operatorname{per}_\omega)$, the group $\widetilde{\mathcal{L}}_\phi(K)$ is 2-connected. □

Definition 3.5. — *Since the rotation action α of $\mathbb{R}$ on $\mathcal{L}_\phi(K)$ lifts uniquely to a smooth action on the central extension $\widetilde{\mathcal{L}}_\phi(K)$* [31, Thm. V.9], *we obtain a 2-connected Fréchet–Lie group*

$$\widehat{\mathcal{L}}_\phi(K) := \widetilde{\mathcal{L}}_\phi(K) \rtimes_\alpha \mathbb{R}.$$

Remark 3.6. — If K is a simple Hilbert–Lie group, then it has a universal complexification $\eta_K\colon K \to K_\mathbb{C}$ which has a polar decomposition, *i.e.*, the map

$$K \times \mathfrak{k} \to K_\mathbb{C}, \quad (k, x) \mapsto k \exp ix$$

is a diffeomorphism (*cf.* [40]). This property is inherited by the group $\mathcal{L}_\phi(K)$, which implies in particular that the inclusion $\mathcal{L}_\phi(K) \to \mathcal{L}_\phi(K_\mathbb{C})$ induces isomorphisms of all homotopy groups. Hence the cocycle ω and its complex bilinear extension to $\mathcal{L}_\phi(\mathfrak{k}_\mathbb{C}) \cong \mathcal{L}_\phi(\mathfrak{k})_\mathbb{C}$ have the same period group. Now [39, Thm. 7.9] implies the existence of a central extension of complex Lie groups

$$\mathbf{1} \to \mathbb{C}^\times \to \widetilde{\mathcal{L}}_\phi(K_\mathbb{C}) \to \mathcal{L}_\phi(K_\mathbb{C}) \to \mathbf{1}$$

for which the inclusion $\widetilde{\mathcal{L}}_\phi(K) \hookrightarrow \widetilde{\mathcal{L}}_\phi(K_\mathbb{C})$ is a universal complexification and a weak homotopy equivalence.

In the preceding theorem we have seen the importance of normalizing the scalar product. To evaluate the period group in all cases, it is thus important to identify the normalized scalar products in all cases.

*Remark 3.7 (*Normalization of the scalar product). — (a) For $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$ all roots in $\Delta(\mathfrak{k}, \mathfrak{t}) = A_J$ have the same length and the coroots correspond to diagonal matrices of the form $E_{jj} - E_{kk}$, so that $\langle x, y\rangle = \operatorname{tr}(xy^*)$ is a scalar product with $\|\check{\alpha}\|^2 = 2$ for all roots α.

(b) For $\mathfrak{k} = \mathfrak{sp}_2(\mathcal{H}_\mathbb{H})$, the long roots are of the form $\pm 2\varepsilon_j$ and their coroots are diagonal matrices of the form $(E_{jj}, -E_{jj})$ with respect to the decomposition of the complex Hilbert space $\mathcal{H}_\mathbb{H} = \ell^2(J, \mathbb{C}) \oplus \ell^2(J, \mathbb{C})$. Therefore $\langle x, y\rangle = \operatorname{tr}_\mathbb{C}(xy^*)$ satisfies $\|\check{\alpha}\|^2 = 2$ for all long roots α.

(c) For $\mathfrak{k} = \mathfrak{o}_2(\mathcal{H}_\mathbb{R})$ and $\Delta(\mathfrak{k}, \mathfrak{t})$ of type B_J or D_J, the long roots are $\pm\varepsilon_j \pm \varepsilon_k$, $j \neq k$. On the complex Hilbert space $\mathcal{H} := (\mathcal{H}_\mathbb{R})_\mathbb{C}$ their coroots correspond to diagonal matrices of the form $\pm E_{jj} \mp E_{-j,-j} \pm E_{kk} \mp E_{-k,-k}$, $j \neq k$ (*cf.* Example 1.13) satisfying

$$\mathrm{tr}_{\mathbb{C}}((\pm E_{jj} \mp E_{-j,-j} \pm E_{kk} \mp E_{-k,-k})^2) = 4,$$

so that $\langle x, y\rangle = \frac{1}{2}\,\mathrm{tr}_{\mathbb{C}}(xy^*) = \frac{1}{2}\,\mathrm{tr}_{\mathbb{R}}(xy^\top)$ satisfies $\|\check{\alpha}\|^2 = 2$ for all long roots α.

(d) In Example 2.8 we have the inclusion $\mathfrak{k}^\phi \cong \mathfrak{o}_2(v_0^\perp) \hookrightarrow \mathfrak{k} = \mathfrak{o}_2(\mathcal{H}_{\mathbb{R}})$, and (c) shows that this is isometric with respect to the normalized scalar products.

(e) In Example 2.9 we have for $\mathcal{H} = \mathcal{H}_0^2 = (\mathcal{H}_0)_{\mathbb{H}}$ the inclusion $\mathfrak{k}^\phi \cong \mathfrak{sp}_2((\mathcal{H}_0)_{\mathbb{H}}) \hookrightarrow \mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$, so that (a) and (b) imply that it is isometric with respect to the normalized scalar products.

(f) In Example 2.10 we have the inclusion $\eta\colon \mathfrak{k}^\phi \cong \mathfrak{o}_2(\mathcal{H}^\tau) \hookrightarrow \mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$, so that (c) implies that $\langle \eta(x), \eta(y)\rangle = 2\langle x, y\rangle$ for $x, y \in \mathfrak{k}^\phi$. In particular, the roots $\alpha = \pm\varepsilon_j \pm \varepsilon_k \in \Delta(\mathfrak{k}^\phi, \mathfrak{t}) = BC_J$ satisfy $\|\check{\alpha}\|_{\mathfrak{k}}^2 = 4$. Accordingly we find $\|\varepsilon_j \pm \alpha_k\|^2 = 1$ and $\|2\varepsilon_j\|^2 = 2$.

At this point we can also make the description of the weight set $\mathcal{P}(\mathfrak{g}, \mathfrak{t})$ from (2.10) more explicit for all 7 locally affine root systems.

Example 3.8. — (a) For the untwisted root systems $X_J^{(1)}$, we have seen in (2.9) that $\lambda \in i\mathfrak{t}_{\mathfrak{g}}^*$ is an integral weight if and only if $\lambda_c \in \frac{\|\alpha\|^2}{2}\mathbb{Z}$ for every root $\alpha \in X_J$ and $\lambda|_{\mathfrak{t}}$ is a weight of X_J. As $\|\alpha\|^2 = 2$ for the long roots, the condition on λ_c is equivalent to $\lambda_c \in \mathbb{Z}$.

(b) For $B_J^{(2)}$ we find the condition $\lambda_c \in \frac{\|\alpha\|^2}{4}\mathbb{Z}$ for every root $\alpha \in B_J$, *i.e.*, $\lambda_c \in \frac{1}{2}\mathbb{Z}$, and that $\lambda(\check{\alpha}) \in \mathbb{Z}$ for $\alpha \in \Delta_0 = B_J$. As $\Delta_1 = \{\pm\varepsilon_j \colon j \in J\} \subseteq \Delta_0$ and $\|\varepsilon_j\| = 1$, this implies

$$\lambda(\check{\alpha}) \in \mathbb{Z} + \frac{2}{\|\alpha\|^2}\lambda_c \quad \text{for} \quad \alpha \in \Delta_1.$$

Therefore

$$\mathcal{P}(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}}) = \big\{\lambda \in i\mathfrak{t}_{\mathfrak{g}}' \colon \lambda_c \in \tfrac{1}{2}\mathbb{Z},\ \lambda \in \mathcal{P}(B_J)\big\}.$$

(c) For $C_J^{(2)}$, the long roots of C_J are of the form $\pm 2\varepsilon_j$, $j \in J$, so that our normalization leads to $\|2\varepsilon_j\|^2 = 2$ (*cf.* Remark 3.7(e)), which in turn implies $\|\varepsilon_j\|^2 = \frac{1}{2}$. The condition $\lambda_c \in \frac{\|\alpha\|^2}{4}\mathbb{Z}$ for every root α leads to $\lambda_c \in \frac{1}{2}\mathbb{Z}$. We further obtain $\lambda(\check{\alpha}) \in \mathbb{Z}$ for $\alpha \in \Delta_0 = C_J$ and this already implies

$$\lambda(\check{\alpha}) \in \mathbb{Z} + \frac{2}{\|\alpha\|^2}\lambda_c = \mathbb{Z} + 2\lambda_c \quad \text{for} \ \ \alpha \in \Delta_1 = D_J.$$

Therefore

$$\mathcal{P}(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}}) = \big\{\lambda \in i\mathfrak{t}_{\mathfrak{g}}' \colon \lambda_c \in \tfrac{1}{2}\mathbb{Z},\ \lambda \in \mathcal{P}(C_J)\big\}.$$

(d) For $BC_J^{(2)}$ we have seen in Remark 3.7(f) that $\|\varepsilon_j \pm \varepsilon_k\|^2 = 1$ and $\|2\varepsilon_j\|^2 = 2$. Hence the condition $\lambda_c \in \frac{\|\alpha\|^2}{4}\mathbb{Z}$ for every root $\alpha \in BC_J$ means that $\lambda_c \in \frac{1}{2}\mathbb{Z}$. We further obtain $\lambda(\check{\alpha}) \in \mathbb{Z}$ for $\alpha \in \Delta_0 = B_J$ which means that $\lambda_j \in \frac{1}{2}\mathbb{Z}$ for every j with $\lambda_j - \lambda_k \in \mathbb{Z}$ for $j \neq k$. An integral weight λ also has to satisfy

$$\lambda(\check{\alpha}) \in \mathbb{Z} + \frac{2}{\|\alpha\|^2}\lambda_c \quad \text{for } \alpha \in \Delta_1 = BC_J.$$

For $\alpha \in B_J$, this is satisfied because $\|\alpha\|^2 \in \{\frac{1}{2}, 1\}$. For $\alpha = \pm 2\varepsilon_j$, it means that

$$\pm\lambda_j = \lambda(\check{\alpha}) \in \mathbb{Z} + \lambda_c.$$

Therefore the parity of $2\lambda_c$ equals the parity of $2\lambda_j$.

If we also take into account that λ should be continuous, *i.e.*, $(\lambda_j) \in \ell^2(J,\mathbb{R})$, then only finitely many λ_j are non-zero, which leads to $\lambda_j \in \mathbb{Z}$ and hence to $\lambda_c \in \mathbb{Z}$. We therefore have

$$\mathcal{P}(\mathfrak{g}, \mathfrak{t}_\mathfrak{g}) = \big\{\lambda \in i\mathfrak{t}_\mathfrak{g}' \colon \lambda_c \in \mathbb{Z},\ \lambda \in \mathcal{P}(B_J)\big\}.$$

3.2. The topology of the fixed point group $\widehat{\mathcal{L}}(K)^{\widetilde{\gamma}}$

Let K be a 1-connected simple infinite-dimensional Hilbert–Lie group and $\widehat{\mathcal{L}}(K) \cong \widetilde{\mathcal{L}}(K) \rtimes \mathbb{R}$ the simply connected Fréchet–Lie group with Lie algebra $\widehat{\mathcal{L}}(\mathfrak{k})$ from Definition 3.5. Let $\widehat{\gamma} \in \mathrm{Aut}(\widehat{\mathcal{L}}(K))$ be the automorphism induced by the automorphism γ of $\mathcal{L}(K)$ given by

$$\gamma(f)(t) = \phi\Big(f\big(t + \frac{2\pi}{N}\big)\Big) \quad \text{and} \quad \mathbf{L}(\widehat{\gamma})(z,\xi,t) = (z, \mathbf{L}(\gamma)\xi, t).$$

Then $\widehat{\mathcal{L}}(K)^{\widehat{\gamma}} = \widetilde{\mathcal{L}}(K)^{\widehat{\gamma}} \rtimes \mathbb{R}$ is a Lie subgroup with the Lie algebra $\widetilde{\mathcal{L}}(\mathfrak{k})^{\mathbf{L}(\widehat{\gamma})} \rtimes \mathbb{R}$. Here we use that the central extension $\widetilde{\mathcal{L}}(K)$ of the locally exponential Lie group $\mathcal{L}(K)$ is again locally exponential (*cf.* [16] and also [41, Thm. IV.2.10]).

Proposition 3.9. — *The inclusion $\widehat{\mathcal{L}}_\phi(\mathfrak{k}) \hookrightarrow \widehat{\mathcal{L}}(\mathfrak{k})$ integrates to a Lie group morphism $\widehat{\mathcal{L}}_\phi(K) \to \widehat{\mathcal{L}}(K)$ whose kernel is the subgroup $C_N := \{z \in Z \cong \mathbb{T} \colon z^N = 1\}$ and whose range is $\widehat{\mathcal{L}}(K)^{\widehat{\gamma}}$. In particular, $\pi_1(\widehat{\mathcal{L}}(K)^{\widehat{\gamma}}) \cong C_N$.*

Proof. — With the normal subgroup $\mathcal{L}(K)_* := \{f \in \mathcal{L}(K) \colon f(0) = \mathbf{1}\} \trianglelefteq \mathcal{L}(K)$ of based loops, we obtain the semidirect decomposition $\mathcal{L}(K) \cong \mathcal{L}(K)_* \rtimes K$ corresponding to the inclusion $K \hookrightarrow \mathcal{L}(K)$ as the constant maps. Then $\widetilde{\mathcal{L}}(K) \cong \widetilde{\mathcal{L}}(K)_* \rtimes K$, where $\widetilde{\mathcal{L}}(K)_*$ is a simply connected central $\mathbb{T}$-extension of $\mathcal{L}(K)_*$.

As γ commutes with the translation action of $\mathbb{R}$ on $\mathcal{L}(K)$, $\widehat{\mathcal{L}}(K)^{\widehat{\gamma}} \cong \widetilde{\mathcal{L}}(K)^{\widehat{\gamma}} \rtimes \mathbb{R}$, and we have a central extension

$$\mathbf{1} \to \mathbb{T} \to \widetilde{\mathcal{L}}(K)^{\widehat{\gamma}} \to \mathcal{L}(K)^\gamma \to \mathbf{1}.$$

In Remark 3.1 we have seen that $\mathcal{L}_\phi(K) = \mathcal{L}(K)^\gamma$ is 1-connected, so that $\widetilde{\mathcal{L}}(K)^{\widehat{\gamma}}$ is a central $\mathbb{T}$-extension of a 1-connected Lie group. Therefore it is connected, and its fundamental group is isomorphic to the cokernel of the corresponding period homomorphism

$$\mathrm{per}\colon \pi_2(\mathcal{L}(K)^\gamma) \to \pi_1(\mathbb{T}) \cong 2\pi\mathbb{Z}$$

(*cf.* [39, Rem. 5.12(b)]).

Using Remark 3.1 again, we see that the triviality of $\pi_3(\phi)$ (Corollary 1.19) implies that we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} \cong \pi_3(K) \cong \pi_2(\mathcal{L}_\phi(K)_*) & \xrightarrow{\cong} & \pi_2(\mathcal{L}_\phi(K)) \\
\downarrow & & \downarrow \\
\mathbb{Z} \cong \pi_3(K) \cong \pi_2(\mathcal{L}(K)_*) & \xrightarrow{\cong} & \pi_2(\mathcal{L}(K)).
\end{array}$$

For the group

$$\mathcal{L}^c_\phi(K) := \Big\{ f \in C(\mathbb{R}, K)\colon (\forall t \in \mathbb{R})\ f\Big(t + \frac{2\pi}{N}\Big) = \phi^{-1}(f(t)) \Big\}$$

of continuous maps, it is easy to see that

$$\mathcal{L}^c_\phi(K)_* \cong \Big\{ f \in C(\mathbb{R}, K)_*\colon (\forall t \in \mathbb{R})\ f\Big(t + \frac{2\pi}{N}\Big) = f(t) \Big\},$$

which coincides with the range of the map

$$\Phi\colon \mathcal{L}^c(K)_* \to \mathcal{L}^c(K)_*, \quad \Phi(f)(t) := f(Nt).$$

Since the inclusion of smooth into continuous maps induced isomorphisms of homotopy groups (*cf.* [48, Cor. 3.4]), [31, Lemma 1.10] now implies that the vertical arrows in the above diagram correspond to the endomorphism $\mathbb{Z} \to \mathbb{Z}, n \mapsto Nn$. We conclude that $\mathrm{coker(per)} \cong \mathbb{Z}/N\mathbb{Z}$ and from that the assertion follows immediately. □

4. *d*-extremal weights

For $\mathfrak{g} = \widehat{\mathcal{L}}_\phi(\mathfrak{k})$, recall its root decomposition and the elements c and d from (2.6) in Example 2.4. We write $\widehat{\mathcal{W}} = \mathcal{W}(\mathfrak{g}, \widehat{\mathfrak{t}})$ for the *Weyl group of* $\mathfrak{g}$. In this section we derive a characterization of the set of those elements

$\lambda \in i\mathfrak{t}_{\mathfrak{g}}^*$ (the space of all linear functionals which are not necessarily continuous) which are d-minimal in the sense that $\lambda(d) = \min(\widehat{\mathcal{W}}\lambda)(d)$. In Sections 5 and 6 below, we see that these elements parametrize the irreducible semibounded representations of $\widehat{\mathcal{L}}_\phi(K)$.

Definition 4.1. — *Let $\widehat{\mathcal{W}}$ denote the Weyl group of the pair $(\mathfrak{g}, \mathfrak{t}_{\mathfrak{g}})$ (cf. Definition 1.9). We call $\lambda \in i\mathfrak{t}_{\mathfrak{g}}^*$ is d-minimal if $\lambda(d) = \min\langle\widehat{\mathcal{W}}\lambda, d\rangle$.*

Lemma 4.2 ([20], Lemma 3.8). — *If $(\widehat{\mathcal{W}}\lambda)(d)$ is bounded from below, then $\lambda_c \geqslant 0$. If, in addition, $\lambda_c = 0$, then λ is fixed by $\widehat{\mathcal{W}}$.*

Proposition 4.3 ([20] Cor. 3.6, Prop 3.9). — *Suppose that $(\Delta_{\mathfrak{g}})_c$ is one of the 7 irreducible locally affine root systems with their natural $\mathbb{Z}$-grading. For $\lambda \in i\mathfrak{t}_{\mathfrak{g}}^*$ with $\lambda_c > 0$, the following are equivalent:*

(i) *λ is d-minimal.*
(ii) $(\forall \alpha \in \Delta(\mathfrak{k}, \mathfrak{t}), n = 1, 2) \quad (\alpha, n) \in (\Delta_{\mathfrak{g}})_c \Rightarrow |\lambda(\check{\alpha})|\frac{(\alpha,\alpha)}{2n} \leqslant \lambda_c$.
(iii) *For $\underline{\alpha} = (\alpha, n) \in (\Delta_{\mathfrak{g}})_c$ with $n > 0$ we have $\lambda(\check{\underline{\alpha}}) \leqslant 0$.*

Theorem 4.4. — *For the seven irreducible locally affine root systems $X_J^{(r)} = (\Delta_{\mathfrak{g}})_c \subseteq i\mathfrak{t}_{\mathfrak{g}}'$ of infinite rank, a linear functional $\lambda \in i\mathfrak{t}_{\mathfrak{g}}^*$ with $\lambda_c > 0$ is d-minimal if and only if the following conditions are satisfied by the corresponding function $\overline{\lambda}\colon J \to \mathbb{R}, j \mapsto \lambda_j$:*

$(A_J^{(1)})$ $\max \overline{\lambda} - \min \overline{\lambda} \leqslant \lambda_c$.

$(B_J^{(1)})$ $|\lambda_j| + |\lambda_k| \leqslant \lambda_c$ *for* $j \neq k$.

$(C_J^{(1)})$ $|\lambda_j| \leqslant \lambda_c$ *for* $j \in J$.

$(D_J^{(1)})$ $|\lambda_j| + |\lambda_k| \leqslant \lambda_c$ *for* $j \neq k$.

$(B_J^{(2)})$ $|\lambda_j| \leqslant \lambda_c$ *for* $j \in J$.

$(C_J^{(2)})$ $|\lambda_j| + |\lambda_k| \leqslant 2\lambda_c$ *for* $j \neq k$.

$(BC_J^{(2)})$ $|\lambda_j| \leqslant \lambda_c$ *for* $j \in J$.

Proof. — In all cases where the normalization of the scalar product is such that $\|\varepsilon_j\|^2 = 1$ for every j, this follows immediately from [20, Thm. 3.10], and this is the case for $A_J^{(1)}$, $B_J^{(1)}$ and $D_J^{(1)}$, where the long roots are of the form $\pm\varepsilon_j \pm \varepsilon_k$, and for $B_J^{(2)}$ it follows from Remark 3.7(d).

In all other cases, the normalization for the scalar product $(\cdot,\cdot)_*$ in [20, Thm. 3.10] by $(\varepsilon_j, \varepsilon_j)_* = 1$ is different and we have to take a closer look at the consequences. For $C_J^{(1)}$ the long roots are of the form $\pm 2\varepsilon_j$, which leads to the normalization $\|\varepsilon_j\|^2 = \frac{1}{2}$. For $C_J^{(2)}$ and $BC_J^{(2)}$ we have the same normalization by Remark 3.7(e),(f), which leads to $(\cdot,\cdot)_* = 2(\cdot,\cdot)$

in these 3 cases. The relation $|\lambda(\check{\alpha})|\frac{(\alpha,\alpha)}{2n} \leqslant \lambda_c$ is therefore equivalent to $|\lambda(\check{\alpha})|\frac{(\alpha,\alpha)_*}{2n} \leqslant 2\lambda_c$. Replacing λ_c in [20, Thm. 3.10] by $2\lambda_c$ now leads to the correct inequalities in these 3 cases. $\square$

Remark 4.5. — (a) The preceding theorem implies that d-minimal weights $\lambda \in i\mathfrak{t}_{\mathfrak{g}}^*$ define bounded functions $\overline{\lambda}\colon J \to \mathbb{R}$ and, moreover, that the boundedness of $\overline{\lambda}$ is equivalent to the existence of a $\lambda_c > 0$ such that a corresponding $\lambda \in \mathfrak{t}_{\mathfrak{g}}^*$ is d-minimal.

(b) If $\lambda \in i\mathfrak{t}_{\mathfrak{g}}^*$ satisfies $\lambda(\check{\alpha}) \in \mathbb{Z}$ for each $\alpha \in (\Delta_{\mathfrak{g}})_c$, then the subset $\lambda + \widehat{\mathcal{Q}} \subseteq i\mathfrak{t}_{\mathfrak{g}}^*$, where $\widehat{\mathcal{Q}} = \langle \Delta_{\mathfrak{g}} \rangle_{\mathrm{grp}}$ is the *root group*, is invariant under the Weyl group $\widehat{\mathcal{W}}$. Therefore $(\widehat{\mathcal{W}}\lambda)(d) \subseteq \lambda(d) + \mathbb{Z}$. If $(\widehat{\mathcal{W}}\lambda)(d)$ is bounded from below, we thus obtain the existence of a d-minimal element in $\widehat{\mathcal{W}}\lambda$. In particular, we obtain $\widehat{\mathcal{P}}^+ = \widehat{\mathcal{W}}\mathcal{P}_d^+$.

5. Semibounded representations of Hilbert loop groups

After the preparations in the preceding sections, we now approach our goal of classifying the irreducible semibounded representations of $G = \widehat{\mathcal{L}}_\phi(K)$. The first major step is Theorem 5.2, asserting that for a semibounded representation $(\pi, \mathcal{H})$, the operator $\mathtt{d}\pi(d)$ is either bounded from below (positive energy representations) or from above. Up to passing to the dual representation, we may therefore assume that we are in the first case. Then the minimal spectral value of $\mathtt{d}\pi(d)$ turns out to be an eigenvalue and the group $Z_G(d)$ acts on the corresponding eigenspace, which leads to a bounded representation (ρ, V) of this group. We then show that $(\pi, \mathcal{H})$ can be reconstructed from (ρ, V) by holomorphic induction and that ρ is irreducible if and only if π is. Since an explicit classification of the bounded irreducible representations of the groups $Z_G(d)_0$ can be given in terms of $\mathcal{W}$-orbits of extremal weights λ (Theorem 5.9), the final step is to characterize those weights λ for which the corresponding representation $(\rho_\lambda, V_\lambda)$ occurs.

5.1. From semibounded to bounded representations

Note that $d = (0, 0, -i)$ satisfies $\exp(2\pi i \cdot d) \in \ker \mathrm{Ad} = Z(G)$. Therefore the following lemma can be used to obtain smooth eigenvectors of $\mathtt{d}\pi(d)$ in irreducible representations of G. The assumption of this lemma implies that $\pi(\exp \mathbb{R}x)\mathbb{T}$ is a torus, so that we know a priori that the Hilbert space decomposes into eigenspaces of this group.

Lemma 5.1. — *For a unitary representation $(\pi, \mathcal{H})$ of G and $x \in \mathfrak{g}$ with $\pi(\exp(Tx)) = e^{i\mu}\mathbf{1}$ for some $T > 0$ and $\mu \in \mathbb{R}$, the space $\mathcal{H}^\infty$ of smooth vectors is invariant under the operators*

$$P_n(v) := \frac{1}{T}\int_0^T e^{-(2\pi n+\mu)it/T}\pi(\exp tx)v\,dt$$

which are orthogonal projections onto the eigenvectors of $-i\mathtt{d}\pi(x)$ for the eigenvalues $(\mu + 2\pi n)/T$, $n \in \mathbb{Z}$.

Proof (*cf.* the proof of [42, Prop. 4.11]). — For $v \in \mathcal{H}^\infty$, we have

$$\pi(g)P_n(v) = \frac{1}{T}\int_0^T e^{-(2\pi n+\mu)it/T}\pi(g\exp tx)v\,dt,$$

which is an integral of a smooth function on $[0, T] \times G$ over the compact factor $[0, T]$, which results in a smooth function on G. □

Theorem 5.2. — *Suppose that either $\phi = \mathrm{id}_K$ or that ϕ is one of the three standard involutions. If $(\pi, \mathcal{H})$ is a semibounded unitary representation of $G = \widehat{\mathcal{L}}_\phi(K)$ for which $\mathtt{d}\pi(c)$ is bounded, then $\mathtt{d}\pi(d)$ is bounded from below, resp., above.*

If, in addition, π is irreducible, then this is the case and, accordingly, the minimal/maximal spectral value of $\mathtt{d}\pi(d)$ is an eigenvalue and the K^ϕ-representation on the corresponding eigenspace is bounded.

Proof. — Since $\alpha_t := e^{t\,\mathrm{ad}\,i\cdot d}$ defines a continuous circle action on $\mathfrak{g}$, the open invariant convex cone W_π intersects the fixed point algebra $\mathfrak{z}_\mathfrak{g}(d)$ of this circle action. Since $[d, (0, \xi, 0)] = -i(0, \xi', 0)$, an element $(0, \xi, 0) \in \widehat{\mathcal{L}}_\phi(\mathfrak{k})$ commutes with d if and only if ξ is constant, *i.e.*, its values are contained in $\mathfrak{k}^\phi$. We thus have

$$\mathfrak{z}_\mathfrak{g}(d) = \ker(\mathrm{ad}\,id) = \mathbb{R}ic \oplus \mathfrak{k}^\phi \oplus \mathbb{R}i \cdot d,$$

which is a Hilbert–Lie algebra.

Since every open invariant cone in the Hilbert–Lie algebra $\mathfrak{z}_\mathfrak{g}(d)$ intersects the center ([36, Prop. A.2]), the non-empty open invariant cone $W_\pi \cap \mathfrak{z}_\mathfrak{g}(d)$ actually intersects the subspace $\mathbb{R}ic \oplus \mathfrak{z}(\mathfrak{k}^\phi) \oplus \mathbb{R}i \cdot d$. For the three types of standard involutions we have $\mathfrak{z}(\mathfrak{k}^\phi) = \{0\}$ (*cf.* Examples 2.8, 2.9 and 2.10), so that

$$W_\pi \cap (\mathbb{R}ic + \mathbb{R}i \cdot d) \neq \emptyset.$$

If, in addition, $\mathtt{d}\pi(c)$ is bounded, then $ic + W_\pi = W_\pi$ leads to $i \cdot d \in W_\pi \cup -W_\pi$. In particular, $\mathtt{d}\pi(d)$ is bounded from below or above.

Now we assume that π is irreducible and w.l.o.g. that $i \cdot d \in W_\pi$. Then $\pi(\exp 2\pi i \cdot d) \in \pi(Z(G)) \subseteq \mathbb{T}\mathbf{1}$ by Schur's Lemma and Lemma 5.1 implies that the minimal spectral value μ of $\mathtt{d}\pi(d)$ is an eigenvalue. Let $V := \ker(\mathtt{d}\pi(d) - \mu\mathbf{1})$ be the corresponding eigenspace. It is invariant under the subgroup $Z_G(d)$ which leads to a unitary representation (ρ, V) of this group. The corresponding open convex cone W_ρ satisfies

$$i \cdot d \in W_\pi \cap \mathfrak{z}_\mathfrak{g}(d) \subseteq W_\rho,$$

but since $\mathtt{d}\rho(d) = \mu\mathbf{1}$, this leads to $0 \in W_\rho - i \cdot d = W_\rho$ and thus to $W_\rho = \mathfrak{z}_\mathfrak{g}(d)$, *i.e.*, ρ is bounded. □

Remark 5.3. — (a) Since $Z(G)_0 \cong \mathbb{T}$ by Theorem 3.4, any unitary representation of G is a direct sum of $\mathtt{d}\pi(c)$-eigenspaces, so that we can easily reduce to the situation where $\mathtt{d}\pi(c) \in \mathbb{R}\mathbf{1}$.

(b) It is a key point in the proof of Theorem 5.2 that $\mathfrak{z}(\mathfrak{k}^\phi) = \{0\}$ which holds for all 3 standard involutions. For any finite order automorphism ϕ for which $\mathfrak{z}(\mathfrak{k}^\phi) = \{0\}$, the argument in the proof of Theorem 5.2 goes through, and even Theorem 5.4 below remains valid. This is in particular the case if $\mathfrak{k}$ is abelian and $\mathfrak{k}^\phi = \{0\}$. With automorphisms of the form $\phi(g) = UgU^{-1}$, where U is unitary of order N with some finite-dimensional eigenspaces, one obtains examples where $\mathfrak{z}(\mathfrak{k}^\phi) \neq \{0\}$.

Our next step is to explain why irreducible semibounded representations with $i \cdot d \in W_\pi$ are uniquely determined by the representations on the minimal eigenspaces of $\mathtt{d}\pi(d)$. This requires the technique of holomorphic induction from Appendix C.

5.2. Holomorphic induction for $\widehat{\mathcal{L}}_\phi(K)$

Let $\mathfrak{g}_B := \widetilde{\mathcal{L}}_\phi^H(\mathfrak{k})$ be the central extension of H^1-loop algebra $\mathcal{L}_\phi^H(\mathfrak{k})$ from Definition A.6 which is a Banach–Lie algebra. According to Theorem A.8, there exists a corresponding 1-connected Banach–Lie group $G_B := \widetilde{\mathcal{L}}_\phi^H(K)$ and also a complex group $(G_B)_\mathbb{C} = \widetilde{\mathcal{L}}_\phi^H(K_\mathbb{C})$ (Remark A.10). Then

$$\mathfrak{p}_B^\pm := \overline{\sum_{\pm n > 0} e_n \otimes \mathfrak{k}_\mathbb{C}}$$

are closed subalgebras of $(\mathfrak{g}_B)_\mathbb{C}$. We also put

$$\mathfrak{h} := \mathfrak{h}_B := \mathbb{R}ic + \mathfrak{k}^\phi \quad \text{and} \quad \mathfrak{q}_B := \mathfrak{h}_\mathbb{C} \ltimes \mathfrak{p}_B^+.$$

The Fourier expansion of H^1-loops implies that $\mathfrak{g}_B$ satisfies the splitting condition (SC) from Appendix C:

$$(\mathfrak{g}_B)_\mathbb{C} = \mathfrak{p}_B^+ \oplus \mathfrak{h}_\mathbb{C} \oplus \mathfrak{p}_B^-.$$

Therefore all assumption of Example C.4(a) are satisfied.

On G_B we now consider the one-parameter group $\alpha\colon \mathbb{R} \to \mathrm{Aut}(\widetilde{\mathcal{L}}_\phi^H(K))$ defining the translation action $(\alpha_T f)(t) = f(t+T)$ of $\mathbb{R}$. Then $\mathfrak{p}_B^\pm$, $\mathfrak{h}$ and $\mathfrak{q}_B$ are α-invariant, so that Example C.4(b) applies. On the Lie algebra level, the subspace $\mathfrak{g} \subseteq \mathfrak{g}_B$ of smooth vectors for α coincides with the Fréchet–Lie algebra $\widetilde{\mathcal{L}}_\phi(\mathfrak{k})$ defined by smooth loops, and $\mathfrak{h}$ corresponds to the Lie subalgebra of α-invariant elements. Therefore the concepts and results from Appendix C concerning holomorphic induction are now available for the pair $(G, Z_G(d)_0)$, resp., the complex homogeneous space

$$\widehat{\mathcal{L}}_\phi(K)/Z_G(d)_0 \cong \widetilde{\mathcal{L}}_\phi(K)/Z_{\widetilde{\mathcal{L}}_\phi(K)}(d)_0 \cong \mathcal{L}_\phi(K)/(K^\phi)_0.$$

In view of Lemma 2.12 we know that the subgroup K^ϕ is connected, so that $\mathcal{L}_\phi(K)/K^\phi \cong \mathcal{L}_\phi(K)/(K^\phi)_0$ is actually simply connected because $\mathcal{L}_\phi(K)$ is 1-connected.

Theorem 5.4. — *Every irreducible semibounded unitary representation $(\pi, \mathcal{H})$ of $\widehat{\mathcal{L}}_\phi(K)$ for which $\mathtt{d}\pi(d)$ is bounded from below is holomorphically induced from the bounded representation (ρ, V) of $H = Z_G(d)_0$ on the minimal eigenspace of $\mathtt{d}\pi(d)$.*

Proof. — We want to apply Theorem C.3. We know from Theorem 5.2 that (ρ, V) is a bounded representation of H, which implies (HI1). So we first use Lemma 5.1 to see that the projection $P_0\colon \mathcal{H} \to V$ onto the eigenspace $V = \ker(\mathtt{d}\pi(d) - \mu\mathbf{1})$ maps $\mathcal{H}^\infty$ onto $\mathcal{H}^\infty \cap V$, and since P_0 is continuous and $\mathcal{H}^\infty$ is dense in $\mathcal{H}$, $\mathcal{H}^\infty \cap V$ is dense in V. Let $P_n\colon \mathcal{H} \to \mathcal{H}_n$, $n \in \mathbb{Z}$, denote the projections onto the other eigenspaces of $\exp(\mathbb{R}id)$ from Lemma 5.1. As V is the minimal eigenspace of the diagonalizable operator $\mathtt{d}\pi(d)$, the fact that $\mathtt{d}\pi(\mathfrak{g}_\mathbb{C}^k)\mathcal{H}_n \subseteq \mathcal{H}_{n+k}$ for $k, n \in \mathbb{Z}$ implies that $V \cap \mathcal{H}^\infty \subseteq (\mathcal{H}^\infty)^{\mathfrak{p}^-}$ for $\mathfrak{p}^- = \mathfrak{p}_B^- \cap \mathfrak{g}_\mathbb{C}$. This proves (HI2). Finally (HI3) follows from the irreducibility of $(\pi, \mathcal{H})$. □

In view of the preceding theorem, a classification of the irreducible semibounded representations of $\widehat{\mathcal{L}}_\phi(K)$ now consists in a classification of the irreducible bounded representations (ρ, V) of $Z_G(d)_0$ which are inducible in the sense of Definition C.1. It is easy to pinpoint a necessary condition for inducibility.

Definition 5.5. — *We say that a representation (ρ, V) of $\mathfrak{z}_\mathfrak{g}(d)$ is d-minimal if $\rho(\mathfrak{t}_\mathfrak{g})$ is diagonalizable and all $\mathfrak{t}_\mathfrak{g}$-weights of ρ are d-minimal.*

Proposition 5.6. — *For $y \in \mathfrak{g}_\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{h}_\mathbb{C} \oplus \mathfrak{p}^-$, we write $y = y_+ + y_0 + y_-$ for the corresponding decomposition. If a bounded unitary representation (ρ, V) of $H = Z_G(d)_0$ is holomorphically inducible for*

$\mathfrak{q} = \mathfrak{p}^+ \rtimes \mathfrak{h}_{\mathbb{C}}$, *then*

$$\mathtt{d}\rho([z^*, z]_0) \geqslant 0 \quad \text{for } z \in \mathfrak{p}^+, \tag{5.1}$$

and this implies that it is d-minimal, provided $\rho(\mathfrak{t}_{\mathfrak{g}})$ is diagonalizable.

Proof. — Suppose that $(\pi, \mathcal{H})$ is obtained from (ρ, V) by holomorphic induction. Then $V \subseteq (\mathcal{H}^\infty)^{\mathfrak{p}^-}$, so that we obtain for $v \in V$ and $z \in \mathfrak{p}^+$

$$\begin{aligned}\langle \mathtt{d}\rho([z^*, z]_0)v, v\rangle &= \langle \mathtt{d}\rho([z^*, z])v, v\rangle = \langle [\mathtt{d}\pi(z^*), \mathtt{d}\pi(z)]v, v\rangle \\ &= \langle \mathtt{d}\pi(z^*)\mathtt{d}\pi(z)v, v\rangle = \|\mathtt{d}\pi(z)v\|^2 \geqslant 0.\end{aligned}$$

This proves the necessity of (5.1).

For every weight vector $v_\mu \in V$ with weight $\mu \in i\mathfrak{t}'_{\mathfrak{g}}$ and $\underline{\alpha} = (\alpha, n) \in (\Delta_{\mathfrak{g}})_c$ with $n > 0$ we pick $x \in \mathfrak{g}_{\mathbb{C}}^{\underline{\alpha}}$ such that $[x, x^*] = \underline{\check{\alpha}}$ (*cf.* Example 2.5). Then (5.1) implies $\mu(\underline{\check{\alpha}}) \leqslant 0$. In view of Proposition 4.3(iii), this is equivalent to the d-minimality of μ. □

5.3. Bounded representations of K^ϕ

Remark 5.7. — Let $\Delta = \Delta(\mathfrak{k}, \mathfrak{t})$ be a root system of type A_J, B_J, C_J or D_J. We represent a corresponding integral weight as a function $\lambda \colon J \to \mathbb{R}$ and observe that

$$A_J \subseteq D_J = B_J \cap C_J.$$

Then the integrality with respect to A_J means that $\lambda_j - \lambda_k \in \mathbb{Z}$ for $j \neq k \in J$. Since J is infinite in our context, the requirement $\lambda \in i\mathfrak{t}' \cong \ell^2(J, \mathbb{R})$ implies that λ is finitely supported with values in $\mathbb{Z}$. This in turn implies that λ is an integral weight for A_J, B_J, C_J and D_J.

Proposition 5.8 (Classification of $\mathcal{W}$-orbits). — *For $\Delta = \Delta(\mathfrak{k}, \mathfrak{t})$ of type A_J, B_J, C_J or D_J the corresponding set of integral weights $\mathcal{P}(\mathfrak{k}, \mathfrak{t}) \subseteq i\mathfrak{t}' \cong \ell^2(J, \mathbb{R})$ coincides with $\ell^2(J, \mathbb{Z}) \cong \mathbb{Z}^{(J)}$. For the $\mathcal{W}$-action on this set, we have the following set of invariants which is complete in the sense that it separates the $\mathcal{W}$-orbits in $\mathcal{P}(\mathfrak{k}, \mathfrak{t})$:*

- A_J : $\quad m_n(\lambda) := |\{j \in J \colon \lambda_j = n\}|$ *for* $0 \neq n \in \mathbb{Z}$.
- B_J, C_J *and* D_J: $\quad m_n(\lambda) := |\{j \in J \colon |\lambda_j| = n\}|$ *for* $n \in \mathbb{N}$.

Proof. — (a) For A_J, the functions m_n are constant on the orbits of $\mathcal{W} \cong S_{(J)}$ and, conversely, if $m_n(\lambda) = m_n(\lambda')$ for $\lambda, \lambda' \in \mathcal{P}(\mathfrak{k}, \mathfrak{t})$, then $\lambda' \in \mathcal{W}\lambda$ follows from the finiteness of the support of λ.

(b) For the root systems B_J, C_J and D_J, the functions m_n, $n \in \mathbb{N}$, are constant on the $\mathcal{W}$-orbits, and since every $\lambda \in \mathcal{P}(\mathfrak{k}, \mathfrak{t})$ is finitely supported, its $\mathcal{W}$-orbit contains a non-negative element. Hence $m_n(\lambda) = m_n(\lambda')$ for $\lambda, \lambda' \in \mathcal{P}(\mathfrak{k}, \mathfrak{t})$ and every $n \in \mathbb{N}$ leads to $\lambda' \in \mathcal{W}\lambda$. □

THEOREM 5.9. — *Let K be a simple Hilbert–Lie group with Lie algebra $\mathfrak{k}$ and $\mathfrak{t} \subseteq \mathfrak{k}$ maximal abelian with root system $\Delta \subseteq i\mathfrak{t}'$. Then every bounded unitary representation of K is a direct sum of irreducible ones. The irreducible representations $(\rho_\lambda, V_\lambda)$ can be parametrized by their extremal weights $\lambda \in \mathcal{P}(\mathfrak{k}, \mathfrak{t})$ as follows. If $\mathcal{Q} := \langle \Delta \rangle_{\rm grp} \subseteq i\mathfrak{t}'$ is the root group, then the weight set $\mathcal{P}_\lambda$ of ρ_λ satisfies*

$$\mathcal{P}_\lambda = \operatorname{conv}(\mathcal{W}\lambda) \cap (\lambda + \mathcal{Q}) \quad \textit{and} \quad \operatorname{Ext}(\operatorname{conv}(\mathcal{P}_\lambda)) = \mathcal{W}\lambda.$$

We have $\rho_\lambda \sim \rho_\mu$ if and only if $\mu \in \mathcal{W}\lambda$, so that the irreducible bounded unitary representations of K are classified by the set $\mathcal{P}(\mathfrak{k}, \mathfrak{t})/\mathcal{W}$ of $\mathcal{W}$-orbits in $\mathcal{P}(\mathfrak{k}, \mathfrak{t})$. All these representations factor through the adjoint group $K/Z(K)$.

Proof. — In view of the classification of simple Hilbert–Lie algebras, the assertion on the classification follows from [37, Thm. III.14] for $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$ and from [45, Thms. D.5, D.6] for $\mathfrak{k} = \mathfrak{o}_2(\mathcal{H})$ and $\mathfrak{sp}_2(\mathcal{H})$.

That all these representations factor through the adjoint group is trivial for $\mathfrak{k} = \mathfrak{sp}_2(\mathcal{H})$ because in this case the center of the corresponding simply connected group $\mathrm{Sp}_2(\mathcal{H})$ is trivial (Theorem 1.5). For $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$ it follows from [45, Rem. D.2], and for $\mathfrak{k} = \mathfrak{o}_2(\mathcal{H})$ the description of the corresponding highest weights (*cf.* Remark 5.7) implies that they are contained in the root group $\mathcal{Q}$, and hence that the corresponding representation is trivial on the center. □

For the 3 standard involutions ϕ (*cf.* Examples 2.8-2.10), the Lie algebra $\mathfrak{k}^\phi$ is simple, so that we obtain an explicit description of the bounded irreducible representations of the groups K^ϕ, resp., $Z_G(d)_0$ in all seven cases. According to Theorem 5.4, any irreducible semibounded representations of $G = \widehat{\mathcal{L}}_\phi(K)$ for which $\mathtt{d}\pi(d)$ is bounded from below is holomorphically induced from the representation (ρ, V) on the minimal eigenspace, hence uniquely determined by this representation (*cf.* Definition C.1). It therefore remains to identify those bounded representations of $Z_G(d)_0$ which are holomorphically inducible.

5.4. Characterization of the inducible bounded representations

In this subsection we show that any d-minimal bounded representation (ρ, V) of $Z_G(d)_0$ is inducible.

THEOREM 5.10. — *An irreducible bounded unitary representation (ρ,V) of $Z_G(d)_0$ is holomorphically inducible if and only if it is d-minimal.*

Proof. — We have already seen in Proposition 5.6 that ρ is d-minimal if it is holomorphically inducible. Now we assume that (ρ, V) is a d-minimal bounded representation of $H = Z_G(d)_0$ of extremal weight $\lambda \in \widehat{T}_G = \mathrm{Hom}(T_G, \mathbb{T})$ (recall $T_G = \exp \mathfrak{t}_\mathfrak{g}$). Then λ is d-minimal, so that $\lambda(\check{\underline{\alpha}}) \geqslant 0$ for $\underline{\alpha} = (\alpha, n)$, $n < 0$. This means that

$$\mathfrak{p}_\lambda := (\mathfrak{t}_\mathfrak{g})_\mathbb{C} + \sum_{\lambda(\underline{\alpha}^\sharp)\geqslant 0} \mathfrak{g}_\mathbb{C}^{\underline{\alpha}} \supseteq \sum_{\alpha\in\Delta, n>0} \mathfrak{g}_\mathbb{C}^{(\alpha,n)}.$$

If $\lambda_c = 0$, then λ vanishes on $\check{\Delta}_\mathfrak{g}$ (Lemma 4.2) which implies that G has a one-dimensional representation $(\pi, \mathcal{H})$ for which $\mathtt{d}\pi\colon \mathfrak{g}_\mathbb{C} \to \mathrm{End}(\mathcal{H}) \cong \mathbb{C}$ extends λ. We may therefore assume that $\lambda_c \neq 0$.

In view of Theorem C.6, it suffices to show that the corresponding linear map

$$\beta\colon U(\mathfrak{g}_\mathbb{C}) \to B(V)$$

that vanishes on $\mathfrak{p}^+ U(\mathfrak{g}_\mathbb{C}) + U(\mathfrak{g}_\mathbb{C})\mathfrak{p}^-$ and satisfies $\beta|_{U(\mathfrak{h}_\mathbb{C})} = \mathtt{d}\rho$ for $\mathfrak{h} = \mathfrak{z}_\mathfrak{g}(d)$, is positive definite on the $*$-algebra $U(\mathfrak{g}_\mathbb{C})$ (*cf.* Definition B.1(c)).

Let $\mathfrak{t}_\mathbb{C}^{\mathrm{alg}} := \mathbb{C}i \cdot d + \mathrm{span}_\mathbb{C}\, \check{\Delta}_\mathfrak{g}$ and

$$\mathfrak{g}_\mathbb{C}^{\mathrm{alg}} := \mathbb{C}d + \langle \mathfrak{g}_\mathbb{C}^\alpha \colon \alpha \in (\Delta_\mathfrak{g})_c \rangle_{\text{Lie alg}},$$

and observe that this is a Lie algebra with a root decomposition with respect to $\mathfrak{t}_\mathbb{C}^{\mathrm{alg}}$. It is a coral locally affine complex Lie algebra in the sense of [43, Def. 3.1] and λ defines an integral weight of $\mathfrak{g}_\mathbb{C}^{\mathrm{alg}}$ for which $\lambda_c \neq 0$. Therefore [43, Thm. 4.11] implies the existence of a unitary extremal weight module $(\pi_\lambda, L(\lambda))$ of $\mathfrak{g}_\mathbb{C}$ generated by a $\mathfrak{p}_\lambda$-eigenvector v_λ of weight λ. Note that [43, Thm. 5.7] shows that $\mathfrak{g}_\mathbb{C}^{\mathrm{alg}}$ is a locally extended affine Lie algebra with root system $\Delta_\mathfrak{g}$ in the sense of [35] (see also [43]).

Now $V_\lambda := U(\mathfrak{h}_\mathbb{C})v_\lambda$ is an $\mathfrak{h}_\mathbb{C}$-module of extremal weight λ, so that we may identify it with a dense subspace of V. For $\mathfrak{p}^\pm_{\mathrm{alg}} := \mathfrak{p}^\pm \cap \mathfrak{g}_\mathbb{C}^{\mathrm{alg}}$, the relation $v_\lambda \in L(\lambda)^{\mathfrak{p}^-_{\mathrm{alg}}}$ implies that $V_\lambda \subseteq L(\lambda)^{\mathfrak{p}^-_{\mathrm{alg}}}$, and

$$L(\lambda) = U(\mathfrak{g}_\mathbb{C}^{\mathrm{alg}})v_\lambda = U(\mathfrak{p}^+_{\mathrm{alg}})V_\lambda \subseteq V_\lambda + \mathfrak{p}^+_{\mathrm{alg}} L(\lambda)$$

shows that V_λ is the minimal d-eigenspace in $L(\lambda)$. Let $p_V\colon L(\lambda) \to V_\lambda \subseteq V$ denote the orthogonal projection. Then

$$\gamma(D) := p_V \pi_\lambda(D) p_V^*$$

satisfies $\mathfrak{p}^+_{\mathrm{alg}} U(\mathfrak{g}_{\mathbb{C}}^{\mathrm{alg}}) + U(\mathfrak{g}_{\mathbb{C}}^{\mathrm{alg}})\mathfrak{p}^-_{\mathrm{alg}} \subseteq \ker \gamma$ and $\gamma|_{U(\mathfrak{h}_{\mathbb{C}}^{\mathrm{alg}})} = \rho_\lambda$. Since the representation π_λ on $L(\lambda)$ is unitary, γ is positive definite. Since all maps

$$\mathfrak{g}_{\mathbb{C}}^k \to B(V), \quad (x_1, \ldots, x_k) \mapsto \beta(x_1 \cdots x_k)$$

are continuous and the restriction of β to the subalgebra $U(\mathfrak{g}_{\mathbb{C}}^{\mathrm{alg}})$ coincides with γ, it follows that β is also positive definite. Now the assertion follows from Theorem C.6. □

Remark 5.11. — (a) Consider the Banach–Lie group $\widetilde{\mathcal{L}}_\phi^H(K)$ from Appendix A and the subgroup $\mathbb{T} \times K^\phi$ corresponding to the centrally extended Lie algebra $\mathbb{R} \times \mathfrak{k}^\phi \subseteq \widetilde{\mathcal{L}}_\phi^H(\mathfrak{k})$. Suppose that (ρ, V) is a bounded representation of $H := \mathbb{T} \times K^\phi$ which is holomorphically inducible to the Fréchet–Lie group $\widetilde{\mathcal{L}}_\phi(K)$. Since $\widetilde{\mathcal{L}}_\phi(\mathfrak{k})$ is dense in $\widetilde{\mathcal{L}}_\phi^H(\mathfrak{k})$, the fact that the conditions in Theorem C.6 are satisfied for $\widetilde{\mathcal{L}}_\phi(K)$ immediately implies that they are also satisfied for the bigger group $\widetilde{\mathcal{L}}_\phi^H(K)$. Therefore the holomorphically induced representation $(\pi, \mathcal{H})$ of $\widetilde{\mathcal{L}}_\phi(K)$ extends to a holomorphically induced representation of the Banach–Lie group $\widetilde{\mathcal{L}}_\phi^H(K)$, and we thus obtain a continuous unitary representation of the topological group $\widehat{\mathcal{L}}_\phi^H(K)$.

(b) The preceding argument also shows that, if ρ is irreducible, then the same holds for the corresponding holomorphically induced representation of $\widetilde{\mathcal{L}}_\phi(K)$ resp., $\widetilde{\mathcal{L}}_\phi^H(K)$.

(c) Assume that $\phi = \mathrm{id}$. Then we can also ask about the restriction of π to the subgroup $L := \widetilde{\mathcal{L}}(K)_*$ corresponding to functions vanishing in $\mathbf{1}$. As $\widetilde{\mathcal{L}}(K) = L \rtimes K$ corresponding to functions vanishing in $\mathbf{1}$, the group L acts transitively on the complex homogeneous space $\mathcal{L}(K)/K$ which implies that $\pi|_L$ is holomorphically induced from the trivial representation of $L \cap K = \{\mathbf{1}\}$ on V. This leads to $\pi(L)' \cong B(V)$ (Theorem C.2(ii)) which implies in particular that $\pi|_L$ is irreducible if and only if $\dim V = 1$.

5.5. Semibounded representations of one-dimensional extensions

In this subsection we provide a few results supporting the point of view that, without the double extension, the representation theory of loop groups is much less interesting. We show that all semibounded unitary representations of the central extension $\widetilde{\mathcal{L}}_\phi(\mathfrak{k})$ are trivial on the center and factor through bounded representations of $\mathcal{L}(\mathfrak{k})$. One can actually show that these are finite-dimensional and tensor products of evaluation representations. For those representation extending to the Lie algebra $\mathcal{L}^c(\mathfrak{k})$ of continuous maps, this follows from [47]. We also show that all semibounded representations of $\mathcal{L}_\phi(K) \rtimes_\alpha \mathbb{R}$ are trivial on $\mathcal{L}_\phi(K)$.

Lemma 5.12. — *Let $(\pi, \mathcal{H})$ be a smooth representation of a Lie group G. Then the following assertions hold:*

(i) *Let $x, y \in \mathfrak{g}$ with $[x,[x,y]] = 0$. If $-i\mathtt{d}\pi(y)$ is bounded from below, then $\mathtt{d}\pi([x,y]) = 0$.*

(ii) *If $\mathfrak{g}$ is 2-step nilpotent and π is semibounded, then $[\mathfrak{g},\mathfrak{g}] \subseteq \ker(\mathtt{d}\pi)$.*

Proof. — (i) For a smooth unit vector $v \in \mathcal{H}^\infty$, we consider the continuous linear functional $\lambda(z) := \langle -i\mathtt{d}\pi(z)v, v\rangle$ on $\mathfrak{g}$. Then our assumption implies that λ is bounded from below on $\mathrm{Ad}(G)y$ which contains $\mathrm{Ad}(\exp \mathbb{R}x)y = y + \mathbb{R}[x,y]$. This leads to $\lambda([x,y]) = 0$. We thus obtain $\mathtt{d}\pi([x,y]) = 0$.

(ii) Pick $y \in W_\pi$. For every $x \in \mathfrak{g}$ we then have $[x,[x,y]] = 0$, so that (i) leads to $\mathtt{d}\pi([x,y]) = 0$ and thus $[W_\pi, \mathfrak{g}] \subseteq \ker(\mathtt{d}\pi)$. As W_π is open, the assertion follows. □

Proposition 5.13. — *Let $\mathfrak{k}$ be a simple Hilbert–Lie algebra and $\phi \in \mathrm{Aut}(\mathfrak{k})$. Then all open invariant cones in $\mathcal{L}_\phi(\mathfrak{k})$ are trivial.*

Proof. — Let $\emptyset \neq W \subseteq \mathcal{L}_\phi(\mathfrak{k})$ be an open invariant convex cone.

(a) First we consider the case $\phi = \mathrm{id}$ and show that, for every compact manifold M with or without boundary, all open invariant cones in $C^\infty(M,\mathfrak{k})$ are trivial. Since $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$ for an infinite dimensional real, complex or quaternionic Hilbert space, and the union of the subalgebras $\mathfrak{su}(\mathcal{H}_F)$, where $\mathcal{H}_F \subseteq \mathcal{H}$ is a finite-dimensional subspace, is dense in $\mathfrak{u}(\mathcal{H})$, the union of the subalgebras $C^\infty(M, \mathfrak{su}(\mathcal{H}_F))$ is dense in $C^\infty(M,\mathfrak{k})$. Hence there exists a subspace $\mathcal{H}_F$ for which $\emptyset \neq W_F := W \cap C^\infty(M, \mathfrak{su}(\mathcal{H}_F))$. Then W_F is invariant under conjugation with the compact group $\mathrm{SU}(\mathcal{H}_F)$, hence contains a $\mathrm{SU}(\mathcal{H}_F)$-fixed point. Since $\mathfrak{su}(\mathcal{H}_F)$ has trivial center, 0 is the only fixed point, and thus $0 \in W_F \subseteq W$. This in turn implies that $W = \mathcal{L}(\mathfrak{k})$.

(b) For the general case, we consider the closed interval $I := [-a,a]$ for $0 < a < \frac{\pi}{N}$. Then we have a continuous restriction map

$$R\colon \mathcal{L}_\phi(\mathfrak{k}) \to C^\infty(I,\mathfrak{k}),$$

which is also surjective (*cf.* [66, Cor. III.7]), hence open by the Open Mapping Theorem. Therefore $R(W)$ is an open invariant cone in $C^\infty(I,\mathfrak{k})$, and (a) implies that $0 \in R(W)$, which in turn implies that $W \cap \ker R \neq \emptyset$.

For $b := \frac{2\pi}{N}$, we have

$$\ker R = \{f \in \mathcal{L}_\phi(\mathfrak{k})\colon f|_{[-a,a]} = 0\} \cong \{f \in C^\infty([0,b],\mathfrak{k})\colon f|_{[0,a]} = 0 = f_{[b-a,b]}\}.$$

For every $k \in K$ there exists an element $f \in \mathcal{L}_\phi(\mathfrak{k})$ restricting to the constant function k on $[a, b-a]$, so that $W \cap \ker R$ is invariant under conjugation with constant functions in K. Passing to a sufficiently large finite dimensional subalgebra $\mathfrak{k}_F \subseteq \mathfrak{k}$ and averaging over the action of the corresponding compact group K_F, it follows as in (a) that $0 \in W \cap \ker R \subseteq W$, so that $W = \mathcal{L}_\phi(\mathfrak{k})$. □

Corollary 5.14. — *If K is a simple Hilbert–Lie group, then all semibounded unitary representations of $\mathcal{L}_\phi(K)$ are bounded.*

Theorem 5.15 (Semibounded representations of central extensions). *If $\mathfrak{k}$ is a simple Hilbert–Lie algebra, then all semibounded unitary representations of the central extension $\widetilde{\mathcal{L}}_\phi(\mathbb{K})$ are trivial on the center and bounded.*

Proof. — Localization on the center reduces the problem to representations which are bounded on the center, so that $\mathbb{R}ic + W_\pi = W_\pi$. Hence W_π defines an open invariant cone in $\mathcal{L}_\phi(\mathfrak{k}) \cong \widetilde{\mathcal{L}}_\phi(\mathfrak{k})/\mathbb{R}ic$, which is trivial by Proposition 5.13. Therefore π is bounded. In particular, the restriction of π to the 2-step nilpotent group $\widetilde{\mathcal{L}}_\phi(\mathfrak{t}_\mathfrak{k})$ is bounded. Since it is 2-step nilpotent, $\mathtt{d}\pi$ is trivial on the commutator algebra (*cf.* Lemma 5.12(ii)), and thus $\mathtt{d}\pi(c) = 0$. We conclude that $\mathtt{d}\pi(c)$ vanishes for all semibounded representations, and hence also that these representations are bounded. □

The preceding result shows that the central extension $\widetilde{\mathcal{L}}_\phi(K)$ and $\mathcal{L}_\phi(K)$ have the same (semi-)bounded representations. In a similar vein, extending $\mathcal{L}_\phi(K)$ to the semidirect product defined by the translation actions only leads to trivial semibounded representations.

Theorem 5.16 (Semibounded representations of semidirect products). *If $\mathfrak{k}$ is a simple Hilbert–Lie algebra and ϕ any finite order automorphism of the corresponding simply connected group K for which $\mathfrak{z}(\mathfrak{k}^\phi) = \{0\}$, then every unitary representation $(\pi, \mathcal{H})$ of $\mathcal{L}_\phi(K) \rtimes_\alpha \mathbb{R}$ for which $-i\mathtt{d}\pi(0,1)$ is bounded from below is trivial on $\mathcal{L}_\phi(K)$.*

Proof. — For any abelian ϕ-invariant subalgebra $\mathfrak{a} \subseteq \mathfrak{k}$, we consider the 2-step solvable Lie algebra $\mathcal{L}_\phi(\mathfrak{a})$ and note that $\mathcal{L}_\phi(\mathfrak{a}) = [d, \mathcal{L}_\phi(\mathfrak{a})] \oplus \mathfrak{z}_{\mathcal{L}_\phi(\mathfrak{a})}(d) = [d, \mathcal{L}_\phi(\mathfrak{a})] \oplus \mathfrak{a}^\phi$. Therefore Lemma 5.12 implies that $\mathcal{L}_\phi(\mathfrak{a}) \subseteq \mathfrak{a}^\phi + \ker(\mathtt{d}\pi)$.

Applying this observation to one-dimensional subalgebras $\mathfrak{a} = \mathbb{R}x \subseteq \mathfrak{k}^\phi$, we obtain $\mathcal{L}_\phi(\mathfrak{k}^\phi) = \mathcal{L}(\mathfrak{k}^\phi) \subseteq \mathfrak{k}^\phi + \ker(\mathtt{d}\pi)$. As $\mathfrak{k}^\phi$ is topologically perfect, it is contained in the ideal of $\mathcal{L}(\mathfrak{k}^\phi)$ generated by $\mathfrak{z}_d(\mathcal{L}(\mathfrak{k}^\phi))$. This leads to $\mathcal{L}(\mathfrak{k}^\phi) \subseteq \ker(\mathtt{d}\pi)$.

For the abelian subalgebra $\mathfrak{t}_\mathfrak{k} \subseteq \mathfrak{k}$ we likewise obtain $\mathcal{L}_\phi(\mathfrak{t}_\mathfrak{k}) \subseteq \mathfrak{t} + \ker(\mathtt{d}\pi) = \mathfrak{t}_\mathfrak{k} + \ker(\mathtt{d}\pi)$. Hence $\mathfrak{t} \subseteq \mathfrak{k}^\phi \subseteq \ker(\mathtt{d}\pi)$ enventually yields $\mathcal{L}_\phi(\mathfrak{t}_\mathfrak{k}) \subseteq \ker(\mathtt{d}\pi)$. We finally arrive at $\mathcal{L}_\phi(\mathfrak{k}) \subseteq \mathfrak{t}_\mathfrak{k} + [\mathfrak{t}, \mathcal{L}_\phi(\mathfrak{k})] \subseteq \ker(\mathtt{d}\pi)$. □

6. Semiboundedness of holomorphically induced representations

We are now ready to complete the picture by showing that the irreducible G-representations $(\pi_\lambda, \mathcal{H}_\lambda)$ obtained by holomorphic induction from d-minimal representations $(\rho_\lambda, V_\lambda)$ are semibounded.

Theorem 6.1. — *Let K be a 1-connected Hilbert–Lie group and $(\pi, \mathcal{H})$ be a unitary representation of $G = \widehat{\mathcal{L}}_\phi(K)$ which is holomorphically induced from the bounded representation (ρ, V) of $H = Z_G(d)_0$ for which $\mathtt{d}\rho(d) = \mu\mathbf{1}$ for some $\mu \in \mathbb{R}$. Then $(\pi, \mathcal{H})$ is semibounded with $i \cdot d \in W_\pi$.*

Proof. — Recall the subalgebras $\mathfrak{p}_B^\pm$ and $\mathfrak{p}^\pm$ from Section 5. We note that the representation $\mathrm{ad}_{\mathfrak{p}^+}$ of the Hilbert–Lie algebra $\mathfrak{h} = \mathfrak{z}_\mathfrak{g}(d)$ on the Hilbert space $\mathfrak{p}_B^+ = \overline{\sum_{n>0} \mathfrak{g}_\mathbb{C}^n} \subseteq (\mathfrak{g}_B)_\mathbb{C}$ is unitary, ic acts trivially, d acts by $n \cdot \mathrm{id}$ on $\mathfrak{g}_\mathbb{C}^n = e_n \otimes \mathfrak{k}_\mathbb{C}^n$, and $\mathfrak{k}^\phi$ acts by the adjoint representation on $\mathfrak{g}_\mathbb{C}^n \cong \mathfrak{k}_\mathbb{C}^n$. Hence an element $x = (z, x_0, t) \in \mathfrak{z}_\mathfrak{g}(d)$ satisfies

$$-i\,\mathrm{ad}_{\mathfrak{g}_\mathbb{C}^n}(z, x_0, t) = tn - i\,\mathrm{ad}_{\mathfrak{k}_\mathbb{C}} x_0 \geqslant 0 \quad \text{for every } n > 0 \quad \text{if } \|\mathrm{ad}\, x_0\| \leqslant t.$$

Therefore the elements $x \in \mathfrak{z}_\mathfrak{g}(d)$ with this property form a closed invariant cone with non-empty interior C.

For the $\mathtt{d}\pi(d)$-eigenspace decomposition

$$\mathcal{H} = \widehat{\bigoplus}_{n \in \mathbb{N}_0} \mathcal{H}_n \quad \text{with} \quad \mathcal{H}_n = \ker(\mathtt{d}\pi(d) - (\mu + n)\mathbf{1}),$$

all subspaces $\mathcal{H}_n$ are invariant under $Z_G(d)_0$. For every $v \in V = \mathcal{H}_0 \subseteq \mathcal{H}^\omega$, the Poincaré–Birkhoff–Witt Theorem shows that

$$U(\mathfrak{g}_\mathbb{C})v = U(\mathfrak{p}^+)U(\mathfrak{h}_\mathbb{C})U(\mathfrak{p}^-)v = U(\mathfrak{p}^+)U(\mathfrak{h}_\mathbb{C})v \subseteq U(\mathfrak{p}^+)V$$

is dense in $\mathcal{H}$. Therefore $\mathcal{H}_n$, as a unitary representation of $Z_G(d)_0$ containing a dense subspace which is a quotient of of the bounded unitary representation on the Hilbert space

$$\left(\bigoplus_{0 \leqslant k \leqslant n} (\mathfrak{p}_B^+)^{\widehat{\otimes} k} \right)_n \widehat{\otimes} V,$$

is a bounded representation of $Z_G(d)_0$. For $x \in C$, the spectrum of $-ix$ on the left hand factor is non-negative, so that

$$\inf(\mathrm{Spec}(-i\mathtt{d}\pi(x)) = \inf(\mathrm{Spec}(-i\mathtt{d}\rho(x)),$$

resp.,

$$s_\pi(x) = \sup(\mathrm{Spec}(i\mathtt{d}\pi(x))) = s_\rho(x) \quad \text{for} \quad x \in C.^{(3)} \tag{6.1}$$

To see that $C \subseteq W_\pi$, it now remains to see that $\mathrm{Ad}(G)C$ has interior points. Let $U^\pm \subseteq \mathfrak{k}^{\pm\phi} = \ker(\mathrm{id} \mp \phi)$ be open convex symmetric 0-neighborhoods for which the map

$$E\colon U^+ \times U^- \to K, \quad (x_+, x_-) \mapsto \exp x_- \exp x_+ \exp x_-$$

is a diffeomorphism onto an open subset of K. The existence of such a 0-neighborhood follows from the Inverse Function Theorem because the differential of E in $(0,0)$ is given by $(x_+, x_-) \mapsto x_+ + 2x_-$.

For the ϕ^{-1}-twisted conjugation action $c^\phi_k(h) := kh\phi(k)^{-1}$ of K on itself we have

$$E(x_+, x_-) = \exp(x_-)\exp(x_+)\exp(x_-) = c^\phi_{\exp x_-}(\exp x_+).$$

Therefore each c^ϕ-orbit meeting $\mathrm{im}(E)$ also meets $\exp(U^+)$.

Next we recall the smooth map $\mathrm{Hol}_{2\pi/N}\colon \mathcal{L}_\phi(\mathfrak{k}) \to K$ from Proposition 2.14 and note that

$$V := \{\xi \in \mathcal{L}_\phi(\mathfrak{k})\colon \mathrm{Hol}_{2\pi/N}(\xi) \in \mathrm{im}(E)\}$$

is an open 0-neighborhood. From Proposition 2.14 we derive that every element in $V \times \{1\} \subseteq \mathcal{L}_\phi(\mathfrak{k}) \rtimes \mathbb{R}$ is conjugate under $\mathrm{Ad}(\mathcal{L}_\phi(K))$ to an element $\widetilde{\xi}$ with $\mathrm{Hol}_{2\pi/N}(\widetilde{\xi}) \in \exp(U^+)$. As $\mathrm{Hol}_{2\pi/N}(x_+) = \exp\left(\frac{2\pi}{N}x_+\right)$ for $x_+ \in \mathfrak{k}^\phi$, this further implies that $(\xi, 1)$ is conjugate to an element in $\frac{N}{2\pi}U^+ \times \{1\}$. We conclude that, for every 0-neighborhood $B \subseteq \mathfrak{k}^\phi$,

$$\mathrm{Ad}(\mathcal{L}_\phi(K))(\mathbb{R} \times B \times \{1\})$$

contains an open subset of the hyperplane $\mathbb{R} \times \mathcal{L}_\phi(\mathfrak{k}) \times \{1\} \subseteq \widehat{\mathcal{L}}_\phi(\mathfrak{k})$. Eventually this shows that $\mathrm{Ad}(G)C$ has interior points, and hence that $W_\pi \neq \emptyset$. □

(3) This is trivial if $\mathrm{ad}\, x$ is diagonalizable on V on each $\mathfrak{g}^n_{\mathbb{C}}$. Let π_n denote the representation of $Z_G(d)_0$ on $\mathcal{H}_n$ and $\pi^x_n(t) := \pi_n(\exp tx)$. For the general case, it is instructive to think of the spectrum of $\mathtt{d}\pi_n(x)$ in terms of Arveson's spectral theory, where $\mathrm{Spec}(-i\mathtt{d}\pi_n(x))$ is the minimal closed subset $S \subseteq \mathbb{R}$ with the property that, for every Schwartz function $f\colon \mathbb{R} \to \mathbb{R}$ with $\mathrm{supp}(f) \cap S = \emptyset$, we have $\pi^x_n(\widehat{f}) = 0$. In this context it is clear that we obtain the same spectrum from the representation on any dense invariant subspace, and that equivariant bilinear maps are compatible with addition of spectra (*cf.* [46, Prop. A.14]).

At this point we are ready to prove Theorem 0.1 stated in the introduction.

Proof of Theorem 0.1. — We have seen in Theorems 5.2 and 5.4 that a semibounded representation $(\pi, \mathcal{H})$ for which $\mathtt{d}\pi(d)$ is bounded from below (which is always the case for π or its dual representation) is holomorphically induced from a bounded representation (ρ, V) of the Hilbert–Lie group $Z_G(d)_0$ whose Lie algebra is $\mathbb{R}ic \oplus \mathfrak{k}^\phi \oplus \mathbb{R}i \cdot d$, where $\mathfrak{k}^\phi$ is a simple Hilbert–Lie algebra in all 7 cases.

Theorem 5.9 now provides a classification of the bounded irreducible representations of the simply connected covering group $\mathbb{R} \times \widetilde{K}^\phi \times \mathbb{R}$ of $Z_G(d)_0 = Z \times K^\phi \times \mathbb{R}$ in terms of extremal weights $\lambda \in \mathcal{P}(\mathfrak{k}^\phi, \mathfrak{t}) \subseteq i\mathfrak{t}'$. From Lemma 2.12 we know that K^ϕ is connected, so that its simply connected covering group is defined. We even know that K^ϕ is 1-connected if $(\Delta_\mathfrak{g})_c$ is not of type $BC_J^{(2)}$, and in the latter case $\pi_1(K^\phi) \cong \mathbb{Z}/2$. Next Theorem 5.10 characterizes the weights λ for which $(\rho_\lambda, V_\lambda)$ is holomorphically inducible as the d-minimal weights and the corresponding G-representation $(\pi_\lambda, \mathcal{H}_\lambda)$ is semibounded by Theorem 6.1.

That the $\mathfrak{t}_\mathfrak{g}$-weight set $\mathcal{P}_\lambda$ of π_λ satisfies

$$\mathcal{P}_\lambda := \mathrm{conv}(\widehat{\mathcal{W}}\lambda) \cap (\lambda + \widehat{\mathcal{Q}}) \quad \text{with} \quad \mathrm{Ext}(\mathrm{conv}(\mathcal{P}_\lambda)) = \widehat{\mathcal{W}}\lambda$$

follows from the corresponding result in [43, Thm. 4.10] for the highest weight module $L(\lambda)$ of $\mathcal{L}_\phi(\mathfrak{k})_\mathbb{C}^{\mathrm{alg}}$. This description implies in particular that the equivalence of π_λ and π_μ implies that $\mu \in \widehat{\mathcal{W}}\lambda$. We also see that the set of weights occurring as extremal weights in this context is contained in set $\mathcal{P}^+ = \widehat{\mathcal{W}}\mathcal{P}_d^+$ of integral weights bounded from below (Remark 4.5). From [20, Thm. 3.5] we further know that

$$\widehat{\mathcal{W}}\lambda \cap \mathcal{P}_d^+ = \widehat{\mathcal{W}}_d\lambda = \mathcal{W}\lambda \quad \text{for} \ \lambda \in \mathcal{P}_d^\pm,$$

where $\widehat{\mathcal{W}}_d \cong \mathcal{W}$ is the stabilizer of d in $\widehat{\mathcal{W}}$. This leads to a bijection

$$\mathcal{P}_d^\pm/\mathcal{W} \to \mathcal{P}^\pm/\widehat{\mathcal{W}}, \quad \mathcal{W}\lambda \mapsto \widehat{\mathcal{W}}\lambda.$$

To complete the proof, it only remains to show that every elements $\lambda \in \mathcal{P}_d^+$ actually is an extremal weight of a bounded representation $(\rho_\lambda, V_\lambda)$ of $Z_G(d)_0 \cong Z \times K^\phi \times \mathbb{R}$. As $\lambda|_\mathfrak{t}$ is a weight for Δ_0, the existence of the corresponding unitary representation of K^ϕ follows from Theorem 5.9. It therefore remains to verify that $N\lambda_c = \lambda(Nc) \in \mathbb{Z}$ (Theorem 3.4). In the untwisted cases the normalization of the scalar product is such that long roots α satisfy $(\alpha, \alpha) = 2$, so that $\lambda_c \in \mathbb{Z}$ follows from (2.9). In the twisted cases $2\lambda_c \in \mathbb{Z}$ follows from Example 3.8. □

Remark 6.2. — Let us take a closer look at the ambiguities arising in our parametrization of irreducible semibounded unitary representations of $G = \widehat{\mathcal{L}}_\phi(K)$ in terms of bounded representations (ρ, V) of $Z_G(d)_0 \cong \mathbb{T} \times K^\phi \times \mathbb{R}$.

If (ρ, V) is d-minimal, the corresponding representation of G is obtained by holomorphic induction with $\mathfrak{q} = \mathfrak{p}^+ \rtimes \mathfrak{h}_\mathbb{C}$ and $V = (\mathcal{H}^\infty)^{\mathfrak{p}^-}$ is the minimal eigenspace of $\mathtt{d}\pi(d)$. If (ρ, V) is d-maximal, then $(\pi, \mathcal{H})$ is obtained by holomorphic induction with $\mathfrak{q} = \mathfrak{p}^- \rtimes \mathfrak{h}_\mathbb{C}$, and $V = (\mathcal{H}^\infty)^{\mathfrak{p}^+}$ is the maximal eigenspace of $\mathtt{d}\pi(d)$.

Therefore the only ambiguity in the parametrization of corresponding irreducible unitary representations of $\widehat{\mathcal{L}}_\phi(K)$ arises for representations for which $\mathtt{d}\pi(d)$ is bounded, which only happens for one-dimensional representations, see Proposition 6.3 below. Hence the ambiguity of the parametrization consists only in twisting with characters of $\widehat{\mathcal{L}}_\phi(K)$, resp., representations vanishing on the codimension 1 subgroup $\widetilde{\mathcal{L}}_\phi(K)$ of $\widehat{\mathcal{L}}_\phi(K)$.

On the level of d-minimal/maximal weights, the corresponding assertion is that a weight $\lambda \in i\mathfrak{t}'_\mathfrak{g}$ is d-minimal and d-maximal at the same time if and only if $n\lambda((\alpha, n)^\vee) = 0$ holds for every root $(\alpha, n) \in (\Delta_\mathfrak{g})_c$, but this implies that the corresponding representation of G is one-dimensional.

Proposition 6.3. — *If $(\pi, \mathcal{H})$ is an irreducible semibounded representation of $\widehat{\mathcal{L}}_\phi(K)$ for which $\mathtt{d}\pi(d)$ is bounded, then it is one-dimensional.*

Proof. — As $i \cdot d \in W_\pi \cup -W_\pi$, the boundedness of $\mathtt{d}\pi(d)$ implies that $0 \in W_\pi$ and hence that π is bounded. From Theorem 5.15 we now obtain that $\mathtt{d}\pi(c) = 0$, so that we obtain a positive energy representation of the semidirect product $\mathcal{L}_\phi(K) \rtimes \mathbb{R}$. From Theorem 5.16 we now derive that $\widetilde{\mathcal{L}}_\phi(K) \subseteq \ker \pi$, so that the image of π is an abelian group and the assertion follows from Schur's Lemma. □

7. Perspectives and open problems

Problem 7.1. — Let $\mathfrak{k}$ be a simple Hilbert–Lie algebra and $\phi \in \mathrm{Aut}(\mathfrak{k})$ be an automorphism of finite order. Then

$$\mathcal{L}_\phi(\mathfrak{k}) \cong \mathcal{L}^1_\phi(\mathfrak{k}) = \big\{ f \in C^\infty(\mathbb{R}, \mathfrak{k}) \colon (\forall t \in \mathbb{R})\ f(t+1) = \phi^{-1}(f(t)) \big\}$$

can be identified with the space of smooth sections of the Lie algebra bundle $\mathbb{L}_\phi \to \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ obtained as the quotient of the trivial bundle $\mathbb{R} \times \mathfrak{k}$ by the equivalence relation generated by $(t+1, x) \sim (t, \phi(x))$. The Lie connections on this bundle lead to covariant derivatives of the form

$$D_A \xi = \xi' + A\xi \quad \text{with} \quad A \in \mathcal{L}^1_{c_\phi}(\mathrm{der}(\mathfrak{k})), \quad c_\phi(B) = \phi^{-1} B \phi,$$

and these operators are continuous derivations on $\mathcal{L}^1_\phi(\mathfrak{k})$ which are skew-symmetric with respect to our scalar product, so that they can all be used to define double extensions.

A map of the form $\Gamma(\xi)(t) = \gamma(t)\xi(t)$ with $\gamma \in C^\infty(\mathbb{R}, \mathrm{Aut}(\mathfrak{k}))$ defines an isomorphism $\mathcal{L}^1_\phi(\mathfrak{k}) \to \mathcal{L}^1_\psi(\mathfrak{k})$ if and only if

$$\gamma(t+1) = \psi^{-1}\gamma(t)\phi \quad \text{for} \quad t \in \mathbb{R}.$$

Let $\gamma\colon \mathbb{R} \to \mathrm{Aut}(\mathfrak{k})$ be the unique smooth curve with $\delta^l(\gamma) = A$ with $\gamma(0) = \mathbf{1}$. Then $A \in \mathcal{L}_{c_\phi}(\mathrm{der}(\mathfrak{k}))$ implies that

$$\gamma(t+1) = \gamma(1)\phi^{-1}\gamma(t)\phi,$$

so that we obtain for $\psi := \phi\gamma(1)^{-1}$ an isomorphism $\Gamma\colon \mathcal{L}^1_\phi(\mathfrak{k}) \to \mathcal{L}^1_\psi(\mathfrak{k})$ satisfying

$$D_0 \circ \Gamma = \Gamma \circ D_A.$$

This means that, by changing the automorphism, we can transform the covariant derivative D_A into the standard one. This has the advantage that the corresponding one-parameter group $(\alpha^A_t)_{t\in\mathbb{R}}$ of automorphisms satisfies

$$\alpha_t \circ \Gamma = \Gamma \circ \alpha^A_t \quad \text{for} \quad t \in \mathbb{R}.$$

We conclude in particular, that α^A is periodic if and only if the translation action on $\mathcal{L}^1_\psi(\mathfrak{k})$ is periodic, which, in view of $\alpha_1\xi = \psi^{-1}\xi$, is equivalent to the order of ψ being finite. This gives a geometric interpretation for a preference of finite order automorphisms for the contructions of double extensions.

It remains to be explored how the representation theory of $\widehat{\mathcal{L}}_\phi(\mathfrak{k})$ changes for other finite order (or even general) automorphisms of $\mathfrak{k}$. Is it possible to classify the semibounded representations of $\widehat{\mathcal{L}}_\phi(\mathfrak{k})$ for any automorphism ϕ of finite order? The present paper covers the case $\phi = \mathrm{id}$ and the three involutions which lead to the three twisted locally affine root systems.

Problem 7.2. — The proof of Theorem 5.10 shows that, for every d-minimal integral weight $\lambda \in \mathfrak{t}^*_\mathfrak{g}$ (continuous or not), we have a unitary extremal weight representation $(\pi_\lambda, L(\lambda))$ of $\mathfrak{g}_\mathbb{C}$ generated by a vector v_λ annihilated by $\mathfrak{p}^-$. Then the representation $(\rho_\lambda, V_\lambda)$ on the minimal d-eigenspace $V_\lambda := \ker(\pi_\lambda(d) - \lambda(d)\mathbf{1})$ is an extremal weight representation of the Lie algebra $Z_\mathfrak{g}(d)$ and for the orthogonal projection $p_V\colon L(\lambda) \to V_\lambda$, we obtain a positive definite linear map

$$\beta\colon U(\mathfrak{g}_\mathbb{C}) \to \mathrm{End}(V_\lambda), \quad D \mapsto p_V\pi_\lambda(D)p_V^*.$$

However, if λ is not continuous, then all these representations need not integrate to representations of $\widehat{\mathcal{L}}_\phi(K)$, resp., $Z_G(d)$.

To deal with the global aspects of these representations, we need to pass from $Z_G(d)_0$ to a suitable central extension to integrate the representation ρ_λ on the completion of V_λ. As the so obtained representation will not be bounded, we need a further refinement of the method of holomorphic induction to derive a corresponding unitary representation of $\widehat{\mathcal{L}}_\phi(K)$, or a suitable modification of this group, on the completion $\mathcal{H}_\lambda$ of $L(\lambda)$ (*cf.* [36]). A natural, but certainly not maximal, candidate for a group to which these representations may integrate is $\widehat{\mathcal{L}}_\phi(\mathrm{U}_1(\mathcal{H}))$ if $K = \mathrm{U}_2(\mathcal{H})$.

Problem 7.3. — Suppose that K is a 1-connected simple Hilbert–Lie group. Is every irreducible positive energy representation of $G = \widehat{\mathcal{L}}_\phi(K)$ holomorphically induced? Using similar arguments as for semibounded representations (*cf.* Theorem 5.4), we obtain a representation (ρ, V) of $H = Z_G(d)$ on the minimal eigenspace $V \neq \{0\}$ for $\mathtt{d}\pi(d)$, but a priori we do not know if this representation is bounded, so that holomorphic induction of (ρ, V) need not make sense.

In this context it would be interesting if there are (irreducible) unitary representations of H which are d-minimal in a suitable sense. These representations would be natural candidates for a "holomorphic induction" to a unitary representation of G to make sense. The representations from Problem 7.2 may lead to interesting examples.

Problem 7.4. — The group $\mathrm{Diff}(\mathbb{S}^1)$ acts naturally by automorphisms on the group $\widetilde{\mathcal{L}}(K)$. Does it also act on the irreducible semibounded representations $(\pi_\lambda, \mathcal{H}_\lambda)$? We expect a unitary representation of the Virasoro group which is positive/negative energy representation because we already have the action of the generator of the subgroup of rigid rotations.

In this context it is important to observe that the restrictions $\widetilde{\pi}_\lambda$ of the representations π_λ to the codimension-one subgroup $\widetilde{\mathcal{L}}_\phi(K)$ remain irreducible because they are holomorphically induced from a bounded irreducible representation (*cf.* Remark 5.11).

The philosophy is that the set $\{[\widetilde{\pi}_\lambda] \colon \lambda \in \mathcal{P}_d\}$ of equivalence classes of irreducible unitary representations of $\widetilde{\mathcal{L}}(K)$ should be "discrete" and therefore fixed pointwise under the action of $\mathrm{Diff}(\mathbb{S}^1)_0$. One way to verify this is to observe that $\mathrm{Diff}(\mathbb{S}^1)_0$ preserves the class of those representations which are "semibounded" in the sense that they extend to semibounded representations of a semidirect product with a compact circle subgroup of $\mathrm{Diff}(\mathbb{S}^1)$. Then one can try to show that such representations are determined by their momentum sets, but here one looses information by restricting to $\widetilde{\mathcal{L}}_\phi(\mathfrak{k})$ on which the representation π_λ is not semibounded.

Appendix A. H^1-curves in Hilbert–Lie groups

In this appendix we briefly introduce the Banach–Lie algebra of H^1-curves with values in a Hilbert–Lie algebra and explain how this can be used to obtain Banach–Lie algebras $\widetilde{\mathcal{L}}_\phi^H(\mathfrak{k})$ whose construction is based on H^1-curves instead of smooth ones. We also obtain corresponding Banach–Lie groups $\widetilde{\mathcal{L}}_\phi^H(K)$ in which the groups $\widetilde{\mathcal{L}}_\phi(K)$ are dense.

A.1. The group of H^1-curves

Definition A.1. — *Let $I = [0,1] \subseteq \mathbb{R}$ denote the unit interval. We write $H^1(I)$ for the space of absolutely continuous functions $f\colon I \to \mathbb{R}$ with $f' \in L^2(I)$, endowed with the scalar product*

$$\langle f, g\rangle := \int_0^1 f(t)g(t) + f'(t)g'(t)\,dt.$$

We recall from [51, Cor. 9.7] *that $H^1(I)$ is a Hilbert space, that the inclusion $H^1(I) \to C(I,\mathbb{R})$ is continuous, and that $H^1(I)$ is a Banach algebra with respect to the pointwise product.*

If $\mathcal{H}$ is a Hilbert space, then we write $H^1(I,\mathcal{H}) := H^1(I)\widehat{\otimes}\mathcal{H}$ for the tensor product of Hilbert spaces.

Remark A.2. — (a) Let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}$ and $f \in H^1(I,\mathcal{H})$. Then $f = \sum_{i\in I} f_i e_i$ with $f_i \in H^1(I)$ satisfying $\|f\|^2 = \sum_i \|f_i\|^2 < \infty$. This implies that for each $t \in I$ we have $\sum_{i\in I} |f_i(t)|^2 < \infty$, so that we obtain a well-defined function

$$f\colon I \to \mathcal{H}, \quad f(t) := \sum_{i\in I} f_i(t)e_i.$$

The sum on the right hand side is actually countable, so that we have a series expansion of f, where each finite sum is an H^1-function with values in some finite-dimensional subspace and the range of f lies in a separable subspace. From the Dominated Convergence Theorem we now derive that f is continuous. One can further show that f is absolutely continuous and that $f'\colon I \to \mathcal{H}$ exists almost everywhere in such a way that the Fundamental Theorem holds (*cf.* [65, Sect. 25] for a detailed treatment of H^1-spaces with values in a (separable) Hilbert space).

(b) If, in addition, $\mathcal{H}$ carries a continuous bilinear product $\mathcal{H}\times\mathcal{H} \to \mathcal{H}$, then the product rule implies that $H^1(I,\mathcal{H}) \subseteq C(I,\mathcal{H})$ is a subalgebra. Now [40, Lemma A.2] implies that the multiplication on $H^1(I,\mathcal{H})$ is continuous, hence turning it into a Banach algebra.

Now let K be a Hilbert–Lie group with Lie algebra $\mathfrak{k}$. Then $C(I,K)$ carries the structure of a Banach–Lie group with Lie algebra $C(I,\mathfrak{k})$ and the inclusion $H^1(I,\mathfrak{k}) \hookrightarrow C(I,\mathfrak{k})$ is a morphism of Banach–Lie algebras (*cf.* Remark A.2). We write

$$H^1(I,K) \subseteq C(I,K)$$

for the corresponding integral subgroup of $C(I,K)$, so that $H^1(I,K)$ is a Banach–Lie group with Lie algebra $H^1(I,\mathfrak{k})$ whose underlying set is the subgroup $\langle \exp H^1(I,\mathfrak{k})\rangle \subseteq C(I,K)$ (*cf.* [2, Sect. I.9], [32]).

Then $H^1(I,K)$ consists of paths $\gamma\colon I \to K$ for which the left and right logarithmic derivative exists almost everywhere and $\delta^r(\gamma), \delta^l(\gamma)\colon I \to \mathfrak{k}$ are L^2-functions (it suffices to verify this for the exponential image of $H^1(I,\mathfrak{k})$).

Proposition A.3. — *For a Hilbert–Lie group K, the following assertions hold:*

(i) $\mathrm{Ad}_{L^2}\colon H^1(I,K) \to \mathrm{O}(L^2(I,\mathfrak{k})), \mathrm{Ad}_{L^2}(f)(\xi)(t) = \mathrm{Ad}(f(t))\xi(t)$ *defines a bounded representation of the Lie group $H^1(I,K)$ on the real Hilbert space $L^2(I,\mathfrak{k})$.*

(ii) *The right logarithmic derivative $\delta^r\colon H^1(I,K) \to L^2(I,\mathfrak{k}), f \mapsto f'\cdot f^{-1}$ is a smooth cocycle whose differential is the Lie algebra cocycle $\mathbf{L}(\delta^r)f = f'$.*

Proof. — (i) Since the multiplication map $H^1(I)\times L^2(I) \to L^2(I), (f,g) \mapsto fg$ is continuous, the Lie bracket induces a continuous bilinear map

$$H^1(I,\mathfrak{k}) \times L^2(I,\mathfrak{k}) \to L^2(I,\mathfrak{k}), \quad (\xi,\eta)\mapsto [\xi,\eta], \tag{A.1}$$

defining a continuous representation of the Banach–Lie algebra $H^1(I,\mathfrak{k})$ on the Hilbert space $L^2(I,\mathfrak{k})$. This representation integrates to the morphism Ad_{L^2} of Banach–Lie groups.

(ii) First we observe that the cocycle property follows from the product rule

$$\delta^r(fg) = \delta^r(f) + \mathrm{Ad}(f)\delta^r(g).$$

Since δ^r is a cocycle with values in the smooth $H^1(I,K)$-module $L^2(I,\mathfrak{k})$, it defines a homomorphism

$$(\delta^r, \mathrm{id})\colon H^1(I,K) \to L^2(I,\mathfrak{k}) \rtimes H^1(I,K)$$

of Banach–Lie groups. Therefore its smoothness follows, once we have shown that it is continuous. As δ^r is a cocycle, it suffices to verify its continuity in an identity neighborhood, so that it suffices to show that the map

$$\delta^r \circ \exp_{H^1(I,K)}\colon H^1(I,\mathfrak{k}) \to L^2(I,\mathfrak{k}), \quad f \mapsto \delta^r(\exp_K \circ f)$$

is continuous. Writing κ_K^r for the right Maurer–Cartan form on K, we find

$$\delta^r(\exp_K \circ f) = (\exp_K \circ f)^*\kappa_K^r = f^*(\exp_K^* \kappa_K^r).$$

The 1-form $\kappa_{\mathfrak{k}} := \exp_K^* \kappa_K^r \in \Omega^1(\mathfrak{k}, \mathfrak{k}) \cong C^\infty(\mathfrak{k}, B(\mathfrak{k}))$ is explicitly given by the analytic function

$$F\colon \mathfrak{k} \to B(\mathfrak{k}), \quad F(x) := \frac{\mathbf{1} - e^{-\operatorname{ad} x}}{\operatorname{ad} x} = \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)!}(\operatorname{ad} x)^n,$$

and we have

$$\delta^r(\exp_K \circ f)(t) = F(f(t))(f'(t)).$$

The evaluation map $B(\mathfrak{k}) \times \mathfrak{k} \to \mathfrak{k}$ is continuous, it induces a continuous bilinear map $C(I, B(\mathfrak{k})) \times L^2(I, \mathfrak{k}) \to L^2(I, \mathfrak{k})$. Further, the map $H^1(I, \mathfrak{k}) \to L^2(I, \mathfrak{k}), f \mapsto f'$, and the inclusion $H^1(I, \mathfrak{k}) \to C(I, \mathfrak{k})$ are continuous. Therefore it remains to observe that the map

$$C(I, \mathfrak{k}) \to C(I, B(\mathfrak{k})), \quad f \mapsto F \circ f$$

is continuous, because for each Banach space X, the topology on the space $C(I, X)$ defined by the sup-norm coincides with the compact open topology. This completes the proof of the smoothness of δ^r.

To calculate its derivative in $\mathbf{1}$, we note that for $s \in \mathbb{R}^\times$, we have

$$\frac{1}{s}\delta^r(\exp_K \circ (sf)) = \big(F \circ (s \cdot f)\big)(f').$$

Therefore $\lim_{s\to 0} F \circ (s \cdot f) = F(0) = \mathrm{id}_{\mathfrak{k}}$ implies that $\mathbf{L}(\delta^r)f := T_{\mathbf{1}}(\delta^r)f = f'$. □

Lemma A.4. — *If a group G acts by isometries on the metric space (X, d), then each open G-orbit is also closed. In particular, the action is transitive if X is connected and G has an open orbit.*

Proof (*cf.* [17]). — Let $\mathcal{O} = Gx_0$ be an open orbit and suppose that the ball $B_\varepsilon(x_0)$ is contained in $\mathcal{O}$. We show that $\mathcal{O}$ is also closed. Let $y \in \overline{\mathcal{O}}$. Then $B_\varepsilon(y)$ intersects $\mathcal{O}$ in some point gx_0. Then $y \in B_\varepsilon(gx_0) = gB_\varepsilon(x_0) \subseteq \mathcal{O}$ shows that $\mathcal{O}$ is closed. □

The following proposition is well known for the case where K is a compact group (*cf.* [61, p. 23]).

Proposition A.5. — *The affine action of the normal subgroup*

$$H^1(I, K)_* := \{f \in H^1(I, K)\colon f(0) = \mathbf{1}\}$$

of $H^1(I, K)$ on $L^2(I, \mathfrak{k})$ by

$$\tau_f(\xi) := \mathrm{Ad}(f)\xi - \delta^r(f)$$

is simply transitive. Each orbit map yields a diffeomorphism $H^1(I,K)_* \to L^2(I,\mathfrak{k})$. In particular, $H^1(I,K)_*$ is contractible.

Proof. — That τ defines a smooth group action follows from the cocycle property and the smoothness of δ^r (Proposition A.3). Moreover, this action is isometric. Further, the derivative in $\mathbf{1}$ of the orbit map τ^0 of 0 is

$$H^1(I,\mathfrak{k})_* \to L^2(I,\mathfrak{k}), \quad f \mapsto -f',$$

which is a topological linear isomorphism of Hilbert spaces. It follows from the Inverse Function Theorem that the orbit $\mathcal{O}_0$ of 0 is open and Lemma A.4 implies that $\mathcal{O}_0 = L^2(I,\mathfrak{k})$.

Since $\delta^r(f) = 0$ implies that f is constant, the stabilizer of 0 in $H^1(I,K)_*$ is trivial and the orbit map

$$\tau^0\colon H^1(I,K)_* \to L^2(\mathfrak{k}), \quad f \mapsto -\delta^r(f)$$

is a smooth equivariant bijection. Since its differential in $\mathbf{1}$ is a topological isomorphism, the equivariance implies that this is everywhere the case, and finally the Inverse Function Theorem shows that τ^0 is a diffeomorphism. □

A.2. The H^1-version of twisted loop groups

To apply the method of holomorphic induction (*cf.* Appendix C) to the group $\widehat{\mathcal{L}}_\phi(K)$ constructed in Section 3, we need a Banach version of this group. Since we shall see that all semibounded representations of $\widehat{\mathcal{L}}_\phi(K)$ extend to various Banach completions of this group (Remark 5.11), it makes sense to use one which is rather large.

To this effect, we observe that the cocycle $\omega_D(\xi,\eta) = \langle \xi', \eta\rangle$ on $\mathcal{L}_\phi(\mathfrak{k})$ extends continuously to the Banach–Lie algebra $\mathcal{L}_\phi^H(\mathfrak{k})$ of twisted loops of class H^1, so that we obtain a central extension $\widetilde{\mathcal{L}}_\phi^H(\mathfrak{k})$ which again is a Banach–Lie algebra. Below we show that this Lie algebra integrates to a 1-connected Banach–Lie group $\widetilde{\mathcal{L}}_\phi^H(K)$ on which we have a continuous $\mathbb{R}$-action α defined by translations.

The Lie algebra $\mathcal{L}^H(\mathfrak{k})$ of H^1-loops is maximal with the property that the cocycle $D\xi := \xi'$ defined by the derivative defines a linear functional on $\mathcal{L}(\mathfrak{k})$ which is continuous with respect to the L^2-norm. This is crucial to define a corresponding cocycle by $\omega(\xi,\eta) = \langle \xi', \eta\rangle$. In particular, there are no non-trivial cocycles for the Lie algebra $C(\mathbb{S}^1,\mathfrak{k})$ of continuous loops (*cf.* [30, Cor. 13, Thm. 16]).

Definition A.6. — *For a Hilbert–Lie algebra* $\mathfrak{k}$ *and an automorphism* $\phi \in \mathrm{Aut}(\mathfrak{k})$ *of order* N*, we write* $\mathcal{L}_\phi^H(\mathfrak{k})$ *for the Hilbert space of local* H^1*-maps* $f\colon \mathbb{R} \to \mathfrak{k}$ *satisfying the condition*

$$(\forall t \in \mathbb{R})\ f\Big(t + \frac{2\pi}{N}\Big) = \phi^{-1}(f(t)),$$

endowed with the Hilbert norm defined by

$$\|\xi\|_{H^1}^2 := \|\xi\|_2^2 + \|\xi'\|_2^2 := \frac{1}{2\pi}\int_0^{2\pi} \|\xi(t)\|^2 + \|\xi'(t)\|^2\, dt.$$

This defines on $\mathcal{L}_\phi^H(\mathfrak{k})$ *the structure of a Banach–Lie algebra. It is NOT a Hilbert–Lie algebra in the sense of Definition 1.1 because the norm is not invariant under the adjoint action. Since the derivative defines a continuous map from* H^1 *to* L^2*,*

$$\omega_D(\xi,\eta) = \langle \xi', \eta\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi'(t), \eta(t)\rangle\, dt$$

defines a continuous 2*-cocycle on* $\mathcal{L}_\phi^H(\mathfrak{k})$*, and we thus obtain the centrally extended Banach–Lie algebra*

$$\widetilde{\mathcal{L}}_\phi^H(\mathfrak{k}) := \mathbb{R} \oplus_{\omega_D} \mathcal{L}_\phi^H(\mathfrak{k}), \quad [(z,\xi),(w,\eta)] := (\omega_D(\xi,\eta), [\xi,\eta]),$$

containing the Fréchet–Lie algebra $\widetilde{\mathcal{L}}_\phi(\mathfrak{k}) = \mathbb{R} \oplus_{\omega_D} \mathcal{L}_\phi(\mathfrak{k})$.

Remark A.7. — Let K be a connected Hilbert–Lie group and $\phi \in \mathrm{Aut}(K)$ be an automorphism of order N. To obtain similar information as in Remark 3.1 on the topology of the Banach–Lie group

$$\begin{aligned}\mathcal{L}_\phi^H(K) &:= \Big\{ f \in H^1_{\mathrm{loc}}(\mathbb{R},K)\colon (\forall t \in \mathbb{R})\ f\Big(t + \frac{2\pi}{N}\Big) = \phi^{-1}(f(t))\Big\}\\ &\cong \Big\{ f \in H^1([0,2\pi/N],K)\colon f(2\pi/N) = \phi^{-1}(f(0))\Big\},\end{aligned}$$

we first claim that the inclusion $\mathcal{L}(K) \to \mathcal{L}^H(K)$ of untwisted loop groups induces isomorphisms

$$\pi_k(\mathcal{L}(K)) \to \pi_k(\mathcal{L}^H(K)) \quad \text{for } k \in \mathbb{N}_0.$$

Consider the commutative diagram

$$\begin{array}{ccccc}\mathcal{L}_\phi(K)_* & \to & \mathcal{L}_\phi(K) & \xrightarrow{\mathrm{ev}_0} & K\\ \downarrow & & \downarrow & & \downarrow \mathrm{id}_K\\ \mathcal{L}_\phi^H(K)_* & \to & \mathcal{L}_\phi^H(K) & \xrightarrow{\mathrm{ev}_0} & K\end{array}$$

in which both rows describe locally trivial fiber bundles. Let $I = [0,a]$ for $a := \frac{2\pi}{N}$ and $\Omega(I,K) \subseteq H^1_*(I,K)$ denote the kernel of the evaluation map

$$\mathrm{ev}_a\colon H^1_*(I,K) \to K, \quad f \mapsto f(a)$$

in a. Then ev_a defines a locally trivial fiber bundle, so that the contractibility of $H^1_*(I,K)$ (Proposition A.5) implies the existence of natural isomorphisms

$$\pi_{k+1}(K) \to \pi_k(\Omega(I,K)), \quad k \in \mathbb{N}_0.$$

Next we observe that restriction to $[0, \frac{2\pi}{N}]$ defines an isomorphism

$$\mathcal{L}^H_\phi(K)_* \to \Omega([0,a],K).$$

Since we also have natural isomorphisms

$$\pi_{k+1}(K) \to \pi_k(\mathcal{L}_\phi(K)_*), \quad k \in \mathbb{N}_0$$

(*cf.* [48, Cor. 3.4]), we conclude that the inclusion $\mathcal{L}_\phi(K)_* \to \mathcal{L}^H_\phi(K)_*$ induces isomorphisms of all homotopy groups. Applying the Five Lemma to the long exact homotopy sequence corresponding to the rows of the above diagram, we see that the inclusion $\mathcal{L}_\phi(K) \to \mathcal{L}^H_\phi(K)$ also induces isomorphisms of all homotopy groups (*cf.* [40] for more details on this technique).

Theorem A.8. — *The assertion of Theorem 3.4 remains true for the Banach–Lie algebra $\widetilde{\mathcal{L}}^H_\phi(\mathfrak{k})$ defined by H^1-maps and the corresponding group $\widetilde{\mathcal{L}}^H_\phi(K)$.*

Proof. — From Remark A.7 it follows that the period homomorphism $\mathrm{per}_{\omega_D} \colon \pi_2(\mathcal{L}^H_\phi(K)) \to \mathbb{R}$ has the same range as the period homomorphism on $\pi_2(\mathcal{L}_\phi(K))$, and since $\mathcal{L}^H_\phi(K)$ is also 1-connected (Remark A.7), [39, Thm. 7.9] applies as in the proof of Theorem 3.4 the existence of a central $\mathbb{T}$-extension $\widetilde{\mathcal{L}}^H_\phi(K)$ of $\mathcal{L}^H_\phi(K)$ which is compatible with the inclusion $\mathcal{L}_\phi(K) \hookrightarrow \mathcal{L}^H_\phi(K)$. □

Definition A.9. — *It is easy to see that the rotation action of $\mathbb{R}$ on $\mathcal{L}^H_\phi(K)$ lifts uniquely to a continuous action on the central extension $\widetilde{\mathcal{L}}^H_\phi(K)$* ([31, Thm. V.9])*, but since the rotation action on $\mathcal{L}^H_\phi(K)$ is not differentiable, the corresponding semidirect product group*

$$\widehat{\mathcal{L}}^H_\phi(K) := \widetilde{\mathcal{L}}^H_\phi(K) \rtimes \mathbb{R}$$

is a topological group but not a Lie group. This is the main difference to the smooth setting, where $\widehat{\mathcal{L}}_\phi(K) = \widetilde{\mathcal{L}}_\phi(K) \rtimes \mathbb{R}$ is a Fréchet–Lie group.

Remark A.10. — As in Remark 3.6, we derive from the polar decomposition $K_\mathbb{C} = K\exp(i\mathfrak{k})$ of the universal complexification of K the existence of a central extension of complex Lie groups

$$\mathbf{1} \to \mathbb{C}^\times \to \widetilde{\mathcal{L}}^H_\phi(K_\mathbb{C}) \to \mathcal{L}^H_\phi(K_\mathbb{C}) \to \mathbf{1}$$

for which the inclusion $\widetilde{\mathcal{L}}^H_\phi(K) \hookrightarrow \widetilde{\mathcal{L}}^H_\phi(K_\mathbb{C})$ is a universal complexification and a weak homotopy equivalence.

Appendix B. Analytic operator-valued positive definite functions

In this appendix we discuss operator-valued positive definite functions on Lie groups. The main result is Theorem B.6, asserting that, for a Hilbert space V, analytic $B(V)$-valued defined in a $\mathbf{1}$-neighborhood of a Fréchet–BCH Lie group G are positive definite if the corresponding linear map $\beta\colon U(\mathfrak{g}_{\mathbb{C}}) \to B(V)$, defined by derivatives in $\mathbf{1}$, is positive definite.

Definition B.1. — *Let X be a set and $\mathcal{K}$ be a Hilbert space.*

(a) *A function $Q\colon X \times X \to B(\mathcal{K})$ is called a $B(\mathcal{K})$-valued kernel. It is said to be hermitian if $Q(z,w)^* = Q(w,z)$ holds for all $z, w \in X$.*

(b) *A hermitian $B(\mathcal{K})$-valued kernel K on X is said to be positive definite if for every finite sequence $(x_1, v_1), \ldots, (x_n, v_n)$ in $X \times \mathcal{K}$ we have*

$$\sum_{j,k=1}^{n} \langle Q(x_j, x_k) v_k, v_j\rangle \geqslant 0.$$

(c) *If $(S, *)$ is an involutive semigroup, then a function $\phi\colon S \to B(\mathcal{K})$ is called positive definite if the kernel $Q_\phi(s,t) := \phi(st^*)$ is positive definite.*

(d) *Positive definite kernels can be characterized as those for which there exists a Hilbert space $\mathcal{H}$ and a function $\gamma\colon X \to B(\mathcal{H}, \mathcal{K})$ such that*

$$Q(x,y) = \gamma(x)\gamma(y)^* \quad \text{for } x, y \in X$$

(cf. [38, Thm. I.1.4]*). Here one may assume that the vectors $\gamma(x)^* v$, $x \in X, v \in \mathcal{K}$, span a dense subspace of $\mathcal{H}$. Then the pair $(\gamma, \mathcal{H})$ is called a realization of K. The map $\Phi\colon \mathcal{H} \to \mathcal{K}^X, \Phi(v)(x) := \gamma(x)v$, then realizes $\mathcal{H}$ as a Hilbert subspace of $\mathcal{K}^X$ with continuous point evaluations $\mathrm{ev}_x\colon \mathcal{H} \to \mathcal{K}$. It is the unique Hilbert subspace in $\mathcal{K}^X$ with this property for which $Q(x,y) = \mathrm{ev}_x \mathrm{ev}_y^*$ for $x, y \in X$. We write $\mathcal{H}_Q \subseteq \mathcal{K}^X$ for this subspace and call it the reproducing kernel Hilbert space with kernel Q.*

Definition B.2. — *Let $\mathcal{K}$ be a Hilbert space, G be a group, and $U \subseteq G$ be a subset. A function $\phi\colon UU^{-1} \to B(\mathcal{K})$ is said to be positive definite if the kernel*

$$Q_\phi\colon U \times U \to B(\mathcal{K}), \quad (x,y) \mapsto \phi(xy^{-1})$$

is positive definite.

Definition B.3. — *A Lie group* G *with Lie algebra* $\mathfrak{g}$ *is said to be locally exponential if it has an exponential function for which there is an open* 0*-neighborhood* U *in* $\mathfrak{g}$ *mapped diffeomorphically by* $\exp_G$ *onto an open subset of* G*. If, in addition,* G *is analytic and the exponential function is an analytic local diffeomorphism in* 0*, then* G *is called a BCH–Lie group (for Baker–Campbell–Hausdorff). Then the Lie algebra* $\mathfrak{g}$ *is a BCH–Lie algebra, i.e., there exists an open* 0*-neighborhood* $U \subseteq \mathfrak{g}$ *such that for* $x, y \in U$ *the Hausdorff series*

$$x * y = x + y + \frac{1}{2}[x, y] + \cdots$$

converges and defines an analytic function $U \times U \to \mathfrak{g}, (x, y) \mapsto x * y$*. The class of BCH–Lie groups contains in particular all Banach–Lie groups* [41, Prop. IV.1.2].

Theorem B.4 (Extension of local positive definite analytic functions (*cf.* [45], Thm. A.7). — *Let* G *be a* 1*-connected Fréchet–BCH–Lie group,* $V \subseteq G$ *an open connected* $\mathbf{1}$*-neighborhood,* $\mathcal{K}$ *be a Hilbert space and* $\phi \colon VV^{-1} \to B(\mathcal{K})$ *be an analytic positive definite function. Then there exists a unique analytic positive definite function* $\widetilde{\phi} \colon G \to B(\mathcal{K})$ *extending* ϕ*.*

Definition B.5. — *Let* U *be an open subset of the Lie group* G *and* E *be a locally convex space. Then we obtain for each* $x \in \mathfrak{g}$ *a differential operator on* $C^\infty(U, E)$ *by*

$$(L_x f)(g) := \frac{d}{dt}\Big|_{t=0} f(g \exp tx).$$

These operators define a representation of the Lie algebra $\mathfrak{g}$ *on* $C^\infty(U, E)$*, so that we obtain a natural extension to a homomorphism*

$$U(\mathfrak{g}) \to \mathrm{End}(C^\infty(U, E)), \quad D \mapsto L_D.$$

We likewise define

$$(R_x f)(g) := \frac{d}{dt}\Big|_{t=0} f(\exp(tx)g)$$

and note that $[R_x, R_y] = R_{[y,x]}$ *for* $x, y \in \mathfrak{g}$*.*

Theorem B.6 (Infinitesimal characterization of positive definite analytic functions). — *Let* G *be a Fréchet–BCH–Lie group,* $V \subseteq G$ *an open connected* $\mathbf{1}$*-neighborhood,* $\mathcal{K}$ *be a Hilbert space and* $\phi \colon V \to B(\mathcal{K})$ *be an analytic function satisfying* $\phi(\mathbf{1}) = \mathbf{1}$*. Then* ϕ *is positive definite on a* $\mathbf{1}$*-neighborhood in* G *if and only if the corresponding linear map*

$$\beta \colon U(\mathfrak{g}_{\mathbb{C}}) \to B(\mathcal{K}), \quad \beta(D) := (L_D\phi)(\mathbf{1})$$

is a positive definite linear function on the $*$-algebra $U(\mathfrak{g}_{\mathbb{C}})$.

Proof.
"$\Rightarrow$": Suppose first that ϕ is positive definite in a $\mathbf{1}$-neighborhood. Then Theorem B.4 provides an extension of the germ of ϕ in $\mathbf{1}$ to an analytic positive definite function on all of G. We may therefore assume that ϕ is defined on G. Then the vector-valued GNS construction provides a unitary representation $(\pi, \mathcal{H})$ of G on a Hilbert space $\mathcal{H}$, containing $\mathcal{K}$ as a closed subspace such that the orthogonal projection $p_{\mathcal{K}} \colon \mathcal{H} \to \mathcal{K}$ satisfies

$$\phi(g) = p_{\mathcal{K}} \pi(g) p_{\mathcal{K}}^* \quad \text{for } \ g \in G.$$

This implies that $\mathcal{K}$ consists of analytic vectors, and for $D \in U(\mathfrak{g}_{\mathbb{C}})$ we find the formula

$$\beta(D) = p_{\mathcal{K}} \mathtt{d}\pi(D) p_{\mathcal{K}}^*.$$

Any function of this form is easily seen to be positive definite.

"$\Leftarrow$": Let $U_{\mathfrak{g}} \subseteq \mathfrak{g}$ be an open symmetric 0-neighborhood which is mapped by exp bianalytically to an open $\mathbf{1}$-neighborhood of G and such that ϕ is defined on $\exp(U_{\mathfrak{g}})$. Then $\phi \circ \exp \colon U_{\mathfrak{g}} \to B(V)$ is also analytic, and, after shrinking $U_{\mathfrak{g}}$, we may assume that

$$\phi(\exp x) = \sum_{n=0}^{\infty} \phi_n(x),$$

where $\phi_n \colon \mathfrak{g} \to B(V)$ is a continuous homogeneous polynomial function of degree n ([8]). Now the relation

$$\phi(\exp tx) = \sum_n \frac{t^n}{n!} (L_x^n \phi)(\mathbf{1}) = \sum_n \frac{t^n}{n!} \beta(x^n) \quad \text{for } \ |t| < \varepsilon$$

implies that $\phi_n(x) = \frac{\beta(x^n)}{n!}$. In particular,

$$\phi(\exp x) = \sum_n \frac{1}{n!} \beta(x^n) \quad \text{for } \ x \in U_{\mathfrak{g}},$$

which implies that β is *analytic* in the sense of [44, Def. 3.2].

Let $\beta_n(x_1, \ldots, x_n) := \beta(x_1 \cdots x_n)$ and

$$\beta_n^s(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \beta(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

be its symmetrization. For a continuous seminorm p on $\mathfrak{g}$, we then define

$$\|\beta_n^s\|_p := \sup\{\|\beta_n^s(x_1, \ldots, x_n)\| \colon x_1, \ldots, x_n \in \mathfrak{g}, p(x_i) \leqslant 1\} \in [0, \infty].$$

From [44, Prop. 3.4] we now obtain the existence of a continuous seminorm p on $\mathfrak{g}$ with $\sum_n \frac{1}{n!}\|\beta_n^s\|_p < \infty$. In particular, there exists a constant $C > 0$ with

$$\|\beta_n^s\|_p \leqslant Cn! \quad \text{for all } n \in \mathbb{N}_0. \tag{B.1}$$

Let $\mathcal{H} \subseteq \mathrm{Hom}(U(\mathfrak{g}_\mathbb{C}), \mathcal{K})$ be the reproducing kernel Hilbert space corresponding to the positive definite function $\beta\colon U(\mathfrak{g}_\mathbb{C}) \to B(\mathcal{K})$. The corresponding positive definite kernel Q and the corresponding evaluation maps $Q_D\colon \mathcal{H} \to \mathcal{K}$ then satisfy

$$Q(D_1, D_2) = \beta(D_1 D_2^*) = Q_{D_1} Q_{D_2}^* \quad \text{and} \quad Q_D f = f(D) \quad \text{for } f \in \mathcal{H}.$$

We have a $*$-representation of $U(\mathfrak{g}_\mathbb{C})$ on the dense subspace

$$\mathcal{H}^0 := \mathrm{span}\{Q_D^* v\colon v \in \mathcal{K}, D \in U(\mathfrak{g}_\mathbb{C})\}$$

by

$$(\rho(D)f)(D') = f(D'D) \quad \text{for } D, D' \in U(\mathfrak{g}_\mathbb{C}).$$

From $\beta(\mathbf{1}) = Q_\mathbf{1} Q_\mathbf{1}^* = \mathbf{1}$ we derive that we may identify $\mathcal{K}$ with its image under the isometric embedding $Q_\mathbf{1}^*\colon \mathcal{K} \to \mathcal{H}$. For $v \in \mathcal{K}$ we then have

$$(\rho(D)v)(D') = v(D'D) = Q_{D'D} Q_\mathbf{1}^* v = \beta(D'D)v = Q_{D'} Q_{D^*}^* v = (Q_{D^*}^* v)(D'),$$

so that

$$\rho(D)v = Q_{D^*}^* v \quad \text{for } D \in U(\mathfrak{g}_\mathbb{C}).$$

In view of $\|Q_D\|^2 = \|Q_D Q_D^*\| = \|\beta(DD^*)\|$, we find for the operators $Q_{x^n} \in B(\mathcal{H}, \mathcal{K})$, $x \in \mathfrak{g}$, the estimates

$$\frac{1}{n!}\|Q_{x^n}\| = \frac{1}{n!}\|\beta(x^{2n})\|^{1/2} \leqslant \frac{p(x)^n}{n!}\|\beta_{2n}^s\|_p^{1/2} \leqslant \frac{p(x)^n}{n!}\sqrt{C}\sqrt{(2n)!}.$$

In view of $\lim_{n\to\infty} \frac{\sqrt{(2n+2)(2n+1)}}{n+1} = 2$, it follows that

$$\sum_n \frac{1}{n!}\|Q_{x^n}\| < \infty \quad \text{for} \quad p(x) < \tfrac{1}{2}.$$

We thus obtain an analytic function

$$\eta\colon \{x \in \mathfrak{g}\colon p(x) < \tfrac{1}{2}\} \to B(\mathcal{H}, \mathcal{K}), \quad \eta(x) := \sum_n \frac{1}{n!} Q_{x^n}.$$

Now let $W \subseteq \{x \in \mathfrak{g}\colon p(x) < \frac{1}{2}\}$ be an open symmetric 0-neighborhood such that all BCH products $x * y$ for $x, y \in W$ are defined and that we thus obtain an analytic function on $W \times W$ with values in the set $\{z \in \mathfrak{g} \colon p(z) < \frac{1}{2}\}$. For $x, y \in W$ we finally derive

$$\begin{aligned}\phi(\exp x \exp(-y)) = \phi(\exp(x * (-y))) &= \sum_n \frac{1}{n!}\beta((x * (-y))^n) \\ = \sum_{k,\ell} \frac{1}{k!\ell!}\beta(x^k(-y)^\ell) &= \sum_{k,\ell} \frac{1}{k!\ell!} Q_{x^k} Q_{y^\ell}^* = \eta(x)\eta(y)^*.\end{aligned}$$

This factorization implies that ϕ is positive definite on $\exp W \exp W$. This completes the proof. □

Remark B.7. — If $\mathcal{K}$ is one-dimensional, then a linear map $\beta \colon U(\mathfrak{g}_{\mathbb{C}}) \to B(\mathcal{K}) \cong \mathbb{C}$ is positive definite if and only if it is a positive functional in the sense that $\beta(DD^*) \geqslant 0$ for every $D \in U(\mathfrak{g}_{\mathbb{C}})$. For general $\mathcal{K}$, it is shown in [53, Ex. 11.2.1, Thm. 11.2.2] that β is positive definite if and only if it is *completely positive* in the sense that every induced map

$$M_n(\beta) \colon M_n(U(\mathfrak{g}_{\mathbb{C}})) \to M_n(B(\mathcal{K})) \cong B(\mathcal{K}^n)$$

obtained by applying β to all matrix entries maps positive elements to positive elements.

Appendix C. Holomorphic induction for BCH–Lie groups

Let G be a Lie group and $M = G/H$ be a homogeneous space of G which carries the structure of a complex manifold so that G acts analytically by holomorphic maps. In [46] we have developed a theory of holomorphic induction for bounded unitary representations of H in the context where G is a Banach–Lie group. To deal with semibounded representations of Fréchet–Lie groups such as the double extension $\widehat{\mathcal{L}}_\phi(K)$ of the Fréchet–Lie group $\mathcal{L}_\phi(K)$ of smooth ϕ-twisted loops, we need an extension of this theory to certain classes of Fréchet–Lie groups. In this appendix we explain which properties of Banach–Lie groups were used in [46, Sects. 2,3] and why $\widehat{\mathcal{L}}_\phi(K)$ also has these properties.

Let G be a connected Fréchet–BCH–Lie group with Lie algebra $\mathfrak{g}$. We further assume that there exists a complex BCH–Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and a natural map $\eta \colon G \to G_{\mathbb{C}}$ for which $\mathbf{L}(\eta)$ is the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$. Let $H \subseteq G$ be a Lie subgroup for which $M := G/H$ carries the structure of a smooth manifold with a smooth G-action and $\mathfrak{h} \subseteq \mathfrak{g}$ be its Lie algebra. We also assume the existence of closed $\mathrm{Ad}(H)$-invariant subalgebras $\mathfrak{p}^\pm \subseteq \mathfrak{g}_\mathbb{C}$ with $\overline{\mathfrak{p}^\pm} = \mathfrak{p}^\mp$ for which we have a topological direct sum decomposition

$$\text{(SC)} \qquad \mathfrak{g}_\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{h}_\mathbb{C} \oplus \mathfrak{p}^-.$$

We put

$$\mathfrak{q} := \mathfrak{p}^+ \rtimes \mathfrak{h}_\mathbb{C} \quad \text{and} \quad \mathfrak{p} := \mathfrak{g} \cap (\mathfrak{p}^+ \oplus \mathfrak{p}^-),$$

so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ is a topological direct sum. We assume that there exist open symmetric convex 0-neighborhoods $U_{\mathfrak{g}_\mathbb{C}} \subseteq \mathfrak{g}_\mathbb{C}$, $U_\mathfrak{p} \subseteq \mathfrak{p} \cap U_{\mathfrak{g}_\mathbb{C}}, U_\mathfrak{h} \subseteq \mathfrak{h} \cap U_{\mathfrak{g}_\mathbb{C}}, U_{\mathfrak{p}^\pm} \subseteq \mathfrak{p}^\pm \cap U_{\mathfrak{g}_\mathbb{C}}$ and $U_\mathfrak{q} \subseteq \mathfrak{q} \cap U_{\mathfrak{g}_\mathbb{C}}$ such that the BCH-product is defined and holomorphic on $U_{\mathfrak{g}_\mathbb{C}} \times U_{\mathfrak{g}_\mathbb{C}}$, and the following maps are analytic diffeomorphisms onto an open subset:

(A1) $U_\mathfrak{p} \times U_\mathfrak{h} \to \mathfrak{g}, (x,y) \mapsto x * y$.
(A2) $U_\mathfrak{p} \times U_\mathfrak{q} \to \mathfrak{g}_\mathbb{C}, (x,y) \mapsto x * y$.
(A3) $U_{\mathfrak{p}^-} \times U_\mathfrak{q} \to \mathfrak{g}_\mathbb{C}, (x,y) \mapsto x * y$.

Then (A1) implies the existence of a smooth manifold structure on $M = G/H$ for which G acts analytically. Condition (A2) implies the existence of a complex manifold structure on M which is G-invariant and for which $T_{\mathbf{1}H}(M) \cong \mathfrak{g}_\mathbb{C}/\mathfrak{q}$. Finally, (A3) makes the proof of [46, Thm. 2.6] work, so that we can associate to every bounded unitary representation (ρ, V) of H a holomorphic Hilbert bundle $\mathbb{V} := G \times_H V$ over the complex G-manifold M by defining $\beta\colon \mathfrak{q} \to \mathfrak{gl}(V)$ by $\beta(\mathfrak{p}^+) = \{0\}$ and $\beta|_\mathfrak{h} = \mathtt{d}\rho$. Now it is easy to check that all results in Sections 2 and 3 of [46] remain valid.

Definition C.1. — *We write $\Gamma(\mathbb{V})$ for the space of holomorphic sections of the holomorphic Hilbert bundle $\mathbb{V} \to M = G/H$ on which the group G acts by holomorphic bundle automorphisms. A unitary representation $(\pi, \mathcal{H})$ of G is said to be holomorphically induced from (ρ, V) if there exists a G-equivariant linear injection $\Psi\colon \mathcal{H} \to \Gamma(\mathbb{V})$ such that the adjoint of the evaluation map $\mathrm{ev}_{\mathbf{1}H}\colon \mathcal{H} \to V = \mathbb{V}_{\mathbf{1}H}$ defines an isometric embedding $\mathrm{ev}_{\mathbf{1}H}^*\colon V \hookrightarrow \mathcal{H}$. If a unitary representation $(\pi, \mathcal{H})$ holomorphically induced from (ρ, V) exists, then it is uniquely determined (*[46, Def. 3.10]*) and we call (ρ, V) (holomorphically) inducible.*

This concept of inducibility involves a choice of sign. Replacing $\mathfrak{p}^+$ by $\mathfrak{p}^-$ changes the complex structure on G/H and thus leads to a different class of holomorphically inducible representations of of H.

Theorem C.2. — *If the unitary representation $(\pi, \mathcal{H})$ of G is holomorphically induced from the bounded H-representation (ρ, V), then the following assertions hold:*

(i) *$V \subseteq \mathcal{H}^\omega$ consists of analytic vectors.*

(ii) $R\colon \pi(G)' \to \rho(H)', A \mapsto A|_V$ *is an isomorphism of von Neumann algebras.*

Proof. — (i) follows from [46, Lemma 3.5] and (ii) from [46, Thm. 3.12]. □

Theorem C.3 ([46], Thm. 3.17). — *Suppose that* $(\pi, \mathcal{H})$ *is a unitary representation of* G *and* $V \subseteq \mathcal{H}$ *is an* H*-invariant closed subspace such that*

(HI1) *The representation* (ρ, V) *of* H *on* V *is bounded.*

(HI2) $V \cap (\mathcal{H}^\infty)^{\mathfrak{p}^-}$ *is dense in* V*.*

(HI3) $\pi(G)V$ *spans a dense subspace of* $\mathcal{H}$*.*

Then $(\pi, \mathcal{H})$ *is holomorphically induced from* (ρ, V)*.*

Examples C.4. — (a) Let G be a simply connected Banach–Lie group for which $\mathfrak{g}_\mathbb{C}$ also is the Lie algebra of a Banach–Lie group and $M = G/H$ is a Banach homogeneous space. If the subalgebras $\mathfrak{p}^\pm \subseteq \mathfrak{g}_\mathbb{C}$ satisfy the splitting condition (SC), then (A1-3) follows directly from the Inverse Function Theorem. This is the context of [46].

(b) Let G_B be a Banach–Lie group with Lie algebra $\mathfrak{g}_B$, $H_B \subseteq G_B$ and $M_B = G_B/H_B$ *etc.* as in (a). We assume that the splitting condition (SC) is satisfied. In addition, let $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G_B)$ be a one-parameter group of automorphisms defining a continuous $\mathbb{R}$-action on G_B and assume that the subalgebras $\mathfrak{p}_B^\pm$, $\mathfrak{q}_B$ and $\mathfrak{h}$ are α-invariant. Then the subgroup

$$G := \{g \in G_B \colon \mathbb{R} \to G_B, t \mapsto \alpha_t(g) \text{ is smooth}\}$$

of G_B carries the structure of a Fréchet–BCH–Lie group with Lie algebra

$$\mathfrak{g} := \{x \in \mathfrak{g}_B \colon \mathbb{R} \to \mathfrak{g}_B, t \mapsto \mathbf{L}(\alpha_t)x \text{ is smooth}\},$$

the Fréchet space of smooth vectors for the continuous $\mathbb{R}$-action on the Banach–Lie algebra $\mathfrak{g}_B$. Likewise $H := G \cap H_B$ is a Lie subgroup of G for which $M := G/H$ is a smooth manifold consisting of the elements of $M_B = G_B/H_B$ with smooth orbit maps with respect to the one-parameter group of diffeomorphisms induced by α via $\alpha_t(gH_B) = \alpha_t(g)H_B$.

Since the automorphisms $\mathbf{L}(\alpha_t)$ of $\mathfrak{g}$ resp., $\mathfrak{g}_\mathbb{C}$ are compatible with the BCH multiplication, it is easy to see with Lemma C.5 below that conditions (A1-3) are inherited by the closed subalgebras

$$\mathfrak{h} = \mathfrak{h}_B \cap \mathfrak{g}, \quad \mathfrak{p}^\pm = \mathfrak{p}_B^\pm \cap \mathfrak{g}_\mathbb{C} \quad \text{and} \quad \mathfrak{q} = \mathfrak{q}_B \cap \mathfrak{g}_\mathbb{C}.$$

Lemma C.5. — *Let* V_1 *and* V_2 *be Banach spaces and* (α_t^1)*, resp.,* (α_t^2) *define continuous* $\mathbb{R}$*-actions on* V_1*, resp.,* V_2*. If* $U \subseteq V_1$ *is an open invariant subset and* $F\colon U \to V_2$ *an equivariant smooth map, then the induced map*

$$F^\infty \colon U^\infty := U \cap V_1^\infty \to V_2^\infty, \quad v \mapsto F(v)$$

is a smooth map on the open subset U^∞ of the Fréchet space V_1^∞.

Proof. — Let $D_j := \alpha_j'(0)$ denote the infinitesimal generator of α_j. Then we have to verify that all maps $F_k\colon U^\infty \to V_2, x \mapsto D_2^k F(x)$ are smooth. Since α^1 defines a smooth $\mathbb{R}$-action on U^∞, the map

$$\Phi\colon \mathbb{R}\times U^\infty \to V_2, \quad (t,x)\mapsto F(\alpha_t^1(x)) = \alpha_t^2 F(x)$$

is smooth. Hence the map $F_k(x) = \frac{\partial^k}{\partial t^k}|_{t=0}\Phi(t,x)$ is also smooth. □

From (A1-3) we derive the existence of open convex symmetric 0-neighborhoods $U_\pm \subseteq \mathfrak{p}^\pm$ and $U_0 \subseteq \mathfrak{h}_\mathbb{C}$ for which the BCH-multiplication map

$$U_+\times U_0\times U_- \to \mathfrak{g}_\mathbb{C}, \quad (x_+,x_0,x_-)\mapsto x_+ * x_0 * x_-$$

is biholomorphic onto an open 0-neighborhood U of $\mathfrak{g}_\mathbb{C}$. For a bounded representation (ρ, V) of H_0 we then define a holomorphic map

$$F_\rho\colon U\to B(V), \quad F_\rho(x_+ * x_0 * x_-) := e^{\mathtt{d}\rho(x_0)}.$$

For the Banach case the equivalence of (i) and (ii) in the following theorem can also be found in [45, Thm. B.1]. Its proof also works without change in our context. We include it for the sake of completeness.

Theorem C.6. — *For a bounded representation (ρ, V) of H, the following are equivalent:*

(i) *(ρ, V) is holomorphically inducible.*

(ii) *$f_\rho(\exp x) := F_\rho(x)$ defines a positive definite analytic function on a $\mathbf{1}$-neighborhood of G.*

(iii) *The corresponding linear map $\beta\colon U(\mathfrak{g}_\mathbb{C})\to B(V), \beta(D) = (L_D f_\rho)(\mathbf{1})$ is positive definite. It is characterized by the property that $\mathfrak{p}^+U(\mathfrak{g}_\mathbb{C}) + U(\mathfrak{g}_\mathbb{C})\mathfrak{p}^- \subseteq \ker\beta$ and $\beta|_{U(\mathfrak{h}_\mathbb{C})} = \mathtt{d}\rho$.*

Proof.

(i) ⇒ (ii): Let $(\pi, \mathcal{H})$ be the unitary representation of G obtained by holomorphic induction from (ρ, V). We identify V with the corresponding closed subspace of $\mathcal{H}$ and write $p_V : \mathcal{H}\to V$ for the corresponding orthogonal projection. For $v\in V\subseteq (\mathcal{H}^\omega)^{\mathfrak{p}^-}$ (Theorem C.2), we let $f_\rho^v\colon U_v \to G$ be a holomorphic map on an open convex 0-neighborhood $U_v\subseteq U$ satisfying $f_\rho^v(x) = \pi(\exp x)v$ for $x\in U_v\cap\mathfrak{g}$. Then $\mathtt{d}\pi(\mathfrak{p}^-)v = \{0\}$ implies that $L_z f_\rho^v = 0$ for $z\in\mathfrak{p}^-$. For $w\in V$ and $z\in\mathfrak{p}^+$, we also obtain

$$\langle (R_z f_\rho^v)(x), w\rangle = \langle \mathtt{d}\pi(z) f_\rho^v(x), w\rangle = \langle f_\rho^v(x), \mathtt{d}\pi(z^*)w\rangle = 0.$$

This proves that $R_z(p_V\circ f_v) = 0$. We conclude that, for $x_\pm$ and x_0 sufficiently close to 0, we have

$$p_V f_\rho^v(x_+ * x_0 * x_-) = f_\rho^v(x_0) = e^{\mathtt{d}\rho(x_0)}v = F_\rho(x_+ * x_0 * x_-)v.$$

Therefore $p_V \circ f^v_\rho$ extends holomorphically to U and

$$\langle \pi(\exp x)v, w\rangle = \langle F_\rho(x)v, w\rangle \quad \text{for } x \in U_v \cap \mathfrak{g}, v, w \in V.$$

We conclude that $F_\rho(x) = p_V \pi(\exp x) p_V$ holds for x sufficiently close to 0, and hence that $f_\rho(\exp x) = p_V \pi(\exp x) p_V$ defines a positive definite function on a **1**-neighborhood of G.

(ii) $\Rightarrow$ (i): From Theorem B.4 it follows that some restriction of f_ρ to a possibly smaller **1**-neighborhood in G extends to a global analytic positive definite function ϕ. Then the vector-valued GNS construction yields a unitary representation of G on the corresponding reproducing kernel Hilbert space $\mathcal{H}_\phi \subseteq V^G$ for which all the elements of $\mathcal{H}^0_\phi = \operatorname{span}(\phi(G)V)$ are analytic vectors. In particular, $V \subseteq \mathcal{H}^\omega_\phi$ consists of smooth vectors, and the definition of f_ρ implies that $\mathtt{d}\pi(\mathfrak{p}^-)V = \{0\}$. Therefore Theorem C.3 implies that the representation $(\pi, \mathcal{H}_\phi)$ is holomorphically induced from (ρ, V).

(ii) $\Leftrightarrow$ (iii) follows from Theorem B.6. The relation $U(\mathfrak{g}_\mathbb{C})\mathfrak{p}^- \subseteq \ker \beta$ follows from the definition of f_ρ which does not depend on the x_--component. In view of $f_\rho(g^{-1}) = f_\rho(g)^*$, we have $\beta(D^*) = \beta(D)^*$ for $D \in U(\mathfrak{g}_\mathbb{C})$, and we thus also obtain $\mathfrak{p}^+ U(\mathfrak{g}_\mathbb{C}) \subseteq \ker \beta$, so that β is determined by its restriction to $U(\mathfrak{h}_\mathbb{C})$, where it coincides with $\mathtt{d}\rho$. □

Appendix D. Finite order automorphisms of Hilbert–Lie algebras

In this appendix we generalize some of the results on finite order automorphisms of complex, resp., compact semisimple Lie algebras ([19, Sec. X.5]) to Hilbert–Lie algebras.

Let $\mathfrak{k}$ be a Hilbert–Lie algebra and $\phi \in \operatorname{Aut}(\mathfrak{k})$ be an automorphism of order N.

Lemma D.1. — *If $\mathfrak{k}$ is semisimple and non-zero, then $\mathfrak{k}^\phi \neq \{0\}$.*

Proof. — We also write ϕ for the complex linear extension of ϕ to $\mathfrak{k}_\mathbb{C}$ and write

$$\mathfrak{k}^n_\mathbb{C} := \{x \in \mathfrak{k}_\mathbb{C} \colon \phi^{-1}(x) = e^{2\pi i n/N} x\}.$$

Assume that $\mathfrak{k}^\phi = \{0\}$. This means that $\mathfrak{k}^0_\mathbb{C} = (\mathfrak{k}^\phi)_\mathbb{C} = \{0\}$. We show by induction that $\mathfrak{k}^k_\mathbb{C} = \{0\}$ for $k = 1, \ldots, N-1$. Assume $1 \leqslant k < N$ and that $\mathfrak{k}^j_\mathbb{C} = \{0\}$ holds for $j = 0, 1, \ldots, k-1$. Pick $x \in \mathfrak{k}^k_\mathbb{C}$. For each $j \in \mathbb{Z}$, there exists an $r \in \mathbb{N}$ such that $j + kr$ is congruent to one of the numbers $0, \ldots, k-1$ modulo N. We then obtain

$$(\operatorname{ad} x)^r \mathfrak{k}_{\mathbb{C}}^j \subseteq \mathfrak{k}_{\mathbb{C}}^{j+rk} = \{0\},$$

and conclude that $\operatorname{ad} x$ is nilpotent. Then $\operatorname{ad} x^*$ is also nilpotent. Moreover, $[x, x^*] \in [\mathfrak{k}_{\mathbb{C}}^k, \mathfrak{k}_{\mathbb{C}}^{-k}] \subseteq \mathfrak{k}_{\mathbb{C}}^0 = \{0\}$ implies that $\operatorname{ad} x$ and $\operatorname{ad} x^* = (\operatorname{ad} x)^*$ commute. Thus $\operatorname{ad} x$ is a normal operator on the complex Hilbert space $\mathfrak{k}_{\mathbb{C}}$, and since it is nilpotent, we obtain $\operatorname{ad} x = 0$. Now $x \in \mathfrak{z}(\mathfrak{k}_{\mathbb{C}}) = \{0\}$ completes our inductive proof of $\mathfrak{k}_{\mathbb{C}}^k = \{0\}$ for $k = 0, \ldots, N-1$. This contradicts the assumption that $\mathfrak{k}$ is non-zero. □

Lemma D.2. — *If $\mathfrak{t} \subseteq \mathfrak{k}^\phi$ is maximal abelian, then $\mathfrak{t}_{\mathfrak{k}} := \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t})$ is maximal abelian in $\mathfrak{k}$.*

Proof. — Clearly, $\mathfrak{t}_{\mathfrak{k}}$ is a closed subalgebra of $\mathfrak{k}$ invariant under ϕ, hence a Hilbert–Lie algebra, endowed with a finite order automorphism $\phi|_{\mathfrak{t}_{\mathfrak{k}}}$. Let $\mathfrak{s} = \mathfrak{z}(\mathfrak{t}_{\mathfrak{k}})^\perp \cap \mathfrak{t}_{\mathfrak{k}}$ denote the commutator algebra of $\mathfrak{t}_{\mathfrak{k}}$. Then $\mathfrak{s}$ is also ϕ-invariant and semisimple. If $\mathfrak{s}$ is non-zero, then Lemma D.1 implies that $\mathfrak{s}^\phi$ is non-zero, but this leads to the contradiction $\mathfrak{s}^\phi \subseteq \mathfrak{z}_{\mathfrak{k}^\phi}(\mathfrak{t}) = \mathfrak{t}$. □

Lemmas D.1 and D.2 imply in particular, that there exists a maximal abelian subalgebra of $\mathfrak{k}$ which is ϕ-invariant. According to [55], $\mathfrak{k}_{\mathbb{C}}$ decomposes into an orthogonal sum of $\mathfrak{t}_{\mathfrak{k}}$-root spaces, and this implies that $\mathfrak{k}_{\mathbb{C}}$ decomposes into $\mathfrak{t}$-weight spaces $\mathfrak{k}_{\mathbb{C}}^\alpha$, $\alpha \in \mathfrak{t}'$.

Let $\Delta := \Delta(\mathfrak{k}, \mathfrak{t}) := \{\alpha \in \mathfrak{t}' \colon \mathfrak{k}_{\mathbb{C}}^\alpha \neq \{0\}\}$ denote the $\mathfrak{t}$-weight set of $\mathfrak{k}$. As $\operatorname{ad} \mathfrak{t}$ and ϕ commute, the weight spaces $\mathfrak{k}_{\mathbb{C}}^\alpha$ are ϕ-invariant, so that we obtain a simultaneous diagonalization of $\mathfrak{t}$ and ϕ by the spaces

$$\mathfrak{k}_{\mathbb{C}}^{(\alpha,n)} := \mathfrak{k}_{\mathbb{C}}^\alpha \cap \mathfrak{k}_{\mathbb{C}}^n, \quad n \in \mathbb{Z}, \alpha \in \Delta.$$

For $x, y \in \mathfrak{k}_{\mathbb{C}}^{(\alpha,n)}$ we then have $[x, y^*] \in \mathfrak{k}_{\mathbb{C}}^{(0,0)} = \mathfrak{t}_{\mathbb{C}}$, and for $h \in \mathfrak{t}_{\mathbb{C}}$

$$\langle h, [x, y^*]\rangle = \langle [h, y], x\rangle = \alpha(h)\langle y, x\rangle = \langle h, \langle x, y\rangle \alpha^\sharp\rangle,$$

where $\alpha^\sharp \in \mathfrak{t}_{\mathbb{C}}$ is the unique element satisfying $\langle h, \alpha^\sharp\rangle = \alpha(h)$ for $h \in \mathfrak{t}_{\mathbb{C}}$. This leads to

$$[x, y^*] = \langle x, y\rangle \alpha^\sharp \quad \text{for } x, y \in \mathfrak{k}_{\mathbb{C}}^{(\alpha,n)}. \tag{D.1}$$

For $\|x\| = 1$ we obtain in particular $[x, x^*] = \alpha^\sharp$ and thus

$$\alpha([x, x^*]) = \alpha(\alpha^\sharp) = \|\alpha^\sharp\|^2 > 0 \quad \text{for } 0 \neq \alpha.$$

We conclude that

$$\mathfrak{k}(\alpha, n) := \operatorname{span}_{\mathbb{R}}\{x - x^*, i(x + x^*), i[x, x^*]\} \cong \mathfrak{su}_2(\mathbb{C})$$

(*cf.* Lemma 1.8). For $y \perp x$ in $\mathfrak{k}_{\mathbb{C}}^{(\alpha,n)}$, we obtain $[x, y^*] = 0$ by (D.1) and thus

$$0 \leqslant \langle [x,y],[x,y]\rangle = \langle [x^*,[x,y]],y\rangle = \langle [[x^*,x],y],y\rangle = -\alpha(\alpha^\sharp)\|y\|^2 \leqslant 0,$$

so that $y = 0$, which means that

$$\dim \mathfrak{k}_{\mathbb{C}}^{(\alpha,n)} = 1. \tag{D.2}$$

Lemma D.3. — *If $\mathfrak{k}$ is simple, then the weight set $\Delta^\times := \Delta \setminus \{0\}$ does not decompose into two mutually orthogonal non-empty subsets.*

Proof. — Suppose that $\Delta = \Delta_1 \dot\cup \Delta_2$ is a decomposition into mutually orthogonal subsets. Then, for $\alpha \in \Delta_1$, $\beta \in \Delta_2$, we have $\alpha + \beta \not\in \Delta$, so that $[\mathfrak{k}_{\mathbb{C}}^\alpha, \mathfrak{k}_{\mathbb{C}}^\beta] = \{0\}$. Therefore the subalgebra $\mathfrak{k}_1$ generated by the weight spaces $\mathfrak{k}_{\mathbb{C}}^\alpha$, $\alpha \in \Delta_1$, is invariant under brackets with all root spaces and with $\mathfrak{t}_{\mathfrak{k}}$, hence an ideal. As $\mathfrak{k}$ is simple and $\Delta_1 \neq \emptyset$, it follows that $\mathfrak{k} = \mathfrak{k}_1$, and this leads to $\Delta_2 = \emptyset$. □

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