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Niels LAURITZEN & Jesper Funch THOMSEN

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# MAXIMAL COMPATIBLE SPLITTING AND DIAGONALS OF KEMPF VARIETIES

**by Niels LAURITZEN & Jesper Funch THOMSEN**

Abstract. — Lakshmibai, Mehta and Parameswaran (LMP) introduced the notion of maximal multiplicity vanishing in Frobenius splitting. In this paper we define the algebraic analogue of this concept and construct a Frobenius splitting vanishing with maximal multiplicity on the diagonal of the full flag variety. Our splitting induces a diagonal Frobenius splitting of maximal multiplicity for a special class of smooth Schubert varieties first considered by Kempf. Consequences are Frobenius splitting of tangent bundles, of blow-ups along the diagonal in flag varieties along with the LMP and Wahl conjectures in positive characteristic for the special linear group.

Résumé. — Lakshmibai, Mehta et Parameswaran (LMP) ont introduit la notion de multiplicité maximale dans le scindage de Frobenius.

Dans cet article, nous définissons l'analogue algébrique de cette notion et nous construisons un scindage de Frobenius avec multiplicité maximale le long de la diagonale de la variété des drapeaux complets.

Notre scindage induit aussi un scindage diagonal avec multiplicité maximale pour une classe particulière de variétés de Schubert lisses introduite par Kempf.

Comme conséquences, nous obtenons des scindages de Frobenius des fibrés tangents et des éclatements le long des diagonales dans les variétés de drapeaux, ainsi que les conjectures de LMP et de Wahl en caractéristique positive pour le groupe spécial linéaire.

## 1. Introduction

In [11], Lakshmibai, Mehta and Parameswaran introduced the notion of multiplicities of Frobenius splittings: if $X$ is a smooth projective algebraic variety over an algebraically closed field $k$ of positive characteristic $p$, duality for the Frobenius morphism identifies Frobenius splittings with certain sections of the $(p-1)$-th power of the anticanonical line bundle $\omega_X^{-1}$ on $X$.

*Keywords:* Special linear group, Schubert variety, Frobenius splitting, maximal multiplicity, Wahl's conjecture.
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If $Y \subseteq X$ is a compatibly split smooth subvariety of codimension $d$ under the section $s$ of $\omega_X^{1-p}$, then $s$ vanishes with multiplicity $\leqslant (p-1)d$ on $Y$. The splitting $s$ is said to split $Y$ compatibly with maximal multiplicity if $s$ vanishes with multiplicity $(p-1)d$ on $Y$ (cf. §2.3 of this paper for an equivalent algebraic notion). A Frobenius splitting vanishing with maximal multiplicity on $Y$ lifts to a Frobenius splitting of the blow-up $\mathrm{Bl}_Y(X)$ splitting the exceptional divisor compatibly.

Let $X = G/P$, where $G$ is a semisimple linear algebraic group and $P \subset G$ a parabolic subgroup. In a beautiful geometric argument Lakshmibai, Mehta and Parameswaran proved that a Frobenius splitting of the blow-up $\mathrm{Bl}_\Delta(X \times X)$ compatibly splitting the exceptional divisor implies Wahl's conjecture in positive characteristic. They conjectured the existence of a Frobenius splitting of $X \times X$ vanishing with maximal multiplicity on the diagonal $\Delta$ (we refer to this as the LMP conjecture, cf. §2.4 in [11] and §2.C in [2]).

Wahl's conjecture predicts that the (generalized) Gaussian map (cf. [16])

$$H^0(X \times X, \mathcal{I}_\Delta \otimes p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2) \to H^0(X, \Omega_X^1 \otimes \mathcal{L}_1 \otimes \mathcal{L}_2) \tag{1.1}$$

is surjective for $\mathcal{L}_1$ and $\mathcal{L}_2$ ample line bundles on $X$. This conjecture was proved by Kumar [8] for complex semisimple groups using detailed information on the decomposition of tensor products. In positive characteristic the conjecture has been proved for Grassmannians by Mehta and Parameswaran [13], for symplectic and orthogonal Grassmannians by Lakshmibai, Raghavan and Sankaran [12] and by Brown and Lakshmibai for minuscule $G/P$ [3]. These positive characteristic results were proved by verifying the LMP conjecture in the specific cases. The LMP conjecture for $G/P$ is implied by the conjecture for the full flag variety $G/B$ (cf. Proposition 2.14 of this paper). Lakshmibai, Mehta and Parameswaran verified their conjecture for $\mathrm{SL}_n/B$ and $n \leqslant 6$.

In this paper we prove the LMP conjecture for $\mathrm{SL}_n/P$ by explicitly constructing a Frobenius splitting of $\mathrm{SL}_n/B \times \mathrm{SL}_n/B$ vanishing with maximal multiplicity on the diagonal for every $n \geqslant 2$. Our splitting compatibly splits $X \times X$, where $X$ is a Kempf variety in $\mathrm{SL}_n/B$ (Kempf varieties are special smooth Schubert varieties introduced by Kempf in [7]. See also §3.2 in this paper for their definition and examples).

Our construction comes from observing in the $\mathrm{SL}_3$-case that the product of the minors from the lower left hand corner in

$$\begin{pmatrix} x_{31} & 0 & x_{32} & 0 & x_{33} & 0 \\ x_{21} & 0 & x_{22} & 0 & x_{23} & 0 \\ x_{11} & 0 & x_{12} & 0 & x_{13} & 0 \\ x_{11} & y_{11} & x_{12} & y_{12} & x_{13} & y_{13} \\ x_{21} & y_{21} & x_{22} & y_{22} & x_{23} & y_{23} \\ x_{31} & y_{31} & x_{32} & y_{32} & x_{33} & y_{33} \end{pmatrix},$$

where

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} \in \mathrm{SL}_3$$

is a section of the anticanonical bundle on $\mathrm{SL}_3/B \times \mathrm{SL}_3/B$ giving a Frobenius splitting vanishing with maximal multiplicity on the diagonal and compatibly splitting $X \times X$, where $X$ is one of the five Kempf varieties in $\mathrm{SL}_3/B$ (cf. Example 5.2 in this paper).

In the last part (§6) of this paper, we enhance the geometric arguments in [11] and show that the Gaussian map (1.1) is surjective, provided that $\mathcal{L}_1 = \mathcal{L} \otimes \mathcal{M}_1$ and $\mathcal{L}_2 = \mathcal{L} \otimes \mathcal{M}_2$, where $\mathcal{L}$ is ample and $\mathcal{M}_1, \mathcal{M}_2$ globally generated line bundles on $X$ (a projective smooth variety) and the diagonal $\Delta \subset X \times X$ is maximally compatibly split. Here we do not need the underlying field to have odd characteristic (as in [11]). This enables us to prove Wahl's conjecture also for Kempf varieties, since they posses unique minimal ample line bundles as Schubert varieties in $G/B$. We do not know, even over the complex numbers, if Wahl's conjecture holds for smooth Schubert varieties.

We have found it very difficult to prove the LMP conjecture in a general Lie theoretic context and hope this paper will add to the inspiration for further research in this direction. We feel nevertheless, that Frobenius splitting of tangent bundles (cf. the already known case of the cotangent bundle [9]), diagonal Frobenius splitting of Kempf varieties along with the LMP and Wahl conjecture for the special linear group are of some interest.

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## 2. Preliminaries

A scheme will refer to a separated scheme of finite type over an algebraically closed field $k$ of characteristic $p > 0$. A variety will refer to a reduced scheme.

### 2.1. The vanishing multiplicity on a smooth subvariety

Let $X$ be a smooth variety of dimension $n$, $\mathcal{L}$ a line bundle on $X$ and $Y \subset X$ a smooth subvariety of codimension $d$. Then the blow-up $B = \mathrm{Bl}_Y(X)$ is a smooth variety and the exceptional divisor $E \subset B$ a prime divisor. Let $s$ be a section of $\mathcal{L}$. The vanishing multiplicity of $s$ on $Y$ is defined as $v_E(\pi^* s)$ (in the notation of [6, II.6]), where $\pi : B \to X$ is the projection. Notice that the vanishing multiplicity of $s$ on $Y$ can be computed locally on an open subset $U \subset X$ with $U \cap Y \neq \emptyset$. Locally this definition is easy to handle: if $P \in Y$, then there exists a regular system of parameters $x_1, \dots, x_n$ in $\mathcal{O}_{X,P}$, such that $Y$ is defined by $I = (x_1, \dots, x_d)$ [17, VIII. Theorem 26]. The vanishing multiplicity of $s$ is the maximal $m \geqslant 0$ with $s_P \in I^m \mathcal{L}$.

### 2.2. Frobenius splitting

We recall the crucial definitions and concepts on Frobenius splitting from [2] with a few added generalizations on Frobenius splitting of $\mathcal{O}_X$-algebras along with the notion of *maximally compatibly split subschemes*.

The *absolute Frobenius morphism* on a scheme $X$ is the morphism $F : X \to X$, which is the identity on point spaces and the Frobenius homomorphism on the structure sheaf $\mathcal{O}_X$. A *Frobenius splitting* of $X$ is an $\mathcal{O}_X$-linear map $\sigma : F_* \mathcal{O}_X \to \mathcal{O}_X$ splitting $F^\# : \mathcal{O}_X \to F_* \mathcal{O}_X$. Another way of saying this, is that $\sigma$ is a group homomorphism $\mathcal{O}_X \to \mathcal{O}_X$ satisfying

- $\sigma(f^p g) = f \sigma(g)$
- $\sigma(1) = 1$

locally on open subsets. A Frobenius split scheme has to be reduced. A closed subscheme $Y \subset X$ is called *compatibly split* under a Frobenius splitting $\sigma$ if

$$\sigma(F_* \mathcal{I}_Y) \subset \mathcal{I}_Y.$$

The following very useful results follow (almost) from first principles (cf. [2, Proposition 1.2.1 and Lemma 1.1.7]).

Proposition 2.1. — *Let $\sigma$ be a Frobenius splitting of a scheme $X$ and let $Y$ and $Z$ be compatibly split subschemes of $X$ under $\sigma$.*

(i) *The irreducible components of $Y$ are compatibly split under $\sigma$.*
(ii) *The scheme theoretic intersection $Y \cap Z$ given by $\mathcal{I}_Z + \mathcal{I}_Y$ is compatibly split under $\sigma$.*

(iii) *The scheme theoretic union $Y \cup Z$ given by $\mathcal{I}_Z \cap \mathcal{I}_Y$ is compatibly split under $\sigma$.*

(iv) *If $U$ is a dense open subscheme of a reduced scheme $X$, and if*

$$\sigma \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$$

*restricts to a splitting of $U$, then $\sigma$ is a splitting of $X$. If, in addition, $Y$ is a reduced closed subscheme of $X$ such that $U \cap Y$ is dense in $Y$ and compatibly split by $\sigma|_U$, then $Y$ is compatibly split by $\sigma$.*

### 2.3. Frobenius splitting of $\mathcal{O}_X$-algebras

The Frobenius homomorphism makes perfect sense for a sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras, where $X$ is a scheme. In analogy with the classical definition we define $\mathcal{A}$ to be Frobenius split if there exists a homomorphism

$$\sigma : F_*\mathcal{A} \to \mathcal{A}$$

of $\mathcal{A}$-modules splitting the Frobenius homomorphism $\mathcal{A} \to \mathcal{A}$. Similarly we call a sheaf of ideals $\mathcal{J}$ in $\mathcal{A}$ compatibly split under $\sigma$ if $\sigma(F_*\mathcal{J}) \subset \mathcal{J}$.

We let

$$\begin{aligned}\mathcal{R}(\mathcal{I}) &= \bigoplus_{m \geqslant 0} \mathcal{I}^m t^m = \mathcal{O}_X[\mathcal{I}t] \\ &= \{a_0 + a_1 t + \cdots + a_n t^n \mid a_j \in \mathcal{I}^j\} \subset \mathcal{O}_X[t]\end{aligned}$$

denote the *Rees algebra* corresponding to a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. The sheaf of ideals $\mathcal{I}\mathcal{R}(\mathcal{I})$ is called the *exceptional ideal*.

A Frobenius splitting $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$ can always be extended to the Frobenius splitting $\sigma[t] : F_*\mathcal{O}_X[t] \to \mathcal{O}_X[t]$ given by

$$\sigma[t](a_0 + a_1 t + \cdots) := \sigma(a_0) + \sigma(a_p)t + \sigma(a_{2p})t^2 + \cdots$$

Definition 2.2. — *Let $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$ be a Frobenius splitting of $X$. A closed subscheme $Y \subset X$ is called maximally compatibly split under $\sigma$ if*

$$\sigma(\mathcal{I}^{np+1}) \subset \mathcal{I}^{n+1}$$

*for every $n \geqslant 0$, where $\mathcal{I}$ is the ideal sheaf defining $Y$.*

Notice that a maximally compatibly split scheme is compatibly split and that $\sigma(\mathcal{I}^{np}) \subset \mathcal{I}^n$ for $n \geqslant 0$. The following result can be checked explicitly by reducing to the affine case.

Proposition 2.3. — *Let $Y \subset X$ be a maximally compatibly split closed subscheme under a Frobenius splitting $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$ and $\mathcal{I}$ the ideal sheaf defining $Y$.*

(i) *Then* $\sigma[t]$ *restricts to a Frobenius splitting of the Rees algebra* $\mathcal{R}(\mathcal{I})$ *compatibly splitting the exceptional ideal* $\mathcal{I}\mathcal{R}(\mathcal{I})$.

(ii) *If furthermore* $Z$ *is a compatibly split closed subscheme under* $\sigma$, *then the induced splitting on* $Z$ *splits* $Y \cap Z$ *maximally, where* $Y \cap Z$ *denotes the scheme theoretic intersection.*

The *blow-up* of a scheme $X$ along a closed subscheme $Y$ given by the ideal sheaf $\mathcal{I}$ is defined as $\mathrm{Bl}_Y(X) := \mathrm{Proj}\,\mathcal{R}(\mathcal{I})$. The exceptional ideal identifies with the inverse image ideal sheaf $\pi^{-1}(\mathcal{I})$, under the canonical morphism $\pi : \mathrm{Bl}_Y(X) \to X$. It is an invertible sheaf defining the *exceptional divisor* of $\pi$. In this setting we will prove the following analogue of Proposition 2.3.

Proposition 2.4. — *Let* $Y \subset X$ *be a maximally compatibly split closed subscheme under a Frobenius splitting* $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$.

(i) *Then* $\sigma$ *extends to a Frobenius splitting of the blow-up* $\mathrm{Bl}_Y(X)$ *compatibly splitting the exceptional divisor.*

(ii) *If the closed subscheme* $Z$ *is compatibly split under* $\sigma$, *then the induced splitting on* $Z$ *extends to a Frobenius splitting of* $\mathrm{Bl}_{Y\cap Z}(Z)$ *splitting the exceptional divisor compatibly, where* $Y \cap Z$ *denotes the scheme theoretic intersection.*

Proposition 2.4 is a consequence of the next subsection, where we give the necessary details for turning a Frobenius splitting of a homogeneous $\mathcal{O}_X$-algebra $\mathcal{A}$ into a Frobenius splitting of the scheme $\mathrm{Proj}\,\mathcal{A}$.

## 2.4. Graded Frobenius splittings

For a commutative ring $R$ of characteristic $p > 0$ and an $R$-module $M$ with scalar multiplication $(r, m) \mapsto rm$, we let $F_*M$ denote the $R$-module coinciding with $M$ as an abelian group but with scalar multiplication $(r, m) \mapsto r^p m$.

Let $S = S_0 \oplus S_1 \oplus \cdots$ be a graded noetherian ring of characteristic $p$, such that $F_*S$ is a finitely generated $S$-module. If $M = M_0 \oplus M_1 \oplus \cdots$ is a graded $S$-module, then we have a direct sum decomposition of $F_*M$ into graded $S$-modules

$$F_*M = F_*M^{(0)} \oplus \cdots \oplus F_*M^{(p-1)},$$

where

$$F_*M^{(j)} = \bigoplus_{i \equiv j \pmod p} M_i,$$

for $j = 0, \ldots, p-1$. An element $m \in M_{np+j} \subset F_*M^{(j)}$ has degree $n$.

LEMMA 2.5. — *Let $X = \operatorname{Proj}(S)$ and $F : X \to X$ be the absolute Frobenius morphism on $X$. Then there is a canonical isomorphism*

$$\widetilde{F_* M^{(0)}} \cong F_* \widetilde{M}.$$

*Proof.* — Let $f \in S$ be a homogeneous element. Then

$$\varphi_f\left(\frac{m}{f^n}\right) = \frac{m}{f^{np}}$$

defines a local isomorphism $(F_* M^{(0)})_{(f)} \to F_*(M_{(f)})$ on $D_+(f)$. The isomorphisms $\varphi_f$ patch up to give the desired global isomorphism. □

*Example 2.6.* — Suppose that $S = k[x_0, x_1, \dots, x_n]$, where $k$ is a field of characteristic $p$. Then there is an isomorphism

$$F_* S^{(0)} \cong S \oplus S(-1)^{\ell_1} \oplus \cdots \oplus S(-n)^{\ell_n},$$

of graded $S$-modules for certain $\ell_1, \dots, \ell_n \in \mathbb{N}$. Lemma 2.5 shows that

$$F_* \mathcal{O}_X \cong \mathcal{O}_X \oplus \mathcal{O}_X(-1)^{\ell_1} \oplus \cdots \oplus \mathcal{O}_X(-n)^{\ell_n}$$

for $X = \mathbb{P}^n_k = \operatorname{Proj}(S)$. In particular, it follows that $\mathbb{P}^n_k$ is Frobenius split. Building a monomial basis for $F_* S^{(0)}$ in degrees $0, p, 2p, \dots, np$ we also have the following recursive formula for $\ell_j$:

$$\ell_j = \binom{jp+n}{n} - \sum_{i=1}^{j} \binom{i+n}{n} \ell_{j-i},$$

where $j = 0, \dots, n$. The fact that $F_* \mathcal{O}_{\mathbb{P}^n}(m)$ splits into a direct sum of line bundles is a classical result due to Hartshorne (cf. [5, §6]).

### 2.4.1. Frobenius splitting of Proj($S$)

For $\sigma \in \operatorname{Hom}_S(F_* S, S)$, we let

$$\sigma_0 \in \operatorname{Hom}_S(F_* S^{(0)}, S)_0 \subset \operatorname{Hom}_S(F_* S^{(0)}, S)$$

denote the degree 0 component of $\sigma$ restricted to $F_* S^{(0)}$. Then

$$\sigma_0 : F_* S^{(0)} \to \; S$$

is a homomorphism of graded $S$-modules. We may view $\sigma_0 \in \operatorname{Hom}_S(F_* S, S)$ satisfying $\sigma_0(S_{np}) \subset S_n$ and $\sigma_0(S_m) = 0$ if $p \nmid m$.

Lemma 2.7. — *Suppose $\sigma \in \mathrm{Hom}_S(F_*S, S)$, where $S = S_0 \oplus S_1 \oplus \cdots$ is a graded ring. Then $\sigma_0$ is a Frobenius splitting if $\sigma$ is a Frobenius splitting. If $I \subset S$ is a homogeneous ideal, then $\sigma_0$ splits $I$ compatibly if $\sigma$ splits $I$ compatibly.*

*If $S$ is Frobenius split, then $X = \mathrm{Proj}(S)$ is Frobenius split. If $I$ is a compatibly split homogeneous ideal, then the closed subscheme $Y = \mathrm{Proj}(S/I)$ is compatibly split in $X$.*

*Proof.* — Let $\sigma : F_*S \to S$ be a Frobenius splitting. Clearly $\sigma(1)_0 = \sigma_0(1)$, so that $\sigma_0$ is a Frobenius splitting if $\sigma$ is. Notice that $\sigma(I) \subset I$ implies $\sigma_0(I) \subset I$, since $\sigma_0(x) = \sigma(x)_n$ for $x \in S_{np}$. Now the statements in the first part of the lemma follow. For the second part let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf defining $Y$. Then $\mathcal{I} = \widetilde{I}$ and $\mathcal{O}_X = \widetilde{S}$. Now Lemma 2.5 gives

$$F_*\mathcal{I} = \widetilde{F_*I^{(0)}},$$
$$F_*\mathcal{O}_X = \widetilde{F_*S^{(0)}}.$$

The graded $S$-homomorphism $\sigma_0 : F_*S^{(0)} \to S$ then gives a Frobenius splitting $\widetilde{\sigma_0} : F_*\mathcal{O}_X \to \mathcal{O}_X$ with $\widetilde{\sigma_0}(F_*\mathcal{I}) \subset \mathcal{I}$. □

Corollary 2.8. — *Let $X$ be a scheme, $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1 \oplus \cdots$ a sheaf of graded $\mathcal{O}_X$-algebras and $\mathcal{I} \subset \mathcal{S}$ a homogeneous ideal. We will assume that $F_*\mathcal{S}$ locally is a finitely generated $\mathcal{S}$-module. If $\mathcal{S}$ is Frobenius split compatibly with $\mathcal{I}$, then* $\mathrm{Proj}(\mathcal{S})$ *is Frobenius split compatibly with* $\mathrm{Proj}(\mathcal{S}/\mathcal{I})$.

*Proof.* — Let $\sigma : F_*\mathcal{S} \to \mathcal{S}$ be a Frobenius splitting of $\mathcal{S}$ with $\sigma(F_*\mathcal{I}) \subset \mathcal{I}$. The construction of $\sigma_0$ globalizes to give a Frobenius splitting $\sigma_0 : F_*\mathcal{S}^{(0)} \to \mathcal{S}$. For an affine open subset $U \subset X$, $\sigma_0$ gives by Lemma 2.7 a Frobenius splitting

$$\sigma_U : F_*\mathcal{O}_{\mathrm{Proj}(\mathcal{S}(U))} \to \mathcal{O}_{\mathrm{Proj}(\mathcal{S}(U))}$$

compatibly splitting the closed subscheme $\mathrm{Proj}(\mathcal{S}(U)/\mathcal{I}(U))$. Coming from the global splitting $\sigma_0$, these splittings patch up to give the desired global splitting of $\mathrm{Proj}(\mathcal{S})$. □

### 2.5. Duality for the Frobenius morphism

On a non-singular variety $X$ duality for the Frobenius morphism $F : X \to X$ is available for the study of Frobenius splitting: there is a functorial isomorphism $F_*\omega_X^{1-p} \to \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, where $\omega_X$ is the canonical line bundle on $X$. In [14], it is shown how geometric properties of the zero divisor of a section of $\omega_X^{1-p}$ translate into properties of compatible Frobenius splitting. To recall this powerful result in more precise terms, we need to introduce some notation.

If $\alpha \in \mathbb{Q} \setminus \mathbb{N}$ and $x$ is a variable, we define $x^\alpha := 0$. Now let $x = (x_1, \dots, x_n)$ denote a regular system of parameters (in a regular local ring) and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n$ a rational vector. Then we define

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

and $x^\gamma := x_1^\gamma \cdots x_n^\gamma$ for $\gamma \in \mathbb{Q}$.

Theorem 2.9 (Mehta and Ramanathan [14]). — *Let $X$ be a non-singular variety of dimension $n$ over an algebraically closed field $k$ of characteristic $p$. Then there is a canonical isomorphism*

$$\partial : F_*\omega_X^{1-p} \to \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$$

*of $\mathcal{O}_X$-modules whose completion*

$$\hat{\partial}_P : F_*\omega_{\hat{R}}^{1-p} \to \mathrm{Hom}_{\hat{R}}(F_*\hat{R}, \hat{R})$$

*at a closed point $P \in X$, is given by*

$$\hat{\partial}_P \left( x^\alpha \frac{1}{(dx)^{p-1}} \right) (x^\beta) = x^{(\alpha+\beta+1)/p-1},$$

*where $\hat{R} = k[[x_1, \dots, x_n]]$ and $x_1, \dots, x_n$ is a regular system of parameters in $R := \mathcal{O}_{X,P}$ with $dx = dx_1 \wedge \cdots \wedge dx_n$.*

*Remark 2.10.* — Notice that $\partial(s)$ in Theorem 2.9 is a Frobenius splitting if and only if $\partial(s)(1) = 1$. This translates into a local condition on the section $s$. Suppose

$$s_P = \left( \sum_\alpha a_\alpha x^\alpha \right) (1/dx)^{p-1}$$

is a local expansion of $s$ at $P \in X$. Let $\mathrm{supp}(s_P)$ denote the exponents of the monomials occurring with non-zero coefficient in $s_P$. For $\partial(s_P)$ to be a Frobenius splitting we must have $p-1 \in \mathrm{supp}(s_P)$ and $p-1+pv \notin \mathrm{supp}(s_P)$ for $v \in \mathbb{N}^n \setminus \{0\}$. If $X$ is complete, then $\partial(s)$ is a Frobenius splitting if and only if $p-1 \in \mathrm{supp}(s_P)$ for some $P \in X$ [14, Proposition 6] .

An important consequence of this result is the following [2, Proposition 1.3.11].

Lemma 2.11. — *Let $X$ be a complete smooth variety. If $\sigma$ is a section of $\omega_X^{-1}$ such that $\partial(\sigma^{p-1})$ is a Frobenius splitting of $X$, then the subscheme of zeros, $Z(\sigma) \subset X$, is compatibly split under $\partial(\sigma^{p-1})$.*

We have the following result analogous to [11, Proposition 2.1]. In the proof we use the notation

$$|\alpha| := \alpha_1 + \cdots + \alpha_n$$

for a vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n$.

LEMMA 2.12. — *Let $Z$ be a non-singular variety of dimension $n$ and $W \subset Z$ a non-singular subvariety of codimension $d$. Let $s$ be a section of $\omega_Z^{1-p}$, such that $\partial(s)$ is a Frobenius splitting of $Z$. Then $s$ vanishes with multiplicity $\leqslant d(p-1)$ on $W$. The section $s$ vanishes with maximal multiplicity $d(p-1)$ on $W$ if and only if $W$ is maximally compatibly split under $\partial(s)$.*

*Proof.* — Let $z_1, \dots, z_n$ be a regular system of parameters in $R := \mathcal{O}_{Z,P}$, where $P \in W$. We may assume that the ideal $I \subset R$ defining $W$ at $P$ is given by $x := (z_1, \dots, z_d)$. Define $y := (z_{d+1}, \dots, z_n)$ and let

$$t = \left(\sum a_{\alpha,\beta} x^\alpha y^\beta\right) \left(\frac{1}{dx \wedge dy}\right)^{p-1} \tag{2.1}$$

be the local expansion of $s$ at $P$ in the completion $k[[z_1, \dots, z_n]]$ of $R$. If $s$ vanishes with multiplicity $> d(p-1)$ on $W$, then the term $x^{p-1}y^{p-1}$ cannot occur with non-zero coefficient in (2.1) contradicting that $\partial(s)$ is a Frobenius splitting. Therefore $s$ vanishes with multiplicity $\leqslant d(p-1)$ on $W$.

Assume that $t$ vanishes with multiplicity $d(p-1)$ on $W$. This means that $|\alpha| \geqslant d(p-1)$ for every $\alpha$ with $a_{\alpha,\beta} \neq 0$ in (2.1). We will prove that $\partial(t)(I^{mp+1}) \subset I^{m+1}$ for $m \geqslant 0$. For this we assume that

$$w = \sum c_{\gamma,\delta} x^\gamma y^\delta \in I^{mp+1}$$

i.e. $|\gamma| \geqslant mp + 1$ for every $\gamma$ with $c_{\gamma,\delta} \neq 0$. Now we have

$$|(\alpha + \gamma + 1)/p - 1| \geqslant \frac{d(p-1) + mp + 1 + d}{p} - d = m + \frac{1}{p}.$$

So if the vector $(\alpha+\gamma+1)/p$ is integral, then $|(\alpha+\gamma+1)/p-1| \geqslant m+1$. This shows that $\partial(t)(w) \in I^{m+1}$ recalling the definition of $\partial(t)$ in Theorem 2.9.

Now assume that $\partial(t)(I^{mp+1}) \subset I^{m+1}$ for $m \geqslant 0$. We will prove that $t$ has to vanish with multiplicity $d(p-1)$ on $W$. Suppose that $|\alpha| < d(p-1)$ for some non-zero $a_{\alpha,\beta}$ in (2.1). Let $m_i \in \mathbb{N}$ be given by

$$m_i(p-1) \leqslant \alpha_i < (m_i + 1)(p-1)$$

for $i = 1, \dots, d$ and similarly $m_j(p-1) \leqslant \beta_j < (m_j + 1)(p-1)$ for $j = d+1, \dots, n$. Define the monomial $x^\gamma y^\delta \in I$ by

$$\gamma = ((m_1 + 1)p - \alpha_1 - 1, \dots, (m_d + 1)p - \alpha_d - 1)$$

and similarly $\delta = ((m_{d+1}+1)p - \beta_{d+1} - 1, \dots, (m_n+1)p - \beta_n - 1)$. Then

$$\partial(x^\alpha y^\beta \, (\frac{1}{dx \wedge dy})^{p-1})(x^\gamma y^\delta) \in I^{m_1+\cdots+m_d} \setminus I^D,$$

where $D = m_1 + \cdots + m_d + 1$. But $x^\gamma y^\delta \in I^{(D-1)p+1}$, since

$$\begin{aligned}\sum_{i=1}^{d}((m_i+1)p - \alpha_i - 1) > \sum_{i=1}^{d}(m_i+1)p - d(p-1) - d &= \sum_{i=1}^{d} m_i p \\ &= (D-1)p.\end{aligned}$$

This contradicts our assumption and we must have $|\alpha| \geqslant d(p-1)$ for every non-zero $a_{\alpha,\beta}$ in (2.1). $\square$

The following remark relates to the issue of Frobenius splitting of the tangent bundle on a Frobenius split variety (cf. our remarks in the end of the introduction).

*Remark 2.13.* — If $X$ is smooth and $X \times X$ is Frobenius split with the diagonal $\Delta_X \subset X \times X$ maximally compatibly split, then the tangent bundle $T_X$ on $X$ is Frobenius split, since the exceptional divisor in $\mathrm{Bl}_{\Delta_X}(X \times X)$ is isomorphic to $\mathbb{P}(T_X)$ [2, Lemma 1.1.11].

We also need the following ([11, Proposition 2.3] and [2, Exercises 1.3.E. (13)]).

Proposition 2.14. — *Let $f : X \to Y$ be a proper morphism of smooth varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let $Z \subset X$ be a smooth subvariety such that $f$ is smooth at some point of $Z$. If $X$ is Frobenius split and $Z$ compatibly split with maximal multiplicity, then the induced splitting of $Y$ has maximal multiplicity along the non-singular locus of $f(Z)$.*

## 2.6. Residual normal crossing

In this section we recall a very important concept introduced by Mehta, Lakshmibai and Parameswaran [11, Definition 1.6].

Definition 2.15. — *A power series $f \in k[[x_1, \dots, x_n]]$ is said to have residual normal crossings if either*

- *$n = 0$ and $f \neq 0$ or*
- *$n > 0$, $x_1 | f$ and $f/x_1 + (x_1) \in k[[x_1, \dots, x_n]]/(x_1) \simeq k[[x_2, \dots, x_n]]$ has residual normal crossing in $k[[x_2, \dots, x_n]]$.*

The definition of residual normal crossings is dependent on the ordering of the variables i.e. when stating that $f \in k[[x_1, \dots, x_n]]$ has residual normal crossing, it is implicitly assumed that the variables $x_1, \dots, x_n$ are ordered.

*Example 2.16.* — The polynomial $f = x(zy - x^2)(w - y) \in k[[x, y, z, w]]$ has residual normal crossing. However, if the variables are ordered $x, w, z, y$, then $f$ does not have residual normal crossing i.e. $f \in k[[x, w, z, y]]$ does not have residual normal crossing.

The minimal term in a residual normal crossing power series

$$f \in k[[x_1, \dots, x_n]]$$

is precisely $x_1 \cdots x_n$, when the monomials are ordered according to the lexicographical ordering $<$ given by $x_n < x_{n-1} < \cdots < x_1$. This implies the following result by Remark 2.10.

Proposition 2.17. — *Let $X$ be a complete smooth variety, $P \in X$ and $x_1, \dots, x_n$ a system of parameters of $\mathcal{O}_{X,P}$. If $s \in \Gamma(X, \omega_X^{-1})$, such that $s_P \in \hat{\mathcal{O}}_{X,P} = k[[x_1, \dots, x_n]]$ has residual normal crossing, then $\partial(s^{p-1})$ is a Frobenius splitting of $X$.*

## 3. Group theory

Let $G$ be a semisimple algebraic group, $B$ a Borel subgroup of $G$ and $P \supset B$ a parabolic subgroup. A *Schubert variety* is defined as the closure of a $B$-orbit in the generalized flag variety $G/P$. The singular locus of a Schubert variety is $B$-stable.

The map $\pi : G \to G/B$ is a locally trivial principal $B$-fibration and $G \times^B E \to G/B$ is a vector bundle of rank $\dim_k E$, where $E$ is a finite dimensional representation of $B$. We let $\Gamma(E)$ denote the global sections of this vector bundle i.e.

$$\Gamma(E) = \{f : G \to E \mid f(xb) = b^{-1} f(x), \text{ for every } x \in G, b \in B\}.$$

For a one dimensional representation $\chi$ of $B$ we get the following explicit description of the global sections of the line bundle $G \times^B \chi$ on $G/B$:

$$\Gamma(\chi) = \{f \in k[G] \mid f(gb) = \chi(b)^{-1} f(g), \text{ for every } g \in G, b \in B\}. \tag{3.1}$$

For the rest of this paper we will assume that $G = \mathrm{SL}_n(k)$ with $B$ equal to the upper triangular matrices containing the diagonal matrices

$$T = \{\mathrm{diag}(t_1, \dots, t_n) \mid t_i \in k,\, t_1 \cdots t_n = 1\},$$

$U$ the unipotent upper triangular matrices and $U^-$ the unipotent lower triangular matrices. The canonical map

$$\pi : U^- \to \pi(U^-) \subset G/B$$

identifies $U^-$ with an (affine) open subset of $G/B$, since $U^- \cap B = \{e\}$. This open subset is isomorphic to affine $n(n-1)/2$ – space.

Furthermore, $B = TU$ and $X(B) = X(T)$, where $X(B)$ denotes the one-dimensional representations (characters) of $B$. Let $\epsilon_i(t) = t_i$ for $t \in T$ and $\omega_i = \epsilon_1 + \cdots + \epsilon_i$ for $1 \leqslant i \leqslant n-1$ be characters of $T$. Then $X(T)$ is a free abelian group of rank $n-1$ with basis $\omega_1, \ldots, \omega_{n-1}$. The canonical line bundle on $G/B$ can be identified with $G \times^B (2\omega_1 + \cdots + 2\omega_{n-1})$.

In the next section we give an example showing the explicit nature of residual normal crossings in constructing Frobenius splittings for $G/B$ vanishing with different multiplicities on $B/B$.

### 3.1. Frobenius splitting of $G/B$ by residual normal crossings

For an $n \times n$-matrix $g \in G$ we let $\delta_i(g)$ denote the $i \times i$ minor from the lower left hand corner i.e. the minor corresponding to the columns $\{1, \ldots, i\}$ and rows $\{n, \ldots, n-i+1\}$ of $g$. Similarly we let $\delta_i'(g)$ denote the (principal) $i \times i$ minor from the upper left hand corner i.e. the minor corresponding to the columns $\{1, \ldots, i\}$ and rows $\{1, \ldots, i\}$.

We let $\delta(g) = \delta_1(g) \cdots \delta_{n-1}(g)$ and similarly $\delta'(g) = \delta_1'(g) \cdots \delta_{n-1}'(g)$. Then $\delta, \delta' \in \Gamma(-\omega_1 - \cdots - \omega_{n-1})$ and

$$s(g) = \delta(g)\delta'(g) \tag{3.2}$$

is a section of the anticanonical line bundle.

*Example 3.1.* — As a global section of the anticanonical line bundle (3.2) identifies by (3.1) with the regular function

$$f = x_{31}(x_{21}x_{32} - x_{31}x_{22})\, x_{11}(x_{11}x_{22} - x_{21}x_{12}) \in k[\mathrm{SL}_3] \tag{3.3}$$

for

$$g = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \in \mathrm{SL}_3$$

and restricts to the function

$$x_{31}(x_{21}x_{32} - x_{31}) \tag{3.4}$$

on $U^-$. This polynomial has residual normal crossing with respect to $x_{31}$, $x_{21}, x_{32}$ proving that $f^{p-1}$ defines a Frobenius splitting of $\mathrm{SL}_3/B$ by Proposition 2.1 (iv) and Proposition 2.17.

However, (3.4) does not vanish with maximal multiplicity on the point $B/B$, since $(x_{21}x_{32} - x_{31}) \notin (x_{21}, x_{32}, x_{31})^2$. There is, however, a section with this maximal vanishing property:

$$s = x_{21}x_{31}(x_{11}x_{32} - x_{31}x_{12})(x_{11}x_{22} - x_{21}x_{12}),$$

Specializing, it follows that $s$ restricts to the (residual) normal crossing polynomial

$$x_{21}x_{31}x_{32}$$

on $U^-$. This idea can be generalized from $\mathrm{SL}_3$ to $\mathrm{SL}_n$ for $n > 3$. See [11] for this and a standard monomial approach to constructing Frobenius splittings of maximal multiplicity.

### 3.2. Kempf varieties

In [7], Kempf inspired many subsequent developments in algebraic groups proving his celebrated vanishing theorem first for the general linear group. Kempf considered a very natural class of (smooth) Schubert varieties as stepping stones in an inductive proof. Here we review the definition of these Schubert varieties from [7].

We let

$$A = \{(a_1, \dots, a_n) \in \mathbb{N}^n \mid a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n = 0,\ n - a_j \geqslant j \text{ for } j = 1, \dots, n\}.$$

For $a \in A$ we let $M(a)$ denote the closed subset of $G$ given by

$$\left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in G \;\middle|\; x_{ij} = 0 \text{ for } i > n - a_j \right\}.$$

This subset is $B \times B$-stable, as it is stable with respect to row operations adding a multiple of a higher index row to a lower index one and similarly adding a multiple of a lower index column to a higher index one. The Schubert variety $K(a) = \pi(M(a)) \subset G/B$ is called a *Kempf variety* (see also [10]). Notice that $K((0, \dots, 0)) = G/B$ and $K((n-1, n-2, \dots, 1, 0)) = B/B$. The codimension of $K(a)$ is $a_1 + \cdots a_n$. In particular the unique codimension one Kempf variety is given by the vanishing of the lower left hand corner i.e. $a = (1, 0, \ldots, 0)$. Kempf varieties are smooth as $U^- \cap K(a)$ is a linear subspace of $U^- \cong \mathbb{A}^{n(n-1)/2}$ and $U^- \cap K(a)$ is an open subset of $K(a)$ containing $B/B$.

*Example 3.2.* — The Kempf varieties corresponding to

$$(1,0,0,0), (2,1,0,0) \text{ and } (2,1,1,0) \text{ in } G = \mathrm{SL}_4$$

are depicted below.

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \end{pmatrix} \quad \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \quad \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Informally a placement of a lower triangular zero implies zeros below and to the left of the zero.

### 3.2.1. Rectangular Kempf varieties

Every Kempf variety arises as the scheme-theoretic intersection of distinguished Kempf varieties, which we call *rectangular Kempf varieties.* A rectangular Kempf variety $K(r)$ of height $t \leqslant n-1$ is given by

$$r \in \{a \in A \setminus \{0\} \mid a_i \in \{0, t\} \text{ for } i = 1, \ldots, n\}.$$

The *width* of a Kempf variety $K(r)$ is the number of non-zero entries in $r$.

*Example 3.3.* — The rectangular Kempf varieties of heights one and two for $\mathrm{SL}_4$ are depicted below:

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix},$$

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

They correspond to the defining vectors

$$(1,0,0,0),\ (1,1,0,0),\ (1,1,1,0),\ (2,0,0,0), (2,2,0,0)$$

and widths $1, 2, 3, 1, 2$ respectively.

Lemma 3.4. — *For $G = \mathrm{SL}_n$ there are $n(n-1)/2$ rectangular Kempf varieties. Every Kempf variety is the scheme-theoretic intersection of rectangular Kempf varieties.*

## 4. Matrix calculations

In this section we outline the rather explicit linear algebra which is the basis of our diagonal Frobenius splitting of $\mathrm{SL}_n / B \times \mathrm{SL}_n / B$.

We let $\delta_i(M)$ denote the $i \times i$ minor from the lower left hand corner in a matrix $M$. For two $n \times n$ matrices

$$g = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \qquad \text{and} \qquad h = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix}$$

in $G$ we define the $2n \times 2n$ matrix

$$M(g,h) = \begin{pmatrix} x_{n1} & 0 & x_{n2} & 0 & \cdots & x_{nn} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{21} & 0 & x_{22} & 0 & \cdots & x_{2n} & 0 \\ x_{11} & 0 & x_{12} & 0 & \cdots & x_{1n} & 0 \\ x_{11} & y_{11} & x_{12} & y_{12} & \cdots & x_{1n} & y_{1n} \\ x_{21} & y_{21} & x_{22} & y_{22} & \cdots & x_{2n} & y_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n1} & y_{n1} & x_{n2} & y_{n2} & \cdots & x_{nn} & y_{nn} \end{pmatrix}$$

with determinant $\pm 1$. Notice that $\delta_i(M(g,h))$ is invariant under right translation by $U \times U$ for $1 \leqslant i \leqslant 2n$. We are interested in the lower $n \times 2n$ submatrix

$$L(g,h) = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ x_{21} & y_{21} & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{n-1,1} & y_{n-1,1} & x_{n-1,2} & y_{n-1,2} & \cdots & 0 & 0 \\ x_{n1} & y_{n1} & x_{n2} & y_{n2} & \cdots & 1 & 1 \end{pmatrix}$$

of $M(g,h)$ for $g, h \in U^-$.

Definition 4.1. — *The following definitions are necessary to introduce our Frobenius splitting.*

(i) *For $1 \leqslant i \leqslant n$ we let $L_i(g,h)$ be the $i \times i$ submatrix in the lower left hand corner.*

(ii) *When $n \leqslant i \leqslant 2n-1$ we define $L_i(g,h)$ to be the $(2n-i)\times(2n-i)$-submatrix of $L(g,h)$ obtained by deleting the first $2(i-n)$ columns and the first $(i-n)$ rows from the first $i$ columns of $L(g,h)$.*

(iii) *For $1 \leqslant i \leqslant 2n-1$, we let $V_i$ denote the variables in the diagonal of $L_i$, $M_i$ the monomial ideal generated by them and $m_i$ the monomial given by their product. For $i=0$, we define $V_0=\emptyset$ and $M_0=(0)$.*

The reader is advised to study the following example illustrating these definitions.

*Example 4.2.* — For $G=\mathrm{SL}_4$ and $g,h\in U^-$,

$$L(g,h)=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ x_{21} & y_{21} & 1 & 1 & 0 & 0 & 0 & 0\\ x_{31} & y_{31} & x_{32} & y_{32} & 1 & 1 & 0 & 0\\ x_{41} & y_{41} & x_{42} & y_{42} & x_{43} & y_{43} & 1 & 1\end{pmatrix}.$$

Here

$$L_1=\begin{pmatrix}x_{41}\end{pmatrix}, \quad L_2=\begin{pmatrix} x_{31} & y_{31}\\ x_{41} & y_{41}\end{pmatrix},$$

$$L_3=\begin{pmatrix} x_{21} & y_{21} & 1\\ x_{31} & y_{31} & x_{32}\\ x_{41} & y_{41} & x_{42}\end{pmatrix}, \quad L_4=\begin{pmatrix} 1 & 1 & 0 & 0\\ x_{21} & y_{21} & 1 & 1\\ x_{31} & y_{31} & x_{32} & y_{32}\\ x_{41} & y_{41} & x_{42} & y_{42}\end{pmatrix}$$

and

$$L_5=\begin{pmatrix} 1 & 1 & 0\\ x_{32} & y_{32} & 1\\ x_{42} & y_{42} & x_{43}\end{pmatrix}, \quad L_6=\begin{pmatrix} 1 & 1\\ x_{43} & y_{43}\end{pmatrix}, \quad L_7=\begin{pmatrix}1\end{pmatrix}.$$

Notice that

$$\begin{aligned} V_1 &= \{x_{41}\}\\ V_2 &= \{x_{31}, y_{41}\}\\ V_3 &= \{x_{21}, y_{31}, x_{42}\}\\ V_4 &= \{y_{21}, x_{32}, y_{42}\}\\ V_5 &= \{y_{32}, x_{43}\}\\ V_6 &= \{y_{43}\}\\ V_7 &= \emptyset \end{aligned}$$

and that

$$\det L_i \equiv m_i \mod M_1+\cdots+M_{i-1}$$

for $i = 1, \ldots, 7$. Notice also that the columns in $L_i$ are pairwise identical in the set of variables $\{x_{ij}\}$ and $\{y_{ij}\}$. This ensures that the determinants of the $L_i$'s will vanish with high multiplicity on the diagonal in $U^- \times U^-$.

To prepare for showing that $\delta(M(g,h))^{p-1}$ is a Frobenius splitting section of the anticanonical bundle on $G/B \times G/B$ we need the following result when restricting to the open affine subset $U^- \times U^-$.

Proposition 4.3. — *For $1 \leqslant i \leqslant 2n-1$ and $g, h \in U^-$, we have*

(i)

$$\delta_i(M(g,h)) = \det L_i(g,h) \tag{4.1}$$

(ii)

$$\det L_i(g,h) \in I_\Delta^{\mu(i)},$$

*where $I_\Delta \subseteq k[U^- \times U^-]$ is the ideal defining the diagonal,*

$$\mu(i) = \min\left(\left\lfloor \frac{i}{2} \right\rfloor, \left\lfloor \frac{2n-i}{2} \right\rfloor\right)$$

*and $\lfloor x \rfloor$ denotes the largest integer $\leqslant x$*

(iii)

$$V_1 \cup \cdots \cup V_{2n-1} = \{x_{n1}, x_{n-1,1}, y_{n1}, \ldots, x_{n,n-1}, y_{n,n-1}\}.$$

(iv)

$$\det L_i(g,h) \equiv m_i \mod M_1 + \cdots + M_{i-1}.$$

*Proof.* — For $1 \leqslant i \leqslant n$, (4.1) is clear. When $i > n$ and $g, h \in U^-$ the $i \times i$-submatrix of $M(g,h)$ in the lower left hand corner will have a lower triangular unipotent structure in the top $2(i-n)$ rows (up to row permutation of these rows). In particular, when computing the determinant $\delta_i(M(g,h))$ one might as well start by deleting the first $2(i-n)$ columns and rows. The connection with $\det(L_i(g,h))$ is then clear.

The proof of (ii) follows from pairwise subtraction of columns, before computing the determinant, using the fact that $\mu(i)$ is the number of identical $x$-columns and $y$-columns in $L_i(g,h)$.

Let

$$\Delta_r(g,h) = \{L(g,h)_{ij} \mid i - j = n - r\} \text{ for } r = 1, \ldots, n, n+1, \ldots, 2n-1$$

denote the $2n-1$ "diagonals" in $L(g,h)$ starting with the lower left hand corner. Then (iii) follows from the fact that $V_i$ picks up the variables in $\Delta_i(g,h)$ for $i = 1, \ldots, 2n-1$.

In evaluating the determinant of $L_i(g,h)$, a term different from the product of the diagonal elements always involves a variable in $V_1 \cup \cdots \cup V_{i-1}$ for $i = 1, \ldots, 2n-1$. This implies (iv). $\square$

## 5. The diagonal Frobenius splitting on $\mathrm{SL}_n/B \times \mathrm{SL}_n/B$

The following simple lemma is the fundamental tool for showing compatible splitting for Kempf varieties.

Lemma 5.1. — *Let $f, g \in k[x_{m+1}, \ldots, x_n]$ be relatively prime polynomials. Then*

$$(x_1, \ldots, x_m, fg) = (x_1, \ldots, x_m, f) \cap (x_1, \ldots, x_m, g)$$

*in $k[x_1, \ldots, x_n]$.*

To get an initial grasp of our diagonal Frobenius splitting, the reader is encouraged to look at the following example.

*Example 5.2.* — For $G = \mathrm{SL}_3$ and $g, h \in U^-$, $f := \delta(M(g,h))$ is

$$f = \delta\left(\begin{pmatrix} x_{31} & 0 & x_{32} & 0 & 1 & 0 \\ x_{21} & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ x_{21} & y_{21} & 1 & 1 & 0 & 0 \\ x_{31} & y_{31} & x_{32} & y_{32} & 1 & 1 \end{pmatrix}\right)$$

Here

$$\begin{aligned} f &= \det L_1(g,h) \det L_2(g,h) \det L_3(g,h) \det L_4(g,h) \\ &= x_{31}(x_{21}y_{31} - x_{31}y_{21})(y_{21}x_{32} - y_{31} - x_{21}x_{32} + x_{31})(y_{32} - x_{32}) \end{aligned}$$

and $f \in k[x_{31}, x_{21}, y_{31}, y_{21}, x_{32}, y_{32}]$ has residual normal crossing. Furthermore $f$ vanishes with multiplicity three on the diagonal $V(y_{31} - x_{31}, y_{32} - x_{32}, y_{21} - x_{21})$ as $\mu(1) + \mu(2) + \mu(3) + \mu(4) = 0 + 1 + 1 + 1 = 3$ (cf. Proposition 4.3 (ii)). Therefore $f^{p-1}$ is a Frobenius splitting of $\mathrm{SL}_3/B \times \mathrm{SL}_3/B$ by Proposition 2.17 vanishing with maximal multiplicity on the diagonal. Lemma 2.11 and Proposition 2.1 (i) show that the ideals

$$(x_{31}), \quad (x_{21}y_{31} - x_{31}y_{21}), \quad (y_{21}x_{32} - y_{31} - x_{21}x_{32} + x_{31}) \quad \text{and} \quad (y_{32} - x_{32})$$

are compatibly split by $f^{p-1}$. Consequently

$$(x_{31}, x_{21}y_{31}) = (x_{31}) + (x_{21}y_{31} - x_{31}y_{21})$$

is compatibly split by Proposition 2.1 (ii) and

$$(x_{31}, x_{21}), \quad (x_{31}, y_{31})$$

are compatibly split by Lemma 5.1. Similarly

$$\begin{aligned}&(x_{31}, y_{31}, x_{21}) + (y_{21}x_{32} - y_{31} - x_{21}x_{32} + x_{31}) =\\&(x_{31}, y_{31}, x_{21}, y_{21}x_{32}) =\\&(x_{31}, y_{31}, x_{21}, y_{21}) \cap (x_{31}, y_{31}, x_{21}, x_{32})\end{aligned}$$

showing that $(x_{31}, y_{31}, x_{21}, y_{21})$ is compatibly split. Along the same lines we get that

$$\begin{aligned}&(x_{31}, y_{31}) + (y_{21}x_{32} - y_{31} - x_{21}x_{32} + x_{31}) =\\&(x_{31}, y_{31}, x_{32}(y_{21} - x_{21})) =\\&(x_{31}, y_{31}, x_{32}) \cap (x_{31}, y_{31}, y_{21} - x_{21})\end{aligned}$$

and $(x_{31}, y_{31}, x_{32})$ is compatibly split showing that

$$(x_{31}, y_{31}, x_{32}) + (y_{32} - x_{32}) = (x_{31}, y_{31}, x_{32}, y_{32})$$

is compatibly split.

We have verified that $X \times X \subset \mathrm{SL}_3 /B \times \mathrm{SL}_3 /B$ is compatibly split, where $X$ is any rectangular Kempf variety.

With this example in mind, we state and prove our main result.

THEOREM 5.3. — *For $g, h \in U^- \subset SL_n$, let*

$$f = \delta(M(g, h)) \in k[U^- \times U^-] \cong k[V_1 \cup \cdots \cup V_{2n-1}].$$

*Then*

(i) *$f$ is a residual normal crossing polynomial when the variables are ordered respecting $V_1, V_2, \ldots, V_{2n-1}$: if $x \in V_i$ and $y \in V_j$ are variables and $i < j$, then $x$ must precede $y$ in the ordering of the variables.*

(ii) *$f$ vanishes with multiplicity $\geqslant n(n-1)/2$ on the diagonal $\Delta_{U^-}$.*

(iii) *Let $\omega$ denote the canonical line bundle on $\mathrm{SL}_n /B \times \mathrm{SL}_n /B$. Then*

$$\delta(M(g, h))^{p-1} \in k[G \times G]$$

*is a Frobenius splitting section of $\omega^{1-p}$ vanishing with maximal multiplicity on $\Delta_{G/B}$ compatibly splitting $X \times X$, where $X$ is a Kempf variety.*

*Proof.* — Proposition 4.3 (iv) shows (i). Since

$$\sum_{i=1}^{2n-1} \mu(i) = \frac{n(n-1)}{2},$$

Proposition 4.3 (i) and Proposition 4.3 (ii) imply *(ii)*.

Let us prove (iii). The regular function $\delta(M(g,h)) \in k[G\times G]$ is invariant under right translation by $U \times U$. This amounts to observing that the column operations on $g$ and $h$ coming from right multiplication by $U \times U$ do not change $\delta_i(M(g,h))$ for $1 \leqslant i \leqslant 2n-1$.

Define $\omega_0 = \omega_n = 0$. Then $\delta_{2i}(M(g,h)) \in \Gamma(-\omega_i, -\omega_i)$ and $\delta_{2i-1}(M(g,h)) \in \Gamma(-\omega_i, -\omega_{i-1})$ for $1 \leqslant i \leqslant n$. This shows that $\delta(M(g,h)) \in k[G \times G]$ is a section of the anticanonical line bundle

$$G \times^B (-2\omega_1 - 2\omega_2 - \cdots - 2\omega_n)$$

on $G/B \times G/B$. Now (i) and (ii) show after restricting to $U^- \times U^-$ that $\delta(M(g,h))^{p-1}$ is a Frobenius splitting vanishing with (maximal) multiplicity $(p-1)n(n-1)/2$ on $\Delta_{G/B}$. We have silently applied Proposition 2.1 (iv), Proposition 2.17 and the fact that vanishing multiplicity can be checked on an open subset (cf. Section 2.1)

It remains to show that $X \times X$ is compatibly split, where $X \subset \mathrm{SL}_n /B$ is a Kempf variety. We can assume by Lemma 3.4 that $X$ is a rectangular Kempf variety (the argument works for general Kempf varieties, but is slightly less clear).

Suppose that $X$ is of height $r$ and width $s$. Then we must show that the monomial ideal generated by the variables

$$V_X = \left\{ x_{ij}, y_{ij} \,\middle|\, n-r < i \leqslant n,\ 1 \leqslant j \leqslant s \right\}$$

is compatibly split under $f^{p-1}$. We will prove that the monomial ideal generated by the variables

$$V_X \cap (V_1 \cup \cdots \cup V_m)$$

is compatibly split by induction on $m$. Since $V_X \cap V_1 = \{x_{n1}\}$ and $x_{n1}^{p-1}$ is the first factor in $f^{p-1}$, compatible splitting holds for $m = 1$. Suppose now that the monomial ideal generated by

$$W := V_X \cap (V_1 \cup \cdots \cup V_m) \subsetneq V_X$$

is compatibly split. Then $(W, \delta_{m+1}(M(g,h))) = (W, D)$, where $D$ is a monomial of the form $dm$, where $m$ is the product of the variables $V_X \cap V_{m+1}$

and $d$ is a monomial. This is a consequence of the formula

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A)\det(C),$$

where $A, B$ and $C$ are compatible block matrices.

It follows by Lemma 5.1 that the ideal generated by

$$V_X \cap (V_1 \cup \cdots \cup V_m \cup V_{m+1})$$

is compatibly split. Since $V_X \subset V_1 \cup \cdots \cup V_N$ for $N \geqslant r+s-1$ the result follows. □

## 6. Wahl's conjecture for Kempf varieties

Let $Z$ denote a smooth projective variety. The sheaf of differentials on $Z$ is defined by

$$\Omega_Z^1 = \mathcal{I}_\Delta / \mathcal{I}_\Delta^2,$$

where $\mathcal{I}_\Delta$ denotes the sheaf of ideals defining the diagonal within $Z \times Z$. In this setup we may consider the quotient morphism

$$\mathcal{I}_\Delta \to \mathcal{I}_\Delta / \mathcal{I}_\Delta^2 = \Omega_Z^1.$$

Fixing line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $Z$ we obtain an induced restriction morphism

$$\mathrm{H}^0\big(Z \times Z, \mathcal{I}_\Delta \otimes (\mathcal{L}_1 \boxtimes \mathcal{L}_2)\big) \to \mathrm{H}^0\big(Z, \Omega_Z^1 \otimes \mathcal{L}_1 \otimes \mathcal{L}_2\big), \tag{6.1}$$

where $\mathcal{L}_1 \boxtimes \mathcal{L}_2 := p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2$ and $p_1, p_2 : X \times X \to X$ are the projections on the first and second factors. In case $Z$ is a flag variety and $\mathcal{L}_1$ and $\mathcal{L}_2$ are ample it has been conjectured by J. Wahl [16] that the map (6.1) is surjective. In characteristic zero this is now a theorem proved by S. Kumar [8]. In positive characteristic only sporadic cases are known as outlined in the introduction.

The aim of the last part of this paper is to obtain the following related and seemingly stronger result

Theorem 6.1. — *Assume that the blow-up* $\mathrm{Bl}_\Delta(Z \times Z)$ *admits a Frobenius splitting which is compatible with* $E_Z$. *Let* $\mathcal{L}$ *denote a very ample line bundle on* $Z$ *and let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *denote globally generated line bundles on* $Z$. *Let* $j > 0$ *denote an integer. Then the natural map from*

$$\mathrm{H}^0\big(Z \times Z, \mathcal{I}_\Delta^j \otimes ((\mathcal{L}^j \otimes \mathcal{M}_1) \boxtimes (\mathcal{L}^j \otimes \mathcal{M}_2))\big),$$

*to*

$$\mathrm{H}^0\big(Z, S^j\Omega_Z^1 \otimes \mathcal{L}^{2j} \otimes \mathcal{M}_1 \otimes \mathcal{M}_2\big),$$

*induced by the identification* $\mathcal{I}_\Delta^j/\mathcal{I}_\Delta^{j+1} = S^j\Omega_Z^1$*, is surjective.*

Notice that when $Z$ admits a minimal ample line bundle $\mathcal{L}$; i.e. an ample line bundle on $Z$ such that every line bundle of the form $\mathcal{M}\otimes\mathcal{L}^{-1}$, with $\mathcal{M}$ ample, is globally generated, then Wahl's conjecture is a consequence of Theorem 6.1. Schubert varieties are examples of varieties admitting minimal ample line bundle. When the Schubert variety is a flag variety this is well known; e.g. in the notation of the previous sections the minimal ample line bundle on $G/B$ is defined by the weight

$$-\rho = -(\omega_1 + \cdots \omega_{n-1}).$$

For a general Schubert variety the claim follows by the fact that any ample line bundle on a Schubert variety may be lifted to an ample line bundle on the flag variety containing the Schubert variety [1, Prop.2.2.8]

With these remarks in place the following corollary now follows from Proposition 2.4 and Theorem 5.3

Corollary 6.2. — *The conjecture of Wahl on the surjectivity of the map (6.1) is satisfied for Kempf varieties $Z$ and ample line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$.*

The rest of this paper is concerned with the proof of Theorem 6.1. The proof is highly inspired by the discussion in Section 3 of [11]. As a side result we obtain certain cohomological vanishing results for smooth varieties admitting various types of Frobenius splitting (cf. Prop. 6.8 and Prop. 6.9); e.g. for Kempf varieties. We start by collecting a number of well known results about blow-ups along diagonals.

## 6.1. Blow-up of $\mathbb{P}^N \times \mathbb{P}^N$ along the diagonal

Consider the variety $\mathbb{P}^N = \mathbb{P}(V)$ with homogeneous coordinates $X_0, \ldots, X_N$. The homogeneous ideal defining the diagonal within the product $\mathbb{P}^N \times \mathbb{P}^N$ is generated by the elements

$$X_{i,j} = X_i \otimes X_j - X_j \otimes X_i \ , 0 \leqslant i < j \leqslant N;$$

all of the same multidegree $(1,1)$. Applying the Rees algebra description of the blow-up this leads to an embedding of $\mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N)$ as a closed subvariety of the product

$$\mathbb{P}^N \times \mathbb{P}^N \times \mathbb{P}^{\binom{N+1}{2}-1}. \tag{6.2}$$

Alternatively one could also obtain this embedding by considering $\mathrm{Bl}_\Delta$ $(\mathbb{P}^N \times \mathbb{P}^N)$ as the graph of the rational morphism

$$\mathbb{P}^N \times \mathbb{P}^N \dashrightarrow \mathbb{P}^{\binom{N+1}{2}-1}, \tag{6.3}$$

defined by the generators $X_{i,j}$ of the diagonal ideal (cf. [4, Ex. 7.18]). The latter description makes it evident that $\mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N)$ is contained within

$$\mathbb{P}^N \times \mathbb{P}^N \times \mathrm{Gr}_2(V), \tag{6.4}$$

where $\mathrm{Gr}_2(V)$ denotes the Grassmannian of planes in $V$ with the Plücker embedding in $\mathbb{P}^{\binom{N+1}{2}-1}$. This also explains the following setwise description of the blow-up

$$\mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N) = \{(l_1, l_2, b) \in \mathbb{P}^N \times \mathbb{P}^N \times \mathrm{Gr}_2(V) \ : \ l_1, l_2 \subset b\}. \tag{6.5}$$

In this setting the exceptional divisor $E$ is determined as the set of points

$$E = \{(l, b) \in \mathbb{P}^N \times \mathrm{Gr}_2(V) \ : \ l \subset b\} \subset \mathbb{P}^N \times \mathrm{Gr}_2(V),$$

where we consider $\mathbb{P}^N$ as being diagonally embedded in $\mathbb{P}^N \times \mathbb{P}^N$.

The projection on the first two coordinates

$$\pi : \mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N) \to \mathbb{P}^N \times \mathbb{P}^N,$$

is the blow-up map. Restricting $\pi$ to the exceptional divisor $E$ defines the map

$$\pi_E : E \to \mathbb{P}^N,$$

coinciding with the projectivized tangent bundle on $\mathbb{P}^N$. Finally we let

$$\tau : \mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N) \to \mathrm{Gr}_2(V),$$

denote the map induced by projection on the third coordinate, while $\tau_E$ denotes its restriction to $E$.

Lemma 6.3. — *Let $\mathcal{O}_{2,V}(1)$ (resp. $\mathcal{O}(1)$) denote the ample generator of the Picard group of $\mathrm{Gr}_2(V)$ (resp. $\mathbb{P}^N$). Then as locally free sheaves*

$$\tau^*(\mathcal{O}_{2,V}(1)) \simeq \mathcal{O}(-E) \otimes \pi^*\big(\mathcal{O}(1) \boxtimes \mathcal{O}(1)\big). \tag{6.6}$$

*Proof.* — This follows from a local calculation but can also be obtained in the following more abstract way : assume, first of all, that $N \geqslant 2$ in which case we have the following identity of Picard groups

$$\mathrm{Pic}\big(\mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N)\big) \simeq \mathrm{Pic}(\mathbb{P}^N \times \mathbb{P}^N) \oplus \mathbb{Z}.$$

In particular, we may find unique integers $c_1, c_2$ and $c_3$ such that

$$\tau^*(\mathcal{O}_{2,V}(1)) \simeq \mathcal{O}(-c_1 E) \otimes \pi^*\big(\mathcal{O}(c_2) \boxtimes \mathcal{O}(c_3)\big).$$

Restricting to the open subset $\mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N) \setminus E \simeq (\mathbb{P}^N \times \mathbb{P}^N) \setminus \Delta$ we determine $(c_2, c_3)$ as the bidegree of the rational morphism (6.3). In particular, we find that $c_2 = c_3 = 1$. To find $c_1$ we fix some line $\mathbb{P}^1$ inside $\mathbb{P}^N$ and consider $\mathbb{P}^1 \times \mathbb{P}^1$ as a closed subset of $\mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N)$ by identifying it with its strict transform. As the rational morphism (6.3) is constant on an open dense subset of $\mathbb{P}^1 \times \mathbb{P}^1$, the same is true for the restriction of $\tau$ to $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, the restriction of the sheaf $\tau^*(\mathcal{O}_{2,V}(1))$ to $\mathbb{P}^1 \times \mathbb{P}^1$ is trivial. Now as the sheaf of ideals of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ equals $\mathcal{O}(-1) \boxtimes \mathcal{O}(-1)$ we conclude

$$-c_1 + c_2 = -c_1 + c_3 = 0.$$

Thus $c_1 = 1$. This ends the proof in case $N \geqslant 2$. For $N = 1$ the map $\tau$ is constant and the claimed isomorphism (6.6) is trivial. □

We claim that $\tau$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle. More precisely, let $b_0 \in \mathrm{Gr}_2(V)$ denote any plane in $V$ and let $P_0$ denote the stabilizer of $b_0$ in the group $\mathrm{SL}(V)$. Then $\mathrm{Gr}_2(V)$ is isomorphic to the quotient $\mathrm{SL}(V)/P_0$ while $\mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N)$ may be described as

$$\mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N) = \mathrm{SL}(V) \times_{P_0} (\mathbb{P}(b_0) \times \mathbb{P}(b_0)), \tag{6.7}$$

where $P_0$ acts by the diagonal action on $\mathbb{P}(b_0) \times \mathbb{P}(b_0)$. Thus $\tau$ is just the natural map

$$\tau : \mathrm{SL}(V) \times_{P_0} (\mathbb{P}(b_0) \times \mathbb{P}(b_0)) \to \mathrm{SL}(V)/P_0.$$

In this notation we may describe the exceptional divisor as

$$E = \mathrm{SL}(V) \times_{P_0} \mathbb{P}(b_0),$$

where we think of $E$ as a subset of (6.7) by embedding $\mathbb{P}(b_0)$ diagonally in the product $\mathbb{P}(b_0) \times \mathbb{P}(b_0)$. It follows that the restriction

$$\tau_E : \mathrm{SL}(V) \times_{P_0} \mathbb{P}(b_0) \to \mathrm{SL}(V)/P_0,$$

is a $\mathbb{P}^1$-bundle over $\mathrm{Gr}_2(V)$.

### 6.2. Blow-up of diagonals in general

Returning to the general case of a smooth projective subvariety $Z$ in $\mathbb{P}(V)$ we may consider the blow-up $\mathrm{Bl}_\Delta(Z \times Z)$ as the strict transform of $Z \times Z$ in $\mathrm{Bl}_\Delta(\mathbb{P}^N \times \mathbb{P}^N)$. In particular, we obtain a closed embedding

$$\mathrm{Bl}_\Delta(Z \times Z) \subset Z \times Z \times \mathrm{Gr}_2(V). \tag{6.8}$$

The exceptional divisor $E_Z$ is thus embedded as

$$E_Z \subset Z \times \mathrm{Gr}_2(V). \tag{6.9}$$

In this setting the blow-up morphism

$$\pi_Z : \mathrm{Bl}_\Delta(Z \times Z) \to Z \times Z,$$

coincides with the projection on the first two coordinates, while its restriction

$$\pi_{E_Z} : E_Z \to Z$$

coincides with the projectivized tangent bundle on $Z$. Thus if we consider $\mathrm{Gr}_2(V)$ as the set of lines in $\mathbb{P}^N = \mathbb{P}(V)$, then $E_Z$ consists of the set of pairs $(l, b) \in Z \times \mathrm{Gr}_2(V)$ such that $b$ is a line tangent to the point $l$ in $Z$.

The projection on the third coordinate is denoted by

$$\tau_Z : \mathrm{Bl}_\Delta(Z \times Z) \to \mathrm{Gr}_2(V),$$

while its restriction to $E_Z$ is denoted by $\tau_{E_Z}$.

### 6.3. Fibres of $\tau_{E_Z}$

By the discussion above the fibre of $\tau_{E_Z}$ over a line $b$ in $\mathbb{P}(V)$ consists of the set of points $l$ in $Z$ such that $b$ is tangent to $Z$ at $l$. Thus the following result is now easy to prove

Lemma 6.4. — *If every nonempty fibre of $\tau_{E_Z}$ has dimension* 1 *then $Z$ coincides with $\mathbb{P}(V')$ for some vector subspace $V'$ of $V$.*

*Proof.* — The assumptions means that every tangent line of $Z$ is contained in $Z$. In particular, $Z$ contains all of its tangent planes. But any tangent plane of $Z$ is of the same dimension as $Z$ and consequently $Z$, and all of its tangent planes, must coincide (this simple argument was suggested by the referee). □

### 6.4. Technical results

For technical reasons we will need the following setup : let $\mathcal{Z}$ denote a projective variety and let

$$f : Z \times Z \to \mathcal{Z},$$

denote a morphism. The projective morphism

$$\tau_f = (\tau_Z, f \circ \pi_Z) : \mathrm{Bl}_\Delta(Z \times Z) \to \mathrm{Gr}_2(V) \times \mathcal{Z}, \tag{6.10}$$

has a Stein factorization for which we use the notation

$$\mathrm{Bl}_\Delta(Z \times Z) \xrightarrow{\mu_f} \mathcal{B}_f \to \mathrm{Gr}_2(V) \times \mathcal{Z}. \tag{6.11}$$

The restriction of $\tau_f$ to $E_Z$ is denoted by

$$\tau_{E,f} : E_Z \to \mathrm{Gr}_2(V) \times \mathcal{Z}.$$

More important is the map

$$\mu_{E,f} : E_Z \to \mathcal{S}_f := \mu_f\big(E_Z\big),$$

induced by the restriction of $\mu_f$. We claim

Lemma 6.5. — *The derived direct images $R^i(\mu_{E,f})_*\mathcal{O}_{E_Z}$ are zero when $i > 0$.*

*Proof.* — As the second map $\mathcal{B}_f \to \mathrm{Gr}_2(V) \times \mathcal{Z}$ of the Stein factorization (6.11) is a finite map it suffices to prove that $R^i(\tau_{E,f})_*\mathcal{O}_{E_Z} = 0$ for $i > 0$. Consider an open affine subset $U$ of $\mathrm{Gr}_2(V)$ such that

$$\tau_E : E \to \mathrm{Gr}_2(V),$$

is a trivial $\mathbb{P}^1$-bundle over $U$. Then we may consider $\tau_{E_Z}^{-1}(U)$ as a closed subvariety of $\mathbb{P}^1 \times U$. Embedding $\tau_{E_Z}^{-1}(U)$ by the graph of $f \circ \pi_Z$ defines a closed embedding

$$\iota : \tau_{E_Z}^{-1}(U) \hookrightarrow Y := \mathbb{P}^1 \times U \times \mathcal{Z}.$$

The map

$$\tau_U : \tau_{E_Z}^{-1}(U) = \tau_{E,f}^{-1}(U \times \mathcal{Z}) \to U \times \mathcal{Z},$$

induced by the projection $p_{2,3}$ of $Y$ on the second and third coordinate, coincides with the restriction of $\tau_{E,f}$ to the inverse image of $U \times \mathcal{Z}$. It thus suffices to prove that $R^i(\tau_U)_*\mathcal{O}_{\tau_{E_Z}^{-1}(U)} = 0$ for $i > 0$. Now apply the identity

$$R^i(\tau_U)_*\mathcal{O}_{\tau_{E_Z}^{-1}(U)} = R^i(p_{2,3})_*(\iota_*\mathcal{O}_{\tau_{E_Z}^{-1}(U)}),$$

and the long exact sequence

$$\cdots \to R^1(p_{2,3})_*\mathcal{I} \to R^1(p_{2,3})_*\mathcal{O}_Y = 0 \to R^1(p_{2,3})_*(\iota_*\mathcal{O}_{\tau_{E_Z}^{-1}(U)}) \to 0 \to \cdots$$

associated to the trivial $\mathbb{P}^1$-bundle $p_{2,3}$, and the short exacts sequence

$$0 \to \mathcal{I} \to \mathcal{O}_Y \to \iota_*\mathcal{O}_{\tau_{E_Z}^{-1}(U)} \to 0,$$

defining $\tau_{E_Z}^{-1}(U)$ as a closed subvariety in $Y$. □

Lemma 6.6. — *Assume that $Z$ does not coincide with a closed subvariety of $\mathbb{P}(V)$ of the form $\mathbb{P}(V')$, for some vector subspace $V'$ of $V$. Then $\mu_{E,f}$ is birational*

*Proof.* — Let $Y \subset \mathrm{Gr}_2(V) \times \mathcal{Z}$ denote the image of $\tau_f$. We claim that there exists a point $y \in Y$ such that the fibre $\tau_f^{-1}(y)$ is nonempty and finite. To see this we use Lemma 6.4 to obtain a point $b \in \mathrm{Gr}_2(V)$ such that the fibre $\tau_{E_Z}^{-1}(b)$ is nonempty and finite. Assume, for a moment, that $\tau_Z^{-1}(b)$ is infinite : then $\pi_Z(\tau_Z^{-1}(b))$ is an infinite closed subvariety of $\mathbb{P}(b) \times \mathbb{P}(b) = \mathbb{P}^1 \times \mathbb{P}^1$ and thus $\mathbb{P}(b)$ is contained in $Z$. As a consequence

$$(\mathbb{P}(b) \times \mathbb{P}(b)) \setminus \Delta(\mathbb{P}(b)) \times \{b\},$$

is a subset of $\mathrm{Bl}_\Delta(Z \times Z)$ and thus, by taking the closure, we find that

$$\mathbb{P}(b) \times \{b\} \subset E_Z.$$

But then $\mathbb{P}(b) \times \{b\}$ is a subset of the finite set $\tau_{E_Z}^{-1}(b)$, which is a contradiction. It follows that $\tau_Z^{-1}(b)$ is finite and nonempty. Choose an element $y$ in $\tau_f(\tau_Z^{-1}(b))$. As a subset of $\tau_Z^{-1}(b)$ the set $\tau_f^{-1}(y)$ is then finite.

Let now $Y_0$ denote the nonempty set of points in $Y$ where the associated fibre of $\tau_f$ is finite. Then $Y_0$ is an open subset of $Y$ ([15, Cor. I.8.3]). It follows that $\mu_f$ induces an isomorphism between $\mathcal{B}_f$ and $\mathrm{Bl}_\Delta(Z \times Z)$ over $Y_0$

$$\mu_{f,0} : {\tau_f}^{-1}(Y_0) \xrightarrow{\simeq} \mu_f({\tau_f}^{-1}(Y_0)).$$

It thus suffices to prove that the intersection of $E_Z$ and ${\tau_f}^{-1}(Y_0)$ is nonempty. But this is clear as $\tau_{E_Z}^{-1}(b)$ is a nonempty subset of ${\tau_f}^{-1}(Y_0)$. □

From now on we will assume that $f : Z \times Z \to \mathcal{Z}$ is the product $(f_1, f_2)$ of two morphisms

$$f_i : Z \to \mathcal{Z}_i,\ i = 1, 2.$$

We can then prove.

Lemma 6.7. — *The fibres of $\mu_{E,f}$ are connected.*

*Proof.* — Let $z$ denote an element in $\mathcal{S}_f$ and let $(b, x)$ denote the image of $z$ in under the second morphism

$$\mathcal{B}_f \to \mathrm{Gr}_2(V) \times \mathcal{Z} \tag{6.12}$$

of the Stein factorization (6.11). As $\mu_{E,f}^{-1}(z) \subset \mu_f^{-1}(z)$ and $\mu_f^{-1}(z)$ is connected we may assume that $\mu_f^{-1}(z)$ is infinite. Consequently the intersection $Z \cap \mathbb{P}(b)$ must also be infinite and thus equal to $\mathbb{P}(b)$. It follows that

$$\mathbb{P}(b) \times \mathbb{P}(b) \times \{b\} \subset \mathrm{Bl}_\Delta(Z \times Z).$$

This leads to the inclusion

$$\mu_f^{-1}(z) \subset \tau_f^{-1}(b, x) = (f_1^{-1}(x_1) \cap \mathbb{P}(b)) \times (f_2^{-1}(x_2) \cap \mathbb{P}(b)) \times \{b\}, \tag{6.13}$$

where we have used the notation $x = (x_1, x_2) \in \mathcal{Z}$, with $x_i \in \mathcal{Z}_i$ for $i = 1, 2$. As $\mu_f$ and $\tau_f$ only differ by a finite morphism it follows that $\tau_f^{-1}(b, x)$ is a disjoint union of $\mu_f^{-1}(z)$ with another closed (possibly empty) subset of $\tau_f^{-1}(b,x)$. At the same time $\mu_f^{-1}(z)$ is connected and thus (6.13) implies that $\mu_f^{-1}(z)$ is of one of the forms

$$\mathbb{P}(b) \times \mathbb{P}(b) \times \{b\},\ \{l\} \times \mathbb{P}(b) \times \{b\},\ \mathbb{P}(b) \times \{l\} \times \{b\},$$

for some line $l$ contained in $b$. We conclude that $\mu_{E,f}^{-1}(z)$ is either equal to

$$\mathbb{P}(b) \times \{b\} \subset E_Z,$$

or of the form

$$\{l\} \times \{b\} \subset E_Z.$$

In both cases $\mu_{E,f}^{-1}(z)$ is connected.

□

## 6.5. Proof of Theorem 6.1

We continue the notation of Section 6.4. The proof of Theorem 6.1 is built from the following two results.

Proposition 6.8. — *Assume that $E_Z$ admits a Frobenius splitting. Let $\mathcal{L}$ (resp. $\mathcal{M}$) denote a very ample (resp. globally generated) line bundle on $Z$ and let $j > 0$ denote an integer. Then*

$$\mathrm{H}^i\big(Z, S^j\Omega_Z^1 \otimes \mathcal{L}^{2j} \otimes \mathcal{M}\big) = 0\ ,\ \textit{for } i > 0. \tag{6.14}$$

*Proof.* — We assume that the embedding $Z \subset \mathbb{P}^N$ is defined by the very ample line bundle $\mathcal{L}$, and that the map $f$, of Section 6.4, is the composition

$$f : Z \times Z \to Z \to \mathcal{Z} := \mathbb{P}\big(\mathrm{H}^0(\mathcal{M})^\vee\big),$$

where the first map is projection on the first coordinate while the second map is the projective morphism defined by the globally generated line bundle $\mathcal{M}$. Let $\mathcal{O}_{\mathcal{M}}(1)$ denote the ample generator of the Picard group of $\mathbb{P}\big(\mathrm{H}^0(\mathcal{M})^\vee\big)$. By (6.6) the pull-back of $\mathcal{O}_{2,V}(j) \boxtimes \mathcal{O}_{\mathcal{M}}(1)$ by

$$\tau_{E,f} : E_Z \to \mathrm{Gr}_2(V) \times \mathcal{Z},$$

is then the line bundle

$$\mathcal{L}_j = \mathcal{O}(-jE_Z)_{|E_Z} \otimes \pi_{E_Z}^*\big(\mathcal{L}^{2j} \otimes \mathcal{M}\big),$$

on $E_Z$. Consider the Stein factorization

$$E_Z \xrightarrow{\tilde{\mu}_{E,f}} \tilde{\mathcal{S}}_f \to \mathcal{S}_f,$$

of $\mu_{E,f}$. By Lemma 6.5 and the definition of the Stein factorization, the map $\tilde{\mu}_{E,f}$ is a rational morphism, i.e.

$$R^i(\tilde{\mu}_{E,f})_*\mathcal{O}_{E_Z} = \begin{cases} \mathcal{O}_{\tilde{\mathcal{S}}_f} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Moreover, the pull back $\tilde{\mathcal{L}}_j$ of $\mathcal{O}_{2,V}(j) \boxtimes \mathcal{O}_{\mathcal{M}}(1)$ by the finite morphism

$$\tilde{\mathcal{S}}_f \to \mathcal{S}_f \to \mathcal{B}_f \to \mathrm{Gr}_2(V) \times \mathcal{Z}, \tag{6.15}$$

is an ample line bundle on $\tilde{\mathcal{S}}_f$ whose pull back by $\tilde{\mu}_{E,f}$ coincides with $\mathcal{L}_j$. As $\tilde{\mathcal{S}}_f$ is Frobenius split (by push-down of the Frobenius splitting on $E_Z$ [2, Lemma 1.1.8] ) it follows that the higher cohomology of $\tilde{\mathcal{L}}_j$, and hence of $\mathcal{L}_j$, is trivial [2, Thm.1.2.8]. Notice finally that by [6, Ex. III.8.4] the cohomology of $\mathcal{L}_j$ and the direct image

$$(\pi_{E_Z})_*\mathcal{L}_j = S^j\Omega^1_Z \otimes \mathcal{L}^{2j} \otimes \mathcal{M},$$

coincide. Here we use that the identification $(\pi_{E_Z})_*\mathcal{O}_{E_Z}(-jE_Z) = S^j\Omega^1_Z$. This ends the proof. □

Proposition 6.9. — *Assume that the blow-up $\mathrm{Bl}_\Delta(Z \times Z)$ admits a Frobenius splitting which is compatible with $E_Z$. Let $\mathcal{L}$ denote a very ample line bundle on $Z$ and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote globally generated line bundles on $Z$. Let $j > 0$ denote an integer. Then*

$$\mathrm{H}^i\big(Z \times Z, \mathcal{I}_\Delta^{j+1} \otimes ((\mathcal{L}^j \otimes \mathcal{M}_1) \boxtimes (\mathcal{L}^j \otimes \mathcal{M}_2))\big) = 0, \ \textit{for } i > 0, \tag{6.16}$$

*where $\mathcal{I}_\Delta$ denotes the sheaf of ideals defining the diagonal in $Z \times Z$.*

*Proof.* — We will assume that $\mathcal{L}$ is the line bundle defining the embedding $Z \subset \mathbb{P}^N$, and that

$$f_i : Z \to \mathcal{Z}_i := \mathbb{P}^0\big(\mathrm{H}^0(\mathcal{M}_i)^\vee\big), \ i = 1, 2,$$

are the maps defined by the globally generated line bundles $\mathcal{M}_1$ and $\mathcal{M}_2$. Let $\mathcal{L}_j$ denote the line bundle

$$\mathcal{L}_j = \mathcal{O}(-jE_Z) \otimes \pi_Z^*\big((\mathcal{L}^j \otimes \mathcal{M}_1) \boxtimes (\mathcal{L}^j \otimes \mathcal{M}_2)\big),$$

on $\mathrm{Bl}_\Delta(Z \times Z)$. We claim that the restriction morphism

$$\mathrm{H}^0\big(\mathrm{Bl}_\Delta(Z \times Z), \mathcal{L}_j\big) \to \mathrm{H}^0\big(E_Z, \mathcal{L}_j\big), \tag{6.17}$$

is surjective. To see this let $\mathcal{O}_i(1)$, for $i = 1, 2$, denote the ample generator of the Picard group of $\mathcal{Z}_i$. Consider the ample line bundle

$$\tilde{M}_j = \mathcal{O}_{2,V}(j) \boxtimes \mathcal{O}_1(1) \boxtimes \mathcal{O}_2(1),$$

on $\mathrm{Gr}_2(V) \times \mathcal{Z}$ and let $\tilde{\mathcal{L}}_j$ denote the ample pull back of $\tilde{\mathcal{M}}_j$ to $\mathcal{B}_f$ by the finite morphism in (6.11). Then by (6.6) the line bundle $\mathcal{L}_j$ is the pull back of $\tilde{\mathcal{L}}_j$ by $\mu_f$. In particular, as $\mu_f$ is part of a Stein factorization we obtain an identification

$$\mathrm{H}^0\big(\mathrm{Bl}_\Delta(Z\times Z),\mathcal{L}_j\big)=\mathrm{H}^0\big(\mathcal{B}_f,\tilde{\mathcal{L}}_j\big).$$

Assume, for a moment, that $Z$ is not of the form $\mathbb{P}(V')$ as in the assumptions of Lemma 6.6. Then $\mu_{E,f}$ is a birational morphism with connected fibres by Lemma 6.6 and Lemma 6.7. Moreover, by push-forward of the Frobenius splitting on $\mathrm{Bl}_\Delta(Z\times Z)$ we know that $\mathcal{B}_f$ is Frobenius split compatibly with $\mathcal{S}_f$ [2, Lemma 1.1.8]. Thus by [2, Ex. 1.2.E(3)] the variety $\mathcal{S}_f$ is normal, and hence

$$\mathrm{H}^0\big(E_Z,\mathcal{L}_j\big)=\mathrm{H}^0\big(\mathcal{S}_f,\tilde{\mathcal{L}}_j\big), \tag{6.18}$$

by Zariski's main theorem. Thus to prove (6.17) it suffices to prove that the restriction map

$$\mathrm{H}^0\big(\mathcal{B}_f,\tilde{\mathcal{L}}_j\big)\to\mathrm{H}^0\big(\mathcal{S}_f,\tilde{\mathcal{L}}_j\big),$$

is surjective. As $\mathcal{S}_f$ is compatibly Frobenius split in $\mathcal{B}_f$ and as $\tilde{\mathcal{L}}_j$ is ample the latter follows by general theory of Frobenius splitting [2, Thm.1.2.8]. Consider next the case $Z=\mathbb{P}(V')$. If either $\mathcal{M}_1$ or $\mathcal{M}_2$ are ample then $\mu_{E,f}$ is easily seen to be an isomorphism and we may argue as above. This leaves us with the case $\mathcal{M}_1=\mathcal{M}_2=\mathcal{O}_Z$. Then $\mathcal{Z}$ is just a 1-point space and thus $\mathcal{S}_f=\mathrm{Gr}_2(V')$ while $\mu_{E,f}$ coincides with $\tau_{E_Z}$ which is a $\mathbb{P}^1$-bundle over $\mathrm{Gr}_2(V')$. So again we obtain the identification (6.18). This proves the claim about the surjectivity of (6.17).

As the blow-up map satisfies

$$R^i(\pi_Z)_*\mathcal{O}(-jE_Z)=\begin{cases}(\mathcal{I}_\Delta)^j & \text{if } i=0,\\ 0 & \text{if } i>0,\end{cases}$$

we may reformulate the statement (6.16) as

$$\mathrm{H}^i\big(\mathrm{Bl}_\Delta(Z\times Z),\mathcal{L}_j\otimes\mathcal{O}(-E_Z)\big)=0,\ \text{for } i>0.$$

To prove the latter we consider the short exact sequence

$$0\to\mathcal{O}(-E_Z)\to\mathcal{O}_{\mathrm{Bl}_\Delta(Z\times Z)}\to\mathcal{O}_{E_Z}\to 0,$$

and apply Proposition 6.8. It follows that it suffices to prove

$$\mathrm{H}^i\big(\mathrm{Bl}_\Delta(Z\times Z),\mathcal{L}_j\big)=0,\ \text{for } i>0.$$

As $E_Z$ is compatibly Frobenius split divisor in $\mathrm{Bl}_\Delta(Z\times Z)$ we have by [2, Lemma 1.4.11] an inclusion (of abelian groups)

$$\mathrm{H}^i\big(\mathrm{Bl}_\Delta(Z\times Z),\mathcal{L}_j\big)\subset\mathrm{H}^i\big(\mathrm{Bl}_\Delta(Z\times Z),\mathcal{L}_j^p\otimes\mathcal{O}((p-1)E_Z)\big).$$

Thus, as $\mathrm{Bl}_\Delta(Z \times Z)$ is Frobenius split, it suffices to show that the line bundle

$$\mathcal{L}_j^p \otimes \mathcal{O}((p-1)E_Z) = \mathcal{O}((p(1-j)-1)E_Z) \otimes \pi_Z^*\big((\mathcal{L}^{pj} \otimes \mathcal{M}_1^p) \boxtimes (\mathcal{L}^{pj} \otimes \mathcal{M}_2^p)\big),$$

is ample on $\mathrm{Bl}_\Delta(Z \times Z)$. But the latter line bundle is by (6.6) isomorphic to the restriction to $\mathrm{Bl}_\Delta(Z \times Z)$ of the line bundle

$$\big(\mathcal{M}_1^p \otimes \mathcal{L}^{(p-1)}\big) \boxtimes \big(\mathcal{M}_2^p \otimes \mathcal{L}^{(p-1)}\big) \boxtimes \mathcal{O}_{2,V}\big(p(j-1)+1\big), \tag{6.19}$$

on $Z \times Z \times \mathrm{Gr}_2(V)$. Here $\mathcal{O}_{2,V}\big(1\big)$ denotes the ample generator of the Picard group of $\mathrm{Gr}_2(V)$. As the line bundle (6.19) is ample this ends the proof. $\square$

Theorem 6.1 is now a direct consequence of Proposition 6.9.

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Niels LAURITZEN & Jesper Funch THOMSEN
Aarhus University
Department of Mathematical Sciences
Bygning 1530
Ny Munkegade 118
8000 Århus C (Denmark)
niels@imf.au.dk
funch@imf.au.dk