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SMALL SUBSETS OF FINITE ABELIAN GROUPS

par Robert KAUFMAN

Varopoulos [2], in constructing certain sets in infinite compact abelian groups, made special use of various properties of finite groups. In this note a method is given by which many of the results of [2] can be recovered in a uniform manner. To explain the utility of the theorem stated below, let $G = \prod_{i=1}^{\infty} G_i$ be the product of infinitely many finite abelian groups, μ_i a probability measure on G_i ($1 \leq i < \infty$). As usual, the Fourier-Stieltjes transform is defined for a measure ν as

$$\hat{\nu}(x) = \int_G \overline{x(t)} \nu(dt), \quad x \in \hat{G}.$$

Then $\hat{\mu}(x) \longrightarrow 0$ as $x \longrightarrow \infty$ in \hat{G} if and only if

$$\max \{ |\hat{\mu}_i(x_i)| : x_i \in \hat{G}_i, x_i \neq 1 \}$$

converges to 0 as $i \longrightarrow \infty$.

THEOREM. — *For each $\varepsilon > 0$ there is an M such that any finite abelian group G of order $> M$ contains a subset S with the properties*

- 1) $\log |S| \leq \varepsilon \log |G|$
- 2) $|\sum_{s \in S} x(s)| < \varepsilon |S|$ for any character $x \neq 1$ of G .

(Here $|A|$ means the number of elements of a set A).

Observe that by 2) S generates G , and by 1) that the r -fold sum $\pm S \pm \dots \pm S$ contains at most $2^r |S|^r$ elements ; this is the kind of estimate needed often in [2].

In the proof it is necessary to decompose finite subgroups of the circle T , the decomposition depending upon a positive integer K . If H is such a subgroup and H has order $r \leq K^2$, H is partitioned into singletons. Otherwise, let ω be an element of $H - \{1\}$ for which the Euclidean distance $|\omega - 1|$ is smallest. Set

$$H_i = \{\omega^j : (i-1)rK^{-1} \leq j < irK^{-1}\}, 1 \leq i < K,$$

$$H_K = \{\omega^j : rK^{-1}(K-1) \leq j < r\}.$$

Note that $rK^{-1} \leq |H_i| + 1 \leq rK^{-1} + 1$.

Each character $x \neq 1$ maps G onto a finite subgroup H of T , decomposed into subsets depending on some K chosen in advance. Suppose that for some $p > 0$ and every H_i , $1 \leq i \leq K^2$, depending on any $x \neq 1$,

$$p \left(1 - \frac{1}{2} \varepsilon\right) |x^{-1}(H_i)| \leq |S \cap x^{-1}(H_i)| \leq p \left(1 + \frac{1}{2} \varepsilon\right) |x^{-1}(H_i)|.$$

Then requirement 2) is met for large enough K . Because $|S| = O(p) \cdot |G|$, requirement 1) for large $|G|$ follows from

$$\log p = \left(\frac{1}{2} \varepsilon - 1\right) \log |G|.$$

S is chosen as a "random" element of 2^G , each element being chosen with probability p . More precisely, let $\{X_g : g \in G\}$ be a set of $|G|$ independent random variables indexed by G , and

$$P\{X_g = 1\} = p, P\{X_g = 0\} = 1 - p, g \in G.$$

Of course $g \in S$ means $X_g = 1$ and so

$$|S \cap x^{-1}(H_i)| = \sum_{g \in x^{-1}(H_i)} X_g.$$

All of this can be stated without reference to any special properties of G . The sums in question are, in number, $\leq K^2 |G|$, and the number of independent random variables in each sum is $\geq K^{-2} |G|$. Write

$Y = \sum_1^N X_j$, for any of these sums ; it is enough to show that for all such Y ,

$$P\left\{|Y - pN| \geq \frac{1}{2} \varepsilon pN\right\} = o(|G|)$$

as $|G| \longrightarrow \infty$.

Writing \bar{E} for expectation, for any number λ

$$\bar{E}(e^{\lambda Y}) = \bar{E}^N(e^{\lambda X_1}) = (1 + p(e^\lambda - 1))^N.$$

If

$$\lambda = \log\left(1 + \frac{1}{2} \varepsilon\right) \text{ and } Y \geq \left(1 + \frac{1}{2} \varepsilon\right) pN, e^{\lambda Y} \geq \left(1 + \frac{1}{2} \varepsilon\right)^{(1 + \frac{1}{2} \varepsilon) pN},$$

so that

$$P\left\{Y \geq \left(1 + \frac{1}{2} \varepsilon\right) pN\right\} \leq \left(1 + \frac{1}{2} \varepsilon p\right)^N / \left(1 + \frac{1}{2} \varepsilon\right)^{(1 + \frac{1}{2} \varepsilon) pN},$$

$$\begin{aligned} N^{-1} \log P\left\{Y \geq \left(1 + \frac{1}{2} \varepsilon\right) pN\right\} &\leq \log\left(1 + \frac{1}{2} \varepsilon p\right) \\ &\quad - p\left(1 + \frac{1}{2} \varepsilon\right) \log\left(1 + \frac{1}{2} \varepsilon\right). \end{aligned}$$

$$\text{Now } \log\left(1 + \frac{1}{2} \varepsilon p\right) - p\left(1 + \frac{1}{2} \varepsilon\right) \log\left(1 + \frac{1}{2} \varepsilon\right) =$$

$$0(p^2) + \frac{1}{2} \varepsilon p - p\left(1 + \frac{1}{2} \varepsilon\right) \log\left(1 + \frac{1}{2} \varepsilon\right),$$

$$\text{while } s < (1 + s) \log(1 + s) \text{ for } s > 0.$$

$$\text{Hence } P\left\{Y \geq \left(1 + \frac{1}{2} \varepsilon\right) pN\right\} \leq c^{pN}$$

for some $c < 1$.

This is clearly $o(|G|)$, and the magnitude of $P\left\{Y \leq \left(1 - \frac{1}{2} \varepsilon\right) pN\right\}$ is subject to entirely similar estimates. The proof is complete.

We indicate briefly how the argument can be applied to the a -adic integers, defined by Hewitt and Ross ([1], pp. 108-111). The analogue of the first result is as follows.

For each $\varepsilon > 0$ there is an M such that for any numbers

$$a_1 > 1, \dots, a_N > 1 \quad \text{with} \quad a_1 \dots a_N > M,$$

there is a subset S of the set $\{(u_1, \dots, u_N), 0 \leq u_i < a_i\}$ with the properties

$$1') \quad \log |S| \leq \varepsilon \log(a_1 \dots a_N).$$

$$2') \quad \text{For any complex number } w \neq 1, \text{ but } w^{a_1 a_2 \dots a_N} = 1,$$

$$\left| \sum_{s \in S} w^{u_1 + a_1 u_2 + \dots + a_1 \dots a_{N-1} u_N} \right| < \varepsilon |S|.$$

In fact, the collection of all possible exponents

$$\{u_1 + a_1 u_2 + \dots + a_1 \dots a_{N-1} u_N : 0 \leq u_i < a_i\}$$

is a complete residue system modulo $a_1 a_2 \dots a_N$, so the previous arguments are valid.

BIBLIOGRAPHIE

- [1] E. HEWITT and K.A. ROSS, *Abstract Harmonic Analysis I*, (1963).
- [2] N. Th. VAROPOULOS, Sets of multiplicity in locally compact abelian groups, *Ann. Inst. Fourier, Grenoble*, XVI (1966), 123-158.

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