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# ON THE S-FUNDAMENTAL GROUP SCHEME

by Adrian LANGER (*)

Abstract. — We introduce a new fundamental group scheme for varieties defined over an algebraically closed (or just perfect) field of positive characteristic and we use it to study generalization of C. Simpson's results to positive characteristic. We also study the properties of this group and we prove Lefschetz type theorems.

Résumé. — Nous introduisons un nouveau schéma en groupes fondamental pour les variétés définies sur un corps algébriquement clos (ou simplement parfait) de caractéristique positive. Nous utilisons ce schéma en groupes pour étudier des généralisations en caractéristique positive des résultats de C. Simpson. Nous étudions également quelques propriétés de ce schéma en groupes fondamental, en particulier nous obtenons des résultats de type "Lefschetz".

## Introduction

A. Grothendieck as a substitute of a topological fundamental group introduced the étale fundamental group, which in the complex case is just a profinite completion of the topological fundamental group. The definition uses all finite étale covers and in positive characteristic it does not take into account inseparable covers. To remedy the situation M. Nori introduced the fundamental group scheme which takes into account all principal bundles with finite group scheme structure group. In characteristic zero this recovers the étale fundamental group but in general it carries more information about the topology of the manifold. Obviously, over complex numbers the topological fundamental group carries much more information than the étale fundamental group. To improve this situation C. Simpson introduced in [38] the universal complex pro-algebraic group (or an algebraic envelope

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of the topological fundamental group in the language of [8, 10.24]). This group carries all the information about finite dimensional representations of the topological fundamental group. On this group Simpson introduced a non-abelian Hodge structure which gives rise to a non-abelian Hodge theory.

The main aim of this paper is to generalize some of his results to positive characteristic. As a first step to this kind of non-abelian Hodge theory we study the quotient of the universal complex pro-algebraic group which, in the complex case, corresponds to the Tannakian category of holomorphic flat bundles that are extensions of unitary flat bundles. Via the well known correspondence started with the work of M. S. Narasimhan and C. S. Seshadri, objects in this category correspond to semistable vector bundles with vanishing Chern classes.

In positive characteristic we take this as a starting point of our theory. In particular, in analogy to [38, Theorem 2] we prove that strongly semistable sheaves with vanishing Chern classes are locally free. We use this to prove that such sheaves are numerically flat (*ie.*, such nef locally free sheaves whose dual is also nef). We also prove the converse: all numerically flat sheaves are strongly semistable and they have vanishing Chern classes (in complex case this equivalence follows from [12, Theorem 1.18]).

This motivates our definition of the S-fundamental group scheme (see Definition 6.1). Namely, we define the S-fundamental group scheme as Tannaka dual to the neutral Tannaka category of numerically flat sheaves. Note that in this definition we do not need neither smoothness nor projectivity of the variety for which we define the S-fundamental group scheme.

However, considering reflexive sheaves with vanishing Chern classes on smooth projective varieties is sometimes much more useful. For example, notion of strong stability can be used to formulate some interesting restriction theorems (see Section 4) that are used in proofs of Lefschetz type theorems. It is also of crucial importance in several other proofs.

The S-fundamental group scheme always allows us to recover Nori's fundamental group scheme. In fact, Nori in [32] considered a closely related category of degree 0 vector bundles whose pull-backs by birational maps from smooth curves are semistable. Recently, the S-fundamental scheme group was defined in the curve case in [4, Definition 5.1] (in this case there are no problems caused, *eg.*, by non-locally free sheaves).

If the cotangent sheaf of the variety does not contain any subsheaves of non-negative slope (with respect to some fixed polarization) then in the complex case the S-fundamental group scheme is equal to Simpson's universal complex pro-algebraic group (note that the corresponding non-abelian Hodge structure is in this case trivial). In positive characteristic, under the same assumption, we prove that the S-fundamental group scheme allows us to recover all known fundamental groups like Deligne-Shiho's pro-unipotent completion of the fundamental group or dos Santos' fundamental group scheme obtained by using all $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules (or stratified sheaves). Note that in this case we also get projective (!) moduli space structure on the non-abelian cohomology set $H^1(\pi_1^S(X,x), \mathrm{GL}_k(n))$, corresponding to the Dolbeaut moduli space (this follows from Theorem 4.1).

A large part of the paper is devoted to study the properties of the S-fundamental group scheme. It satisfies the same properties as Nori's fundamental group scheme. Many of the properties are quite easy to prove but some as in the case of Nori's fundamental group scheme are quite difficult. For example, the behavior under tensor products for Nori's fundamental group scheme was studied only in [31]. The corresponding result for the S-fundamental group scheme uses completely different techniques and it is subject of the second part of this paper.

One of the main results of this paper are Lefschetz type theorems for the S-fundamental group scheme. As a corollary get the corresponding results for Nori's (and étale) fundamental groups. This corollary was proved in [3] in a much more cumbersome way using Grothendieck's Lefschetz theorems for the étale fundamental group. Our proofs are quite quick and they depend on some vanishing of cohomology proven using the techniques described by Szpiro in [39].

Our proof of the Lefschetz type theorems for the S-fundamental group scheme is quite delicate as we need to extend vector bundles from ample divisors and this usually involves vanishing of cohomology that even in characteristic zero we cannot hope for (see the last part of Section 11). A similar problem occurred in Grothendieck's proof of Lefschetz theorems for Picard groups. In this case the Picard scheme of a smooth surface in $\mathbb{P}^3$ is not isomorphic to $\mathbb{Z}$ (for example for a cubic surface) and Lefschetz theorem for complete intersection surfaces says that the component of the numerically trivial divisors in the Picard scheme is trivial (see [11, Exposé XI, Théorème 1.8]). Our theorem gives information about the Picard scheme not only in case of hypersurfaces in projective spaces but for ample divisors in arbitrary projective varieties (also if the Picard scheme of the ambient variety is non-reduced). One just needs to notice that the component of the numerically trivial divisors in the Picard scheme is equal to the group of characters of the S-fundamental group scheme.

In the higher rank case there also appears another problem: extension of a vector bundle on a divisor need not be a vector bundle. This is taken care of by Theorem 4.1 (which partially explains why we bother with semistable sheaves and not just numerically flat vector bundles).

In the last section we use the lemma of Deligne and Illusie to give a quick proof of Lefschetz type theorems for the S-fundamental group scheme for varieties which admit a lifting modulo $p^2$.

We should note that a strong version of boundedness of semistable sheaves (see [24] and [26]) is frequently used in proofs in this paper (although we could do without it in many but not all places).

To prevent the paper to grow out of a reasonable size we decided to skip many interesting topics. In future we plan to treat the (full) universal pro-algebraic fundamental group and a tame version of this group for non-proper varieties. We also plan to add some applications to the study of varieties with nef tangent bundle (for this purpose the results of this paper are already sufficient).

The structure of the paper is as follows. In Section 1 we recall a few well known results. In particular, Subsection 1.3 motivates the results of Section 4. In Section 2 we recall some boundedness results used in later proofs. We also use them to prove some results on deep Frobenius descent generalizing H. Brenner's and A. Kaid's results [20]. These results are of independent interest and they are not used later in the paper. In Section 3 we prove a restriction theorem for strongly stable sheaves with vanishing discriminant. The results of this section are used in Sections 4, 5 and 10. In Section 4 we prove the analogue of Simpson's theorem in positive characteristic. In Section 5 we prove that reflexive strongly semistable sheaves with vanishing Chern classes are numerically flat locally free sheaves. In Section 6 we finally define the S-fundamental group scheme and we compare it to other fundamental group schemes. In Section 7 we study numerically flat principal bundles and we state some results generalizing the results on the monodromy group proved in [4]. In Section 8 we study basic properties of the S-fundamental group scheme. In Section 9 we prove some vanishing theorems for the first and second cohomology groups of sheaves associated to twists of numerically flat sheaves. Finally, in Section 10 and 11 we prove Lefschetz type theorems for the S-fundamental group scheme.

After this paper was written, there appeared preprint [1] of V. Balaji and A.J. Parameswaran. In this paper the authors introduce another graded Tannaka category of vector bundles with filtrations whose quotients are degree 0 stable, strongly semistable vector bundles. The zeroth graded piece of their construction corresponds to our S-fundamental group scheme. However, unlike our group scheme their group scheme depends on the choice of polarization.

After the author send this paper to V. B. Mehta, he obtained in return another preprint [28]. In this paper Mehta also introduces the S-fundamental group scheme (using numerically flat bundles and calling it the "big fundamental group scheme"). He proves that if $G$ is semisimple then principal $G$-bundles whose pull backs to all curves are semistable come from a representation of the S-fundamental group scheme (see [Theorem 5.8, loc. cit.]). He also shows that for a smooth projective variety defined over an algebraic closure of a finite field the S-fundamental group scheme is isomorphic to Nori's fundamental group scheme (see [Remark 5.11, loc. cit.]).

## 0.1. Notation and conventions

For simplicity all varieties in the paper are defined over an algebraically closed field $k$. We could also assume that $k$ is just a perfect field but in this case our fundamental group, similarly to Nori's fundamental group, is not a direct generalization of Grothendieck's fundamental group as it ignores the arithmetic part of the group. Let us also recall that if a variety is defined over a non-algebraically closed field $k$, then the notions of stability and semistability can be defined using subsheaves defined over $k$. In case of semistability this is equivalent to geometric semistability (*ie.*, we can pass to the algebraic closure and obtain the same notion), but this is no longer the case for stability (see [20, Corollary 1.3.8 and Example 1.3.9]).

We will not need to distinguish between absolute and geometric Frobenius morphisms.

Let $E$ be a rank $r$ torsion free sheaf on a smooth $n$-dimensional projective variety $X$ with an ample line bundle $L$. Then one can define the *slope* of $E$ by $\mu(E) = c_1E \cdot c_1L^{n-1}/r$. The *discriminant* of $E$ is defined by $\Delta(E) = 2rc_2(E) - (r-1)c_1^2(E)$.

One can also define a generalized slope for pure sheaves for singular varieties but the notation becomes more cumbersome and for simplicity of notation we restrict only to the smooth case.

Semistability will always mean slope semistability with respect to the considered ample line bundle (or a collection of ample line bundles). The slope of a maximal destabilizing subsheaf of $E$ is denoted by $\mu_{\max}(E)$ and that of minimal destabilizing quotient by $\mu_{\min}(E)$.

In the following we identify locally free sheaves and corresponding vector bundles.

Let us recall that an affine $k$-scheme $\operatorname{Spec} A$ is called *algebraic* if $A$ is finitely generated as a $k$-algebra.

In this paper all representations of groups are continuous. In other words, all groups in the paper are pro-algebraic so we have a structure of a group scheme and the homomorphism defining the representation is required to be a homomorphism of group schemes.

## 1. Preliminaries

In this section we gather a few auxiliary results.

### 1.1. Numerical equivalence

Let $X$ be a smooth complete $d$-dimensional variety defined over an algebraically closed field $k$. Then an $e$-cycle $\alpha$ on $X$ is *numerically equivalent to zero* if and only if $\int_X \alpha\beta = 0$ for all $(d-e)$-cycles $\beta$ on $X$. Let $\operatorname{Num}_* X$ be the subgroup of the group of cycles $Z_*X$ generated by cycles numerically equivalent to 0. Then $N_*X = Z_*X/\operatorname{Num}_* X$ is a finitely generated free abelian group (see [14, Examples 19.1.4 and 19.1.5]).

In this paper, Chern classes of sheaves will be considered only as elements of $N_*X$.

Similarly as above one defines the *numerical Grothendieck group* $K(X)_{\mathrm{num}}$ as the Grothendieck group (ring) $K(X)$ of coherent sheaves modulo numerical equivalence, *ie.*, modulo the radical of the quadratic form given by the Euler characteristic $(a,b) \to \chi(a\cdot b) = \int_X \operatorname{ch}(a)\operatorname{ch}(b)\operatorname{td}(X)$. Here $\operatorname{ch} : K(X)_{\mathrm{num}} \otimes \mathbb{Q} \to N_*(X) \otimes \mathbb{Q}$ is the map given by the Chern character. By $\operatorname{ch}_i$ we denote the degree $i$ part of this map.

The following result is well known but the author was not able to provide a reference to its proof and hence we give it below:

Lemma 1.1. — *If a family of isomorphism classes of sheaves on $X$ is bounded then the set of Chern classes of corresponding sheaves is finite.*

*Proof.* — By definition a family is bounded if there exists a $k$-scheme $S$ of finite type and a coherent $\mathcal{O}_{S\times X}$-module $\mathcal{F}$ such that $\{\mathcal{F}_{s\times X}\}_{s\in S}$ contains all members of this family. Passing to the flattening stratification of $S$ for $\mathcal{F}$ (see, *eg.*, [20, Theorem 2.15]) we can assume that $\mathcal{F}$ is $S$-flat. Let

$q : S\times X \to X$ be the projection. For a flat family $\mathcal{F}$ the Euler characteristic $s \to \chi((\mathcal{F} \otimes q^*\alpha)_s)$ is locally constant for all classes $\alpha \in K(X)$. This implies that there are only finitely many classes of $[\mathcal{F}_s]$ in $K(X)_{\mathrm{num}}$. Since $\mathrm{ch} : K(X)_{\mathrm{num}} \otimes \mathbb{Q} \to N_*(X) \otimes \mathbb{Q}$ is an isomorphism and $N_*(X)$ is torsion free we get the required assertion. □

### 1.2. Nefness

Let us recall that a locally free sheaf $E$ on a complete $k$-scheme is called *nef* if and only if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on the projectivization $\mathbb{P}(E)$ of $E$. We say that $E$ is *numerically flat* if both $E$ and $E^*$ are nef.

A locally free sheaf $E$ is nef if and only if for any finite morphism $f : C \to X$ from a smooth projective curve $C$ we have $\mu_{\min}(f^*E) \geqslant 0$ (see, *eg.*, [2, Theorem 2.1 and p. 437]). Hence, quotients of a nef bundle are nef.

Let $f : X \to Y$ be a surjective morphism of complete $k$-varieties. Then $E$ on $Y$ is nef if and only if $f^*E$ is nef. Similarly, since pull back commutes with dualization, $E$ is numerically flat if and only if $f^*E$ is numerically flat.

### 1.3. Flatness and complex fundamental groups

Let us recall that a *flat bundle* on a complex manifold is a $\mathcal{C}^\infty$ complex vector bundle together with a flat connection. One can also look at it as a complex representation of the topological fundamental group $\pi_1(X, x)$ or a bundle associated to a local system of complex vector spaces. We say that a flat bundle is *unitary* if it is associated to a representation that factors through the unitary group. For unitary flat bundles (and extensions of unitary flat bundles) the holomorphic structure is preserved in the identification of flat bundles and Higgs bundles.

The following theorem was proven in the curve case by Narasimhan–Seshadri, and then generalized by Donaldson, Uhlenbeck–Yau and Mehta–Ramanathan to higher dimension:

Theorem 1.2 (see [30], Theorem 5.1). — *Let $X$ be a smooth $d$-dimensional complex projective manifold with an ample divisor $H$. Let $E$ be a vector bundle on $X$ with $c_1(E) = 0$ and $c_2(E)H^{d-2} = 0$. Then $E$ comes from an irreducible unitary representation of $\pi_1(X, x)$ if and only if $E$ is slope $H$-stable.*

Later C. Simpson generalized this statement to correspondence between flat bundles and semistable Higgs bundles. As a special case he obtained the following result:

Theorem 1.3 ([38], Corollary 3.10 and the following remark). — *There exists an equivalence of categories between the category of holomorphic flat bundles which are extensions of unitary flat bundles and the category of $H$-semistable bundles with $ch_1 \cdot H^{d-1} = ch_2 \cdot H^{d-2} = 0$. In particular, the latter category does not depend on the choice of ample divisor $H$.*

Let us fix a point $x \in X$. Then the above category of $H$-semistable bundles $E$ with $ch_1(E)H^{d-1} = ch_2(E)H^{d-2} = 0$ can be given the structure of a neutral Tannakian category (*cf.* [38, p. 70]) with a fibre functor defined by sending bundle $E$ to its fibre $E(x)$.

Definition 1.4. — *The affine group scheme over $\mathbb{C}$ corresponding to the above Tannakian category is called the* S-fundamental group scheme *and denoted by $\pi_1^S(X,x)$.*

In [38, Section 5] Simpson defined the *universal complex pro-algebraic group* $\pi_1^a(X,x)$ as the inverse limit of the directed system of representations $\rho : \pi_1(X,x) \to G$ for complex algebraic groups $G$, such that the image of $\rho$ is Zariski dense in $G$ (in the language of [8, 10.24] $\pi_1^a(X,x)$ is an *algebraic envelope* of the topological fundamental group). This group is Tannaka dual to the neutral Tannaka category of semistable Higgs bundles with vanishing (rational) Chern classes (and with the obvious fibre functor of evaluation at $x$). Therefore by [10, Proposition 2.21 (a)] we get the following corollary which solves the problem posed in [4, Remark 5.2]:

Corollary 1.5. — *We have a surjection $\pi_1^a(X,x) \to \pi_1^S(X,x)$ of pro-algebraic groups (or, more precisely, a faithfully flat morphism of complex group schemes).*

In general, the surjection $\pi_1^a(X,x) \to \pi_1^S(X,x)$ is not an isomorphism. For example, it is not an isomorphism for all curves of genus $g \geqslant 2$ because $\mathcal{O}_C \oplus \omega_C$ with the Higgs field given by the identity on $\omega_C$ is Higgs semistable but not semistable (after twisting by an appropriate line bundle this gives a representation of $\pi_1^a(X,x)$ not coming from $\pi_1^S(X,x)$).

If $\mu_{\max}(\Omega_X) < 0$ then $\pi_1^a(X,x) \to \pi_1^S(X,x)$ is an isomorphism. This follows from the fact that if $\mu_{\max}(\Omega_X) < 0$ then all (Higgs) semistable Higgs bundles have vanishing Higgs field and they are semistable in the usual sense. In fact, $\pi_1^a(X,x)$ and $\pi_1^S(X,x)$ are both zero by the following lemma:

Lemma 1.6. — *If $X$ is a complex manifold with $\mu_{\max}(\Omega_X) < 0$ then* $\pi_1^a(X, x) = 0$.

*Proof.* — By assumption $h^i(X, \mathcal{O}_X) = h^0(X, \Omega_X^i) = 0$ for $i > 0$. Therefore $\chi(X, \mathcal{O}_X) = 1$. Let $\pi : Y \to X$ be an étale cover. Then $\mu_{\max}(\Omega_Y) < 0$ (because $\Omega_Y = \pi^*\Omega_X$) so $\chi(Y, \mathcal{O}_Y) = 1$. But $\chi(Y, \mathcal{O}_Y) = \deg \pi \cdot \chi(X, \mathcal{O}_X)$ so $\pi$ is an isomorphism. This implies that the étale fundamental group of $X$ is trivial. But by Malcev's theorem a finitely generated linear group is residually finite so any non-trivial representation $\pi_1(X, x) \to G$ in an algebraic complex affine group gives rise to some non-trivial representation in a finite group. Therefore $\pi_1^a(X, x)$ is also trivial. □

Note that assumption immediately implies that $H^0(X, \Omega_X^{\otimes m}) = 0$ for $m > 0$. There is a well-known Mumford's conjecture (see, *eg.*, [23, Chapter IV, Conjecture 3.8.1]) saying that in this case $X$ should be rationally connected. Since rationally connected complex manifolds are simply connected we expect that all varieties in the lemma are simply connected.

## 2. Deep Frobenius descent in higher dimensions

The aim of this section is to recall some boundedness results used later in several proofs, and to generalize some results of H. Brenner and A. Kaid [7] to higher dimensions.

Let $f : \mathcal{X} \to S$ be a smooth projective morphism of relative dimension $d \geqslant 1$ of schemes of finite type over a fixed noetherian ring $R$. Let $\mathcal{O}_{\mathcal{X}/S}(1)$ be an $f$-very ample line bundle on $\mathcal{X}$. Let $\mathcal{T}_{\mathcal{X}/S}(r, c_1, \Delta; \mu_{\max})$ be the family of isomorphism classes of sheaves $E$ such that

(1) $E$ is a rank $r$ reflexive sheaf on a fibre $\mathcal{X}_s$ over some point $s \in S$.
(2) Let $H_s$ be some divisor corresponding to the restriction of $\mathcal{O}_{\mathcal{X}/S}(1)$ to $\mathcal{X}_s$. Then $c_1(E)H_s^{d-1} = c_1$ and $(\Delta(E) - (c_1(E) - r/2K_X)^2)H_s^{d-2} \leqslant \Delta$.
(3) $\mu_{\max}(E) \leqslant \mu_{\max}$.

The following theorem is a special case of [26, Theorem 3.4]. The only difference is that we allow mixed characteristic. The proof of the theorem holds in this more general case because the dependence on the characteristic is very simple (see the proof of [24, Theorem 4.4]).

Theorem 2.1. — *The family $\mathcal{T}_{\mathcal{X}/S}(r, c_1, \Delta; \mu_{\max})$ is bounded. In particular, the set of Hilbert polynomials of sheaves in $\mathcal{T}_{\mathcal{X}/S}(r, c_1, \Delta; \mu_{\max})$ is finite. Moreover, there exist polynomials $P_{\mathcal{X}/S}$, $Q_{\mathcal{X}/S}$ and $R_{\mathcal{X}/S}$ such that for any $E \in \mathcal{T}_{\mathcal{X}/S}(r, c_1, \Delta; \mu_{\max})$ we have:*

(1) $E(m)$ *is $m$-regular for* $m \geqslant P_{\mathcal{X}/S}(r, c_1, \Delta, \mu_{\max})$,
(2) $H^1(X, E(-m)) = 0$ *for* $m \geqslant Q_{\mathcal{X}/S}(r, c_1, \Delta, \mu_{\max})$,
(3) $h^1(X, E(m)) \leqslant R_{X/S}(r, c_1, \Delta, \mu_{\max})$ *for all* $m$.

*Example 2.2.* — Let $C$ be a smooth projective curve of genus $g \geqslant 1$. Let $p_1, p_2$ denote projections of $C \times C$ on the corresponding factors. Let us fix a point $x \in C$ and put $H = p_1^* x + p_2^* x$. Let $\Delta \subset C \times C$ be the diagonal. Finally, set $L_n = \mathcal{O}_{C\times C}(n(H - \Delta))$. Then $c_1(L_n)H = 0$ and $\Delta(L_n) = 0$ but the family $\{L_n\}_{n\in\mathbb{Z}}$ is not bounded. This shows that in the definition of the family $\mathcal{T}(r, c_1, \Delta; \mu_{\max})$ we cannot replace the bound on $(\Delta(E) - (c_1(E) - r/2K_X)^2)H^{d-2}$ with the bound on $\Delta(E)H^{n-2}$ as the family need not be bounded.

The following corollary generalizes [7, Lemma 3.2]:

Corollary 2.3. — *There exists some constant $c = c_{\mathcal{X}/S}(r, c_1, \Delta; \mu_{\max})$ such that for any (possibly non-closed) point $s \in S$ the number of reflexive sheaves $E$ of rank $r$ with fixed $c_1(E)H^{d-1} = c_1$, $(\Delta(E) - (c_1(E) - r/2K_X)^2)H^{d-2} \leqslant \Delta$ and $\mu_{\max}(E) \leqslant \mu_{max}$ is bounded from above by $|k(s)|^c$.*

*Proof.* — By the above theorem there are only finitely many possibilities for the Hilbert polynomial of $E$, so we can fix it throughout the proof. Let us take $E$ as above on the fibre $X_s$ over a point $s \in S$ with finite $k(s)$ (if $k(s)$ is infinite then our assertion is trivially satisfied). By the above theorem if we take $m = P_{\mathcal{X}/S}(r, c_1, \Delta, \mu_{\max}) + 1$ then $E(m)$ is globally generated by $a = P(E)(m)$ sections. Let us define $E'$ using the sequence

$$0 \to E' \to \mathcal{O}_{X_s}(-mH_s)^a \to E \to 0.$$

Clearly, the Hilbert polynomial of $E'$ depends only on the Hilbert polynomials of $E$ and $H_s$. Since $\mu_{\max}(E') \leqslant \mu(\mathcal{O}_{X_s}(-mH_s)) = -mH_s^d$ we can again use the above theorem to find some explicit $m'$ such that $E'(m')$ is globally generated by $b = P(E')(m') = a\chi(\mathcal{O}_{X_s}((m'-m)H)) - P(E)(m')$ sections. Therefore $E$ is a cokernel of some map

$$\mathcal{O}_{X_s}(-m'H_s)^b \to \mathcal{O}_{X_s}(-mH_s)^a.$$

Then we can conclude similarly as in the proof of [7, Lemma 3.2]. Namely, we can assume that the dimension of $H^0(\mathcal{O}_{X_s}((m'-m)H_s))$ is computed by the Hilbert polynomial of $\mathcal{O}_{X_s}$ (possibly we need to increase $m'$ but only by some function depending on $\mathcal{X}/S$: for example we can apply the above theorem to the rank 1 case). Then the number of the sheaves that we consider is bounded from the above by $|k(s)|^c$, where $c = ab\chi(\mathcal{O}_X((m'-m)H))$. □

Let $R$ be a $\mathbb{Z}$-domain of finite type containing $\mathbb{Z}$. Let $f : \mathcal{X} \to S = \operatorname{Spec} R$ be a smooth projective morphism of relative dimension $d \geqslant 1$ and let $\mathcal{O}_{\mathcal{X}}(1)$ be an $f$-very ample line bundle.

Let $K$ be the quotient field of $R$. Let $\mathcal{X}_0 = \mathcal{X} \times_S \operatorname{Spec} K$ be the generic fibre of $f$. Let $\mathcal{E}$ be an $S$-flat family of rank $r$ torsion free sheaves on the fibres of $f$. Let us choose an embedding $K \subset \mathbb{C}$. Then for the restriction $\mathcal{E}_0$ of $\mathcal{E}$ to $\mathcal{X}_0$ we consider $\mathcal{E}_{\mathbb{C}} = \mathcal{E}_0 \otimes \mathbb{C}$.

We say that $(s_n, e_n)_{n\in\mathbb{N}}$, where $s_n \in S$ are closed points and $e_n$ are positive integers, is a *Frobenius descent sequence* for $\mathcal{E}$ if there exist coherent sheaves $\mathcal{F}_n$ on the fibres $\mathcal{X}_{s_n}$ such that $\mathcal{E}_{\mathcal{X}_{s_n}} \simeq (F^{e_n})^*\mathcal{F}_n$.

The following theorem generalizes [7, Theorem 3.4] to higher dimensions and relates the notion of flatness in positive characteristic to the one coming from complex geometry:

Theorem 2.4. — *Let us assume that there exists a Frobenius descent sequence $(s_n, e_n)_{n\in\mathbb{N}}$ for $\mathcal{E}$ with $(e_n - |k(s_n)|^c)_{n\in\mathbb{N}} \to \infty$, where $c$ is the constant from Corollary 2.3. Then the restriction $\mathcal{E}_0$ of $\mathcal{E}$ to the generic fibre of $f$ is an extension of stable (with respect to an arbitrary polarization) locally free sheaves with vanishing Chern classes. Moreover, $\mathcal{E}_{\mathbb{C}}$ is also an extension of slope stable locally free sheaves with vanishing Chern classes (note that these stable sheaves need not be extensions of sheaves defined over $K$). In particular, $\mathcal{E}_{\mathbb{C}}$ has structure of a holomorphic flat bundle on $X_{\mathbb{C}}$ which is an extension of unitary flat bundles.*

*Proof.* — Note that we can assume that $S$ is connected. Then by $S$-flatness of $\mathcal{E}$ the numbers $c_i = c_i(\mathcal{E}_s) \cdot c_1(\mathcal{O}_{\mathcal{X}_s}(1))^{d-i}$ are independent of $s \in S$. Since

$$c_i(\mathcal{E}_{s_n}) \cdot c_1(\mathcal{O}_{\mathcal{X}_{s_n}}(1))^{d-i} = (\operatorname{char} k(s_n))^{e_n} c_i(\mathcal{F}_n) \cdot c_1(\mathcal{O}_{\mathcal{X}_{s_n}}(1))^{d-i}$$

and $e_n \to \infty$ we see that $c_i = 0$. The rest of the proof is the same as the proof of [7, Theorem 3.4] using Corollary 2.3 instead of [7, Lemma 3.2]. The final part of the theorem follows from [38, Theorem 2] and [38, Lemma 3.5].

Alternatively, we can use Theorem 4.1 as for large $n$ the sheaves $\mathcal{E}_{s_n}$ are strongly semistable as follows from the proof. Hence by Theorem 4.1 $\mathcal{E}_{s_n}$ are locally free for large $n$ which implies that $\mathcal{E}_0$ is locally free by openness of local freeness. Then one can consider the Jordan–Hölder filtration of $\mathcal{E}_0$, extend it to some filtration over nearby fibres and use induction on the rank as in the proof of Theorem 4.1. □

## 3. Restriction theorem for strongly stable sheaves with vanishing discriminant

In this section we prove the restriction theorem for strongly stable sheaves. It is used in the next section and it also plays an important role in proofs of the Lefschetz type theorems for the S-fundamental group (see, *eg.*, proof of Theorem 10.2).

Let us consider $\mathbb{P}^2$ over an algebraically closed field of characteristic $p > 0$. In [6] H. Brenner showed that the restriction of $\Omega_{\mathbb{P}^2}$ to a curve $x^d + y^d + z^d = 0$, where $p^e < d < 3/2p^e$ for some integer $e$, is not strongly stable. Hence the restriction of a strongly stable sheaf to a smooth hypersurface of large degree need not be strongly stable. But by [24, Theorem 5.2] restriction of a strongly stable sheaf with trivial discriminant to a hypersurface of large degree is still strongly stable (the bound on the degree of this hypersurface depends on the rank of the sheaf). However, in this case we have the following stronger version of restriction theorem (valid in arbitrary characteristic):

Theorem 3.1. — *Let $D_1, \dots, D_{d-1}$ be a collection of ample divisors on $X$ of dimension $d \geqslant 2$. Let $E$ be a rank $r \geqslant 2$ torsion free sheaf with $\Delta(E)D_2 \dots D_{d-2} = 0$. Assume that $E$ is strongly $(D_1, \dots, D_{d-1})$-stable. Let $D \in |D_1|$ be any normal effective divisor such that $E_D$ has no torsion. Then $E_D$ is strongly $(D_2, \dots, D_{d-1})_D$-stable.*

*Proof.* — For simplicity of notation we proof the result in case when all the divisors $D_1, \dots, D_{d-1}$ are equal to one ample divisor denoted by $H$. The general proof is exactly the same.

Let $\Delta(E)H^{d-2} = 0$ and assume that $E$ is strongly $H$-stable. Let $D \in |H|$ be any normal effective divisor such that $E_D$ has no torsion. We need to prove that $E_D$ is strongly $H_D$-stable. Suppose that there exists a non-negative integer $k_0$ such that the restriction of $\tilde{E} = (F^{k_0})^*E$ to $D$ is not stable. Let $S$ be a rank $\rho$ saturated destabilizing subsheaf of $\tilde{E}_D$. Set $T = (\tilde{E}_D)/S$. Let $G$ be the kernel of the composition $\tilde{E} \to \tilde{E}_D \to T$. From the definition of $G$ we get a short exact sequence:

$$0 \to G \to \tilde{E} \to T \to 0.$$

Applying the snake lemma to the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & S \arrow[d] & \\
0 \arrow[r] & \tilde{E}(-D) \arrow[r] \arrow[d] & \tilde{E} \arrow[r] \arrow[d] & \tilde{E}_D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] & \tilde{E} \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

we also get the following exact sequence:

$$0 \to \tilde{E}(-D) \to G \to S \to 0.$$

Computing $\Delta(G)$ we get

$$\Delta(G)H^{d-2} = -\rho(r-\rho)H^d + 2(rc_1(T) - (r-\rho)Dc_1(\tilde{E}))H^{d-2}.$$

By assumption $(rc_1(T) - (r-\rho)Dc_1(\tilde{E}))H^{d-2} \leqslant 0$, so

$$\Delta(G)H^{d-2} \leqslant -\rho(r-\rho)H^d.$$

By [24, Theorem 2.7], for large $l$ we have $\mu_{\max}((F^l)^*G) = L_{\max}((F^l)^*G)$ and similarly for $\mu_{\min}$. Using strong $H$-stability of $\tilde{E}$ and $\tilde{E}(-D)$ we get for large integers $l$

$$\begin{aligned} L_{\max}((F^l)^*G) - \mu((F^l)^*G) &= \mu_{\max}((F^l)^*G) - \mu((F^l)^*\tilde{E}) + \frac{r-\rho}{r}p^lH^d \\ &\leqslant \frac{r-\rho}{r}p^lH^d - \frac{1}{r(r-1)} \end{aligned}$$

and

$$\begin{aligned} \mu((F^l)^*G) - L_{\min}((F^l)^*G) &= \mu((F^l)^*\tilde{E}(-D)) - \mu_{\min}((F^l)^*G) + \frac{\rho}{r}p^lH^d \\ &\leqslant \frac{\rho}{r}p^lH^d - \frac{1}{r(r-1)}. \end{aligned}$$

Hence, applying [24, Theorem 5.1] to $(F^l)^*G$ gives

$$\begin{aligned} 0 &\leqslant H^d \cdot \Delta((F^l)^*G)H^{d-2} + r^2(L_{\max}((F^l)^*G) - \mu(F^l)^*G))(\mu((F^l)^*G) \\ &\quad - L_{\min}((F^l)^*G)) \\ &\leqslant -\rho(r-\rho)p^{2l}(H^d)^2 + r^2\left(\frac{r-\rho}{r}p^lH^d - \frac{1}{r(r-1)}\right)\left(\frac{\rho}{r}p^lH^d - \frac{1}{r(r-1)}\right). \end{aligned}$$

Therefore

$$\frac{r}{r-1}p^lH^d \leqslant \frac{1}{(r-1)^2},$$

which gives a contradiction. $\square$

Later we show a much stronger restriction theorem for semistability (see Corollary 5.3) but we need this weaker result to establish Theorem 4.1 used in the proof of this stronger result.

## 4. Strongly semistable sheaves with vanishing Chern classes

In this section we show that strongly semistable torsion free sheaves with vanishing Chern classes are locally free and that they are strongly semistable with respect to all polarizations.

The following theorem is an analogue of [38, Theorem 2] in positive characteristic. However, we need a different proof as Simpson's proof uses Lefschetz hyperplane theorem for topological fundamental groups and the correspondence between flat (complex) bundles and semistable Higgs bundles with vanishing Chern classes (see [38, Lemma 3.5]). We reverse his ideas and we use this result to prove Lefschetz type theorems for étale, Nori and S-fundamental groups.

Theorem 4.1. — *Let $X$ be a smooth $d$-dimensional projective variety over an algebraically closed field $k$ of characteristic $p > 0$ and let $H$ be an ample divisor on $X$. Let $E$ be a strongly $H$-semistable torsion free sheaf on $X$ with $\mathrm{ch}_1(E) \cdot H^{d-1} = 0$ and $\mathrm{ch}_2(E) \cdot H^{d-2} = 0$. Assume that either $E$ is reflexive or the reduced Hilbert polynomial of $E$ is equal to the Hilbert polynomial of $\mathcal{O}_X$. Then $E$ is an extension of stable and strongly semistable locally free sheaves with vanishing Chern classes. Moreover, there exists $n$ such that $(F^n)^*E$ is an extension of strongly stable locally free sheaves with vanishing Chern classes.*

*Proof.* — Before starting the proof of the theorem let us prove the following lemma:

Lemma 4.2. — *Let $E$ be a strongly $H$-semistable torsion free sheaf on $X$ with $\mathrm{ch}_1(E) \cdot H^{d-1} = 0$ and $\mathrm{ch}_2(E) \cdot H^{d-2} = 0$. Then the 1-cycle $c_1(E)H^{d-2}$ is numerically trivial and $\Delta(E)H^{d-2} = 0$.*

*Proof.* — By [24, Theorem 3.2] we have $\Delta(E)H^{d-2} \geqslant 0$. Therefore by the Hodge index theorem

$$0 = 2r(\mathrm{ch}_2(E)H^{d-2}) = (c_1(E)^2 - \Delta(E))H^{d-2} \leqslant c_1(E)^2H^{d-2} \leqslant \frac{(c_1(E)H^{d-1})^2}{H^d} = 0,$$

which implies the required assertions. □

In case of curves the theorem follows from the existence of the Jordan–Hölder filtration. The proof is by induction on the dimension starting with dimension 2.

If $X$ is a surface then we prove that a strongly semistable torsion free sheaf $E$ on $X$ with $\mathrm{ch}_1(E)\cdot H=0$ and $\mathrm{ch}_2(E)=0$ is an extension of stable and strongly semistable locally free sheaves with vanishing Chern classes. This part of the proof is well known and analogous to the proof of [38, Theorem 2]. Namely, the reflexivization $E^{**}$ is locally free and strongly semistable. Hence by [24, Theorem 3.2] $\Delta(E^{**})\geqslant 0$. Since $\Delta(E^{**})\leqslant\Delta(E)$ and by the above lemma $\Delta(E)=0$, we have $c_2(E^{**}/E)=0$. This implies that $E^{**}/E$ is trivial and $E$ is locally free. The required assertion follows easily from this fact (it will also follow from the proof below).

Now fix $d\geqslant 3$ and assume that the theorem holds in dimensions less than $d$. Let $E$ be a strongly stable reflexive sheaf on $d$-dimensional $X$ with $\mathrm{ch}_1(E)\cdot H^{d-1}=0$ and $\mathrm{ch}_2(E)\cdot H^{d-2}=0$. Then by the above lemma all the sheaves $\{(F^n)^*E\}_{n\in\mathbb{N}}$ are in the family $\mathcal{T}_{X/k}(r,0,0;0)$. This family is bounded by Theorem 2.1. Therefore, since by Lemma 1.1 there are only finitely many classes among $c_i((F^n)^*E)=p^nc_i(E)$, we see that the Chern classes of $E$ vanish. In particular, for any smooth divisor $D$ on $X$ the reduced Hilbert polynomial of $E_D$ is equal to the Hilbert polynomial of $\mathcal{O}_D$. Let us also remark that $E_D$ is torsion free (see, *eg.*, [20, Corollary 1.1.14]).

Let us first assume that $E$ is strongly stable. By Theorem 3.1 the restriction $E_D$ is also strongly stable for all smooth divisors $D\in|mH|$ and all $m\geqslant 1$. In particular, $E_D$ is locally free by the induction assumption. Note that if $x\in D$ then $E\otimes k(x)\simeq E_D\otimes k(x)$ is an $r$-dimensional vector space over $k(x)\simeq k$. Therefore by Nakayama's lemma $E$ is locally free at $x$. By Bertini's theorem (see, *eg.*, [13, Theorem 3.1]) for any closed point $x\in X$ there exists for large $m$ a smooth hypersurface $D\in|mH|$ containing $x$. Therefore $E$ is locally free at every point of $X$, *ie.*, it is locally free.

Now let us consider the general case. Let us choose $m$ such that all quotients in a Jordan-Hölder filtration of $(F^m)^*E$ are strongly stable (clearly such $m$ exists). Then we can prove the result by induction on the rank $r$. Namely, if

$$0=E_0\subset E_1\subset\cdots\subset E_l=(F^m)^*E$$

is the Jordan-Hölder filtration then $E_1$ is reflexive with $c_1(E_1)H^{d-1}=0$ and $\Delta(E_1)H^{d-2}=0$. The last equality follows from Bogomolov's inequality for strongly semistable sheaves (see [24, Theorem 3.2]) and from the inequality $\Delta(E_1)H^{d-2}\leqslant\Delta(E)H^{d-2}$ obtained from the Hodge index theorem (see, *eg.*, [20, Corollary 7.3.2]). So by the above we know that $E_1$ is locally free with vanishing Chern classes. Note that $\{(F^n)^*(((F^m)^*E)/E_1)\}_{n\in\mathbb{N}}$ are semistable torsion free quotients of the sheaves from a bounded family. Therefore by Grothendieck's lemma (see [20, Lemma 1.7.9]) they also form a bounded family and by the previous argument they have vanishing Chern classes. Hence the reduced Hilbert polynomial of $((F^m)^*E)/E_1$ is equal to the Hilbert polynomial of $\mathcal{O}_X$ and we can apply the induction assumption to conclude that $((F^m)^*E)/E_1$ is locally free. This implies that all the quotients in the Jordan-Hölder filtration of $(F^m)^*E$ are locally free, which proves the last assertion of the theorem. Then the first assertion follows just by taking any Jordan-Hölder filtration of $E$.

Now we assume that the reduced Hilbert polynomial of $E$ is equal to the Hilbert polynomial of $\mathcal{O}_X$ but we do not assume that $E$ is reflexive. Then the reflexivization $E^{**}$ of $E$ satisfies the previous assumptions and hence it is locally free with vanishing Chern classes. Therefore the reduced Hilbert polynomial of $E^{**}$ is also equal to the Hilbert polynomial of $\mathcal{O}_X$. In particular, the Hilbert polynomial of the quotient $T = E^{**}/E$ is trivial and hence $T = 0$ and $E$ is reflexive. So we reduced the assertion to the previous case (without changing the rank which is important because of the induction step). □

Note that the theorem fails if $d \geqslant 3$ and we do not make any additional assumptions on the Hilbert polynomial or reflexivity of $E$. For example one can take the ideal sheaf of a codimension $\geqslant 3$ subscheme. This sheaf is strongly stable and torsion free with $\mathrm{ch}_1(E)\cdot H^{d-1}=0$ and $\mathrm{ch}_2(E)\cdot H^{d-2}=0$ but it is not locally free.

Corollary 4.3. — *Let $E$ be a locally free sheaf with* $\mathrm{ch}_1(E)\cdot H^{d-1} = 0$ *and* $\mathrm{ch}_2(E) \cdot H^{d-2} = 0$. *Let $D \in |H|$ be any normal effective divisor. If $E$ is strongly semistable then $E_D$ is strongly semistable.*

*Proof.* — By the above theorem we can choose $m$ such that all quotients in a Jordan-Hölder filtration of $(F^m)^*E$ are locally free and strongly stable. Then by Theorem 3.1 the restriction of each quotient is strongly stable which proves the corollary. □

*Remark 4.4.* — Let us remark that in general a strongly semistable locally free sheaf on a smooth projective variety does not restrict to a semistable sheaf on a general smooth hypersurface of large degree passing through a fixed point (not even in characteristic 0).

For example one can take a non-trivial extension $E$ of $m_x$ by $\mathcal{O}_{\mathbb{P}^2}$ for some $x \in \mathbb{P}^2$. Then $E$ is a strongly semistable locally free sheaf but the restriction of $E$ to any curve passing through $x$ is not semistable.

This shows that one cannot generalize the proof of Mehta–Ramanathan's theorem to prove stability of the restriction of a stable sheaf to a general hyperplane passing through some fixed points (the proof for restriction of stable sheaves uses restriction of semistable sheaves).

The following theorem says that strong semistability for locally free sheaves with vanishing Chern classes does not depend on the choice of polarization:

Proposition 4.5. — *Let $D_1, \dots, D_{d-1}$ be ample divisors on $X$. Let $E$ be a strongly $(D_1, \dots, D_{d-1})$-semistable reflexive sheaf on $X$ with $\mathrm{ch}_1(E) \cdot D_1 \dots D_{d-1} = 0$ and $\mathrm{ch}_2(E) \cdot D_2 \dots D_{d-1} = 0$. Then it is locally free with vanishing Chern classes and it is strongly semistable with respect to an arbitrary collection of ample divisors.*

*Proof.* — The first assertion can be proven as in the above theorem. So it is sufficient to prove that for any ample divisor $A$ the sheaf $E$ is strongly $(A, D_2, \dots, D_{d-1})$-semistable. We can assume that $D_2, \dots D_{d-1}$ are very ample. Taking a general complete intersection of divisors in $|D_2|, \dots, |D_{d-1}|$ and using Theorem 3.1 we see that it is sufficient to prove the assertion in the surface case. In the following we assume that $d = 2$ and set $H = D_1$. Taking the Jordan–Hölder filtration of some Frobenius pull-back of $E$ we can also assume that $E$ is in fact strongly $H$-stable.

Let us consider the family $\mathcal{F}$ of all sheaves $E'$ such that $\mu_A(E') > 0$ and there exists a non-negative integer $n$ such that $E' \subset (F^n)^*E$ and the quotient $(F^n)^*E/E'$ is torsion free. Let us set $H_t = (1-t)H + tA$ for $t \in [0,1]$. Since the family $\{(F^n)^*E\}_n$ is bounded, the family $\mathcal{F}$ is also bounded by Grothendieck's lemma [20, Lemma 1.7.9]. Therefore there exists the largest rational number $0 < t < 1$ such that for all sheaves $E' \in \mathcal{F}$ we have $\mu_{H_t}(E') \leqslant 0$ (note that by assumption $\mu_H(E') < 0$). Then there exists a sheaf $E' \in \mathcal{F}$ such that $\mu_{H_t}(E') = 0$.

If $E'$ is not strongly $H_t$-semistable then there exist an integer $l$ and a saturated subsheaf $E'_1 \subset (F^l)^*E'$ such that $\mu_{H_t}(E'_1) > \mu_{H_t}((F^l)^*E') = 0$. But $E'_1 \in \mathcal{F}$ so we have a contradiction with our choice of $t$. Therefore $E'$ is strongly $H_t$-semistable.

Let us take integer $n_0$ such that $E' \subset (F^{n_0})^*E$. Similarly as above we can prove that the quotient $E'' = (F^{n_0})^*E/E'$ is strongly $H_t$-semistable. Namely, if $E''$ is not strongly $H_t$-semistable then there exist an integer $l$ and a quotient sheaf $(F^l)^*E'' \to E''_1$ such that $\mu_{H_t}(E''_1) < \mu_{H_t}((F^l)^*E'') = 0$. But then the kernel of $(F^{l+n_0})^*E \to E''_1$ is in $\mathcal{F}$ and it has positive slope with respect to $H_t$ which contradicts our choice of $t$.

Therefore all the sheaves in the following exact sequence

$$0 \to E' \to (F^{n_0})^* E \to E'' \to 0$$

are strongly $H_t$-semistable with $H_t$-slope equal to 0. Now let us recall that by the Hodge index theorem we have

$$\begin{aligned} 0 = \frac{\Delta((F^{n_0})^* E)}{r} &= \frac{\Delta(E')}{r'} + \frac{\Delta(E'')}{r''} - \frac{r'r''}{r}\left(\frac{c_1 E'}{r'} - \frac{c_1 E''}{r''}\right)^2 \\ &\geqslant \frac{\Delta(E')}{r'} + \frac{\Delta(E'')}{r''} - \frac{r'r''}{rH_t^2}(\mu_{H_t}(E') - \mu_{H_t}(E'')). \end{aligned}$$

But by [24, Theorem 3.2] we have $\Delta(E') \geqslant 0$, $\Delta(E'') \geqslant 0$. Since $\mu_{H_t}(E') = \mu_{H_t}(E'') = 0$ we see that both $\Delta(E')$ and $\Delta(E'')$ are equal to 0. Therefore by Theorem 4.1 both $E'$ and $E''$ have vanishing Chern classes which contradicts strong $H$-stability of $E$. □

*Remark 4.6.* — Note that nefness of $D_1$ is not sufficient to get the assertion of the theorem. For example, if $X$ is a surface and $D_1$ is a numerically non-trivial nef divisor with $D_1^2 = 0$ (*eg.*, a fibre of a morphism of $X$ onto a curve) then the family $\{\mathcal{O}_X(nD_1) \oplus \mathcal{O}_X(-nD_1)\}_n$ is not bounded although it consists of strongly $D_1$-semistable locally free sheaves with $\mathrm{ch}_1 \cdot D_1 = 0$ and $\mathrm{ch}_2 = 0$.

## 5. Comparison with numerically flat bundles

In this section we compare strongly semistable vector bundles with vanishing Chern classes with numerically flat vector bundles and we show that they can be used to define a Tannaka category.

Let $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ denotes the full subcategory of the category of coherent sheaves on $X$, having as objects all strongly $H$-semistable reflexive sheaves with $\mathrm{ch}_1(E) \cdot H^{d-1} = 0$ and $\mathrm{ch}_2(E) \cdot H^{d-2} = 0$. By Proposition 4.5, $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ does not depend on the choice of $H$ so we do not include it into notation.

Let us mention that in the complex case the above category can be identified with the category of numerically flat vector bundles (see Theorem 1.3 and [12, Theorem 1.18]). The author does not know a direct purely algebraic proof of this equivalence (over $\mathbb{C}$). A similar characterization holds also in positive characteristic:

Proposition 5.1. — *Let $X$ be a smooth projective $k$-variety. Let $E$ be a coherent sheaf on $X$. Then the following conditions are equivalent:*

1. $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$,
2. *$E$ is numerically flat (in particular, it is locally free),*
3. *$E$ is nef of degree* $0$ *with respect to some ample divisor (in particular, it is locally free).*

*Proof.* — First we prove that 1 implies 2. If $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ then the family $\{(F^n)^*E\}_n$ is bounded, so there exists an ample line bundle $L$ on $X$ such that $(F^n)^*E \otimes L$ is globally generated for $n = 0, 1, \dots$ Therefore for any smooth projective curve $C$ and a finite morphism $f : C \to X$ the bundles $f^*((F^n)^*E \otimes L)$ are globally generated. In particular, $\mu_{\min}(f^*((F^n)^*E \otimes L)) \geqslant 0$. Therefore for all $n \geqslant 0$

$$-\deg f^*L \leqslant \mu_{\min}(f^*((F^n)^*E)) \leqslant p^n \mu_{\min}(f^*E).$$

Dividing by $p^n$ and passing with $n$ to infinity we get $\mu_{\min}(f^*E) \geqslant 0$. Therefore $E$ is nef. Since $E^* \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$, $E^*$ is also nef.

To prove that 2 implies 3 we take $E$ such that both $E$ and $E^*$ are nef. Let us fix some ample divisor $H$ on $X$. Let us remark that if some polynomial in Chern classes of ample vector bundle is positive (see [15, p. 35] for the definition) then it is also non-negative for nef vector bundles. Therefore by [15, Theorem I] $c_1 \cdot H^{d-1}, c_2 \cdot H^{d-2}, (c_1^2 - c_2) \cdot H^{d-2}$ are non-negative for all nef vector bundles. In particular, from $c_1(E)H^{d-1} \geqslant 0$ and $c_1(E^*)H^{d-1} \geqslant 0$ we get $c_1(E)H^{d-1} = 0$.

To prove that 3 implies 1 note that $E$ is strongly semistable with respect to all polarizations. By assumption and the Hodge index theorem we have

$$0 \leqslant c_1^2(E)H^{d-2} \cdot H^d \leqslant (c_1(E)H^{d-1})^2 = 0.$$

Hence from non-negativity of $c_2 \cdot H^{d-2}$,$(c_1^2 - c_2) \cdot H^{d-2}$ we see that $c_2(E)H^{d-2}$ is equal to 0. Therefore by definition $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$. □

*Remark 5.2.* — Note that the condition that a locally free sheaf $E$ is numerically flat is equivalent to the condition that for all smooth curves $C$ and all maps $f : C \to X$ the pull back $f^*E$ is semistable of degree 0. In this case one sometimes says that $E$ is *Nori semistable* (see, *eg.*, [28]). Obviously, this is completely fair as Nori made huge contributions in the subject but it should be noted that in [32] Nori considered a slightly different condition. Namely, he considered locally free sheaves $E$ such that for all smooth curves $C$ and all birational maps $f : C \to X$ onto its image, the pull back $f^*E$ is semistable of degree 0 (see [32, p. 81, Definition]). In positive characteristic such sheaves do not form a tensor category.

Note that the proof of the above proposition gives another proof of Proposition 4.5. As in the proof of Proposition 4.5 we can restrict to the surface case so that we deal with only one ample divisor when the above proof shows the assertion (in general however, there are technical problems with boundedness with respect to collection of polarizations).

Proposition 5.1 allows us to define $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ for complete $k$-schemes. Then $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ denotes the full subcategory of the category of coherent sheaves on $X$, which as objects contains all numerically flat locally free sheaves. If $X$ is smooth and projective then by Proposition 5.1 this gives the same category as before.

The following corollary is a generalization of Theorem 3.1:

Corollary 5.3. — *[Very strong restriction theorem] Let $X$ be a complete $k$-scheme and let $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$. Then for any closed subscheme $Y \subset X$ the restriction $E_Y$ is in $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{Y})$.*

Clearly, $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ if and only if the restriction of $E$ to every curve $C$ belongs to the category $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{C})$. This gives relation with the category considered by Nori in [32].

By [2, Proposition 3.5] tensor product of two nef sheaves is nef. Therefore we have the following corollary:

Corollary 5.4. — *Let $X$ be a complete $k$-scheme. If $E_1, E_2 \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ then $E_1 \otimes E_2 \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$.*

Proposition 5.5. — *Let $X$ be a complete connected reduced $k$-scheme. Then $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ is a rigid $k$-linear abelian tensor category.*

*Proof.* — By the above corollary $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ is a tensor category. To check that it is abelian, it is sufficient to check that for any homomorphism $\varphi : E_1 \to E_2$ between objects $E_1$ and $E_2$ of $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ its kernel and cokernel is still in the same category. Restricting to curves it is easy to see that $\ker \varphi$, $\operatorname{im} \varphi$ and $\operatorname{coker} \varphi$ are locally free (see, *eg.*, [32, proof of Lemma 3.6]). Since quotients of nef bundles are nef and since we have surjections $E_1 \to \operatorname{im} \varphi$ and $E_2^* \to (\operatorname{im} \varphi)^*$, $\operatorname{im} \varphi$ is numerically flat. This implies that $\ker \varphi$ and $\operatorname{coker} \varphi$ are also numerically flat. □

For definitions and basic properties of rigid, tensor and Tannakian categories we refer the reader to [10].

## 6. Fundamental groups in positive characteristic

In this section we generalize the notion of S-fundamental group scheme, defined in the curve case by Biswas, Parameswaran and Subramanian in

[4, Section 5], and we compare it with other known fundamental group schemes.

Let $X$ be a complete connected reduced $k$-scheme and let $x \in X$ be a $k$-point. Let us define the fibre functor $T_x : \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X}) \to \mathrm{k}-\mathrm{mod}$ by sending $E$ to its fibre $E(x)$. Then $(\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X}), \otimes, \mathrm{T_x}, \mathcal{O}_{\mathrm{X}})$ is a neutral Tannaka category (see Proposition 5.5). Therefore by [10, Theorem 2.11] the following definition makes sense:

Definition 6.1. — *The affine $k$-group scheme Tannaka dual to this neutral Tannaka category is denoted by $\pi_1^S(X,x)$ and it is called the* S-fundamental group scheme *of $X$ with base point $x$.*

By definition, there exists an equivalence of categories $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X}) \to \pi_1^S(X,x) - \mathrm{mod}$ such that $T_x$ becomes a forgetful functor for $\pi_1^S(X,x)$-modules. Inverse of this equivalence defines a principal $\pi_1^S(X,x)$-bundle $\tilde{X}^S \to X$, called the *S-universal covering*, which to a $\pi_1^S(X,x)$-module associates a numerically flat vector bundle.

Let $\pi_1^N(X,x)$ and $\pi_1^{Et}(X,x)$ denote Nori and étale fundamental group schemes, respectively. Using [10, Proposition 2.21 (a)] it is easy to see that the following lemma holds:

Lemma 6.2. — *There exist natural faithfully flat homomorphisms $\pi_1^S(X,x) \to \pi_1^N(X,x) \to \pi_1^{Et}(X,x)$ of affine group schemes.*

Since on curves there exist strongly stable vector bundles of degree zero and rank $r > 1$ (such vector bundles are numerically flat but not essentially finite), $\pi_1^S(X,x) \to \pi_1^N(X,x)$ is usually not an isomorphism. In fact, already non-torsion line bundles of degree 0 show that the S-fundamental group scheme is usually much larger that Nori or étale fundamental group schemes.

By definition and [10, Corollary 2.7] $\pi_1^S(X,x)$ is isomorphic to the inverse limit of the directed system of representations $\rho : \pi_1^S(X,x) \to G$ in algebraic $k$-group schemes $G$, such that the image of $\rho$ is Zariski dense in $G$. If we restrict to representations of $\pi_1^S(X,x)$ in finite group schemes or in étale finite group schemes then we obtain $\pi_1^N(X,x)$ and $\pi_1^{Et}(X,x)$, respectively. We can summarize this using the following obvious lemma. The formulation for the étale fundamental group is left to the reader.

Lemma 6.3. — *$\pi_1^N(X,x)$ is characterized by the following universal property: for any representation $\rho : \pi_1^S(X,x) \to G$ in a finite $k$-group scheme $G$, there is a unique extension to $\overline{\rho} : \pi_1^N(X,x) \to G$ such that the* *diagram*

$$\begin{tikzcd} \pi_1^S(X,x) \arrow[r, "\rho"] \arrow[d] & G \\ \pi_1^N(X,x) \arrow[ur, "\overline{\rho}"'] & \end{tikzcd}$$

*is commutative.*

In [36] dos Santos used [17] to introduce another fundamental group scheme, which we denote by $\pi_1^F(X,x)$. It is defined as the group scheme Tannaka dual to the Tannakian category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules (corresponding to the so called flat or stratified bundles; see [17]).

Let us recall that there exist $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules $(E,\nabla)$ for which $E$ is not semistable (see [16, proof of Theorem 1]). Similarly, not every numerically flat bundle descends infinitely many times under the Frobenius morphism. Therefore, in general, we cannot expect any natural homomorphism between $\pi_1^S(X,x)$ and $\pi_1^F(X,x)$. But as expected from the complex case (see Corollary 1.5), if $\mu_{\max}(\Omega_X) < 0$ then the S-fundamental group scheme carries all the algebraic information about the fundamental group. So in this case we can obtain $\pi_1^F(X,x)$ from this group scheme:

Proposition 6.4. — *Let $X$ be a smooth projective $k$-variety. If $\mu_{\max}(\Omega_X) < 0$ then there exist a natural faithfully flat homomorphism $\pi_1^S(X,x) \to \pi_1^F(X,x)$.*

*Proof.* — We will need the following lemma:

Lemma 6.5. — *If $\mu_{\max}(\Omega_X) < 0$ then for any semistable locally free sheaf $E$ of degree zero the canonical map $H^0(X,E) \to H^0(X,F^*E)$ is an isomorphism.*

*Proof.* — To prove the lemma we use the exact sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to F_*\Omega_X.$$

Tensoring it with $E$ and taking sections we get

$$0 \to H^0(X,E) \to H^0(X,F_*(F^*E)) \to H^0(X,F_*(F^*E\otimes\Omega_X)).$$

Note that

$$H^0(X,F_*(F^*E\otimes\Omega_X)) = H^0(X,F^*E\otimes\Omega_X) = \operatorname{Hom}(F^*(E^*),\Omega_X).$$

Now let us recall that if $\mu_{\max}(\Omega_X) < 0$ then a semistable sheaf is strongly semistable (this fact is due to Mehta and Ramanathan; see [25, Theorem 2.9]). In particular, $F^*(E^*)$ is semistable of degree larger than $\mu_{\max}(\Omega_X)$.

Therefore there are no nontrivial $\mathcal{O}_X$-homomorphisms between $F^*(E^*)$ and $\Omega_X$. Then the assertion follows from equality $H^0(X, F_*(F^*E)) = H^0(X, F^*E)$. □

Now we can begin the proof of the proposition. Let us recall that a *flat bundle* $\{E_i, \sigma_i\}$ (which is equivalent to an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module) is a sequence of locally free sheaves $E_i$ and $\mathcal{O}_X$-isomorphisms $\sigma_i : F^*E_{i+1} \to E_i$. Since $E_i$ is semistable for large $i$, by the above lemma $E_0$ is also semistable. Let us define the functor between the neutral Tannaka category of flat bundles and numerically flat bundles by sending $\{E_i, \sigma_i\}$ to $E_0$. Let $\{E_i, \sigma_i\}$ and $\{E_i', \sigma_i'\}$ be flat bundles. Then by the above lemma applied to the sheaf $\mathcal{H}\mathrm{om}(E_{i+1}, E_{i+1}')$ we get a canonical isomorphism

$$\mathrm{Hom}(E_{i+1}, E_{i+1}') \simeq \mathrm{Hom}(E_i, E_i')$$

for every $i \geqslant 0$. This shows that

$$\mathrm{Hom}(\{E_i, \sigma_i\}, \{E_i', \sigma_i'\}) = \mathrm{Hom}(E_0, E_0').$$

Therefore by [10, Proposition 2.21 (a)] to finish the proof it is sufficient to show that if $E'$ is a numerically flat subbundle of a bundle $E_0$ coming from the flat bundle $\{E_i, \sigma_i\}$ then there exists the flat subbundle $\{E_i', \sigma_i'\}$ with $E_0' \simeq E'$. Let us recall that $E_0$ has a canonical connection $\nabla_{\mathrm{can}} : E_0 \to E_0 \otimes \Omega_X$. Since $\mathrm{Hom}_{\mathcal{O}_X}(E', (E_0/E') \otimes \Omega_X) = 0$, as follows from our assumption, the sheaf $E'$ is preserved by the above connection. Hence by Cartier's theorem $E' \subset F^*E_1$ descends under the Frobenius morphism. This way we constructed $E_1'$ and we can proceed by induction to construct the required flat bundle. □

In [8, 10.25 and Proposition 10.32] and [37, Definition 3.1.3] Deligne and Shiho introduced a pro-unipotent completion of the fundamental group (Shiho called this group the de Rham fundamental group scheme but it takes care only of the unipotent part of such a hypothetical de Rham fundamental group). Let us call this group $\pi_1^U(X, x)$. In our case, it is defined as Tannaka dual to the neutral Tannaka category $\mathcal{D}$ consisting of such sheaves $E$ with an integrable connection $\nabla : E \to E \otimes \Omega_X$, which have a filtration

$$0 = E_0 \subset (E_1, \nabla_1) \subset \cdots \subset (E_n, \nabla_n) = (E, \nabla)$$

such that we have short exact sequences

$$0 \to (E_{i-1}, \nabla_{i-1}) \to (E_i, \nabla_i) \to (\mathcal{O}_X, d) \to 0.$$

Let us note that usually the connection is not uniquely determined by the sheaf. For example, for any closed 1-form $\gamma$ the pair $(\mathcal{O}_X, d+\gamma)$ is an object of $\mathcal{D}$. Also, not every numerically flat bundle has a filtration with quotients isomorphic to $\mathcal{O}_X$ (for example, no strongly stable numerically flat bundle of rank $r \geqslant 2$ has such a filtration). So, in general, we cannot expect any natural homomorphism between $\pi_1^U(X,x)$ and $\pi_1^S(X,x)$. However, as before, if $\mu_{\max}(\Omega_X) < 0$ then we can obtain $\pi_1^U(X,x)$ from the S-fundamental group scheme (as a pro-unipotent completion, although we will not try to prove it as it is likely to be a trivial statement; see the remark at the end of the section).

Proposition 6.6. — *Let $X$ be a smooth projective $k$-variety. If $\mu_{\max}(\Omega_X) < 0$ then there exist a natural faithfully flat homomorphism $\pi_1^S(X,x) \to \pi_1^U(X,x)$.*

*Proof.* — Let us construct a functor $\Phi$ from $\mathcal{D}$ to the Tannaka category of numerically flat bundles by associating to an object $(E,\nabla)$ of $\mathcal{D}$ the sheaf $E$. Clearly, $E$ is numerically flat so this makes sense. Let $(E_1,\nabla_1)$ and $(E_2,\nabla_2)$ be objects of $\mathcal{D}$. Let us take an $\mathcal{O}_X$-homomorphism $\varphi : E_1 \to E_2$ and consider the diagram

$$\begin{tikzcd} E_1 \arrow[r, "\nabla_1"] \arrow[d, "\varphi"] & E_1 \otimes \Omega_X \arrow[d, "\varphi\otimes \mathrm{id}_{\Omega_X}"] \\ E_2 \arrow[r, "\nabla_2"] & E_2 \otimes \Omega_X \end{tikzcd}$$

Then $(\varphi \otimes \mathrm{id}_{\Omega_X}) \circ \nabla_1 - \nabla_2 \circ \varphi \in \mathrm{Hom}_{\mathcal{O}_X}(E_1, E_2 \otimes \Omega_X)$. But $E_1, E_2$ are strongly semistable and $\mu_{\max}(\Omega_X) < 0$, so $\mathrm{Hom}_{\mathcal{O}_X}(E_1, E_2 \otimes \Omega_X) = 0$. Therefore the above diagram is commutative which shows that the functor $\Phi$ is fully faithful.

By [10, Proposition 2.21 (a)] to finish the proof we need to show that if $E'$ is a numerically flat subbundle of a bundle $E$ coming from $(E,\nabla)$ then $\nabla$ induces an integrable connection on $E'$. Then, automatically, $E'$ has a filtration as in the definition of $\mathcal{D}$, so it is a subobject of $(E,\nabla)$. Note that if $\nabla$ does not preserve $E'$ then it induces a non-trivial $\mathcal{O}_X$-homomorphism $E' \to (E/E') \otimes \Omega_X$. Again one can easily see that there are no such homomorphisms, which proves the required assertion. □

Finally let us formulate the following easy lemma whose proof is left to the reader:

Lemma 6.7. — *Let $X$ be a smooth projective $k$-variety. If $\mu_{\max}(\Omega_X) < 0$ then every semistable locally free sheaf $E$ of degree zero admits at most one connection. If $E$ admits a connection $\nabla$ then it is integrable (ie., $\nabla^2 = 0$) and its $p$-curvature vanishes. In particular, there exists $E'$ such that $(E,\nabla) \simeq (F^*E', \nabla_{\mathrm{can}})$.*

Let us note that if $h^1(X, \mathcal{O}_X) \neq 0$ then $\pi_1^S(X, x)$ is non-trivial because $\mathrm{Pic}^0(X) \neq 0$. Nevertheless, the author does not know any examples of varieties in positive characteristic with $\mu_{\max}(\Omega_X) < 0$ and a non-trivial S-fundamental group scheme. One can show that there is no such example in dimension $\leqslant 2$.

## 7. Monodromy groups

In this section we recall a few results, mostly from [4], generalizing them to higher dimension. Since the proofs, using our definitions, are either the same as in [4] or simpler we usually skip them.

Let $G$ be a connected reductive $k$-group and let $E_G \to X$ be a principal $G$-bundle on $X$.

Definition 7.1 ([5], Definition 2.2). — *$E_G$ is called* numerically flat *if for every parabolic subgroup $P \subset G$ and every character $\chi : P \to \mathbb{G}_m$ dominant with respect to some Borel subgroup of $G$ contained in $P$, the dual line bundle $L(\chi)^*$ over $E_G/P$ is nef.*

If $X$ is a smooth projective curve then $E_G$ is numerically flat if and only if it is a strongly semistable principal $G$-bundle of degree zero. Note that if $G$ is semisimple then a principal $G$-bundle has always degree zero.

Lemma 7.2. — *The following conditions are equivalent:*

1. *$E_G$ is numerically flat,*
2. *for every representation $G \to \mathrm{GL}(V)$ the associated vector bundle $E_G(V)$ is numerically flat,*
3. *$E_G(\mathfrak{g})$, associated to $E_G$ via the adjoint representation, is numerically flat.*

*Proof.* — It is sufficient to prove the lemma when $X$ is a smooth projective curve. Then 1 implies 2 because of [35, Theorem 3.23]. This needs a small additional argument if $G$ is not semisimple as the radical of $G$ is not necessarily mapped to the centre of $\mathrm{GL}(V)$ (the only problem is with degree of associated bundles but this is zero as $E_G$ is numerically flat). Obviously, 2 implies 3 and 3 is equivalent to 1 by [25, Corollary 2.8]. $\square$

Let $E_G$ be a numerically flat principal $G$-bundle. Let $\mathcal{E}_G : G-\mathrm{mod} \to \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ be the functor corresponding to $E_G$ (see, *eg.*, [32, Lemma 2.3 and Proposition 2.4]). Let us set $T_G = T_x \circ \mathcal{E}_G$. Then $(G-\mathrm{mod}, \otimes, \mathrm{T_G}, \mathrm{k})$ is a neutral Tannakian category. The affine group scheme corresponding to this category is $\mathrm{Ad}(E_G)_x \simeq G$. Therefore the functor $(G-\mathrm{mod}, \otimes, \mathrm{T_G}, \mathrm{k}) \to (\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X}), \otimes, \mathrm{T_x}, \mathcal{O}_{\mathrm{X}})$ defines a homomorphism of groups

$$\rho(E_G) : \pi_1^S(X,x) \to \mathrm{Ad}(E_G)_x.$$

The image $M$ of this homomorphism is called the *monodromy group scheme* of $E_G$. One can see that $E_G$ has a reduction of structure group to $M$ and it is the smallest such subgroup scheme (*cf.* [4, Proposition 4.9]).

Let us recall that a subgroup of a group is called *irreducible* if it is not contained in any proper parabolic subgroup. By [4, Lemma 4.13] $E_G$ is strongly stable if and only if the reduced monodromy group $M_{\mathrm{red}}$ is an irreducible subgroup of $\mathrm{Ad}(E_G)_x \simeq G$. It is well known that irreducible subgroups of reductive groups are reductive, so if $E_G$ is strongly stable then by [4, Lemma 4.12] for large $m$ the monodromy group of $(F^m)^*E_G$ is a reductive group (this is analogous to the complex case; see [4, Proposition 8.1]).

It follows that if $E_G$ is numerically flat then for large $m$ there exists a reduction $E'_P$ of $(F^m)^*E_G$ to a parabolic subgroup $P \subset G$ such that the monodromy group of the extension $E_L$ of $E_P$ to the Levi quotient $q : P \to L = P/R_u(P)$ is reduced and it is an irreducible subgroup of $L$. In fact, the monodromy group $M$ of $E_G$ is a reduced subgroup of $P$ and $q(M)$ is the monodromy group of $E_L$.

## 8. Basic properties of the S-fundamental group scheme

In this section we prove a few basic properties of the S-fundamental group scheme: behavior under morphisms and field extension (in arbitrary characteristic).

Let $f : X \to Y$ be a $k$-morphism of complete $k$-varieties. Since pull-backs of nef bundles are nef for a $k$-point $x \in X$ there exists an induced map $\pi_1^S(X,x) \to \pi_1^S(Y,y)$, where $y = f(x)$.

Lemma 8.1. — *Let $f : X \to Y$ be a surjective flat morphism of complete $k$-varieties. If $f_*\mathcal{O}_X = \mathcal{O}_Y$ then $\pi_1^S(X,x) \to \pi_1^S(Y,y)$ is a faithfully flat surjection.*

*Proof.* — By [10, Proposition 2.21 (a)] we need to show that:

(a) the functor $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{Y},\mathrm{y}) \to \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X},\mathrm{x})$ is fully faithful,

(b) if $E' \subset f^*E$ is a numerically flat subbundle for $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{Y})$ then $E'$ is isomorphic to pull back of a numerically flat subbundle of $E$.

(a) follows immediately from the projection formula:

$$\mathcal{H}\mathrm{om}_Y(E',E'')\simeq f_*\mathcal{H}\mathrm{om}_X(f^*E',f^*E'')$$

by taking sections.

To prove (b) let us set $E''=(f^*E)/E'$ and denote by $r,r',r''$ the ranks of $E,E',E''$ respectively and let $X_y$ be the fibre over a $k$-point $y\in Y$. Then $E'_y=E'_{X_y}$ is a numerically flat subbundle of the trivial bundle $(f^*E)_{X_y}\simeq\mathcal{O}^r_{X_y}$. But $(E'_y)^*$ is also globally generated. Since a section of such bundle has no zeroes $E'_y$ is trivial. Similarly, $E''_y$ is trivial. In particular, since $E'$ is $Y$-flat and $h^0(X_y,E'_y)=r'$ does not depend on $y\in Y$ we see that $f_*E'$ is locally free of rank $r'$ by Grauert's theorem. In the same way we prove that $f_*E''$ is locally free of rank $r''$. Since the surjective map $f^*E\to E''$ factors through $f^*f_*E''\to E''$ we see that $f^*f_*E''\to E''$ is a surjective map of rank $r''$ vector bundles and hence it is an isomorphism. Therefore $f^*f_*E'\to E'$ is also an isomorphism. Let us remind that if the pull back of a bundle is nef then the bundle is nef. Therefore $f_*E'$ is numerically flat. □

Proposition 8.2. — *For any $k$-point $x$ of $\mathbb{P}^n_k$ we have $\pi_1^S(\mathbb{P}^n_k,x)=0$.*

*Proof.* — For $n=1$ the assertion is easy as every strongly semistable vector bundle of degree 0 is trivial.

Let $E$ be a stable vector bundle on $\mathbb{P}^2$. Then by a standard argument $\mathrm{Hom}(E,E)=k$, $\mathrm{ext}^2(E,E)=\mathrm{hom}(E,E(-3))=0$ and

$$\chi(E,E)=1-\mathrm{ext}^1(E,E)=r^2-\Delta(E)\leqslant 1.$$

Therefore if $E$ has vanishing Chern classes then $r=1$ and $E\simeq\mathcal{O}_{\mathbb{P}^2}$. Since extensions of trivial bundles on $\mathbb{P}^2$ are trivial, by Theorem 4.1 every $E\in\mathrm{Vect}_0^{\mathrm{s}}(\mathbb{P}^2)$ is trivial.

It is well known that a vector bundle on $\mathbb{P}^n$ splits if and only if its restriction to some plane splits (see [33, Chapter I, Theorem 2.3.2]; the proof given in [33] works in arbitrary characteristic). Therefore if $E\in\mathrm{Vect}_0^{\mathrm{s}}(\mathbb{P}^{\mathrm{n}})$ then by restriction theorem the restriction of $E$ to a plane belongs to $\mathrm{Vect}_0^{\mathrm{s}}(\mathbb{P}^2)$, hence it is trivial, which proves that $\pi_1^S(\mathbb{P}^n_k,x)=0$. □

Lemma 8.3. — *Let $Y$ be a smooth complete $k$-variety and let $f:X\to Y$ be the blow-up of $Y$ along a smooth subvariety $Z\subset Y$. Then $\pi_1^S(X,x)\to\pi_1^S(Y,y)$ is an isomorphism.*

*Proof.* — Let $E\in\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$. Then by Proposition 8.2 restriction of $E$ to each fibre of $f$ is trivial. Then by [21, Theorem 1] (which can be easily adapted to our setting) $f_*E$ is locally free and $E\simeq f^*f_*E$. By [10, Proposition 2.21 (b)] this shows that $\pi_1^S(X,x)\to\pi_1^S(Y,y)$ is a closed immersion.

Then the proof that it is faithfully flat is an easier version of the proof of Lemma 8.1. □

The above lemma strongly suggests that the S-fundamental group scheme is a birational invariant. This would follow from the above lemma and a version of Włodarczyk's result [40] in positive characteristic. [(1)].

The proof of the following lemma was motivated by the proof of [31, Proposition 3.1].

Lemma 8.4. — *Let $X$ be a complete variety defined over an algebraically closed field $k$. Let $k'$ be an algebraically closed extension of $k$. Let $x'$ be the $k'$-point of $X_{k'} = X \times_k \operatorname{Spec} k'$ corresponding to a $k$-point of $x$ of $X$. Then $\pi_1^S(X_{k'}, x') \to \pi_1^S(X, x) \times_k \operatorname{Spec} k'$ is faithfully flat (in particular, it is surjective).*

*Proof.* — Let us note that if $E$ on $X_k$ is numerically flat then $E \otimes_k k'$ is also numerically flat. By definition it is sufficient to check this in case of smooth projective curves. But in case of curves this follows immediately from the fact that if $E$ on $X_k$ is stable (semistable or strongly semistable) then $E \otimes_k k'$ is also stable (semistable or strongly semistable, respectively); see [20, Corollary 1.3.8 and Corollary 1.5.11].

Let $\mathcal{T}$ be the Tannakian subcategory of $\mathcal{C}' = (\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X}_{\mathrm{k}'}), \otimes, \mathrm{T}_{\mathrm{x}'}, \mathcal{O}_{\mathrm{X}_{\mathrm{k}'}})$ whose objects are numerically flat vector bundles $E'$ on $X_{k'}$ such that there exists $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X}_{\mathrm{k}})$ such that $E' \subset E \otimes_k k'$.

Let us set $G = \pi_1^S(X, x)$ and consider the category $\mathcal{T}'$ of finite dimensional $k'$-representations of $G_{k'} = G \times_k \operatorname{Spec} k'$. Let $G_{k'} \to \mathrm{GL}(V')$ be a $k'$-representation. Then by [22, I 3.9 and 3.10] there exists an inclusion of $G_{k'}$-modules $V' \subset k'[G_{k'}]^{\oplus m} = (k[G]^{\oplus m}) \otimes k'$. Therefore there exists a $k$-vector subspace $W' \subset k[G]^{\oplus m}$ such that $V' \subset W' \otimes k'$. But there exists a finite dimensional $G$-module $W \subset k[G]^{\oplus m}$ containing $W'$. Let $\tilde{X}^S_{k'}$ be the base change of the S-universal covering of $X$. Then the vector bundle $E'$ associated to $V'$ via this principal $G'$-bundle is a vector subbundle of the base change of the vector bundle $E$ associated to $W$ via the S-universal covering of $X$.

This shows that we have a natural functor $\mathcal{T}' \to \mathcal{T}$ of neutral Tannakian categories. It is easy to see that this functor is an equivalence of Tannakian categories. Then by [10, Proposition 2.21 (a)] $\mathcal{T} \subset \mathcal{C}'$ defines the faithfully flat homomorphism $\pi_1^S(X_{k'}, x) \to \pi_1^S(X, x) \times_k \operatorname{Spec} k'$. □

(1) *Added in proof:* Very recently, A. Hogadi and V. Mehta proved birational invariance of the S-fundamental group scheme.

As in [31, Proposition 3.1] one can easily see that if $\pi_1^S(X_{k'},x)\to\pi_1^S(X,x)\times_k\operatorname{Spec}k'$ is a closed immersion then every stable strongly semistable vector bundle on $X_{k'}$ must be defined over $k$. Since this is not true already for stable $F$-trivial bundles (see [34] for an example when $X$ is a smooth curve), the above homomorphism is usually not a closed immersion.

Let $X$ and $Y$ be complete $k$-varieties. Let us fix $k$-points $x\in X$ and $y\in Y$. Then we have a natural homomorphism

$$\pi_1^S(X\times Y,(x,y))\to\pi_1^S(X,x)\times\pi_1^S(Y,y).$$

Using embeddings of $X\times\{y\}$ and $Y\times\{x\}$ into $X\times Y$ and Lemma 8.1 one can easily see that this homomorphism is faithfully flat. Unfortunately, it is not clear if this is an isomorphism [(2)].

Note that the question is non-trivial even at the level of characters of S-fundamental groups. For example, it is true but a non-trivial fact that

$$\operatorname{Pic}^0(X)\times\operatorname{Pic}^0(Y)\to\operatorname{Pic}^0(X\times Y)$$

is an isomorphism on the level of $k$-points (*ie.*, it is an isomorphism of the corresponding reduced schemes). But this is not yet sufficient to conclude that a line bundle on $X\times Y$ with a (numerically) trivial first Chern class is of the form $p_X^*L\otimes p_Y^*M$ for some line bundles $L$ on $X$ and $M$ on $Y$. Here we should recall that a line bundle has vanishing first Chern class if and only if certain tensor power of this line bundle is algebraically equivalent to zero in $\operatorname{Pic}X$ (see, *eg.*, [14, Example 19.3.3]).

## 9. Some vanishing theorems for $H^1$ and $H^2$

In this section we prove a few basic vanishing theorems for the cohomology groups of strongly semistable sheaves with vanishing Chern classes.

We assume that $X$ is a smooth $d$-dimensional projective variety defined over an algebraically closed field $k$ and $H$ is an ample divisor on $X$ (we consider slopes only with respect to this divisor).

If $E\in\operatorname{Vect}_0^{\mathrm{s}}(\mathrm{X})$ then for any effective divisor $D$ we have $H^0(X,E(-D))=0$, as $E(-D)$ is semistable with negative slope. In this section we will find similar vanishing theorems for $H^1$ and $H^2$.

Theorem 9.1 (Vanishing theorem for $H^1$). — *Assume that $X$ has dimension $d\geqslant 2$. Let $E\in\operatorname{Vect}_0^{\mathrm{s}}(\mathrm{X})$ and let $D$ be any ample divisor. If $DH^{d-1}>\frac{\mu_{\max}(\Omega_X)}{p}$ then $H^1(X,E(-D))=0$.*

(2) In "On the S-fundamental group scheme II" we prove that this map is an isomorphism.

*Proof.* — First let us prove the following

Lemma 9.2 (see [39] 2.1, Critère). — *Let $E$ be a torsion free sheaf on $X$ such that $H^0(X, F^*E(-pD)\otimes\Omega_X)=0$ and $H^1(X, F^*E(-pD))=0$. Then $H^1(X, E(-D))=0$.*

*Proof.* — Let $B^1_X$ be the sheaf of exact 1-forms. By definition we have an exact sequence

$$0\to \mathcal{O}_X\to F_*\mathcal{O}_X\to F_*B^1_X\to 0.$$

By assumptions and the projection formula we have

$$H^0(X, E(-D)\otimes F_*\Omega_X)=H^0(X, F^*E(-pD)\otimes\Omega_X)=0.$$

But $F_*B^1_X$ is a subsheaf of $F_*\Omega^1_X$, so $H^0(X, E(-D)\otimes F_*B^1_X)=0$.

Tensoring the above sequence with $E(-D)$ and using the projection formula we get the following exact sequence

$$0\to E(-D)\to F_*(F^*E(-pD))\to E(-D)\otimes F_*B^1_X\to 0.$$

Since

$$H^1(X, F_*(F^*E(-pD)))=H^1(X, F^*E(-pD))=0$$

we get $H^1(X, E(-D))=0$. □

The family of all strongly $H$-semistable locally free sheaves $G$ of fixed rank with vanishing Chern classes is bounded. Hence by Serre's vanishing theorem there exists such $m_0$ that for all $m\geqslant m_0$ and all such $G$ we have $H^1(X, G(-p^mD))=0$. Let us also remark that

$$H^0(X, G(-pD)\otimes\Omega_X)=\mathrm{Hom}(G^*, \Omega_X(-pD))=0,$$

since $G^*$ is semistable with slope 0 and by assumption $\mu_{\max}(\Omega_X(-pD))<0$. Therefore applying Lemma 9.2 to $E, F^*E, (F^2)^*E, \dots$ we easily get vanishing of $H^1(X, E(-D))$. □

Corollary 9.3. — *Let $\alpha$ be a non-negative integer such that $T_X(\alpha H)$ is globally generated. Assume that $X$ has dimension $d\geqslant 2$. Let $E\in$ $\mathrm{Vect}^{\mathrm{s}}_0(\mathrm{X})$ and let $D$ be any divisor such that $D-\alpha H$ is ample. If*

$$DH^{d-1}>\max\left((d+1)\alpha H^d-K_XH^{d-1}, \left(1+\frac{1}{p}\right)\alpha H^d\right)$$

*then $H^1(X, E\otimes\Omega_X(-D))=0$.*

*Proof.* — Since $T_X(\alpha H)$ is globally generated there exists a torsion free sheaf $K$ and an integer $N$ such that we have an exact sequence

$$0\to\Omega_X\to\mathcal{O}_X(\alpha H)^N\to K\to 0.$$

In particular, we have $\mu_{\max}(\Omega_X)\leqslant \alpha H^d$ and $\mu_{\min}(K)\geqslant \mu_{\min}(\mathcal{O}_X(\alpha H)^N) = \alpha H^d$. Since $K$ has rank $(N-d)$ we also have

$$\mu_{\max}(K)+(N-d-1)\mu_{\min}(K)\leqslant \deg K = N\alpha H^d - K_XH^{d-1}.$$

Hence $\mu_{\max}(K)\leqslant (d{+}1)\alpha H^d - K_XH^{d-1} < DH^{d-1}=\mu_H(E^*(D))$. Because $E^*(D)$ is semistable we have

$$H^0(X,E(-D)\otimes K)=\mathrm{Hom}(E^*(D),K)=0.$$

Our assumptions imply that

$$\frac{\mu_{\max}(\Omega_X)}{p}<\frac{\alpha H^d}{p}\leqslant (D-\alpha H)H^{d-1}.$$

Therefore by Theorem 9.1 we get vanishing of $H^1(X,E(\alpha H - D))$. Together with the above this implies vanishing of $H^1(X,E(-D)\otimes\Omega_X)$. □

Theorem 9.4 (Vanishing theorem for $H^2$). — *Let $\alpha$ be a non-negative integer such that $T_X(\alpha H)$ is globally generated. Assume that $X$ has dimension $d\geqslant 3$. Let $E\in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$. Let $D$ be any divisor such that $pD-\alpha H$ is ample. If*

$$DH^{d-1}>\max\left(\alpha H^d,\frac{(d+1)\alpha H^d-K_XH^{d-1}}{p}\right)$$

*then $H^2(X,E(-D))=0$.*

*Proof.* — First let us prove the following

Lemma 9.5 (*cf.* [26], Proposition 2.31). — *Let $E$ be a torsion free sheaf on $X$ such that $H^0(X,E(-D)\otimes\Omega_X)=0$, $H^0(X,F^*E(-pD)\otimes\Omega_X^2)=0$, $H^1(X,F^*E(-pD)\otimes\Omega_X)=0$ and $H^2(X,F^*E(-pD))=0$. Then $H^2(X,E(-D))=0$.*

*Proof.* — Let $B_X^1$ and $Z_X^1$ be the sheaves of exact and closed 1-forms, respectively. Then we have the following short exact sequence

$$0\to F_*B_X^1\to F_*Z_X^1\xrightarrow{C}\Omega_X\to 0,$$

where $C$ is the Cartier operator. Tensoring it with $E(-D)$ and using the projection formula we get the following short exact sequence

$$0\to E(-D)\otimes F_*B^1_X\to E(-D)\otimes F_*Z^1_X\to E(-D)\otimes\Omega_X\to 0.$$

Using definition of $Z_X^1$ we also have an exact sequence

$$0\to F_*Z_X^1\to F_*\Omega_X\to F_*\Omega_X^2.$$

Again tensoring it with $E(-D)$ and using the projection formula we get the following exact sequence

$$0 \to E(-D) \otimes F_* Z^1_X \to F_*(F^*E(-pD) \otimes \Omega_X) \to F_*(F^*E(-pD) \otimes \Omega^2_X).$$

Using this sequence we see that vanishing of $H^0(X, F^*E(-pD) \otimes \Omega^2_X)$ and $H^1(X, F^*E(-pD) \otimes \Omega_X)$ implies vanishing of $H^1(E(-D) \otimes F_*Z^1_X)$. Vanishing of this group together with vanishing of $H^0(X, E(-D) \otimes \Omega_X)$ implies vanishing of $H^1(X, E(-D) \otimes F_*B^1_X)$. But from the long cohomology exact sequence this, together with vanishing of $H^2(X, F^*E(-pD))$ implies vanishing of $H^2(X, E(-D))$. □

As before the family of all strongly $H$-semistable locally free sheaves $G$ of fixed rank with vanishing Chern classes is bounded and by Serre's vanishing theorem there exists such $m_0$ that for all $m \geqslant m_0$ and all such $G$ we have $H^2(X, G(-p^m D)) = 0$.

Since $DH^{d-1} > \alpha H^d \geqslant \mu_{\max}(\Omega_X)$ we get vanishing of $H^0(X, G(-D) \otimes \Omega_X)$.

Now note that $\Omega^2_X$ is a subsheaf of $\bigwedge^2(\mathcal{O}_X(\alpha H)^N) = \mathcal{O}_X(2\alpha H)^{\binom{N}{2}}$. This implies that

$$\mu_{\max}(\Omega^2_X) \leqslant 2\alpha H^d < pDH^{d-1} = \mu(G^*(pD)).$$

Therefore

$$H^0(X, G(-pD) \otimes \Omega^2_X) = \mathrm{Hom}(G^*(pD), \Omega^2_X) = 0.$$

By assumption we have

$$pDH^{d-1} > \max\left((d+1)\alpha H^d - K_X H^{d-1}, \left(1 + \frac{1}{p}\right)\alpha H^d\right).$$

Therefore by Corollary 9.3 we also have $H^1(X, G(-pD) \otimes \Omega_X) = 0$.

Now we finish proof of the theorem by applying Lemma 9.5 to $E, F^*E, (F^2)^*E, \dots$ □

## 10. Lefschetz type theorems for the S-fundamental group scheme

In this section we prove Lefschetz type theorems for the S-fundamental group scheme.

Let us recall the following example. It appeared essentially in [39, p.181] and then it reappeared with the interpretation below in [3, Section 2].

*Example 10.1.* — Let $D$ be an ample effective divisor violating the Kodaira vanishing theorem in positive characteristic (*ie.*, such that $H^1(\mathcal{O}_X(-D)) \neq 0$). Let us recall that a non-zero element $c$ of $H^1(\mathcal{O}_X)$ gives rise to a non-trivial extension $E$ of $\mathcal{O}_X$ by $\mathcal{O}_X$. If the class $c$ of $H^1(\mathcal{O}_X)$ is in the kernel of $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_D)$ then $E_D \simeq \mathcal{O}_D \oplus \mathcal{O}_D$. By Serre's vanishing theorem, action of the Frobenius morphism on elements of the kernel of $H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_D)$ is nilpotent. Therefore $(F^m)^*E \simeq \mathcal{O}_X^2$ for large $m$.

This gives an example of a non-trivial representation of $\pi_1^S(X, x)$ which is trivial on the image of $\pi_1^S(D, x)$ (obviously this holds already on the level of Nori's fundamental group scheme). In particular, $\pi_1^S(D, x) \to \pi_1^S(X, x)$ is not surjective.

We can also interpret the above example in the following way which explains connection with [39]. Let $\alpha_{p^n}$ denotes the group scheme on $X$ defined by

$$\alpha_{p^n}(U) = \{t \in \Gamma(U, \mathcal{O}_U) : t^{p^n} = 0\}.$$

Then we have the following exact sequence (only in fppf topology)

$$0 \to \alpha_{p^n} \to \mathbb{G}_a \xrightarrow{F^n} \mathbb{G}_a \to 0,$$

where the last map is given by $t \to t^{p^n}$. Using this one can easily see that

$$H^1_{fl}(X, \alpha_{p^n}) = \ker \left( H^1(X, \mathcal{O}_X) \xrightarrow{F^n} H^1(X, \mathcal{O}_X) \right).$$

But $H^1_{fl}(X, \alpha_{p^n})$ is the set of $\alpha_{p^n}$-torsors on $X$ and each such torsor gives an element of Nori's fundamental group. Therefore the example says that there exists a nontrivial element of $H^1_{fl}(X, \alpha_{p^n})$ whose restriction to $D$ gives a trivial $\alpha_{p^n}$-torsor. But we know that the action of the Frobenius on $H^1(X, \mathcal{O}_X(-D))$ is nilpotent so any non-zero element of $H^1(X, \mathcal{O}_X(-D))$ gives such a torsor for some $n \geqslant 1$.

In this section $X$ is a smooth $d$-dimensional projective variety defined over an algebraically closed field $k$ and $H$ is an ample divisor on $X$.

THEOREM 10.2. — *Let $D \subset X$ be any ample smooth effective divisor. If $d \geqslant 2$ and*

$$DH^{d-1} > \mu_{\max}(\Omega_X)$$

*then $\pi_1^S(D, x) \to \pi_1^S(X, x)$ is a faithfully flat homomorphism.*

*Proof.* — By [10, Proposition 2.21 (a)] we need to show that:

(a) the functor $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X}, \mathrm{x}) \to \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{D}, \mathrm{x})$ is fully faithful,
(b) every subbundle of degree 0 in the restriction $E_D$ of $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ is isomorphic to the restriction of a subbundle of $E$.

To show (a) we need to prove that for $E', E'' \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ the restriction

$$\mathrm{Hom}_X(E', E'') \to \mathrm{Hom}_D(E'_D, E''_D)$$

is an isomorphism. But from the short exact sequence

$$0 \to \mathcal{H}\mathrm{om}_X(E', E'') \otimes \mathcal{O}_X(-D) \to \mathcal{H}\mathrm{om}_X(E', E'') \to \mathcal{H}\mathrm{om}_D(E'_D, E''_D) \to 0$$

we see that it is sufficient to show that $H^i(X, \mathcal{H}\mathrm{om}_X(E', E'') \otimes \mathcal{O}_X(-D)) = 0$ for $i = 0, 1$. Since $\mathcal{H}\mathrm{om}_X(E', E'') \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$, this follows from Theorem 9.1 and the remark preceding it.

To prove (b) let us note that for every degree 0 subbundle of $E_D$ there exists a Jordan–Hölder filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E_D$ and some index $j$ such that this subbundle is equal to $E_j$. So it is sufficient to lift this filtration to a filtration of $E$.

First we prove this for sheaves such that all quotients in any Jordan–Hölder filtration of $E$ are strongly stable. More precisely, let us consider the following assertion: for all sheaves $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ of rank $\leqslant r$ and such that all quotients in any Jordan–Hölder filtration of $E$ are strongly stable if $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E_D$ is a Jordan–Hölder filtration of $E_D$ then $E_i$ lifts to a subsheaf of $E$. We prove it by induction on $r$. The case $r = 1$ is obvious. So assume that we know it for $r - 1$ and consider a rank $r$ sheaf $E$ satisfying the above condition. Note that it is sufficient to lift the first subsheaf $E_1$ to a subsheaf $E' \subset E$ and use the induction assumption for $E/E'$.

To lift $E_1$ let us take an arbitrary Jordan–Hölder filtration $0 = E'_0 \subset E'_1 \subset \cdots \subset E'_n = E$ of $E$. By Theorem 4.1 each quotient $E^j = E'_j/E'_{j-1}$ is locally free and by Theorem 3.1 the restriction $E^j_D$ is strongly stable. In particular, we have $n > 1$ (unless $m = 1$, in which case $E$ is the required lift). Therefore there exists some $j_0$ such that $E_1$ is isomorphic to $E^{j_0}_D$ (every non-zero map from $E_1$ to any of the sheaves $E^j_D$ is an isomorphism). But we already know by (a) that the restriction map

$$\mathrm{Hom}_X(E^{j_0}, E) \to \mathrm{Hom}_D(E_1, E_D)$$

is an isomorphism so we can lift the inclusion $E_1 \subset E_D$ and it clearly lifts to an inclusion.

Now let us consider the general case. Let us choose $m$ such that all quotients in any Jordan-Hölder filtration of $\tilde{E} = (F_X^m)^* E$ are strongly stable. The restriction $\tilde{E}_D \simeq (F_D^m)^*(E_D)$ contains $(F_D^m)^*(E_1)$ which by the above is isomorphic to the restriction $\tilde{E}'_D$ of some subsheaf $\tilde{E}'$ of $\tilde{E}$. We claim that for every $0 \leqslant i \leqslant m$ there exists a subsheaf $E'_i \subset (F_X^{m-i})^* E$ such that $\tilde{E}' = (F^i)^* E'_i$ and $(E'_i)_D \simeq (F_D^{m-i})^*(E_1)$. In particular for $i = m$

we get the subsheaf of $E$ that we were looking for. We prove the above assertion by induction on $i$. For $i=0$ the claim is clear as we already have $E'_0 = E'$. Assume that we constructed $E'_i$ for some $i<m$. Let us set $E''_i = ((F_X^{m-i})^*E)/E'_i$. We only need to show that there exists $E'_{i+1}\subset (F_X^{m-i-1})^*E$ such that $F_X^*E'_{i+1}\simeq E'_i$. If such a sheaf does not exist then the $\mathcal{O}_X$-homomorphism $E'_i\to E''_i\otimes\Omega_X$, induced from the canonical connection $\nabla_{\mathrm{can}}: (F_X^{m-i})^*E\to (F_X^{m-i})^*E\otimes\Omega_X$ coming from Cartier's descent, is non-zero (see, *eg.*, [24, Theorem 2.1]; see also [24, Lemma 2.3] for a similar assertion). But we have a commutative diagram

$$\begin{tikzcd} E'_i \arrow[r] \arrow[d] & E''_i\otimes\Omega_X \arrow[d] \\ (E'_i)_D \arrow[r, "0"] & (E''_i)_D\otimes\Omega_D \end{tikzcd}$$

where the lower map is similarly induced from the canonical connection and it is zero because $(E'_i)_D$ descends to a subsheaf of $(F_D^{m-i-1})^*(E_D)$ by construction. Now using the exact sequence

$$0\to\Omega_X(-D)\to\Omega_X\to\Omega_X|_D\to 0$$

we see that if $E'_i\to E''_i\otimes\Omega_X\otimes\mathcal{O}_D$ is zero, then $E'_i\to E''_i\otimes\Omega_X$ induces a non-zero map $E'_i\to E''_i\otimes\Omega_X(-D)$ or equivalently a non-zero map $E'_i\otimes(E''_i)^*\to\Omega_X(-D)$ . But $E'_i$ and $E''_i$ are strongly semistable of slope 0, so $E'_i\otimes(E''_i)^*$ is also strongly semistable. Since by assumption $\mu_{\max}(\Omega_X(-D))<0$ the above map is zero, a contradiction. Therefore $(E'_i)_D\to(E''_i)_D\otimes\Omega_X|_D$ is non-zero. So using the exact sequences

$$0\to\mathcal{O}_D(-D)\to\Omega_X|_D\to\Omega_D\to 0$$

we see that this map lifts to a non-zero map $(E'_i)_D\to(E''_i)_D\otimes\mathcal{O}_D(-D)$. But there are no non-zero maps between $(E'_i)_D$ and $(E''_i)_D\otimes\mathcal{O}_D(-D)$ because both sheaves are semistable and the second one has smaller slope. This finishes the proof the theorem. □

As a corollary of the above proof of (b) we get the following:

Corollary 10.3. — *Let $E\in\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$, $d\geqslant 2$. Let $D$ be any ample smooth effective divisor such that $DH^{d-1}>\mu_{\max}(\Omega_X)$. If $E$ is stable then $E_D$ is also stable.*

Theorem 10.4. — *Let us assume that $d\geqslant 3$ and $T_X(\alpha H)$ is globally generated for some non-negative integer $\alpha$. Let $D\subset X$ be any ample smooth effective divisor such that $D-\alpha H$ is ample. If*

$$DH^{d-1}>\max\left(p\alpha H^d,(d+1)\alpha H^d-K_XH^{d-1}\right)$$

*then $\pi_1^S(D,x)\to\pi_1^S(X,x)$ is an isomorphism.*

*Proof.* — It is sufficient to show that for every strongly semistable locally free sheaf $E'$ on $D$ with $\mathrm{ch}_1(E')\cdot H^{d-1}=0$ and $\mathrm{ch}_2(E')\cdot H^{d-2}=0$ there exists a locally free sheaf $E$ on $X$ such that $E'\simeq E_D$. Then $E$ is also strongly semistable and $\pi_1^S(D,x)\to\pi_1^S(X,x)$ is a closed immersion by [10, Proposition 2.21 (b)]. Then the assertion follows from the previous theorem.

Let $D_n$ denote the scheme whose topological space is $D$ and the structure sheaf is $\mathcal{O}_X/I_D^n$ (so $D_n$ is just the divisor $nD$ with a natural scheme structure induced from $X$).

Lemma 10.5. — *Let $S$ be a $k$-scheme of finite type. Let $\mathcal{S}$ be a bounded set of coherent sheaves on $D$. There exists $n_0$ such that for all $n\geqslant n_0$ the following holds. Let $\mathcal{F}$ be an $S$-flat family of locally free sheaves on $D_{n_0}$ such that $\mathcal{F}|_{D\times\{s\}}\in\mathcal{S}$ for every $s\in S$. Then the set $S_n\subset S$ of points $s\in S$ such that $\mathcal{F}_s$ can be extended to a locally free sheaf on $D_n\subset X$ is closed. Moreover, for large $n$, $\mathcal{F}|_{D_{n_0}\times S_n}$ can be extended to an $S_n$-flat family of locally free sheaves on the formal completion of $X$ along $D$.*

*Proof.* — Let $p:D\times S\to S$ and $q:D\times S\to D$ be the natural projections. Let $\mathcal{E}\mathrm{xt}^j_p(E,\cdot)$ be the $j$th derived functor of $\mathcal{H}\mathrm{om}_p(E,\cdot)=p_*\circ\mathcal{H}\mathrm{om}(E,\cdot)$ (see, *eg.*, [20, 10.1.7] for definition and basic properties of these functors). Let us set

$$\mathcal{G}=\mathcal{E}\mathrm{xt}^2_p(\mathcal{F},\mathcal{F}\otimes q^*\mathcal{O}_D(-nD)).$$

Let us take $n_0$ such that for all $n\geqslant n_0$, $\mathrm{Ext}^i_D(\mathcal{F}_s,\mathcal{F}_s\otimes\mathcal{O}_D(-nD))$ are for all $k$-points $s\in S$ equal to zero for $i\leqslant 1$ and have the same dimension for $i=2$ (existence of such $n_0$ follows, *eg.*, from [19, Chapter III, Proposition 6.9]; note that we use the fact that $D$ has dimension $\geqslant 2$). Then $\mathcal{G}$ is locally free and it commutes with base-change. In particular, applying the base change for the map $s:\operatorname{Spec}k\to S$ mapping the point $(0)$ to $s\in S$ we get an isomorphism

$$\mathcal{G}_s\simeq\mathrm{Ext}^2_D(\mathcal{F}_s,\mathcal{F}_s\otimes\mathcal{O}_D(-nD)).$$

Using induction, it is sufficient to prove the assertion from the lemma for $n=n_0+1$ (then in the same way one can prove it for $n_0+2$ and so on).

Let $\mathrm{ob}'(\mathcal{F})\in\mathrm{Ext}^2_{D\times S}(\mathcal{F},\mathcal{F}\otimes q^*\mathcal{O}_D(-nD))$ be an obstruction to extend $\mathcal{F}$ from $D_{n_0}\times S$ to $D_n\times S$. Let $\mathrm{ob}(\mathcal{F})$ be the image of $\mathrm{ob}'(\mathcal{F})$ under the map

$$\mathrm{Ext}^2_{D\times S}(\mathcal{F},\mathcal{F}\otimes q^*\mathcal{O}_D(-nD))\to H^0(S,\mathcal{E}\mathrm{xt}^2_p(\mathcal{F},\mathcal{F}\otimes q^*\mathcal{O}_D(-nD)))$$

obtained from the global to local spectral sequence $H^i(S, \mathcal{E}\mathrm{xt}^i_p) \Rightarrow \mathrm{Ext}^{i+j}_{D\times S}$ (note that by our assumptions the beginning of the spectral sequence degenerates and the above map is in fact an isomorphism). Then for every $k$-point $s \in S$ the germ $\mathrm{ob}(\mathcal{F})_s = \mathrm{ob}(\mathcal{F}_s) \in \mathrm{Ext}^2_D(\mathcal{F}_s, \mathcal{F}_s \otimes \mathcal{O}_D(-nD))$ is an obstruction to extend $\mathcal{F}_s$ from $D_{n_0}$ to $D_n$. So $S_n$ is just the zero set of section $\mathrm{ob}(\mathcal{F})$ in $S$. $\square$

Let us take a flat family $\mathcal{F}$ of sheaves on $D$ parameterized by some $k$-scheme $S$ of finite type and such that it contains all sheaves $\{(F_D^n)^*E'\}_n$. Let $s_n \in S$ be such that $\mathcal{F}_{s_n} \simeq (F_D^n)^*E'$. Consider $\mathcal{F}$ as a sheaf on $X \times S$ extending it by zero (this sheaf is no longer locally free on $X \times S$). Taking $\mathcal{F}' = (F_X^{n_0} \times \mathrm{id}_S)^*\mathcal{F}$ we get a sheaf on $X \times S$, whose restriction to $D \times S$ is $(F_D^{n_0} \times \mathrm{id}_S)^*\mathcal{F}$. But we can consider $\mathcal{F}'$ as an $S$-flat family of locally free sheaves on $D_{n_0}$ and hence we can apply the above lemma. Note that $\mathcal{F}'_{s_m} \simeq (F_D^{m+n_0})^*E'$ can be extended to $D_{p^{m+n_0}}$ so $s_m$ belongs to $S_{p^{m+n_0}}$. But the sequence $\dots \subset S_{n+1} \subset S_n \subset \dots \subset S_{n_0} = S$ stabilizes starting with some $n_1$: $S' = S_{n_1} = S_{n_1+1} = \dots$ of $S$. By the above there exists $m_0$ such that for all $m \geqslant m_0$ we have $s_m \in S_{n_1} = S'$. Therefore for large $m$ we can extend $(F_D^m)^*E'$ to a locally free sheaf $\hat{E}_m$ on the formal completion of $X$ along $D$. By [18, Exposé X, Exemple 2.2] the pair $(X, D)$ satisfies the effective Lefschetz condition. In particular, there exists an open set $U \supset D$ and a locally free sheaf $E'_m$ on $U$ such that the formal completion of $E'_m$ is isomorphic to $\hat{E}_m$. Now set $E_m = j_*E'_m$, where $j : U \hookrightarrow X$ denotes the open embedding. This is a reflexive sheaf on $X$ such that $(F_D^m)^*E' \simeq (E_m)_D$. Therefore $E_m$ is strongly semistable and by Theorem 4.1 it is also locally free.

Let us take the smallest $m \geqslant 0$ such that $(F_D^m)^*E'$ can be extended to a locally free sheaf $E_m$ on $X$. We need to prove that $m = 0$. Let us assume that $m \geqslant 1$. Replacing $E'$ with $(F_D^{m-1})^*E'$ we can assume that $m = 1$. Then $F_D^*E'$ extends to a vector bundle $E_1$ on $X$ and it has the canonical connection $\nabla_{\mathrm{can}} : F_D^*E' \to F_D^*E' \otimes \Omega_D$.

Let us recall that an obstruction to existence of a connection on a vector bundle $E$ on a smooth variety $X$ is given by the Atiyah class $A(E) \in \mathrm{Ext}^1_X(E, E \otimes \Omega_X) = H^1(X, \mathcal{E}\mathrm{nd}\, E \otimes \Omega_X)$.

In our case we have a sequence of maps

$$H^1(X, \mathcal{E}\mathrm{nd}\, E_1 \otimes \Omega_X) \xrightarrow{\alpha_0} H^1(X, \mathcal{E}\mathrm{nd}\, E_1 \otimes \Omega_X|_D) \xrightarrow{\beta_0} H^1(D, \mathcal{E}\mathrm{nd}(E_1)_D \otimes \Omega_D)$$

mapping $A(E_1)$ to $A((E_1)_D) = A(F_D^*E') = 0$. Let us set $G = \mathcal{E}\mathrm{nd}\, E_1$. Note that $\alpha_0$ is injective if $H^1(X, G \otimes \Omega_X(-D)) = 0$ and $\beta_0$ is injective if $H^1(D, G_D(-D)) = 0$. Since $G$ is strongly semistable, vanishing of the first cohomology group follows from Corollary 9.3 and our assumptions on $DH^{d-1}$. To get vanishing of the second group we can use the sequence

$$0 \to G(-2D) \to G(-D) \to G_D(-D) \to 0$$

from which we see that it is sufficient to prove that $H^1(X, G(-D)) = H^2(X, G(-2D)) = 0$. This follows from Theorem 9.1, Theorem 9.4 and our assumptions on $D$ and $H$. Therefore $A(E_1) = 0$ and $E_1$ has some connection $\nabla^1$.

We need to show that $E_1$ has a connection $\nabla$ such that on $D$ it induces the connection $\nabla_{\mathrm{can}}$ of $F_D^* E'_D$. Let $\nabla^1_D$ denotes the connection induced from $\nabla_1$ on $D$. As above we have a sequence of maps

$$H^0(X, G \otimes \Omega_X) \xrightarrow{\alpha_1} H^0(X, G \otimes \Omega_X|_D) \xrightarrow{\beta_1} H^0(D, G_D \otimes \Omega_D).$$

Since $H^0(X, G \otimes \Omega_X(-D)) = H^1(X, G \otimes \Omega_X(-D)) = 0$, $\alpha_1$ is an isomorphism. Similarly, $\beta_1$ is an isomorphism since $H^0(D, G_D(-D)) = H^1(D, G_D(-D)) = 0$. Therefore $\nabla_{\mathrm{can}} - \nabla^1_D \in H^0(D, G_D \otimes \Omega_D)$ lifts to a unique class $\gamma \in H^0(X, G \otimes \Omega_X)$. Then $\nabla = \nabla^1 + \gamma$ is the required connection of $E_1$.

Again we have a sequence of maps

$$H^0(X, G \otimes F_X^* \Omega_X) \xrightarrow{\alpha_2} H^0(D, G_D \otimes F_D^*(\Omega_X|_D)) \xrightarrow{\beta_2} H^0(D, G_D \otimes F_D^* \Omega_D)$$

mapping the $p$-curvature of $\nabla$ to the $p$-curvature of $\nabla_{\mathrm{can}}$ which is 0.

Let us recall that by assumption $\Omega_X \hookrightarrow \mathcal{O}_X(\alpha H)^N$ for some integer $N$. Therefore $G \otimes (F_X^* \Omega_X)(-D) \hookrightarrow G(p\alpha H - D)^N$ and since $(p\alpha H - D)H^{d-1} < 0$ we have vanishing of $H^0(X, G \otimes (F_X^* \Omega_X)(-D))$. Since $F_D^*(\Omega_X|_D)) = (F_X^* \Omega_X)_D$ this implies that the map $\alpha_2$ is injective. Since

$$H^0(D, G \otimes F_D^*(\mathcal{O}_D(-D))) = H^0(D, G(-pD)) = 0,$$

the map $\beta_2$ is injective. This proves that the $p$-curvature of $\nabla$ is equal to 0 and hence by Cartier's descent there exists a sheaf $E$ on $X$ such that $E_1 = F_X^* E$ and $E_D \simeq E'$. This contradicts our assumption. □

*Remark 10.6.* — Let us note that we do not really need Theorem 10.2 in the proof of Theorem 10.4. First as above we prove that for any $E' \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{D})$ there exists $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ such that $E_D \simeq E'$. Then we can go back to the proof of Theorem 10.2. Point (a) is proven in the same way as before but now point (b) is much easier. Namely, let $E' \subset E_D$ be a subbundle of degree 0 in the restriction $E_D$ of $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$. Then we can lift $E'$ to some bundle $E'' \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$. But by (a) the restriction map

$$\mathrm{Hom}_X(E'', E) \to \mathrm{Hom}_D(E', E_D)$$

is an isomorphism, so inclusion $E' \subset E_D$ can be lifted to an inclusion $E'' \subset E$, which finishes the proof of (b).

The following corollary strengthens [3, Theorem 1.1]. Note that in their paper Biswas and Holla used Grothendieck's Lefschetz theorem to prove this theorem. In our case the corollary follows immediately from Theorems 10.2 and 10.4 and the universal property of the fundamental group schemes (see Lemma 6.3).

Corollary 10.7 (Lefschetz theorem for Nori's and étale fundamental groups). — *Let $X$ be a smooth $d$-dimensional projective variety defined over an algebraically closed field $k$ and let $H$ be an ample divisor on $X$. Let $D \subset X$ be any ample smooth effective divisor.*

1. *Let us assume that $d \geqslant 2$ and*
$$DH^{d-1} > \mu_{\max}(\Omega_X).$$
*Then $\pi_1^N(D,x) \to \pi_1^N(X,x)$ and $\pi_1^{Et}(D,x) \to \pi_1^{Et}(X,x)$ are faithfully flat.*
2. *Let us assume that $d \geqslant 3$ and $T_X(\alpha H)$ is globally generated for some non-negative integer $\alpha$. Let us also assume that $D - \alpha H$ is ample and*
$$DH^{d-1} > \max\left(p\alpha H^d, (d+1)\alpha H^d - K_X H^{d-1}\right).$$
*Then $\pi_1^N(D,x) \to \pi_1^N(X,x)$ and $\pi_1^{Et}(D,x) \to \pi_1^{Et}(X,x)$ are isomorphisms.*

In case of the local fundamental group of Nori, the Grothendieck–Lefschetz type theorem was also proved in [28], but without the precise bounds on the degrees of the hypersurfaces.

Corollary 10.8. — *Let $G$ be a reduced, connected linear algebraic group and let $X$ be a projective homogeneous $G$-space such that the scheme-theoretic stabilizers of the action of $G$ on $X$ are reduced. Assume that $X$ has dimension $\geqslant 3$. Then for any smooth ample effective divisor $D \subset X$ and any $k$-point $x \in D$ the group $\pi_1^S(D,x)$ is trivial. In particular, if $D$ is a smooth hypersurface in $\mathbb{P}^d$, $d \geqslant 3$ then $\pi_1^S(D,x) = 0$.*

*Proof.* — We can take $\alpha = 0$ in the above theorem so that we get an isomorphism $\pi_1^S(D,x) \simeq \pi_1^S(X,x)$. But by [29, Theorem 1] the S-fundamental group scheme of $X$ is trivial, which proves the first assertion. The last assertion also follows from Proposition 8.2. □

## 11. Lefschetz type theorems in presence of lifting modulo $p^2$ and in characteristic zero

We fix the following notation. Let $X$ be a smooth $d$-dimensional complete variety defined over a perfect field $k$ of characteristic $p > 0$. We assume throughout that $X$ has a lifting to $W_2(k)$. Under this assumption Deligne and Illusie (and Raynaud) showed in [9] that the Kodaira vanishing theorem is still valid in positive characteristic. We can use their method to give stronger Lefschetz type theorems for varieties with lifting modulo $p^2$.

Let us recall the following lemma which is a small variation of [9, Lemma 2.9] (to simplify exposition we avoid the log version):

Lemma 11.1. — *For any locally free sheaf $E$ and an integer $l < p$ we have*

$$\sum_{i+j=l} h^j(X, E \otimes \Omega^i_X) \leqslant \sum_{i+j=l} h^j(X, F^*E \otimes \Omega^i_X).$$

The above lemma allows us to obtain, in presence of lifting, strong vanishing theorems for numerically flat bundles:

Corollary 11.2. — *For any ample divisor $D$ and any $E \in \mathrm{Vect}^{\mathrm{s}}_0(\mathrm{X})$ we have*

$$H^j(X, E(-D) \otimes \Omega^i_X) = 0$$

*if $i + j < \min(p, d)$.*

*Proof.* — Let us note that since the family $\{(F^l)^*E\}$ is bounded we have for large $l$

$$H^j(X, (F^l)^*E(-p^lD) \otimes \Omega^i_X) = 0.$$

Therefore the assertion follows by induction from the lemma applied to the sheaves $(F^{l-1})^*E(-p^{l-1}D)$, $(F^{l-2})^*E(-p^{l-2}D)$, $\dots, E(-D)$. □

Theorem 11.3. — *Let $D$ be any smooth ample effective divisor on $X$.*

1. *If $d \geqslant 2$ then $\pi_1^S(D, x) \to \pi_1^S(X, x)$ is faithfully flat.*
2. *If $d \geqslant 3$ and $p \geqslant 3$ then $\pi_1^S(D, x) \to \pi_1^S(X, x)$ is an isomorphism.*

*Proof.* — Using the above corollary one can follow the proofs of Theorems 10.2 and 10.4 without changes (except for the fact that vanishing of cohomology groups is much simpler). □

Clearly, we get the same result also for Nori and étale fundamental groups.

Now let $X$ be a complex projective manifold. Using Lefschetz theorems for the topological fundamental group and the universal property of S-fundamental group scheme we get the following theorem:

THEOREM 11.4. — *Let $D$ be any smooth ample effective divisor on $X$.*

1. *If $d \geqslant 2$ then $\pi_1^S(D,x) \to \pi_1^S(X,x)$ is faithfully flat.*
2. *If $d \geqslant 3$ then $\pi_1^S(D,x) \to \pi_1^S(X,x)$ is an isomorphism.*

Let us note that a similar theorem holds also for the universal complex pro-algebraic group $\pi_1^a(X,x)$. We sketch now an algebraic proof (in 2 we assume that $d \geqslant 4$).

*Proof.* — Manivel's vanishing theorem (see [27, Theorem A]) implies that for any ample divisor $D$ and any $E \in \mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ we have

$$H^j(X, E(-D) \otimes \Omega_X^i) = 0$$

if $i + j < d$ (note that the proof by reducing to characteristic $p$ and using Corollary 11.2 does not quite work as we do not know if the reduction of $E$ modulo $p$ is still in $\mathrm{Vect}_0^{\mathrm{s}}(\mathrm{X})$ for some $p$). Therefore we can also give an algebraic proof of the above Lefschetz type theorem following the proofs of Theorems 10.2 and 10.4 (replacing the Frobenius morphism with identity). In this case, in proof of Theorem 10.4, we cannot use the Frobenius morphism to extend $E_D$ from the divisor $D$ to $X$. But by the above vanishing theorem we have

$$H^2(D, \mathcal{E}\mathrm{nd}\, E_D \otimes \mathcal{O}_D(-iD)) = 0$$

for $i > 0$. This allows us to extend $E_D$ to a vector bundle on the formal completion of $X$ along $D$ and then we can go back to the proof. □

Note that the above proof works only if $d \geqslant 4$ (as with Grothendieck's proof of the Lefschetz theorem for the Picard group). If $d = 3$ then, as one can see using Serre's duality, the above obstruction space is never equal to zero for large $i$. Nevertheless, in positive characteristic we could go around this problem.

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