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REPELLING PERIODIC POINTS AND LANDING OF RAYS FOR POST-SINGULARLY BOUNDED EXPONENTIAL MAPS

by Anna Miriam BENINI & Mikhail LYUBICH

Abstract. — We show that repelling periodic points are landing points of periodic rays for exponential maps whose singular value has bounded orbit. For polynomials with connected Julia sets, this is a celebrated theorem by Douady, for which we present a new proof. In both cases we also show that points in hyperbolic sets are accessible by at least one and at most finitely many rays. For exponentials this allows us to conclude that the singular value itself is accessible.

Résumé. — On montre que, pour toute fonction exponentielle avec orbite singulière bornée, chaque point périodique répulsif est le point d'atterrissage d'au moins un rayon externe avec adresse périodique. Cela généralise un théorème de Douady qui s'applique au polynômes complexes avec ensemble de Julia connexe, dont on donne une nouvelle démonstration.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Then the dynamical plane $\mathbb{C}$ splits into two completely invariant subsets, the Fatou set $\mathcal{F}(f)$, on which the dynamics is stable, and its complement, the Julia set $J(f)$, on which the dynamics is chaotic. More precisely, the Fatou set is defined as

$$\mathcal{F}(f) := \{z \in \mathbb{C} : \{f^n\} \text{ is normal in a neighborhood of } z\}.$$

In this paper, we will consider the case in which f is either a polynomial or a complex exponential map $e^z + c$. An important role is played by the *escaping set*

$$I(f) := \{z \in \mathbb{C} : \ f^n(z) \to \infty \text{ as } n \to \infty\}.$$

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For polynomials, $I(f) \subset \mathcal{F}(f)$, while for exponentials $I(f) \subset J(f)$ (see [1], [6]). However, in both cases the escaping set can be described as an uncountable collection of injective curves, called *dynamic rays* or just *rays*, which tend to infinity on one side and are equipped with some symbolic dynamics (see Sections 2 and 3). While in the polynomial case either a point is escaping or its orbit is bounded, in the exponential case it might well be that orbits form an unbounded set without tending to infinity.

For a polynomial of degree D with connected Julia set, $I(f)$ is an open topological disk centered at infinity, and the dynamics of f on $I(f)$ is conjugate to the dynamics of z^D on $\mathbb{C}\setminus\overline{\mathbb{D}}$ via the Böttcher map (see Section 2). In this case, dynamic rays are defined simply as preimages of straight rays under the Böttcher map, and the symbolic dynamics on them is inherited from the symbolic dynamics of z^D on the unit circle $\mathbb{R}/\mathbb{Z}$ (see *e.g.* [11]). For exponentials, we refer the reader to [19] and to Section 3 of this paper. A ray is called *periodic* if it is mapped to itself under some iterate of the function.

It is important to understand the interplay between the rays and the set of non-escaping points. A ray $g_{\mathbf{s}}$ is said to *land* at a point z if $\overline{g_{\mathbf{s}}}\setminus g_{\mathbf{s}} = \{z\}$; conversely, a point is *accessible* if it is the landing point of at least one ray.

Ideally every ray lands and every non-escaping point in the Julia set is accessible, like for hyperbolic maps (in both polynomial and exponential setting). One weaker, but very relevant, question to ask is whether all periodic rays land and whether all repelling/parabolic periodic points are accessible by periodic rays. By the Snail Lemma (see *e.g.* [11]) if a periodic ray lands it has to land at a repelling or parabolic periodic point.

Periodic rays are known to land in both the polynomial and the exponential case (see Theorem 18.10 in [11] and [13]), unless one of their forward images contains the singular value.

The question whether repelling periodic points are accessible is harder and still open in the exponential case; in this paper, we give a positive answer to this problem for an exponential map $f(z) = e^z + c$ whose *postsingular set*

$$\mathcal{P}(f) := \overline{\bigcup_{n>0} f^n(c)}$$

is bounded. Observe that in this case the singular value is *non-recurrent*, *i.e.*, $c \notin \mathcal{P}(f)$.

Theorem A. — *Let f be either a polynomial or an exponential map, with bounded postsingular set; then any repelling periodic point is the*

landing point of at least one and at most finitely many dynamic rays, all of which are periodic of the same period.

For polynomials with connected Julia set, all repelling periodic points are known to be accessible by a theorem due to Douady (see [8]). Another proof due to Eremenko and Levin can be found in [5]: their proof covers also the case in which the Julia set is disconnected. However, neither proof can be generalized to the exponential family, because both use in an essential way the fact that the basin of infinity is an open set. Our proof of Theorem A also gives a new proof in the polynomial setting (see Section 2).

Our second result is about accessibility of hyperbolic sets. A forward invariant compact set Λ is called *hyperbolic* (with respect to the Euclidean metric) if there exist $k \in \mathbb{N}$ and $\eta > 1$ such that $|(f^k)'(x)| > \eta$ for all $x \in \Lambda$, $k > \overline{k}$. Observe that hyperbolic sets are subsets of the Julia set.

Theorem B. — *Let f be either a polynomial or an exponential map, with bounded postsingular set. Then any point that belongs to a hyperbolic set is accessible.*

Theorem B for polynomials is a special case of Theorem C in [12]. Under an additional combinatorial assumption (only needed for polynomials, since having bounded postsingular set is a stronger assumption in the exponential case than in the polynomial case) we also show that there are only finitely many rays landing at each point belonging to a hyperbolic set (see Propositions 2.11 and 4.5). This is a new result not only in the exponential case but also in the polynomial case.

For an exponential map whose postsingular set is bounded and contained in the Julia set, the postsingular set itself is hyperbolic (see [17], Theorem 1.2). We obtain hence the following corollary of Theorem B:

Corollary C. — *Let f be an exponential map with bounded postsingular set. Then every point in the postsingular set, and in particular the singular value itself, are accessible.*

Part of the importance of Theorem A is that it gives indirect insight on the structure of the parameter plane. For example, for unicritical polynomials, it implies that there are no irrational subwakes attached to hyperbolic components (see Section I.4 in [8], Theorem 4.1 in [18]). The proof in [18] is combinatorial and can be applied to the exponential family (see Section 4.4). The results in this paper are also used in [2] to show rigidity for non-parabolic exponential parameters with bounded postsingular set.

The structure of this article is as follows: in Section 2 we introduce some background about polynomial dynamics; we then present the new proof

of Douady's theorem about accessibility of repelling periodic points, followed by the proof of Theorem B in the polynomial case. The proof in this paper only uses quite weak information about the structure of dynamic rays, opening up this result to be generalized to other families of functions beyond the exponential family. In Section 3 we recollect some facts on exponential dynamics, including existence and properties of dynamic rays in this case. In Section 4 we state and prove Theorems A and B in the exponential setting. More precisely, in Section 4.2 we make some estimates about the geometry of rays near infinity that are needed to prove Theorems A and B for exponentials; the proofs themselves are presented in Section 4.3.

We denote by $\ell_{\mathrm{eucl}}(\gamma)$ the Euclidean length of a curve γ and by $\ell_\Omega(\gamma)$ its hyperbolic length in a region Ω admitting the hyperbolic metric with density ρ_Ω. A ball of radius r centered at a point z is denoted by either $B_r(z)$ or $B(z,r)$.

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2. Accessibility of repelling periodic orbits for polynomials with connected Julia set

In this section we give a new proof of Douady's theorem for polynomials with connected Julia set, showing that any repelling periodic orbit is the landing point of finitely many periodic rays.

Let f be a polynomial of degree D with connected Julia set $J(f)$ and filled Julia set $K(f)$. As $K(f)$ is full and contains more than one point, $\Omega = \mathbb{C} \setminus K(f)$ is a domain that admits a hyperbolic metric with some density $\rho_\Omega(z)$. Since $f : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus K(f)$ is a covering map, it locally preserves the hyperbolic metric.

Since $K(f)$ is connected there exists a unique conformal isomorphism (see [11, Chapter 18] and [11, Theorem 9.5]), called the Böttcher function B, which conjugates the dynamics of f on $\mathbb{C} \setminus K(f)$ to the dynamics of z^D on $\mathbb{C} \setminus \overline{\mathbb{D}}$ and which is asymptotic to the identity map at infinity.

Points in $\mathbb{C}\setminus\overline{\mathbb{D}}$ can be written as $z = e^t e^{2\pi i \mathbf{s}}$ with $t \in (0,\infty)$ and $\mathbf{s} \in \mathbb{R}/\mathbb{Z}$. We refer to t as the *potential* of z and to $\mathbf{s}$ as the *angle* of z. Consider the

straight ray of angle $\mathbf{s}$ parametrized as $R_{\mathbf{s}}(t) = e^t e^{2\pi i \mathbf{s}}$, with fixed $\mathbf{s} \in \mathbb{R}/\mathbb{Z}$ and $t \in (0, \infty)$; we define the curve $g_{\mathbf{s}}(t) := B^{-1}(R_{\mathbf{s}}(t))$ to be the *dynamic ray* of angle $\mathbf{s}$, and we keep referring to t as the potential.

The space Σ_D of infinite sequences over D symbols projects naturally to $\mathbb{R}/\mathbb{Z}$ via the D-adic expansion of numbers in $\mathbb{R}/\mathbb{Z}$, and this projection is one-to-one except for countably many points corresponding to D-adic numbers. The same projection semiconjugates multiplication by D on angles in $\mathbb{R}/\mathbb{Z}$ to the left-sided shift map σ acting on Σ_D. In this paper we will use both models. The sequences in Σ_D in our paper represent angles, so we will often use the term 'angle' when referring to them.

For $\mathbf{s} = s_0 s_1 s_2 \dots$ and $\mathbf{s}' = s'_0 s'_1 s'_2 \dots$ in Σ_D we consider the distance

$$|\mathbf{s} - \mathbf{s}'|_D = \sum_i \frac{|s_i - s'_i|}{D^{i+1}}. \tag{2.1}$$

We will denote by $|\mathbf{s} - \mathbf{s}'|$ the usual distance between angles in $\mathbb{R}/\mathbb{Z}$. A continuous map ϕ acting on a compact metric space (X, d) is called *locally expanding* if there exists $\rho > 1$ and $\epsilon > 0$ such that $d(\phi(x), \phi(y)) \geqslant \rho\, d(x, y)$ for any two points $x, y \in X$ with $d(x, y) < \epsilon$. Both the shift map σ on Σ_D and multiplication by D on $\mathbb{R}/\mathbb{Z}$ are locally expanding by a factor D.

Define the radial growth function as $F : t \mapsto Dt$. Then since $R_{\mathbf{s}}(t) \mapsto R_{\sigma\mathbf{s}}(F(t))$ under the map $z \mapsto z^D$ we also have

$$f(g_{\mathbf{s}}(t)) = g_{\sigma\mathbf{s}}(F(t)). \tag{2.2}$$

A *fundamental domain* starting at t for a ray $g_{\mathbf{s}}$ is the arc $g_{\mathbf{s}}([t, F(t)))$, and is denoted by $I_t(g_{\mathbf{s}})$.

The following lemma relates convergence of angles to convergence of dynamic rays.

Lemma 2.1. — *Let f be a polynomial of degree D. For each $t_*, t^* > 0$, the rays $g_{\mathbf{s}_n}(t)$ converge uniformly to the ray $g_{\mathbf{s}}(t)$ on $[t_*, t^*]$ as $\mathbf{s}_n \to \mathbf{s}$.*

Proof. — The inverse of the Böttcher map is uniformly continuous on the closed annulus $\{z \in \mathbb{C} : z = e^t e^{2\pi i \mathbf{s}} \text{ with } t \in [t_*, t^*],\ \mathbf{s} \in \mathbb{R}/\mathbb{Z}\}$. Since the straight rays $R_{\mathbf{s}_n}(t)$ converge uniformly to the ray $R_{\mathbf{s}}(t)$ on the compact set $[t_*, t^*]$, the claim follows. □

The next lemma gives a sufficient condition to determine when the limit dynamic ray lands. The proof is the same in both the polynomial and in the exponential cases, once dynamic rays for the latter have been defined (see Section 3 for dynamic rays in the exponential family).

Lemma 2.2. — *Let $x_0 \in \mathbb{C}$, $t_0 > 0$, and for all $m \in \mathbb{N}$ define $t_m := F^{-m}(t_0)$. Also let $g_{\mathbf{s}_n}$ be a sequence of dynamic rays such that $\mathbf{s}_n \to \mathbf{s}$, and*

such that $I_{t_m}(g_{\mathbf{s}_n}) \subset B(x_0, \frac{A}{\nu^m})$ *for some* $A > 0, \nu > 1$ *and for all* $n > N_m$. *Then* $g_{\mathbf{s}}$ *lands at* x_0.

Proof. — To show that $g_{\mathbf{s}}$ lands at x_0 it is enough to show that for each m,

$$I_{t_m}(g_{\mathbf{s}}) \subset B\left(x_0, \frac{A}{\nu^m}\right).$$

For any fixed $m > 0$, $g_{\mathbf{s}_n} \to g_{\mathbf{s}}$ uniformly on $[t_m, t_{m-1}]$ by Lemma 2.1. As $I_{t_m}(g_{\mathbf{s}_n})$ is eventually contained in $B\left(x_0, \frac{A}{\nu^m}\right)$, taking the limit for $n \to \infty$ gives the claim. □

The following lemma is a consequence of the fact that for points tending to the boundary of a hyperbolic domain, the hyperbolic density tends to infinity.

Lemma 2.3. — *Let* $\Omega \subset \mathbb{C}$ *be a hyperbolic region. Let* $\gamma_n : [0,1] \to \Omega$ *be a family of curves with uniformly bounded hyperbolic length and such that* $\gamma_n(0) \to \partial\Omega$. *Then* $\ell_{\mathrm{eucl}}(\gamma_n) \to 0$.

For reader's convenience, we will outline a proof.

Proof. — Passing to a subsequence, we let $\gamma_n(0)$ converge to some point $z_0 \in \partial\Omega$. Let us take two more boundary points, $z_1, z_2 \in \partial\Omega$, such that $\mathrm{dist}(z_i, z_j) \geqslant \delta > 0$, $0 \leqslant i < j \leqslant 2$, where dist is the spherical distance and $\delta = \delta(\Omega)$. By the Schwarz Lemma, the hyperbolic metric on Ω is dominated by the hyperbolic metric on $\hat{\mathbb{C}} \setminus \{z_0, z_1, z_2\}$, so it is enough to prove the statement for the latter domain.

Let us move the points (z_0, z_1, z_2) to $(0, 1, \infty)$ by a Möbius transformation A. Since the triple $\{z_0, z_1, z_2\}$ is δ-separated, the spherical derivative $\|DA(z)\|$ is bounded from below by some constant $c(\delta) > 0$. Hence it is enough to prove the statement for $\Omega = \mathbb{C} \setminus \{0, 1\}$. In this case, the universal covering $\mathbb{H} \to \Omega$ is the classical triangle modular function λ with fundamental domain

$$\{z \in \mathbb{H} : \ 0 \leqslant \mathrm{Re}\, z \leqslant 2, |z \pm 1/2| \geqslant 1/2\}.$$

Near ∞, this function admits a Fourier expansion of form $\phi(e^{\pi i z})$, where ϕ is a univalent function near 0. Pushing the hyperbolic metric on $\mathbb{H}$ down, we conclude that the hyperbolic metric on Ω near 0 is comparable with

$$\rho(z)|dz| = \frac{|dz|}{|z| \, |\log|z||}.$$

Since $\int_0^\epsilon \rho(x)dx = \infty$ for any $\epsilon > 0$ (*i.e.*, the cusp has infinite hyperbolic diameter), the curves γ_n uniformly converge to 0. Hence $\inf_{\gamma_n} \rho(z) \to \infty$

as $n \to \infty$, and

$$l(\gamma_n) = \int_{\gamma_n} \frac{\|dz\|}{\rho(z)} \to 0 \quad \text{as } n \to \infty,$$

where $\|\cdot\|$ stands for the infinitesimal length of the hyperbolic metric. $\square$

2.1. Proof of Theorem A in the polynomial case

In this section we give a proof of Theorem A in the polynomial case. Up to taking an iterate of f, we can assume that the repelling periodic point in question is a repelling fixed point α. Let $\mu > 1$ be the modulus of its multiplier, and let L be a linearizing neighborhood for α. For the branch ψ of f^{-1} fixing α, it is easy to show using linearizing coordinates that there exists a $C > 0$ such that for all $x \in L$

$$\frac{1}{C\mu^n} < |(\psi^n)'(x)| < \frac{C}{\mu^n}. \tag{2.3}$$

Before proceeding to the proof of Theorem A let us observe the following:

Proposition 2.4. — *The Euclidean length $\ell_{\mathrm{eucl}}(I_t(g_{\mathbf{s}}))$ tends to 0 uniformly in $\mathbf{s}$ as $t \to 0$.*

Proof. — Let $t_* > 0$, $t^* > F^2(t_*)$, $\Omega = \mathbb{C} \setminus K(f)$. By uniform continuity of the inverse of the Böttcher map on compact sets, the Euclidean length $\ell_{\mathrm{eucl}}(g_{\mathbf{s}}(t_*, t^*))$, and hence the hyperbolic length $\ell_\Omega(g_{\mathbf{s}}(t_*, t^*))$, are uniformly bounded in $\mathbf{s}$. So by the Schwarz Lemma, $\ell_\Omega(f^{-n}(g_{\mathbf{s}}(t_*, t^*)))$ is also uniformly bounded in $\mathbf{s}$ and n, and hence $\ell_\Omega(I_t(g_{\mathbf{s}}))$ is uniformly bounded in $\mathbf{s}$ for $t \leqslant t_*$ (because for any such t, $I_t(g_{\mathbf{s}}) \subset f^{-n}(g_{\widetilde{\mathbf{s}}}(t_*, t^*))$ for some $\widetilde{\mathbf{s}} \in \mathbb{R}/\mathbb{Z}$ and some $n \geqslant 0$). Since the inverse of the Böttcher map is proper, $\mathrm{dist}(g_{\mathbf{s}}(t), J(f)) \to 0$ uniformly in $\mathbf{s}$ as $t \to 0$, hence by Lemma 2.3 $\ell_{\mathrm{eucl}}(I_t(g_{\mathbf{s}})) \to 0$ uniformly in s as $t \to 0$. $\square$

Theorem 2.5. — *Let f be a polynomial with connected Julia set, and let α be a repelling fixed point for f. Then there is at least one dynamic ray $g_{\mathbf{s}}$ landing at α.*

Proof. — Let L be a linearizing neighborhood for α and μ be as in (2.3). Let $U' \subset L$ be a neighborhood of α, and let U be its preimage under the inverse branch ψ of f which fixes α. Let ϵ be the distance between ∂U and $\partial U'$. By Proposition 2.4, there exists t_ϵ such that

$$\ell_{\mathrm{eucl}}(I_t(g_{\mathbf{s}})) < \epsilon \quad \text{for all } \ \mathbf{s} \in \Sigma_D, t < t_\epsilon. \tag{2.4}$$

As α is in the Julia set, it is approximated by escaping points with arbitrary small potential t, hence there exists a dynamic ray $g_{\mathbf{s}_0}$ such that $g_{\mathbf{s}_0}(t_0)$ belongs to U for some $t_0 < t_\epsilon$. By (2.4), $\ell_{\mathrm{eucl}}(I_{t_0}(g_{\mathbf{s}_0})) \leqslant \epsilon$, hence $I_{t_0}(g_{\mathbf{s}_0}) \subset U'$ (See Figure 2.1).

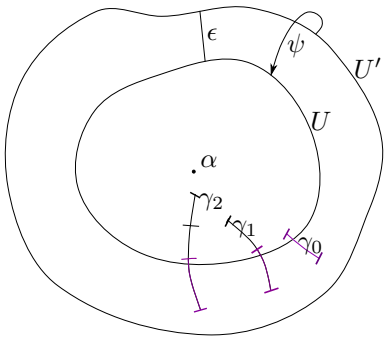

Figure 2.1. Construction of the curves γ_n in the proof of Theorem 2.5.

For any $n > 0$ recall that ψ^n is the branch of f^{-n} fixing α, and let $g_{\mathbf{s}_n}$ be the ray containing $\psi^n(I_{t_0}(g_{\mathbf{s}_0}))$. Observe that $\sigma^n \mathbf{s}_n = \mathbf{s}_0$ for all n. Let us define inductively a sequence of curves $\gamma_n \subset g_{\mathbf{s}_n}$ as

$$\begin{aligned} \gamma_0 &:= I_{t_0}(g_{\mathbf{s}_0}) \\ \gamma_n &:= \psi(\gamma_{n-1}) \cup I_{t_0}(g_{\mathbf{s}_n}). \end{aligned}$$

We show inductively that the curves γ_n are well defined and that the following properties hold:

(1) $\gamma_n = g_{\mathbf{s}_n}(t_n, F(t_0))$, where $t_n := F^{-n}(t_0)$;
(2) $\gamma_n \subset U'$ for all n;
(3) $I_{t_m}(g_{\mathbf{s}_n}) \subset B\left(\alpha, \frac{C \operatorname{diam} U'}{\mu^m}\right)$, for all $m \leqslant n$.

All properties are true for γ_0, so let us suppose that they hold for γ_{n-1} and show that they also hold for γ_n. We have that

$$\psi(\gamma_{n-1}) = \psi(g_{\mathbf{s}_{n-1}}(t_{n-1}, F(t_0))) = g_{\mathbf{s}_n}(t_n, t_0)$$

by the functional equation (2.2) and by the definition of $g_{\mathbf{s}_n}$. Also, $\psi(\gamma_{n-1}) \subset U$ because by the inductive assumption $\gamma_{n-1} \subset U'$ and $\psi(U') = U$. As $\ell_{\mathrm{eucl}}(I_{t_0}(g_{\mathbf{s}_n})) \leqslant \epsilon$, and $g_{\mathbf{s}_n}(t_n, t_0) \subset U$, we have that $\gamma_n \subset U'$.

If $x \in I_{t_m}(g_{\mathbf{s}_n})$ for $m \leqslant n$, then $x = \psi^m y$ for some $y \in I_{t_0}(g_{\mathbf{s}_{n-m}}) \subset U'$, hence by (2.3) we have

$$|x - \alpha| \leqslant \frac{C|y - \alpha|}{\mu^m} \leqslant \frac{C \operatorname{diam} U'}{\mu^m},$$

proving Property 3.

As the sequence $\{\mathbf{s}_n\}$ of angles of the rays $g_{\mathbf{s}_n}$ is contained in $\mathbb{R}/\mathbb{Z}$, there is a subsequence converging to some angle $\mathbf{s}$. As the Julia set is connected, no singular value is escaping, hence the ray $g_{\mathbf{s}}$ of angle $\mathbf{s}$ is well defined for all potentials $t > 0$. Landing of $g_{\mathbf{s}}$ at α follows from Property 3 together with Lemma 2.2. □

To prove periodicity of the landing ray constructed in Theorem 2.5 we use the following lemma.

Lemma 2.6. — *Let (X, d) be a compact metric space, $f : X \to X$ be a locally expanding map and let $A \subset X$ be a closed invariant subset. If $f : A \to A$ is invertible then A is finite.*

Proof. — Since A is compact and $f : A \to A$ is invertible, it is a homeomorphism, so $f^{-1} : A \to A$ is uniformly continuous. Let us take $\epsilon > 0$ from the definition of the local expanding property. Then there exists $\delta \in (0, \epsilon)$ such that if $d(x, y) < \delta$ for some points $x, y \in A$ then $d(f^{-1}x, f^{-1}y) < \epsilon$. Moreover, by the local expanding property

$$d(f^{-1}x, f^{-1}y) \leqslant \rho^{-1} d(x, y) < \rho^{-1}\delta < \delta.$$

Iterating this estimate, we obtain

$$d(f^{-n}x, f^{-n}y) \leqslant \rho^{-n} d(x, y) < \rho^{-n}\delta \to 0 \quad \text{as } n \to \infty.$$

Let us now cover A by N δ-balls $B_i \subset A$. Then the last estimate shows that A is covered by N $(\rho^{-n}\delta)$-balls $f^{-n}(B_i)$, which implies that A contains at most N points. □

Proposition 2.7. — *Any dynamic ray $g_{\mathbf{s}}$ obtained from the construction of Theorem 2.5 is periodic.*

Proof. — Let $\mathcal{B} := \{\mathbf{s}_n\}_{n\in\mathbb{N}}$ be the set of angles of the rays $\psi^n(g_{\mathbf{s}_0})$ constructed in the proof of Theorem 2.5, and $\mathcal{A}$ be their limit set defined as

$$\mathcal{A} := \{\mathbf{s} \in \Sigma_D : \mathbf{s}_{n_k} \to \mathbf{s} \ \text{ for } \mathbf{s}_{n_k} \in \mathcal{B}, n_k \to \infty\}.$$

The set $\mathcal{A}$ is closed and forward invariant by definition, and for any $\mathbf{s} \in \mathcal{A}$ the ray $g_{\mathbf{s}}$ lands at α by the construction of Theorem 2.5. In order to use Lemma 2.6, we begin by showing that $\sigma : \mathcal{A} \to \mathcal{A}$ is a homeomorphism. We first show surjectivity of σ. For any $\mathbf{s}_{n+1} \in \mathcal{B}, \sigma\mathbf{s}_{n+1} = \mathbf{s}_n$ by definition of $\mathbf{s}_n$. If $\mathbf{s} \in \mathcal{A}$, there is a sequence $\mathbf{s}_{n_k} \in \mathcal{B}$ converging to $\mathbf{s}$; then the sequence $\mathbf{s}_{n_k+1} \in \mathcal{B}$ has at least one accumulation point $\tilde{\mathbf{s}} \in \mathcal{A}$ and since $\sigma\mathbf{s}_{n_k+1} = \mathbf{s}_{n_k}$ for all n_k, by continuity of σ we have that $\sigma\tilde{\mathbf{s}} = \mathbf{s}$. To show injectivity of $\sigma|_{\mathcal{A}}$ suppose by contradiction that there exist $\mathbf{s}, \mathbf{s}' \in \mathcal{A}$ such that $\sigma\mathbf{s} = \sigma\mathbf{s}'$.

Since α is a repelling fixed point it has a simply connected neighborhood U such that $f : U \to U' \supset U$ is a homeomorphism, and since both $g_{\mathbf{s}}$ and $g_{\mathbf{s}'}$ land at α, there is a sufficiently small potential t such that $g_{\mathbf{s}}(t)$ and $g_{\mathbf{s}'}(t)$ are both in U. Then $f(g_{\mathbf{s}}(t)) = f(g_{\mathbf{s}'}(t)) = g_{\sigma\mathbf{s}}(F(t))$ contradicting injectivity of f on U. The claim then follows from Lemma 2.6. $\square$

The next lemma can be found in [11], Lemma 18.12. The proof holds also in the exponential case; see at the end of Section 3 for more details.

Lemma 2.8. — *If a periodic ray lands at a repelling periodic point z_0, then only finitely many rays land at z_0, and these rays are all periodic of the same period.*

Corollary 2.9. — *All the rays landing at a repelling fixed point are periodic.*

This concludes the proof of Theorem A in the polynomial case.

2.2. Proof of Theorem B in the polynomial case

In this subsection we prove Theorem B in the polynomial case and we show that under an additional combinatorial condition, every point in a hyperbolic set is the landing point of only finitely many dynamic rays.

Proof of Theorem B. — Let Λ be a hyperbolic set. Up to taking an iterate of f, we can assume that there is a δ-neighborhood U of Λ such that $|f'(x)| > \eta > 1$ for all $x \in U$.

Fix some x_0 in Λ, and let us construct a dynamic ray landing at x_0. Let $x_n := f^n(x_0)$, $B'_n := B_\delta(x_n)$, $B_n := B_{\delta/\eta}(x_n)$. Observe that for each n there is a branch ψ of f^{-1} such that $\psi(B'_n) \subset B_{n-1}$. We refer to ψ^m as the composition of such branches, mapping x_n to x_{n-m}. Let $\epsilon := \delta - \delta/\eta$, and let t_ϵ be such that the length of fundamental domains starting at $t < t_\epsilon$ is smaller than ϵ (see Proposition 2.4). Let us define a family of rays to which we will apply the construction of Theorem 2.5.

Let $t_0 < t_\epsilon$ be such that each B_n contains a point of potential t_0. For each n, let $\mathcal{A}_{x_n}$ be the family of angles $\mathbf{s}$ such that $g_{\mathbf{s}}(t_0) \in B_n$. By Proposition 2.4, and because $\operatorname{dist}(\partial B_n, \partial B'_n) = \epsilon$, $I_{t_0}(g_{\mathbf{s}}) \subset B'_n$ for any $\mathbf{s} \in \mathcal{A}_{x_n}$. For each $\mathbf{s} \in \mathcal{A}_{x_n}$, denote by $\psi^m_* g_{\mathbf{s}}$ the ray to which $\psi^m(g_{\mathbf{s}})(t_0)$ belongs to (see Figure 2.2). For any $m \in \mathbb{N}$ let $t_m := F^{-m}(t_0)$; following the construction of Theorem 2.5, we obtain that

$$(\psi^m_* g_{\mathbf{s}_n})(t_m, F(t_0)) \subset B'_{n-m} \quad \text{for any} \quad m \leqslant n,\ \mathbf{s}_n \in \mathcal{A}_{x_n}.$$

Also,

$$I_{t_m}(\psi_*^n g_{\mathbf{s}_n}) \subset B\left(x_0, \frac{\delta}{\eta^m}\right) \text{ for any } m \leqslant n,\ \mathbf{s}_n \in \mathcal{A}_{x_n}. \tag{2.5}$$

Consider now any sequence $\{\mathbf{s}_n\}$ of angles such that $\mathbf{s}_n \in \mathcal{A}_{x_n}$, and let $\mathbf{a}_n$ be the sequence of angles such that $g_{\mathbf{a}_n} := \psi_*^n g_{\mathbf{s}_n}$. Observe that $\mathbf{a}_n \in \mathcal{A}_{x_0}$, that $\sigma^n \mathbf{a}_n = \mathbf{s}_n$, and that $I_{t_m}(g_{\mathbf{a}_n}) \subset B\left(x_0, \frac{\delta}{\eta^m}\right)$ for any $m \leqslant n,\ n \in \mathbb{N}$ by (2.5). Then by Lemma 2.2, for any accumulation point $\mathbf{a}$ of the sequence $\{\mathbf{a}_n\}$ the ray $g_{\mathbf{a}}$ lands at x_0. □

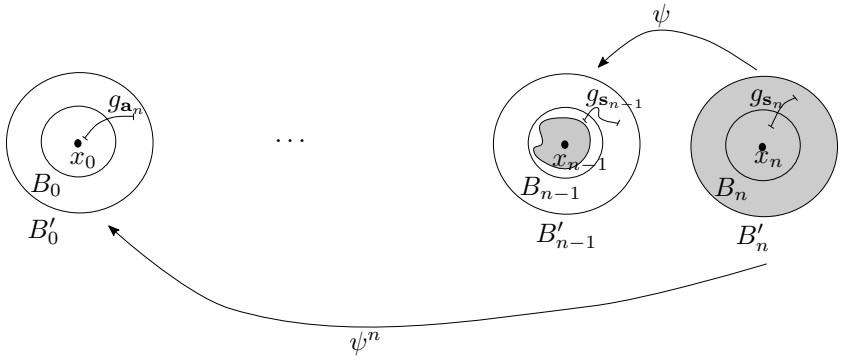

Figure 2.2. Construction of a landing ray for a point $x_0 \in \Lambda$. Here only the pieces of rays $g_{\mathbf{s}_n}$ and $g_{\mathbf{s}_{n-1}}$ between potentials t_0 and $F(t_0)$ are shown in B_n' and B_{n-1}' respectively. Similarly, only the piece of ray $g_{\mathbf{a}_n}$ between potential $F^{-n(t_0)}$ and t_0 is shown inside B_0.

If we assume f to be a unicritical polynomial satisfying some combinatorial conditions, we can show that there are only finitely many rays landing at each $x \in \Lambda$. We say that two points z_1, z_2 are *combinatorially separated* if there is a curve Γ formed by two dynamic rays together with a common landing point such that z_1, z_2 belong to different components of $\mathbb{C} \setminus \Gamma$.

Remark 2.10. — Let f be a unicritical polynomial of degree D. Then there is a rotational symmetry given by $f(e^{2\pi i/D} z) = f(z)$, implying that two dynamic rays of angles $\mathbf{s}, \mathbf{s}'$ land together if and only if the dynamic rays of angles $\mathbf{s} + j/D, \mathbf{s}' + j/D$ (obtained from the dynamic rays of angles $\mathbf{s}, \mathbf{s}'$ through a rotation of angle $2\pi j/D$) do for $j = 1 \dots D-1$. Then by the cyclic order at infinity, two dynamic rays of angles $\mathbf{s}, \mathbf{s}'$ can land together only if $|\mathbf{s} - \mathbf{s}'| \leqslant 1/D$; otherwise, the two dynamic rays of angles $\mathbf{s} + 1/D, \mathbf{s}' + 1/D$ would intersect them, giving a contradiction.

Proposition 2.11. — *Let $f(z) = z^D + c$ be a unicritical polynomial of degree D, and let Λ be a hyperbolic set. Suppose moreover that either 0 is accessible or that any $x \in \Lambda$ is combinatorially separated from 0. Then there are only finitely many dynamic rays landing at each $x \in \Lambda$.*

Proof. — Since Λ is a hyperbolic set, up to proving the statement for $f^k(\Lambda)$ we can assume that $0 \notin \Lambda$. For $x \in \Lambda$, let $\mathcal{A}_x$ be the set of angles of the rays landing at x. By Theorem B, each $\mathcal{A}_x$ is non empty. Near x, f is locally a homeomorphism, so the set $\mathcal{A}_x$ is mapped bijectively to the set $\mathcal{A}_{f(x)}$ by the shift map σ and there is a well defined inverse $\sigma^{-1} : \mathcal{A}_{f(x)} \to \mathcal{A}_x$. Recall that σ is locally expanding by a factor D. Let us first assume that σ^{-1} is uniformly continuous on the sets $\{\mathcal{A}_x\}_{x\in\Lambda}$, in the sense that there exists an $\epsilon > 0$ such that for any $x \in \Lambda$, and for any $\mathbf{a}, \mathbf{a}' \in \mathcal{A}_{f(x)}$ such that $|\mathbf{a} - \mathbf{a}'| < \epsilon$ we have

$$|\sigma^{-1}\mathbf{a} - \sigma^{-1}\mathbf{a}'| < \frac{1}{D}\,|\mathbf{a} - \mathbf{a}'|. \tag{2.6}$$

Fix a point $y_0 \in \Lambda$ and let $y_n := f^n(y_0)$. Consider a finite cover of $\mathbb{R}/\mathbb{Z}$ by ϵ-balls, say N balls, where ϵ is given by uniform continuity of σ^{-1}. Then for any n, $\mathcal{A}_{y_n}$ is covered by N balls of radius ϵ. By (2.6), their n-th preimages have diameter ϵ/D^n and cover $\mathcal{A}_{y_0}$. Passing to the limit, we conclude that $\mathcal{A}_{y_0}$ contains at most N points.

So it is only left to show that σ^{-1} is uniformly continuous in the sense of (2.6). Suppose by contradiction that there is a sequence of points $x_n \in \Lambda$, and two sequences of angles $\mathbf{a}_n, \mathbf{a}_n{}' \in \mathcal{A}_{f(x_n)}$ such that $|\mathbf{a}_n - \mathbf{a}_n{}'| \to 0$, but $|\sigma^{-1}\mathbf{a}_n - \sigma^{-1}\mathbf{a}_n{}'| \to \delta > 0$. Call $\mathbf{s}_n, \mathbf{s}_n{}'$ the angles $\sigma^{-1}\mathbf{a}_n, \sigma^{-1}\mathbf{a}_n{}'$, and assume for definiteness that $\mathbf{s}_n < \mathbf{s}_n{}'$. Since $\mathbf{a}_n$ and $\mathbf{a}_n{}'$ converge to the same angle and the preimages under σ of any angle are spaced at $1/D$ apart, $\delta = k/D$ for some integer $k \in [1, D-1]$; since the rays $g_{\mathbf{s}_n}$ and $g_{\mathbf{s}_n{}'}$ land together at the point x_n, $k = 1$ by Remark 2.10. By the D-fold symmetry of the Julia set, the rays of angles $\mathbf{s}_n + j/D, \mathbf{s}_n{}' + j/D$ for $j = 1 \dots D-1$ also land together at the points $e^{2\pi i j/D}x_n$ (see Figure 2.3).

Altogether, these D pairs of dynamic rays divide $\mathbb{C}$ into $D+1$ connected component. Let V_n be the one that contains the critical point 0. Since for each i we have:

$$(\mathbf{s}_n{}' + j/D) - (\mathbf{s}_n + j/D) \to 1/D \quad \text{as } n \to \infty,$$

we conclude that

$$(\mathbf{s}_n + (j+1)/D) - (\mathbf{s}_n{}' + j/D) \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

Let $Q = \bigcap \overline{V_n}$. Then $0 \in Q$ by definition, and $Q \cap \Lambda \neq \emptyset$ because Λ is compact and $\overline{V_n} \cap \Lambda \neq \emptyset$ for all n. Let $z \in Q \cap \Lambda$. By (2.7), there are exactly

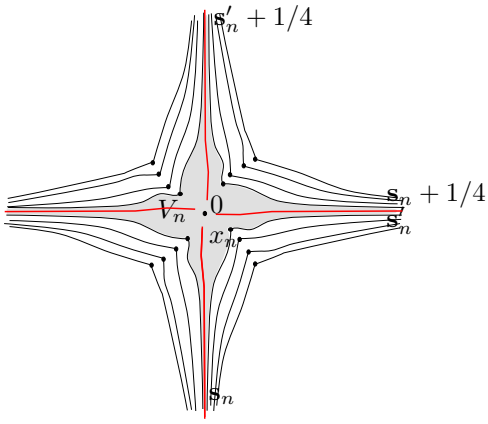

Figure 2.3. Illustration to the proof of Proposition 2.11 for $D = 4$; the region V_n is shaded in gray. Its boundary consists of the rays $g_{\mathbf{s}_n}, g_{\mathbf{s}'_n}$ and their rotations by $\pi/2$. The limiting rays are colored in red, and they all either land at 0 or land at four different points which are rotations of each other by $\pi/2$.

D limiting rays entering Q. By symmetry, if these rays land, they either all land at 0, or they land at D different points belonging to some orbit of the $2\pi/D$-rotation. This gives an immediate contradiction with either of our assumptions:

- Since there are no pairs of rays which could separate 0 from points of Q, the point $z \in \Lambda$ is not combinatorially separated from 0, giving a contradiction with the first assumption;
- In the case when 0 is accessible, this implies that no other accessible point can belong to Q; in particular $z \in \Lambda$ is not accessible, contradicting Theorem B.

This concludes the proof of (2.6) and hence of the proposition. □

3. Dynamic rays in the exponential family

For the remaining two sections, $f(z) = e^z + c$ will be an exponential function, and $J(f)$ will be its Julia set. Any two exponential maps whose singular values differ by $2\pi i$ are conformally conjugate, so we can assume $-\pi \leqslant \operatorname{Im} c < \pi$. Let $\arg(z)$ be defined on $\mathbb{C} \setminus \mathbb{R}_-$ so as to take values in $(-\pi, \pi)$, and let

$$R := \{z \in \mathbb{C} : \ \operatorname{Im} z = \operatorname{Im} c, \ \operatorname{Re} z \leqslant \operatorname{Re} c\}.$$

We define a family $\{L_n\}$ of inverse branches for $f(z) = e^z + c$ on $\mathbb{C} \setminus R$ as

$$L_n(w) := \log|w - c| + i \arg(w - c) + 2\pi i n.$$

Observe that L_n maps $\mathbb{C} \setminus R$ biholomorphically to the strip

$$S_n := \{z \in \mathbb{C} : 2\pi n - \pi < \operatorname{Im} z < 2\pi n + \pi\}.$$

Observe also that $|(L_n)'(w)| = \frac{1}{|w-c|}$.

Dynamic rays have been first introduced by Devaney and Krych in [4] (with the name *hairs*) and studied for example in [3]. A full classification of the set of escaping points in terms of dynamic rays has been then completed by Schleicher and Zimmer in [19]. For points whose iterates never belong to R, we can consider itineraries with respect to the partition of the plane into the strips $\{S_n\}$, *i.e.*,

$$\operatorname{itin}(z) = s_0 s_1 s_2 \dots \quad \text{if and only if} \quad f^j(z) \in S_{s_j}.$$

For a point z whose itinerary with respect to this partition exists, we refer to it as the *address* of z. The set $\mathcal{S}$ of all allowable itineraries is called the *set of addresses*; by construction, is is a subset of $\mathbb{Z}^{\mathbb{N}}$. Addresses as sequences can be characterized also in terms of the growth of their entries. According to the construction, addresses of points cannot have entries growing faster than iterates of the exponential function. Let us use the function $F : t \mapsto e^t - 1$ to model real exponential growth. A sequence $\mathbf{s} = s_0 s_1 s_2 \dots$ is called *exponentially bounded* if there exists constants $A \geqslant 1/2\pi$, $x \geqslant 0$ such that for all $k \geqslant 0$.

$$|s_k| < AF^k(x) \tag{3.1}$$

This growth condition turns out to be not only necessary but also sufficient for a sequence to be realized as an address (see [19]), so that $\mathcal{S}$ also equals the set of all exponentially bounded sequences in $\mathbb{Z}^{\mathbb{N}}$ (see also Theorem 3.1 below).

An address is called *periodic* (resp. *preperiodic*) if it is a periodic (resp. preperiodic) sequence. If $\mathbf{s} = s_0 s_1 s_2 \dots$, let $\|\mathbf{s}\|_\infty = \sup_i |s_i|/2\pi$. We call $\mathbf{s}$ *bounded* if $\|\mathbf{s}\|_\infty < \infty$. For $j \in \mathbb{Z}$ and $\mathbf{s} \in \mathcal{S}$, $j\mathbf{s}$ stands for the sequence $j s_0 s_1 s_2 \dots$.

Given an address $\mathbf{s}$ we define its *minimal potential*

$$t_{\mathbf{s}} := \inf\left\{t > 0 : \limsup_{k\to\infty} \frac{|s_k|}{F^k(t)} = 0\right\}.$$

Observe that if $\mathbf{s}$ is bounded, $t_{\mathbf{s}} = 0$.

Definition, existence and properties of *dynamic rays* for the exponential family are summarized in the following theorem ([19], Proposition 3.2 and Theorem 4.2; the quantitative estimates are taken from Proposition 3.4).

THEOREM 3.1 (Existence of dynamic rays). — *Let $f(z) = e^z + c$ be an exponential map such that c is non-escaping, and K be a constant such that $|c| \leqslant K$. Let $\mathbf{s} = s_0 s_1 s_2 \ldots \in \mathbb{Z}^{\mathbb{N}}$ be a sequence satisfying (3.1), and let $t_{\mathbf{s}}$ be the minimal potential for $\mathbf{s}$. Then there exists a unique maximal injective curve $g_{\mathbf{s}} : (t_{\mathbf{s}}, \infty) \to \mathbb{C}$, consisting of escaping points, such that*

(1) *$g_{\mathbf{s}}(t)$ has address $\mathbf{s}$ for $t > x + 2\log(K+3)$;*
(2) *$f(g_{\mathbf{s}}(t)) = g_{\sigma\mathbf{s}}(F(t))$;*
(3) *$|g_{\mathbf{s}}(t) - 2\pi i s_0 - t - t_{\mathbf{s}}| \leqslant 2e^{-t}(K + 2 + 2\pi|s_1| + 2\pi AC)$ for large t, and for a universal constant C.*

The curve $g_{\mathbf{s}}$ is called the *dynamic ray of address* $\mathbf{s}$. As for polynomials, a dynamic ray is periodic or preperiodic if and only if its address is a periodic or preperiodic sequence respectively. Likewise, the *fundamental domain* starting at t for $g_{\mathbf{s}}$ is defined as the arc $I_t(g_{\mathbf{s}}) := g_{\mathbf{s}}(t, F(t))$.

Remark 3.2. — Fix a parameter c which is non-escaping. If $\mathbf{s}$ is bounded, the constant x in Theorem 3.1 can be taken arbitrarily small, so Property (1) holds for $t > 2\log(K+3)$. Moreover, $g_{\mathbf{s}}(t)$ is approximately straight, *i.e.*, there exists a constant $C'(K, t_{\mathbf{s}})$ such that $|g'_{\mathbf{s}}(t) - 1| < C'(K, t_{\mathbf{s}})$ for large t and $C'(K, t_{\mathbf{s}})$ does not depend on $\mathbf{s}$ if $\mathbf{s}$ is bounded (see Proposition 4.6 in [7]). It then follows from the asymptotic estimates in Theorem 3.1 that for any t there exists a constant $B(t, \|\mathbf{s}\|_\infty)$ such that

$$\ell_{\mathrm{eucl}}(I_t(g_{\mathbf{s}})) \leqslant B(t, \|\mathbf{s}\|_\infty)) \sim e^t - t. \tag{3.2}$$

The notion of minimal potential is important to have a parametrization of the rays respecting some kind of transversal continuity. The following lemma holds, and will play the role of Lemma 2.1 for the exponential family. A proof can be found in [15]; we will use the formulation from [14], Lemma 4.7).

LEMMA 3.3 (Transversal continuity). — *Let f be an exponential map, $\{\mathbf{s}_n\}$ be a sequence of addresses, $\mathbf{s}_n \to \mathbf{s} \in \mathcal{S}$ such that $t_{\mathbf{s}_n} \to t_{\mathbf{s}}$. Then $g_{\mathbf{s}_n} \to g_{\mathbf{s}}$ uniformly on $[t_*, \infty)$ for all $t_* > t_{\mathbf{s}}$.*

It will be useful to keep in mind the next observation, which moreover concurs to the validity of Lemma 2.8 in the exponential case.

Remark 3.4. — The addresses of two rays $g_{\mathbf{s}}, g_{\mathbf{s}'}$ landing together cannot differ by more than one in any entry. Suppose by contradiction that they do:

up to taking forward iterates of the two rays (which keep landing together by continuity of f) we can assume that $\mathbf{s} = s_0 s_1 s_2 \dots$ and $\mathbf{s}' = s_0' s_1' s_2'$ differ by more than one in their first entry. Suppose for definiteness that $s_0 < s_0'$. By the $2\pi i$ symmetry of the dynamical plane, the rays with addresses $\widetilde{\mathbf{s}}' := (s_0' - 1)s_1' s_2' \dots$ and $\widetilde{\mathbf{s}} := (s_0 - 1)s_1 s_2 \dots$ land together. Since $|s_0 - s_0'| \geqslant 2$, we have that $s_0 < \widetilde{s}_0 < s_0'$, so by the asymptotic estimates in Theorem 3.1 the two curves $\Gamma = \overline{g_{\mathbf{s}} \cup g_{\mathbf{s}'}}$ and $\widetilde{\Gamma} = \overline{g_{\widetilde{\mathbf{s}}} \cup g_{\widetilde{\mathbf{s}}'}}$ would intersect contradicting the fact that dynamic rays are disjoint.

We now give a sketch of the proof of Lemma 2.8 in the exponential case, referring to the proof of Lemma 18.12 in [11] for polynomials.

Proof of Lemma 2.8 in the exponential case. — Let z_0 be a repelling periodic point which is the landing point of a periodic dynamic ray $g_{\mathbf{s}}$, and let $\mathcal{A}$ be the set of addresses of the rays landing at z_0. Since $\|\mathbf{s}\|_\infty$ is bounded, by Remark 3.4 the norm of all addresses in $\mathcal{A}$ is also bounded by some constant $M > 0$. Then $\mathcal{A} \subset \Sigma_D$ for $D = 2M + 1$, where Σ_D is represented as $\Sigma_D = \{-M, \dots, M\}^{\mathbb{N}}$. The proof of Lemma 18.12 in [11] can then be repeated. □

4. Accessibility for exponential parameters with bounded postsingular set

4.1. Statement of theorems and some basic facts

From now on we will consider an exponential function f with bounded postsingular set, which implies that the singular value is non-recurrent; this excludes the presence of Siegel disks (see Corollary 2.10, [17]). The strategy used for the proof of Theorem A and B for polynomials can be extended to the exponential family to prove the following theorems.

The first theorem is Theorem A stated for the exponential family.

Theorem 4.1 (Accessibility of periodic orbits for non-recurrent parameters). — *Let f be an exponential map with bounded postsingular set. Then any repelling periodic point is the landing point of at least one and at most finitely many dynamic rays, all of which are periodic of the same period.*

Corollary 4.2. — *For Misiurewicz parameters, the postsingular periodic orbit and hence the singular value are accessible.*

For hyperbolic, parabolic and Misiurewicz parameters Theorem 4.1 has been previously proven in [20]. The next theorem is Theorem B stated for the exponential family, with the additional property that the set of rays landing on a hyperbolic set have uniformly bounded addresses.

THEOREM 4.3 (Accessibility of hyperbolic sets). — *Let f be an exponential map with bounded postsingular set, and Λ be a hyperbolic set. Then any point in Λ is accessible; moreover, the dynamic rays landing at $x \in \Lambda$ all have uniformly bounded addresses.*

Remark 4.4. — The family of rays constructed in Theorem 4.3 form a lamination. Continuity of the family of rays on compact sets is a consequence of Lemma 3.3. Continuity up to the endpoints follows from the estimates in (4.9).

We also prove that there are only finitely many rays landing at each $x \in \Lambda$.

PROPOSITION 4.5. — *Let f be an exponential map with bounded postsingular set, and Λ be a hyperbolic set. Then there are only finitely many dynamic rays landing at each $x \in \Lambda$; more precisely there exists a constant N_0 such that for each $x \in \Lambda$ there are at most N_0 rays landing at x.*

In the case in which the postsingular set is bounded and contained in the Julia set, Rempe and van Strien ([17], Theorem 1.2) have shown that it is hyperbolic (see also [10], Theorem 3, for a different perspective). Together with Theorem 4.3, this implies accessibility of the postsingular set.

COROLLARY 4.6. — *Let f be an exponential map with bounded postsingular set. Then any point in the postsingular set is accessible.*

We will prove Theorem 4.1, Theorem 4.3 and Proposition 4.5 in Section 4.3. Like in the polynomial case, the strategy is to first prove a uniform bound on the length of fundamental domains $I_T(g_{\mathbf{a}})$ for some fixed T and some specific family of addresses, then to translate this into a uniform shrinking for fundamental domains $I_t(g_{\mathbf{a}})$ as $t \to 0$, and finally to study the local dynamics near a repelling periodic orbit. The main difficulties compared to the polynomial case are to find an analogue of Proposition 2.4, and to show that the dynamic rays obtained by pullbacks near the repelling fixed point have uniformly bounded addresses.

In the following, let $\mathcal{P}(f)$ be the postsingular set, and $\Omega := \mathbb{C} \setminus \mathcal{P}(f)$. As $\mathcal{P}(f)$ is forward invariant, Ω is backward invariant, *i.e.*, $f^{-1}(\Omega) \subset \Omega$.

PROPOSITION 4.7. — *If $\mathcal{P}(f)$ is bounded, Ω is connected.*

Proof. — As $\mathcal{P}(f)$ is bounded, there are no Siegel disks, hence either $J(f) = \mathbb{C}$ or f is parabolic or hyperbolic. In the last two cases $\mathcal{P}(f)$ is a totally disconnected set and the claim follows. If $J(f) = \mathbb{C}$, consider the connected components V_i of Ω; as $\mathcal{P}(f)$ is bounded, there is only one unbounded V_i. On the other side, by density of escaping points, each V_i contains escaping points; as dynamic rays are connected sets, each V_i has to be unbounded, hence there is a unique connected component. □

As Ω is connected, and omits at least three points because c cannot be a fixed point, it admits a well defined hyperbolic density ρ_Ω. As $\mathcal{P}(f)$ is bounded, we have

$$\lim_{|z|\to\infty} \frac{\rho_\Omega(z)}{\rho_{\mathrm{eucl}}(z)} = 0. \tag{4.1}$$

To see this, observe that, since $\mathcal{P}(f)$ is bounded, ρ_Ω is bounded from above by the hyperbolic density of the set $\Omega' := \mathbb{C}\setminus\mathbb{D}_R$, for some sufficiently large R. The latter density $\rho_{\Omega'}$ can be computed directly by using that Ω' is the image under the exponential map of the right half plane $\{z \in \mathbb{C} : \operatorname{Re} z > \log R\}$. This yields $\rho_\Omega \leqslant \frac{1}{|z|}$ as $z \to \infty$ giving (4.1).

4.2. Bounds on fundamental domains for exponentials

In this section we prove a uniform bound on the length of fundamental domains for an appropriate family of dynamic rays (Proposition 4.11). Observe first that since $\mathcal{P}(f)$ is bounded, by the asymptotic estimates in Theorem 3.1 for any ray $g_{\mathbf{s}}$ there exists T sufficiently large such that all inverse branches of f^n are well defined and univalent in a neighborhood of $g_{\mathbf{s}}(T, \infty)$.

Proposition 4.8 (Bounded fundamental domains for exponentials). *Let $g_{\widetilde{\mathbf{s}}}$ be a dynamic ray with bounded address and let C be a sufficiently large positive constant. Let*

$$\mathcal{A} := \{\mathbf{a} \in \mathcal{S} : \sigma^m(\mathbf{a}) = \widetilde{\mathbf{s}} \text{ for some } m \geqslant 0\},$$

and let $\{g_{\mathbf{a}}\}_{\mathbf{a}\in\mathcal{A}}$ be the collection of pullbacks of $g_{\widetilde{\mathbf{s}}}$. Then there exists T such that for all $t > T, \mathbf{a} \in \mathcal{A}$:

(P1) *If $\mathbf{a} = a_m ... a_2 a_1 \widetilde{\mathbf{s}}$, then $g_{\mathbf{a}}(t) = L_{a_m} \circ \cdots \circ L_{a_1} g_{\widetilde{\mathbf{s}}}(F^m(t))$;*
(P2) $\operatorname{Re} g_{\mathbf{a}}(t) > C$;
(P3) *$\ell_{\mathrm{eucl}}(I_t(g_{\mathbf{a}})) \leqslant B(t)$, with $B(t)$ independent of $\mathbf{a}$.*

For any positive constant C let us denote by $\mathbb{H}_C$ the right half plane

$$\mathbb{H}_C := \{z \in \mathbb{C} : \operatorname{Re} z > C\}.$$

Proposition 4.8 is a consequence of the following proposition.

Proposition 4.9 (Branches of the logarithm). — *Let $g_{\widetilde{\mathbf{s}}}$ be a dynamic ray with bounded address and fix a small $\epsilon > 0$. Let $C > \max(2, \operatorname{Re} c + 4, \frac{8\pi^2}{\epsilon})$, and such that $\mathcal{P}(f) \cap \mathbb{H}_C = \emptyset$. Then there exists $T > 0$ such that for any $m > 0$, for any $z = g_{\widetilde{\mathbf{s}}}(t)$ with $t > F^m(T)$ and for any finite sequence $a_m \dots a_1$ we have the following properties:*

(1_m) $\operatorname{Re} L_{a_m} \circ \cdots \circ L_{a_1}(z) - \operatorname{Re} c \geqslant \operatorname{Re} L_0^m(z) - \operatorname{Re} c - \epsilon/2 > C$

(2_m) $|L_{a_m} \circ \cdots \circ L_{a_1}(z) - c| \geqslant |L_0^m(z) - c| - \epsilon$.

Moreover, for some $C' > 0$

$$|(L_{a_m} \circ \cdots \circ L_{a_1})'(z)| \leqslant e^{\epsilon C'} |(L_0^m)'(z)|. \tag{4.2}$$

Proof. — Denote by 0^m the finite sequence formed by m zeroes, and let

$$\mathcal{A}' = \{\mathbf{a} \in \mathcal{A} : \mathbf{a} = 0^m \widetilde{\mathbf{s}},\ m \in \mathbb{N}\}.$$

As $\|\widetilde{\mathbf{s}}\|_\infty < M$ for some M, we have $\|\mathbf{a}\|_\infty < M$ for any $\mathbf{a} \in \mathcal{A}'$. It follows from (3) in Theorem 3.1 that for any ϵ there exists T_ϵ such that

$$|g_{\mathbf{a}}(t) - t| < \epsilon \tag{4.3}$$

for any $\mathbf{a} \in \mathcal{A}'$, $t > T_\epsilon$. By Remark 3.2, we have that

$$g_{\widetilde{0^m \mathbf{s}}}(t) = L_0^m g_{\widetilde{\mathbf{s}}}(F^m(t)) \quad \text{for} \quad t \geqslant 2\log(K+3) \tag{4.4}$$

and that

$$\ell_{\mathrm{eucl}}(I_t(g_{\mathbf{a}})) < B(t) \sim e^t - t \ \forall a \ \in \mathcal{A}'. \tag{4.5}$$

We first show that (1_m) implies (2_m), and then that (2_m) implies (1_{m+1}). Let $T > T_\epsilon$ be large enough. By (4.3) for all $t > T$,

$$\operatorname{Re} L_0^m(g_{\widetilde{\mathbf{s}}}(F^m(t))) > T - \epsilon > C + \operatorname{Re} c \quad \text{for all} \quad m. \tag{4.6}$$

For $m = 1$, we have that $\operatorname{Re} L_{a_1}(z) = \operatorname{Re} L_0(z)$, so (1_1) holds by (4.6).

Now let us show that (1_m) implies (2_m).

$$\begin{aligned}
|L_0^m(z) - c| &= \sqrt{|\operatorname{Re} L_0^m(z) - \operatorname{Re} c|^2 + |\operatorname{Im} L_0^m(z) - \operatorname{Im} c|^2} \\
&\leqslant (\operatorname{Re} L_0^m(z) - \operatorname{Re} c)\sqrt{1 + \frac{\epsilon}{|\operatorname{Re} L_0^m(z) - \operatorname{Re} c|}} \\
&\leqslant \operatorname{Re} L_0^m(z) - \operatorname{Re} c + \frac{\epsilon}{2} \leqslant \operatorname{Re} L_{a_m} \circ \cdots \circ L_{a_1}(z) - \operatorname{Re} c + \epsilon \\
&\leqslant |L_{a_m} \circ \cdots \circ L_{a_1}(z) - c| + \epsilon.
\end{aligned}$$

To show that (2_m) implies (1_{m+1}) for any m, observe that

$$\begin{aligned}
&\operatorname{Re} L_{a_{m+1}} \circ \cdots \circ L_{a_1}(z) = \log |L_{a_m} \circ \cdots \circ L_{a_1}(z) - c| \\
&\geqslant \log(|L_0^m(z) - c| - \epsilon) \geqslant \log |L_0^m(z) - c| - \frac{2\epsilon}{|L_0^m(z) - c|} \\
&\geqslant \log |L_0^m(z) - c| - \epsilon/2 = \operatorname{Re} L_0^{m+1}(z) - \epsilon/2 > C + \operatorname{Re} c.
\end{aligned}$$

We now check Equation (4.2). We need a preliminary observation. Let w be a point with $\operatorname{Re} w > C$ whose image $f(w)$ is in S_0. By computing the intersections of ∂S_0 with the circle of radius $e^{\operatorname{Re} w}$ centered at c (and using the fact that $|\operatorname{Im} c| \leqslant \pi$) we obtain

$$\operatorname{Re} f(w) \geqslant \operatorname{Re} c + e^{\operatorname{Re} w}/2 \geqslant (\operatorname{Re} w)^2 \geqslant C^2. \tag{4.7}$$

Since if $w := L_0^m(z)$ we have $f^j(w) \in S_0$ for $j \leqslant m-1$, it follows by induction that

$$\operatorname{Re} f^j(w) = \operatorname{Re} L_0^{m-j}(z) \geqslant (\operatorname{Re} w)^{2^j} \geqslant C^{2^j}$$

hence

$$|L_0^{m-j}(z) - c| \geqslant \operatorname{Re} L_0^{m-j}(z) - \operatorname{Re} c \geqslant C^{2^j} - \operatorname{Re} c.$$

Let $C' = 2\sum_0^\infty \frac{1}{C^{2^j} - \operatorname{Re} c} < \infty$, and let $\alpha_i := |L_0^i(z) - c|$. Then by direct computation using Property (2_m) and Taylor expansions we have that

$$\begin{aligned}
&|(L_{a_m} \circ \cdots \circ L_{a_1})'(z)| = \frac{1}{|z-c|} \frac{1}{|L_{a_1}(z) - c|} \cdots \frac{1}{|(L_{a_{m-1}} \circ \cdots \circ L_{a_1})(z) - c|} \\
&\leqslant \frac{1}{|z-c|} \frac{1}{|L_0(z) - c| - \epsilon} \cdots \frac{1}{|(L_0^{m-1}(z) - c| - \epsilon} \\
&\leqslant |(L_0^m)'(z)| \prod_{i=1}^{m-1} \frac{1}{1 - \epsilon/\alpha_i} \leqslant |(L_0^m)'(z)| e^{-\sum_{i=1}^{m-1} \log(1-\epsilon/\alpha_i)} \\
&\leqslant |(L_0^m)'(z)| e^{\epsilon C'}.
\end{aligned}$$

□

Proof of Proposition 4.8. All addresses in $\mathcal{A}$ are bounded, so by Remark 3.2, $g_{\mathbf{a}}(t)$ has address $\mathbf{a}$ for $t > 2\log(K+3)$, proving Property (P1). Take T such that Proposition 4.9 holds. From Property (1_m), $\operatorname{Re} g_{\mathbf{a}}(t) \geqslant \operatorname{Re} L_0^m(g_{\mathbf{s}}(F^m(t))) - \epsilon/2 > C + \operatorname{Re} c$ proving Property (P2).

Finally, $\ell_{\text{eucl}}(I_t(g_{\mathbf{a}})) \leqslant B(t)$ for all $\mathbf{a} \in \mathcal{A}'$ by (4.5), so $\ell_{\text{eucl}}(I_t(g_{\mathbf{a}})) \leqslant B'(t)$ for all $\mathbf{a} \in \mathcal{A}$ and for some B' by (4.2); Property (P3) follows. □

Let us now show that the length of fundamental domains $\{I_t(g_{\mathbf{a}})\}_{\mathbf{a}\in\mathcal{A}}$ shrinks as $t \to 0$ for the family of pullbacks of the ray $g_{\widetilde{\mathbf{s}}}$. The next proposition follows from classical arguments for normal families, see *e.g.* [9], Proposition 3. We include a proof for completeness.

Proposition 4.10 (Shrinking under inverse iterates). — *Let f be an exponential map, and U be a simply connected domain not intersecting the postsingular set. Let K be any compact set and let L be another compact set which is compactly contained in U and such that $L \subset J(f)$. Denote by $\{f_\lambda^{-m}\}$ the family of branches of f^{-m} such that $f_\lambda^{-m}(L) \cap K \neq \emptyset$. Then*

$$\operatorname{diam}(f_\lambda^{-m}(L)) \to 0 \quad \text{as} \quad m \to \infty$$

uniformly in λ.

Proof. — As U is simply connected, and does not intersect the postsingular set, inverse branches are well defined on U. Suppose by contradiction that there is $\epsilon > 0$, $m_k \to \infty$, and branches $f_{\lambda_k}^{-m_k}$ of f^{-m_k} such that

(1) $\operatorname{diam}_{\mathrm{eucl}}(f_{\lambda_k}^{-m_k}(L)) > \epsilon$ for any m_k, λ_k;
(2) $f_{\lambda_k}^{-m_k}(L) \cap K \neq \emptyset$ for any m_k, λ_k.

By normality of inverse branches, there is a subsequence converging to a univalent function ϕ, which is non-constant by (1). By (2) there is a sequence of points $\{x_k\} \in L$ such that $f_{\lambda_k}^{-m_k}(x_k) \in K$, and by compactness of K, the $f_{\lambda_k}^{-m_k}(x_k)$ accumulate on some point $y \in K$. As ϕ is not constant, there is a neighborhood V of y and infinitely many m_k such that $f^{m_k}(V) \subset U$, contradicting the fact that $y \in J(f)$. □

Proposition 4.11 (Fundamental domains shrinking for exponentials). *Let $g_{\widetilde{\mathbf{s}}}$ be a dynamic ray with bounded address, $\{g_{\mathbf{a}}\}_{\mathbf{a}\in\mathcal{A}}$ be its family of pullbacks as defined in Proposition 4.8. Given an $\epsilon > 0$ and a compact set K there exists $t_\epsilon = t_\epsilon(K)$ such that $\ell_{\mathrm{eucl}}(I_t(g_{\mathbf{a}})) < \epsilon$ whenever $t < t_\epsilon$ and $I_t(g_{\mathbf{a}}) \cap K \neq \emptyset$.*

Proof. — Let $\Omega = \mathbb{C} \setminus \mathcal{P}(f)$, and consider the hyperbolic metric on Ω. Let $C > 0$ such that Proposition 4.8 holds and such that $\mathcal{P}(f) \cap \mathbb{H}_C = \emptyset$. Let T be as in Proposition 4.8 so that $I_t(g_{\mathbf{a}}) \subset \mathbb{H}_C$ for any $\mathbf{a} \in \mathcal{A}$, $t > T$. Inverse branches of f^n are well defined on the family $\{I_T(g_{\mathbf{a}})\}$ because it is contained in a right half plane not intersecting the postsingular set.

We first show that, for fundamental domains which are pullbacks of the family $\{I_T(g_{\mathbf{a}})\}$ and which intersect K, in order to have small Euclidean length it is sufficient to have small hyperbolic length with respect to the hyperbolic metric on Ω. The arcs $\{g_{\mathbf{a}}(T, F^2(T))\}$ have uniformly

bounded Euclidean length by $(P3)$ in Proposition 4.8, so they have uniformly bounded hyperbolic length because they are contained in $\mathbb{H}_C$ and $\mathbb{H}_C \cap \mathcal{P}(f) = \emptyset$ (together with the asymptotic estimates in (4.1)). Then by the Schwarz Lemma the hyperbolic length of the arcs in the families $\{I_t(g_{\mathbf{a}})\}_{t<T}$ is bounded uniformly as well. Since K is compact all fundamental domains which intersect K have also uniformly bounded Euclidean length, so there exists a compact set $K' \supset K$ such that $I_t(g_{\mathbf{a}}) \subset K'$ whenever $I_t(g_{\mathbf{a}}) \cap K \neq \emptyset$. By compactness of K' there exists ϵ' such that $\ell_{\mathrm{eucl}}(\gamma) < \epsilon$ for any curve $\gamma \subset (K' \cap \Omega)$ with $\ell_\Omega(\gamma) < \epsilon'$.

So it remains to show that there exists t_ϵ such that, for all $\mathbf{a} \in \mathcal{A}$ and $t < t_\epsilon$, $\ell_\Omega(I_t(g_{\mathbf{a}})) < \epsilon'$ whenever $I_t(g_{\mathbf{a}}) \cap K \neq \emptyset$. Since the length $\ell_{\mathrm{eucl}}(I_T(g_{\mathbf{a}}))$ is bounded for every $\mathbf{a} \in \mathcal{A}$ by Proposition 4.8, and the family $\{I_T(g_{\mathbf{a}})\}$ is contained in $\mathbb{H}_C$, by the asymptotic estimates in (4.1) there exists a closed disk D of sufficiently large radius such that $\ell_\Omega(I_T(g_{\mathbf{a}})) < \epsilon'$ for any $\mathbf{a} \in \mathcal{A}$ with $I_T(g_{\mathbf{a}}) \not\subset D$. For the pullbacks of any such $\mathbf{a}$ the claim then follows by the Schwarz Lemma (and in fact, for such $\mathbf{a}$ $t_\epsilon = T$). Let us now consider any $\mathbf{a} \in \mathcal{A}$ such that $I_T(g_{\mathbf{a}}) \subset D \cap \mathbb{H}_C$. By Proposition 4.10 there is n_ϵ such that $\mathrm{diam}_{\mathrm{eucl}}\, f^{-n}(I_T(g_{\mathbf{a}})) < \epsilon$ for any $n > n_\epsilon$ and any branch such that $f^{-n}(I_T(g_{\mathbf{a}}) \cap K \neq \emptyset$; the claim then holds for any $t < t_\epsilon = F^{-n_\epsilon}(T)$. $\square$

4.3. Proof of Accessibility Theorems

In this Section we prove Theorems 4.1 and 4.3 as well as Proposition 4.5. To ease the following proofs we introduce the proposition below.

Proposition 4.12. — *Let K be a compact set, $g_{\mathbf{s}_0}$ be a dynamic ray with bounded address, $\mathcal{A}$ as in Proposition 4.8, and $\mathcal{A}'$ be any subset of $\mathcal{A}$ such that for some $t_0 > 0$ and for all $\mathbf{a} \in \mathcal{A}'$ the following properties hold:*

(1) *$g_{\mathbf{a}}(t_0) \in K$;*
(2) *If $\mathbf{a} \neq \mathbf{s}_0$, then $\sigma\mathbf{a} \in \mathcal{A}'$.*

Then there exists $M > 0$ such that for all $\mathbf{a} \in \mathcal{A}'$, $\|\mathbf{a}\|_\infty < M$. The constant M depends on K, t_0 and $\mathbf{s}_0$.

Proof. — Let T be as in Proposition 4.8, and let N be such that $F^N(t_0) > T$. The set $f^N(K)$ is compact, so there exists $M' > \|\mathbf{s}_0\|_\infty$ such that $|\operatorname{Im} z| < 2\pi M'$ for all $z \in f^N(K)$. For any $\mathbf{a} \in \mathcal{A}'$, $g_{\mathbf{a}}(t_0) \in K$, so $g_{\sigma^N\mathbf{a}}(F^N(t_0)) \in f^N(K)$ and by Property (P1) in Proposition 4.8, the first entry of $\sigma^N\mathbf{a}$ is bounded by M'. From (2), we get that $\|\sigma^N\mathbf{a}\|_\infty < M'$ for any $\mathbf{a} \in \mathcal{A}$.

We now show that the points $g_{\mathbf{a}}(F^N(t_0))$ are also contained in finitely many of the strips S_n; then, by Property (P1) in Proposition 4.8, the first entry of $\mathbf{a}$ is bounded by some M for all $\mathbf{a} \in \mathcal{A}$, and the claim follows by (2). Using (3.2) it follows that $\ell_{\mathrm{eucl}}(g_{\sigma^N \mathbf{a}}(F^N(t_0), F^{2N}(t_0))) \leqslant B$ for some $B > 0$, and for all $\mathbf{a} \in \mathcal{A}'$. Since $|(f^N)'|$ is bounded on K because K is compact, there exists $B' > 0$ such that $\ell_{\mathrm{eucl}}(g_{\mathbf{a}}(t_0, F^N(t_0))) \leqslant B'$ for all $\mathbf{a} \in \mathcal{A}'$ so that the points $g_{\mathbf{a}}(F^N(t_0))$ are also contained in finitely many of the strips S_n (since they are a finite distance away from the compact set K). □

We now prove Theorem 4.1.

Proof of Theorem 4.1. — Let us first assume that the repelling periodic point under consideration is a fixed point α. Let U, U', ψ and ϵ be as in the proof of Theorem 2.5. Let $g_{\tilde{\mathbf{s}}}$ be a dynamic ray with bounded address, $\{g_{\mathbf{a}}\}_{\mathbf{a}\in\mathcal{A}}$ be its family of pullbacks and C, T be such that Proposition 4.8 holds. Let $K := \overline{U'}$ and t_ϵ be given by Proposition 4.11 so that, for any $t < t_\epsilon$ and for any $I_t(g_{\mathbf{a}})$ intersecting $\overline{U'}$, we have that $\ell_{\mathrm{eucl}}(I_t(g_{\mathbf{a}})) < \epsilon$.

Consider a dynamic ray $g_{\mathbf{s}_0}$ such that $g_{\mathbf{s}_0}(t_0) \in U$ for some $t_0 < t_\epsilon$. To show that such a point exists, observe that since $U \cap J(f) \neq \emptyset$ there exists a sufficiently large N_0 with $f^{N_0}(U) \cap I_{t_\epsilon}(g_{\tilde{\mathbf{s}}}) \neq \emptyset$, so there exists $y \in I_{t_\epsilon}(g_{\tilde{\mathbf{s}}})$ such that $f^{-N_0}(y) \in U$ giving the desired $g_{\mathbf{s}_0}(t_0)$.

Now we can use the inductive construction from Theorem 2.5 to obtain a sequence of dynamic rays $g_{\mathbf{s}_n}$ such that the arcs $\gamma_n := g_{\mathbf{s}_n}(t_n, F(t_0))$ are well defined and satisfy Properties 1-3 in the proof of Theorem 2.5. By Proposition 4.12, $\|\mathbf{s}_n\|_\infty$ is uniformly bounded, so there exists at least one limiting address $\mathbf{s}$ for the addresses $\mathbf{s}_n$; by Lemma 2.2 and Lemma 3.3, the ray $g_{\mathbf{s}}$ lands at α.

In the case of a repelling periodic point of period $p > 1$, the construction can be repeated using as fundamental domains the arcs between potential t_0 and $F^p(t_0)$.

The proof of periodicity of the landing rays is the same as the proof for polynomials, using the fact that the family of addresses $\{\mathbf{s}_n\}$ is uniformly bounded hence Lemma 2.8 holds. □

The proof of accessibility of hyperbolic sets for an exponential map in the exponential case (Theorem 4.3) is a bit more complicated than in the polynomial case, essentially because Proposition 2.4 holds for all fundamental domains while Proposition 4.11 holds only for the family of pullbacks of a given ray with bounded address. So, when constructing a ray landing at a point x_0, we will need to consider fundamental domains all of which belong to the family of pullbacks of some initial ray with bounded address. To do

this, we will consider a subsequence of the iterates of x_0 all of which have near by a fundamental domain $I_{t_0}(g_{\mathbf{s}_0})$(with sufficiently small potential t_0) of some ray $g_{\mathbf{s}_0}$ with bounded address; then, we will pull back $I_{t_0}(g_{\mathbf{s}_0})$ and use the inductive construction from Theorem 2.5 in order to obtain arcs of rays near x_0 satisfying (4.8) and (4.9).

Proof of Theorem 4.3. — Up to taking an iterate of f, we can assume that there is a δ-neighborhood U of Λ such that $|f'(x)| > \eta > 1$ for all $x \in U$. Let $g_{\tilde{\mathbf{s}}}$ be a dynamic ray with bounded address and $\{g_{\mathbf{a}}\}_{\mathbf{a}\in\mathcal{A}}$ be its the family of pullbacks. For $\epsilon := \delta - \delta/\eta$, let t_ϵ be such that $\ell_{\mathrm{eucl}}(I_t(g_{\mathbf{a}})) < \epsilon$ for $\mathbf{a} \in \mathcal{A}$ and $t < t_\epsilon$ whenever $I_t(g_{\mathbf{a}}) \cap \overline{U} \neq \emptyset$ (see Proposition 4.11).

Let $t_0 < t_\epsilon$ be such that for any $x \in \Lambda$, $B_{\delta/\eta}(x)$ contains $g_{\mathbf{a}}(t_0)$ for some $\mathbf{a} \in \mathcal{A}$ depending on x. To show that such a t_0 exists, consider a finite covering $\mathcal{D}$ of Λ by balls of radius $\frac{\delta}{3\eta}$. For any $D \in \mathcal{D}$, there is N_D such that $f^{N_D}(D) \supset g_{\tilde{\mathbf{s}}}(0, t_\epsilon)$, hence for each D there is some $\mathbf{a} \in \mathcal{A}$ such that $g_{\mathbf{a}}(0, F^{-N_D}(t_\epsilon)) \subset D$. By letting $N = \max N_D$, we have that $g_{\mathbf{a}}(0, F^{-N}t_\epsilon) \subset D$ for any D, hence that each $B_{\delta/\eta}(x)$ contains $g_{\mathbf{a}}(F^{-N}(t_\epsilon))$ for some $\mathbf{a} \in \mathcal{A}$.

Now fix x_0 in Λ, and let us construct a dynamic ray landing at x_0. Let $x_n := f^n(x_0)$, $B'_n := B_\delta(x_n)$, $B_n := B_{\delta/\eta}(x_n)$. For each n there is a branch ψ of f^{-1} such that $\psi(B'_n) \subset B_{n-1}$. Take a disk $D \in \mathcal{D}$ containing infinitely many x_n, say $\{x_n\}_{n\in\mathcal{N}} \in D$ for some infinite set $\mathcal{N} \subset \mathbb{N}$. Observe that for any such n, $D \subset B_n$. Let $\mathbf{s}_0 \in \mathcal{A}$ be such that $g_{\mathbf{s}_0}(t_0) \in D$; by Proposition 4.11, $\ell_{\mathrm{eucl}}(I_{t_0}(g_{\mathbf{s}_0})) < \epsilon$, so for all $n \in \mathcal{N}$ we have that $I_{t_0}(g_{\mathbf{s}_0}) \subset B'_n$.

For $n \in \mathcal{N}$, let ψ^n be the branch of f^{-n} mapping x_n to x_0; ψ^n can be extended analytically to B'_n and hence to $I_{t_0}(g_{\mathbf{s}_0})$. Let $g_{\mathbf{s}_n}$ be the sequence of rays containing $\psi^n(I_{t_0}(g_{\mathbf{s}_0}))$. Let $t_n = F^{-n}(t_0)$. Following the inductive construction from the proof of Theorem 2.5, the arcs $g_{\mathbf{s}_n}(t_n, F(t_0))$ satisfy the following properties:

$$g_{\mathbf{s}_n}(t_n, F(t_0)) \subset B'_0 \quad \text{and} \tag{4.8}$$

$$I_{t_m}(g_{\mathbf{s}_n}) \subset B\left(x_0, \frac{\delta}{\eta^m}\right) \quad \text{for all} \quad n > m,\ n \in \mathcal{N}. \tag{4.9}$$

Consider now the family of addresses $\mathbf{s}_n$ constructed for x_0 and the set

$$\widetilde{\mathcal{A}}_{x_0} = \left\{\mathbf{a} \in \mathcal{S} : \mathbf{a} = \sigma^j \mathbf{s}_n \ \text{ for some } j \leqslant n \ \text{ and } \ n \in \mathcal{N}\right\}.$$

By definition, if $\mathbf{a} \in \widetilde{\mathcal{A}}_{x_0}$ then $\sigma\mathbf{a} \in \widetilde{\mathcal{A}}_{x_0}$ unless $\mathbf{a} = \mathbf{s}_0$, and for each $\mathbf{a} \in \widetilde{\mathcal{A}}_{x_0}$, $g_{\mathbf{a}}(t_0) \in \overline{U}$. It follows by Proposition 4.12 that $\|\mathbf{a}\|_\infty < M$ for all $\mathbf{a} \in \widetilde{\mathcal{A}}_{x_0}$. In particular, $\|\mathbf{s}_n\|_\infty < M$, and there is a limiting address $\mathbf{s}$

such that $g_{\mathbf{s}}$ lands at x_0 by Lemma 2.2. By finiteness of $\mathcal{D}$, there exists $M' > 0$ such that the addresses of all rays coming from the construction and landing at $x \in \Lambda$ are bounded by M'. Finally, the addresses of any two rays landing together cannot differ by more than one in any entry (see Remark 3.4), so the address of any dynamic ray (not necessarily coming from this construction) landing at any $x \in \Lambda$ is bounded by $M' + 1$. $\square$

Let us conclude by showing that also in the exponential case, there are only finitely many rays landing at each point in a hyperbolic set, thus proving Proposition 4.5. The proof is very similar to the polynomial case.

Proof of Proposition 4.5. — For $x \in \Lambda$, let $\mathcal{A}_x$ be the set of addresses of the rays landing at x. By Theorem 4.3, each $\mathcal{A}_x$ is non-empty. Since f is locally a homeomorphism near x, the set $\mathcal{A}_x$ is mapped bijectively to the set $\mathcal{A}_{f(x)}$ by the shift map σ, so there is a well defined inverse $\sigma^{-1} : \mathcal{A}_{f(x)} \to \mathcal{A}_x$. By Theorem 4.3, the norm of the addresses belonging to the set $\mathcal{A}_x$ is bounded by a constant M, so $\bigcup_{x\in\Lambda} \mathcal{A}_x \subset \Sigma_D \subset \mathbb{R}/\mathbb{Z}$ for $D = 2M + 1$ preserving the dynamics.

Assume first that σ^{-1} is uniformly continuous in the sense of (2.6). Then the proof is the same as in the polynomial case by considering a finite cover of Σ_D by ϵ-balls.

Let us now prove (2.6) in the exponential setting. Assume by contradiction that there is a sequence of points $x_n \in \Lambda$, and two sequences of addresses $\mathbf{a}_n, \mathbf{a}_n{}' \in \mathcal{A}_{f(x_n)}$ such that $|\mathbf{a}_n - \mathbf{a}_n{}'|_D \to 0$, but $|\sigma^{-1}\mathbf{a}_n - \sigma^{-1}\mathbf{a}_n{}'|_D \to \delta \neq 0$. Let $\mathbf{s}_n = \sigma^{-1}\mathbf{a}_n$, $\mathbf{s}_n{}' = \sigma^{-1}\mathbf{a}_n{}'$, and assume for definiteness that $\mathbf{s}'_n < \mathbf{s}_n$. Since $|\mathbf{a}_n - \mathbf{a}_n{}'|_D \to 0$, the sequences $\mathbf{a}_n$ and $\mathbf{a}_n{}'$ converge to the same address, say $\mathbf{a}$. Since $\{\sigma^{-1}(\mathbf{a})\} = \{j\mathbf{a} \text{ with } j \in \mathbb{Z}\}$, we have that $\mathbf{s}_n$ and $\mathbf{s}_n{}'$ converge to the same sequence up to the first entry, so that $\delta = k/D$ for some $k \in \mathbb{N}$. Since $g_{\mathbf{s}_n}$ and $g_{\mathbf{s}_n{}'}$ land together, by Remark 3.4 we have that $k = 1$. This means that the dynamic rays of addresses $\mathbf{s}_n = j\mathbf{a}_n$ and $\mathbf{s}_n{}' = (j+1)\mathbf{a}'_n$ belong to the same $\mathcal{A}_{x_n}$, *i.e.*, land together for all $j \in \mathbb{Z}$.

These pairs of rays divide $\mathbb{C}$ into infinitely many regions exactly one of which contains a left half plane, and which we call V_n (see Figure 4.1). Let $Q = \bigcap \overline{V_n}$. Since for each $n \in \mathbb{N}$ there is $x_n \in \Lambda \cap \overline{V_n}$ and Λ is compact we have that $Q \cap \Lambda \neq \emptyset$ and contains at least a point $y \in \mathbb{C}$. Observe that by (3) in Theorem 3.1 and since rays do not intersect, a dynamic ray of address $\mathbf{s}$ is contained in V_n if and only if $j\mathbf{a}_n{}' < \mathbf{s} < j\mathbf{a}_n$ for some $j \in \mathbb{Z}$. Taking the limit as $n \to \infty$, and recalling that $\lim \mathbf{a}_n = \lim \mathbf{a}'_n = \mathbf{a}$, we have that no dynamic rays can intersect Q except for the dynamic rays of address $j\mathbf{a}$, $j \in \mathbb{N}$. However, by Corollary 4.6 the singular value is accessible

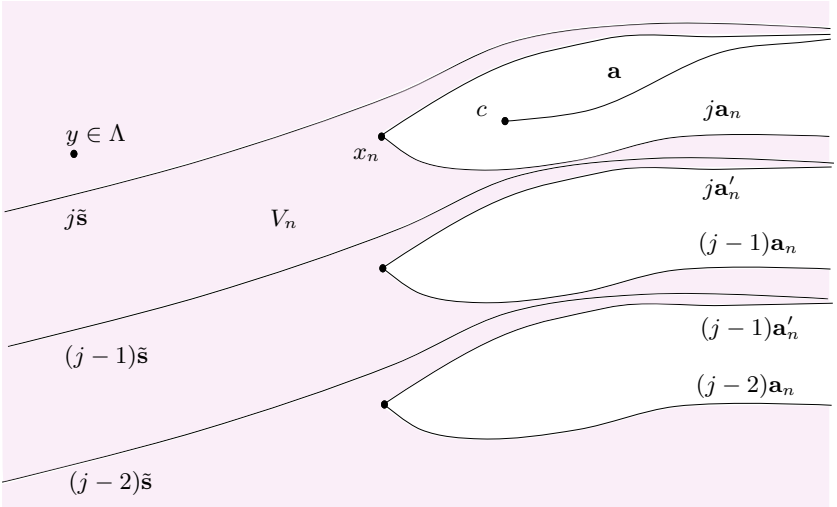

Figure 4.1. Illustration to the proof of Proposition 4.5; the true picture is invariant under translation by $2\pi i$. The rays are labeled by their addresses. The region V_n is shaded.

by some dynamic ray $g_{\widetilde{\mathbf{s}}}$, whose countably many preimages $g_{j\widetilde{\mathbf{s}}}$ intersect any left half plane: it follows that $\widetilde{\mathbf{s}} = \mathbf{a}$ and that none of the dynamic rays of address $j\mathbf{a}$ is landing. This contradicts the fact that $\Lambda \cap Q \neq \emptyset$ and that points in Λ are accessible by Theorem 4.3. $\square$

4.4. A remark about parabolic wakes

Theorem 4.1 implies that non-recurrent parameters with bounded postsingular set always belong to parabolic wakes (see [13], Propositions 4 and 5, for the definition of parabolic wakes and the relation between parabolic wakes and landing of rays in the dynamical plane; see also [16]).

There is a combinatorial proof of this fact (see *e.g.* in [18]) which can be adapted to the exponential case once Theorem 4.1 is known.

Corollary 4.13 (Corollary of Theorem 4.1). — *A non-recurrent parameter c with bounded postsingular set for the exponential family is contained in a parabolic wake attached to the boundary of the period one hyperbolic component W_0.*

Sketch of proof. — Let $c_0 \in W_0$ be a hyperbolic parameter with an attracting fixed point $\alpha(c_0)$. Observe that for any fixed address $\mathbf{s}$, the ray

$g_{\mathbf{s}}^{c_0}$ lands at a fixed point $z_{\mathbf{s}}(c_0)$. Let γ be a curve joining c to c_0 such that all $z_{\mathbf{s}}(c_0)$ can be continued analytically together with their landing rays. Call $\alpha(c)$ the analytic continuation along γ of the attracting fixed point $\alpha(c_0)$, $z_{\mathbf{s}}(c)$ the analytic continuation of $z_{\mathbf{s}}(c_0)$. Suppose that c does not belong to any parabolic wake attached to W_0. By Theorem A, $\alpha(c)$ is the landing point of at least one periodic ray, which is necessarily fixed (otherwise, there would be at least two periodic rays landing at $\alpha(c)$ determining a parabolic wake). But this gives a contradiction, because for any fixed address $\mathbf{s}$ the fixed ray $g_{\mathbf{s}}^{c}$ lands at the point $z_{\mathbf{s}}(c) \neq \alpha(c)$. $\square$

BIBLIOGRAPHY

[1] I. N. Baker & P. J. Rippon, "Iteration of exponential functions", *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9** (1984), p. 49-77.

[2] A. M. Benini, "Expansivity properties and rigidity for non-recurrent exponential maps", arXiv:1210.1353 [math.DS].

[3] C. Bodelón, R. L. Devaney, M. Hayes, G. Roberts, L. R. Goldberg & J. H. Hubbard, "Hairs for the complex exponential family", *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **9** (1999), no. 8, p. 1517-1534.

[4] R. L. Devaney & M. Krych, "Dynamics of exp(z)", *Ergodic Theory Dynam. Systems* **4** (1984), no. 1, p. 35-52.

[5] A. È. Erëmenko & G. M. Levin, "Periodic points of polynomials", *Ukrain. Mat. Zh.* **41** (1989), no. 11, p. 1467-1471, 1581.

[6] A. È. Erëmenko & M. Y. Lyubich, "Dynamical properties of some classes of entire functions", *Ann. Inst. Fourier (Grenoble)* **42** (1992), no. 4, p. 989-1020.

[7] M. Förster & D. Schleicher, "Parameter rays in the space of exponential maps", *Ergodic Theory Dynam. Systems* **29** (2009), no. 2, p. 515-544.

[8] J. H. Hubbard, "Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz", in *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, Publish or Perish, Houston, TX, 1993, p. 467-511.

[9] M. J. Ljubich, "Entropy properties of rational endomorphisms of the Riemann sphere", *Ergodic Theory Dynam. Systems* **3** (1983), no. 3, p. 351-385.

[10] P. Makienko & G. Sienra, "Poincaré series and instability of exponential maps", *Bol. Soc. Mat. Mexicana (3)* **12** (2006), no. 2, p. 213-228.

[11] J. Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006, viii+304 pages.

[12] F. Przytycki, "Accessibility of typical points for invariant measures of positive Lyapunov exponents for iterations of holomorphic maps", *Fund. Math.* **144** (1994), no. 3, p. 259-278.

[13] L. Rempe, "A landing theorem for periodic rays of exponential maps", *Proc. Amer. Math. Soc.* **134** (2006), no. 9, p. 2639-2648 (electronic).

[14] ———, "Topological dynamics of exponential maps on their escaping sets", *Ergodic Theory Dynam. Systems* **26** (2006), no. 6, p. 1939-1975.

[15] ———, "On nonlanding dynamic rays of exponential maps", *Ann. Acad. Sci. Fenn. Math.* **32** (2007), no. 2, p. 353-369.

[16] L. Rempe & D. Schleicher, "Combinatorics of bifurcations in exponential parameter space", in *Transcendental dynamics and complex analysis*, London Math. Soc. Lecture Note Ser., vol. 348, Cambridge Univ. Press, Cambridge, 2008, p. 317-370.

[17] L. Rempe & S. Van Strien, "Absence of line fields and Mañé's theorem for non-recurrent transcendental functions", *Trans. Amer. Math. Soc.* **363** (2011), no. 1, p. 203-228.

[18] D. Schleicher, "On fibers and local connectivity of Mandelbrot and Multibrot sets", in *Fractal geometry and applications: a jubilee of Benoît Mandelbrot. Part 1*, Proc. Sympos. Pure Math., vol. 72, Amer. Math. Soc., Providence, RI, 2004, p. 477-517.

[19] D. Schleicher & J. Zimmer, "Escaping points of exponential maps", *J. London Math. Soc. (2)* **67** (2003), no. 2, p. 380-400.

[20] ———, "Periodic points and dynamic rays of exponential maps", *Ann. Acad. Sci. Fenn. Math.* **28** (2003), no. 2, p. 327-354.

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