ANNALES

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Tome 61, nº 4 (2011), p. 1633-1653.

<http://aif.cedram.org/item?id=AIF_2011__61_4_1633_0>

Ann. Inst. Fourier, Grenoble
61, 4 (2011) 1633-1653

SPECTRUM AND MULTIPLIER IDEALS OF ARBITRARY SUBVARIETIES

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Abstract. — We introduce a spectrum for arbitrary subvarieties. This generalizes the definition by Steenbrink for hypersurfaces. In the isolated complete intersection singularity case, it coincides with the one given by Ebeling and Steenbrink except for the coefficients of integral exponents. We show a relation to the graded pieces of the multiplier ideals by using the filtration V of Kashiwara and Malgrange. This implies a partial generalization of a theorem of Budur in the hypersurface case. The key point is to consider the direct sum of the graded pieces of the multiplier ideals as a module over the algebra defining the normal cone of the subvariety. We also give a combinatorial description in the case of monomial ideals.

Résumé. — Nous introduisons un spectre pour des sous-variétés arbitraires.Ceci généralise la définition de Steenbrink pour les hypersurfaces. Dans le cas d'une singularité isolée d'intersection complète, il coïncide au spectre donné par Ebeling et Steenbrink, sauf pour les coefficients des exposants entiers. Nous montrons une relation avec les gradués des idéaux multiplicateurs en utilisant la filtration V de Kashiwara et Malgrange. Ceci implique une généralisation partielle d'un théorème de Budur dans le cas des hypersurfaces. Le point clef est de considérer la somme directe des gradués d'un idéal multiplicatif comme un module sur l'algèbre définissant le cône normal de la sous-variété. Nous donnons aussi une description combinatoire dans le cas des idéaux monomiaux.

Introduction

In [26], [27], Steenbrink introduced the spectrum for hypersurface singularities. Its relations with b-function and multiplier ideals have been studied in [3], [6], [24], [25], *etc.* The multiplier ideals were originally defined for any subvariety of a smooth variety (see *e.g.* [16]), and the b-function for an

Keywords: Spectrum, V-filtration, multiplier ideal.
Math. classification: 32S40, 32S35.

arbitrary variety has been defined in [4]. In this paper we introduce the spectrum for an arbitrary variety generalizing Steenbrink's definition [26], [27].

Let X be a closed subvariety of a smooth complex algebraic variety or a complex manifold Y. Let $(N_XY)_x$ be the fiber of the normal cone $N_XY \to X$ over $x \in X$. For each irreducible component Λ of $(N_XY)_x$, set $n_\Lambda = \dim Y - \dim \Lambda$. We have the nonreduced spectrum and the (reduced) spectrum

$$\widehat{\mathrm{Sp}}(X,\Lambda) = \sum_{\alpha>0} m_{\Lambda,\alpha} t^\alpha \in \mathbb{Z}[t^{1/e}], \quad \mathrm{Sp}(X,\Lambda) = \widehat{\mathrm{Sp}}(X,\Lambda) - (-1)^{n_\Lambda} t^{n_\Lambda+1},$$

where $e \in \mathbb{Z}_{>0}$, see 1.2 for the definition of $m_{\Lambda,\alpha}$. Note that $\widehat{\mathrm{Sp}}(X,\Lambda)$, $\mathrm{Sp}(X,\Lambda)$ are essentially independent of Y as a corollary of a product formula, see Cor. 3.3 and 3.4. In case $(N_XY)_x$ is irreducible, set $\widehat{\mathrm{Sp}}(X,x) = \widehat{\mathrm{Sp}}(X,\Lambda)$ for $\Lambda = (N_XY)_x$, and similarly for $\mathrm{Sp}(X,x)$, $m_{x,\alpha}$. This generalizes Steenbrink's definition in the hypersurface case where N_XY is a line bundle over X. We use $\mathrm{Sp}(X,x)$ mainly in this case. The difference between $\widehat{\mathrm{Sp}}(X,\Lambda)$ and $\mathrm{Sp}(X,\Lambda)$ comes from the one between cohomology and reduced cohomology.

In the isolated complete intersection singularity case, N_XY is a vector bundle over X (in particular, $(N_XY)_x$ is irreducible), and the spectrum is associated to the mixed Hodge structure on the Milnor cohomology where the action of the monodromy is given by choosing a sufficiently general line passing through the origin in the base space of the Milnor fibration, see also Remark 1.3, (i). In this case Ebeling and Steenbrink [10] defined the spectrum in a different way where they consider also the contribution of the Milnor cohomology associated to the singularity of the total space of a generic 1-parameter smoothing of X so that the symmetry and the semicontinuity hold. Their spectrum differs from ours in general ([10], [19], [28]), but they coincide for $m_{x,\alpha}$ with $\alpha \notin \mathbb{Z}$, see Remark 1.3, (iv). So we can apply Theorem 1 below to their spectrum except for the case $\beta = 1$.

Let $\mathcal{J}(Y,\alpha X)$ denote the multiplier ideals of $X \subset Y$ for $\alpha \in \mathbb{Q}_{>0}$. They can be defined by the local integrability of

$$|g|^2 / \Big(\sum_i |f_i|^2\Big)^\alpha \quad \text{for} \quad g \in \mathcal{O}_Y,$$

where $f_1, \dots, f_r$ are local generators of the ideal of X, see [16]. They are closely related to the filtration V of Kashiwara [14] and Malgrange [18], see [4]. Set

$$\mathcal{G}(Y,\alpha X) = \mathcal{J}(Y,(\alpha-\varepsilon)X)/\mathcal{J}(Y,\alpha X) \quad \text{for} \quad 0 < \varepsilon \ll 1.$$

If $\mathcal{G}(Y,\alpha X)_x \neq 0$, then α is called a jumping coefficient of $X \subset Y$ at x.

Let $\beta \in (0,1] \cap \mathbb{Q}$, and $\mathcal{I}_X$ be the ideal sheaf of $X \subset Y$. Define

$$\mathcal{M}(\beta) = \bigoplus_{i\in\mathbb{N}} \mathcal{G}(Y,(\beta+i)X), \quad \overline{\mathcal{A}} = \bigoplus_{i\in\mathbb{N}} \mathcal{I}_X^i/\mathcal{I}_X^{i+1}.$$

Then $\mathcal{M}(\beta)$ is a graded $\overline{\mathcal{A}}$-module since $\mathcal{I}_X\mathcal{J}(Y,\alpha X) \subset \mathcal{J}(Y,(\alpha+1)X)$. Set

$$\widetilde{Z}_\beta = \operatorname{supp}_{\overline{\mathcal{A}}} \mathcal{M}(\beta) \subset N_XY = \mathcal{S}pec_X\,\overline{\mathcal{A}},$$

i.e. $\widetilde{Z}_\beta$ is the support of the associated sheaf on N_XY. Here $\mathcal{S}pec_X\,\overline{\mathcal{A}}$ is replaced by $\mathcal{S}pecan_X\,\overline{\mathcal{A}}$ in the analytic case. For an irreducible component E of N_XY, let E^0 be the complement in E of the intersection of E with the union of the other irreducible components of N_XY. Let $\widetilde{Z}_{\beta,E}$ be the closure of $\widetilde{Z}_\beta \cap E^0$. Let $Z_{\beta,E}$ be the image of $\widetilde{Z}_{\beta,E}$ in X. Set $d_{\beta,E} = \dim Z_{\beta,E}$.

For $x \in X$, let $\mathfrak{m}_{Y,x}$ be the maximal ideal of $\mathcal{O}_{Y,x}$, and set

$$\mathcal{M}(\beta,x) = \mathcal{M}(\beta)/\mathfrak{m}_{Y,x}\mathcal{M}(\beta), \quad \overline{\mathcal{A}}(x) = \overline{\mathcal{A}}/\mathfrak{m}_{Y,x}\overline{\mathcal{A}}.$$

Then $\mathcal{M}(\beta,x)$ is a graded $\overline{\mathcal{A}}(x)$-module. For each irreducible component Λ of $(N_XY)_x = Spec\,\overline{\mathcal{A}}(x)$, let $\mu_{\Lambda,\beta}$ be the multiplicity of $\mathcal{M}(\beta,x)$ at the generic point of Λ, *i.e.*

$$\mu_{\Lambda,\beta} = \dim_{K(\Lambda)} \mathcal{M}(\beta,x) \bigotimes_{\overline{\mathcal{A}}(x)} K(\Lambda),$$

where $K(\Lambda)$ is the function field of Λ which is a localization of $\overline{\mathcal{A}}(x)_{\mathrm{red}}$.

Theorem 1. — *Let $\beta \in (0,1] \cap \mathbb{Q}$. Then*

(i) *We have in general*

$$0 \leqslant m_{\Lambda,\beta} \leqslant \mu_{\Lambda,\beta}.$$

In particular, $m_{\Lambda,\beta} = 0$ if $x \notin \operatorname{supp} \mathcal{M}(\beta) \subset X$.

(ii) *If $\widetilde{Z}_\beta$ is contained in Λ on a neighborhood of ξ in N_XY, then*

$$m_{\Lambda,\beta} = \mu_{\Lambda,\beta}.$$

(iii) *If x is a sufficiently general point of $Z_{\beta,E}$, then for any irreducible component Λ of $(N_XY)_x$ contained in E, we have $m_{\Lambda,\beta+i} = 0$ for any integer $i < d_{\beta,E}$, and*

$$m_{\Lambda,\beta+d_{\beta,E}} = (-1)^{d_{\beta,E}}\mu_{\Lambda,\beta}.$$

Here a sufficiently general point means that it belongs to a (sufficiently small) non-empty Zariski-open subset of $Z_{\beta,E}$. This gives a partial generalization of a theorem of Budur [3] in the hypersurface case where $\mathcal{M}(\beta)$ is a free $\overline{\mathcal{A}}$-module of rank 1 over $\mathcal{G}(Y,\beta X)$. Theorem 1 was found during an attempt to extend some assertions about the spectrum in [9]. The following is a generalization of Cor 1.5 and 1.6 in loc. cit.

Theorem 2. — *Let $\{S_i\}$, $\{S'_j\}$ be Whitney stratifications of X, N_XY such that the restriction of the projection $N_XY \to X$ to each S'_j is a smooth morphism to some S_i, and the restriction of $\mathcal{H}^\bullet \operatorname{Sp}_X \mathbb{C}_Y$ to S'_j are local systems where $\operatorname{Sp}_X \mathbb{C}_Y$ denotes the Verdier specialization* [31]. *Then*

(i) *The spectrum $\widehat{\operatorname{Sp}}(X, \Lambda_x)$ remains constant if $\{\Lambda_x\}_{x\in U}$ is a locally trivial family of irreducible components of $(N_XY)_x$ for $x \in U \subset S_i$ where U is an analytically open subset of S_i.*

(ii) *If T is a transversal slice to S_i in Y such that $S_i \cap T = \{x\}$, then we have*
$$\widehat{\operatorname{Sp}}(X, \Lambda) = (-t)^{d_i}\, \widehat{\operatorname{Sp}}(X \cap T, \Lambda),$$
for an irreducible component Λ of $(N_{X\cap T}T)_x = (N_XY)_x$ where $d_i = \dim S_i$.

Here a locally trivially family $\{\Lambda_x\}_{x\in U}$ in (i) means a local section of the sheaf defined by the sets of the irreducible components of $(N_XY)_x$ for $x \in S_i$, which is a locally constant sheaf. Theorem 2 implies that Theorem 1 (iii) is reduced to (ii) (where $d_{\beta,E} = 0$) by restricting to a sufficiently general member of a family of transversal slices. Note that for $\beta \notin \mathbb{Z}$, we have $Z_{\beta,E} \subset \operatorname{supp} \mathcal{M}(\beta) \subset \operatorname{Sing} X$, and hence $d_{\beta,E} = 0$ in the isolated singularity case.

Let $f = (f_1, \dots, f_r)$ be a set of local generators of $\mathcal{I}_X$, and $b_f(s)$ be the b-function of f in the sense of [4] (see also [11], [20]). This is independent of the choice of f and r, but depends on the choice of Y. By [4] and Theorem 1 we get

Theorem 3. — *Let $\alpha \in \mathbb{Q}_{>0}$, and assume $m_{\Lambda,\alpha} \neq 0$ for some irreducible component Λ of $(N_XY)_x$. Then $\alpha + i$ is a root of $b_f(-s)$ for some $i \in \mathbb{Z}$. If furthermore $\alpha < 1$, then there is a nonnegative integer j_0 such that $\alpha + j$ is a jumping coefficient of $X \subset Y$ at x for any integer $j \geqslant j_0$.*

It is not easy to determine i and j_0 in Theorem 3 (see Example 4.6 below) except for j_0 in the hypersurface case (here $j_0 = 0$ since $\mathcal{G}(Y, \alpha X)_x = \mathcal{G}(Y, (\alpha + j)X)_x$ for $\alpha > 0$ and $j \in \mathbb{N}$). In the monomial ideal case, j_0 is bounded by $\dim Y - 1$, see Remark 4.8 below.

In the monomial ideal case, there is a combinatorial description for the jumping coefficients [13] and for the roots of the b-function [5]. We give here one for the spectrum, see Th. 4.4.

In Section 1, we review the specializations and define the spectrum. In Section 2, we prove Theorems 1–3. In Section 3, we show a product formula which implies the well-definedness of the spectrum. In Section 4, we treat the monomial ideal case, and prove Th. 4.4.

1. Spectrum

In this section we review the specializations and define the spectrum.

1.1. Specialization. — Let Y be a smooth complex algebraic variety or a complex manifold, and X be a subvariety or a analytic subspace of Y. We do not assume X reduced nor irreducible. Let N_XY denote the normal cone. Let $f=(f_1,\dots,f_r)$ be a set of local generators of the ideal $\mathcal{I}_X$ of $X\subset Y$. We denote the graph embedding by

$$i_f : Y \to \widetilde{Y} := Y\times\mathbb{C}^r.$$

Let $z_1,\dots,z_r$ be the coordinates of $\mathbb{C}^r$. We have the canonical surjection

$$\overline{\mathcal{A}}' := \mathcal{O}_Y[z_1,\dots,z_r] \to \overline{\mathcal{A}} := \bigoplus_{i\in\mathbb{N}} \mathcal{I}_X^i/\mathcal{I}_X^{i+1},$$

sending z_i to f_i. This implies the inclusion $N_XY\subset\widetilde{Y}$ such that the projection $\pi' : \widetilde{Y}\to Y$ induces $\pi : N_XY\to X$. Let $\partial_i=\partial/\partial z_i$, and define

$$(M,F) := (i_f)_*(\mathcal{O}_Y,F) = (\mathcal{O}_Y[\partial_1,\dots,\partial_r],F).$$

Here the Hodge filtration F on $\mathcal{O}_Y$ is defined so that $\mathrm{Gr}_p^F\mathcal{O}_Y=0$ for $p\neq -\dim Y$, and the direct image $(i_f)_*$ is defined as a filtered $\mathcal{D}$-module. Setting $\partial^\nu=\prod_i\partial_i^{\nu_i}$, the Hodge filtration F on the direct image is defined by

$$F_pM = \sum_{|\nu|\leqslant p+\dim Y} \mathcal{O}_Y\otimes\partial^\nu. \tag{1.1.1}$$

Let V denote the filtration of Kashiwara [14] and Malgrange [18] on M along $Y\times\{0\}$ indexed by $\mathbb{Q}$. The specialization of $(\mathcal{O}_Y,F)$ along X is defined by

$$\mathrm{Sp}_X(\mathcal{O}_Y,F) = \pi'^{-1}\Big(\bigoplus_{\alpha\in\mathbb{Q}} \mathrm{Gr}_V^\alpha(M,F)\Big) \bigotimes_{\pi'^{-1}\overline{\mathcal{A}}'} \mathcal{O}_{\widetilde{Y}}.$$

By [4], [9] this is compatible with the definition of specialization [31] in the category of perverse sheaves [1] or mixed Hodge modules [23]

$$\mathrm{Sp}_X\,\mathbb{Q}_Y[\dim Y] = \psi_t\mathbf{R}j_*\mathbb{Q}_{Y\times\mathbb{C}^*}[\dim Y], \tag{1.1.2}$$

where

$$j : Y\times\mathbb{C}^* = \mathcal{S}pec_Y(\mathcal{O}_Y[t,t^{-1}]) \to \mathcal{S}pec_Y\Big(\bigoplus_{i\in\mathbb{Z}} \mathcal{I}_X^{-i}\otimes t^i\Big),$$

denotes the inclusion to the deformation to the normal cone, see [31]. Here $\mathcal{I}_X^{-i} = \mathcal{O}_Y$ for $i \geqslant 0$, and $\mathcal{S}pec_Y$ is replaced by $\mathcal{S}pecan_Y$ in the analytic case. Note that the action of the semisimple part T_s of the monodromy associated to the nearby cycle functor ψ_t corresponds to the multiplication by $\exp(-2\pi i\alpha)$ on $\mathrm{Gr}_V^{\alpha}(M,F)$. These follow from the fact (see [4], [9]) that the filtration V corresponding to the functor ψ_t is essentially given by

$$\bigoplus_{i\in\mathbb{Z}} V^{\alpha-i}M\otimes t^i.$$

By [4] (see also [9] for the analytic case) we have the isomorphisms

$$F_{-\dim Y}\,\mathrm{Gr}_V^{\alpha} M = \mathcal{G}(Y,\alpha X) \quad \text{for} \quad \alpha\in\mathbb{Q}. \tag{1.1.3}$$

For $\mathcal{M}(\beta)$ as in the introduction, this implies

$$\begin{aligned} F_{-\dim Y}\,\mathrm{Sp}_X(\mathcal{O}_Y) &= \pi'^{-1}\Big(\bigoplus_{0<\beta\leqslant 1}\mathcal{M}(\beta)\Big)\bigotimes_{\pi'^{-1}\overline{\mathcal{A}}'}\mathcal{O}_{\widetilde{Y}} \\ &= \pi'^{-1}\Big(\bigoplus_{0<\beta\leqslant 1}\mathcal{M}(\beta)\Big)\bigotimes_{\pi^{-1}\overline{\mathcal{A}}}\mathcal{O}_{N_XY}. \end{aligned}$$

since $\mathcal{O}_{N_XY} = \pi^{-1}\overline{\mathcal{A}}\bigotimes_{\pi'^{-1}\overline{\mathcal{A}}'}\mathcal{O}_{\widetilde{Y}}$ for the last isomorphism. These imply that $\mathcal{M}(\beta)$ is locally finitely generated over $\overline{\mathcal{A}}'$ or $\overline{\mathcal{A}}$.

1.2. Definition. — *With the above notation, set*

$$(N_XY)_x = \pi^{-1}(x) \text{ for } x\in X.$$

Let Λ be an irreducible component of $(N_XY)_x$, and take a sufficiently general point ξ of Λ. Let $i_\xi : \{\xi\}\to N_XY$ denote the inclusion morphism. Set

$$d_Y = \dim Y,\quad d_\Lambda = \dim\Lambda,\quad n_\Lambda = d_Y - d_\Lambda.$$

We denote the pull-back of $\mathrm{Sp}_X(\mathcal{O}_Y,F)[-d_Y]$ as a complex of filtered $\mathcal{D}$-modules by

$$i_\xi^*\,\mathrm{Sp}_X(\mathcal{O}_Y,F)[-d_Y],$$

see Remark 1.3 (ii) below. This corresponds to $i_\xi^*\,\mathrm{Sp}_X\,\mathbb{C}_Y$, and underlies a complex of mixed Hodge modules on $\{\xi\}$, which is identified with a complex of mixed Hodge structures [8]*, see* [23]*. Combined with the action of T_s, it defines the nonreduced spectrum as in* [26], [27]*:*

$$\widehat{\mathrm{Sp}}(X,\Lambda) = \sum_{\alpha} m_{\Lambda,\alpha}t^{\alpha},$$

where

$$m_{\Lambda,\alpha}=\sum_j(-1)^j\dim \mathrm{Gr}_F^p H^{j+n_\Lambda}(i_\xi^*\,\mathrm{Sp}_X\,\mathcal{O}_Y[-d_Y])_\lambda,\quad \text{with}\quad p=[n_\Lambda+1-\alpha],\ \lambda=\exp(-2\pi i\alpha). \tag{1.2.1}$$

Here $H^\bullet(i_\xi^*\,\mathrm{Sp}_X\,\mathcal{O}_Y[-d_Y])_\lambda$ *denotes the* λ*-eigenspace of the cohomology group by the action of* T_s*. Set*

$$\mathrm{Sp}(X,\Lambda)=\widehat{\mathrm{Sp}}(X,\Lambda)-(-1)^{n_\Lambda}t^{n_\Lambda+1}.$$

If $(N_XY)_x$ *is irreducible, set for* $\Lambda=(N_XY)_x$

$$\widehat{\mathrm{Sp}}(X,x)=\widehat{\mathrm{Sp}}(X,\Lambda),\quad \mathrm{Sp}(X,x)=\mathrm{Sp}(X,\Lambda),\quad m_{x,\alpha}=m_{\Lambda,\alpha}.$$

1.3. Remarks.

(i) This generalizes Steenbrink's definition in the hypersurface case ([26], [27]). Assume X is a hypersurface or an isolated complete intersection singularity. Then N_XY is a line bundle or a vector bundle over X, and the Milnor cohomology is given by

$$H^\bullet\big(i_\xi^*\,\mathrm{Sp}_X(\mathcal{O}_Y,F)[-d_Y]\big)\quad\text{for}\quad \xi\in(N_XY)_x\quad\text{sufficiently general,}$$

where the pull-back i_ξ^* is explained in Remark 1.3 (ii) below. In the isolated complete intersection singularity case, the action of the monodromy is given by taking a sufficiently general line C passing through the origin in the base space of the Milnor fibration and corresponding to $\xi\in\Lambda=(N_XY)_x$. Indeed, if the complete intersection X is defined by

$$f=(f_1,\dots,f_r):Y\to\mathbb{C}^r,$$

then f induces the projection

$$N_XY=X\times\mathbb{C}^r\to N_{\{0\}}\mathbb{C}^r=\mathbb{C}^r,$$

and the inverse image Z of a sufficiently general line $C\subset\mathbb{C}^r$ by f is a 1-parameter smoothing of X. Moreover, its Milnor cohomology is isomorphic to the cohomology of $(\mathrm{Sp}_X\,\mathbb{C}_Y)_\xi$ with $\mathbb{C}\,\xi\subset(N_XY)_x$ corresponding to C by the above projection, and the restriction of $\mathrm{Sp}_X\,\mathbb{C}_Y$ to $N_XZ\subset N_XY$ is isomorphic to $\mathrm{Sp}_X\,\mathbb{C}_Z$ on a neighborhood of $\xi\in(N_XZ)_x$ if ξ is sufficiently general. (Note also that we can replace Y with the total space of the miniversal deformation of f by Cor. 3.4.)

(ii) Under a closed embedding of smooth complex varieties or complex manifolds $i:X\to Y$, the pull-back of a complex of filtered $\mathcal{D}$-modules (M,F) underlying a complex of mixed Hodge modules is locally defined as follows. By factorizing i locally on Y, we may assume X is defined by

$y_1 = 0$ where $y_1, \dots, y_m$ are local coordinates of Y. Let V be the filtration of Kashiwara [14] and Malgrange [18] along X. Then the pull-back $i^*(M,F)$ as a complex of filtered $\mathcal{D}$-modules is defined to be the mapping cone of

$$\partial/\partial y_1 : \mathrm{Gr}_V^1(M, F[1]) \to \mathrm{Gr}_V^0(M,F), \tag{1.3.1}$$

where $(F[1])_p = F_{p-1}$. This is the same as in the case of right $\mathcal{D}$-modules, and the transformation between the corresponding left and right $\mathcal{D}$-modules is done without shifting the filtration in this paper. (Note that the usual definition of pull-back as in [2] does not work for the Hodge filtration.)

(iii) Define

$$p(M,F) = \min\big\{p \mid F_pM \neq 0\big\}. \tag{1.3.2}$$

By Remark (ii), $p(M,F)$ does not decrease under the cohomological pull-back functor $\mathcal{H}^k i^*$ for filtered $\mathcal{D}$-modules by a closed embedding i, and increases by the codimension under a non-characteristic restriction to a transversal slice (since $\mathrm{Gr}_V^0 M = 0$ in the non-characteristic case).

Under the pull-back by a smooth morphism, $p(M,F)$ decreases by the relative dimension where the complex is also shifted. This is compatible with the definition of F on $\mathcal{O}_Y$, and we have

$$p(\mathcal{O}_Y, F) = p(\mathrm{Sp}_X \mathcal{O}_Y, F) = -\dim Y. \tag{1.3.3}$$

Then we get by Remark (ii)

$$m_{\Lambda,\alpha} = 0 \quad \text{if} \quad \alpha \leqslant 0, \qquad m_{\Lambda,\beta} \geqslant 0 \quad \text{if} \quad \beta \in (0,1]. \tag{1.3.4}$$

(iv) The normalization of spectrum (1.2.1) is dual of the one used by Ebeling and Steenbrink [10]. Indeed, for a Hodge structure H with an automorphism T of finite order, they use

$$\mathrm{Sp}'(H,T) = \sum_\alpha m_{H,\alpha} t^\alpha \quad \text{with} \quad m_{H,\alpha} = \dim_{\mathbb{C}} \mathrm{Gr}_F^p H_{\mathbb{C},\lambda},$$
$$\text{where} \quad p = [\alpha], \quad \lambda = \exp(2\pi i \alpha).$$

Here $H_{\mathbb{C},\lambda} = \mathrm{Ker}(T-\lambda) \subset H_{\mathbb{C}}$. This definition is somewhat dual of (1.2.1). In the case of isolated hypersurface singularities, their definition of spectrum coincides with ours by the symmetry of the spectrum which follows from the self-duality of the mixed Hodge structure on the Milnor cohomology [26]. In the case of isolated complete intersection singularities, they apply the above definition to the mixed Hodge structure

$$\varphi_f \psi_g \mathbb{Q}_{\mathcal{X}}[d_X],$$

where $f : X' \to \mathbb{C}$ is a generic 1-parameter smoothing of X, $g : \mathcal{X} \to \mathbb{C}$ is a generic 1-parameter smoothing of X', T is the semisimple part of the monodromy associated to φ_f, and $d_X = \dim X$, see [10] for details. Then the symmetry of their spectrum follows from the self-duality of $\varphi_f \psi_g \mathbb{Q}_{\mathcal{X}}[d_X]$ in [23], 2.6.2.

Denoting the Milnor fibers of f, g by F_f, F_g, we have a short exact sequence

$$0 \to \widetilde{H}^{d_X}(F_f, \mathbb{C}) \to \varphi_f \psi_g \mathbb{Q}_{\mathcal{X}}[d_X] \to H^{d_X+1}(F_g, \mathbb{C}) \to 0,$$

since $\psi_g \mathbb{Q}_{\mathcal{X}}|_{X' \setminus \{0\}} = \mathbb{Q}_{X' \setminus \{0\}}$ and $(\psi_g \mathbb{Q}_{\mathcal{X}})_0 = \mathbf{R}\Gamma(F_g, \mathbb{C})$. This means that we have to consider also the contribution of $H^{d_X+1}(F_g, \mathbb{C})$ in order to satisfy the symmetry and the semicontinuity. Since the action of the monodromy on $H^{d_X+1}(F_g, \mathbb{C})$ is associated to the function f, it is the identity. So their definition coincides with ours for the $m_{x,\alpha}$ with $\alpha \notin \mathbb{Z}$. However, it seems rather difficult to generalize the construction in [10] to the case of arbitrary singularities.

(v) By the definition of specialization (1.1.2) using the deformation to the normal cone, we have

$$\operatorname{supp} \operatorname{Sp}_X \mathbb{C}_Y = N_X Y \quad \text{with} \quad \dim N_X Y = \dim Y.$$

The assumption of the next proposition is satisfied in case X has a 0-dimensional embedded component so that $\dim(N_X Y)_x = \dim Y$, see also Th. 4.4 and Ex. 4.6 below.

1.4. Proposition. — *If* $\dim \Lambda = \dim Y$, *then* $m_{\Lambda,\alpha} = 0$ *unless* $\alpha \in (0, 1]$.

Proof. — It is enough to show that the restriction of $\operatorname{Sp}_X(\mathcal{O}_Y, F)$ to a sufficiently small open subvariety of Λ is a variation of mixed Hodge structure of level 0 (*i.e.* the Hodge filtration is trivial). Let $n =: \dim \Lambda = \dim Y$, and

$$(H, F) = i_\xi^* \operatorname{Sp}_X(\mathcal{O}_Y, F)[-n].$$

This underlies a mixed Hodge structure (where $F^p = F_{-p}$) since ξ is sufficiently general. By Remark 1.3 (iii) we have

$$\min\{p \mid F_p H \neq 0\} \geqslant 0.$$

Moreover, the self-duality $\mathbb{D}(\mathcal{O}_Y, F) = (\mathcal{O}_Y, F[n])$ implies

$$\mathbb{D}(\operatorname{Sp}_X(\mathcal{O}_Y, F)) = \operatorname{Sp}_X(\mathcal{O}_Y, F[n]),$$
$$\mathbb{D}(H, F) = (H, F).$$

Indeed, the second isomorphism follows from the first by using the duality

$$i_\xi^* \circ \mathbb{D} = \mathbb{D} \circ i_\xi^!$$

together with

$$i_\xi^! \operatorname{Sp}_X(\mathcal{O}_Y, F) = i_\xi^* \operatorname{Sp}_X(\mathcal{O}_Y, F[n])[2n].$$

Here the last isomorphism follows from the fact that $\operatorname{Sp}_X(\mathcal{O}_Y, F)[-n]$ underlies a variation of mixed Hodge structure on a neighborhood of ξ in Λ. So the assertion follows. □

2. Proof of main theorems

In this section we prove Theorems 1–3. We first show the following proposition which will be used in the proof of Theorem 1.

2.1. Proposition. — *Let (M, F) be a filtered $\mathcal{D}_Y$-module on a smooth variety or a complex manifold Y, which underlies a mixed Hodge module $\mathcal{M}$. Let $p_0 = p(M, F)$ in the notation of* (1.3.2)*. Let $i : X \to Y$ be a closed immersion of a smooth subvariety. Let $i^*(M, F)$ denote of the pull-back as a complex of filtered $\mathcal{D}$-modules. Then*

(i) *We have $F_{p_0}\mathcal{H}^k i^* M = 0$ for $k \neq 0$, and $F_{p_0}\mathcal{H}^0 i^* M$ is locally a quotient of $\mathcal{O}_X \bigotimes_{\mathcal{O}_Y} F_{p_0} M$.*

(ii) *If $\operatorname{supp} F_{p_0} M \subset X$, then $F_{p_0}\mathcal{H}^0 i^* M = F_{p_0} M$.*

Proof. — By the definition of the pull-back, we may assume that X is defined by $y_1 = 0$ as in Remark 1.3 (ii). Then the assertion (i) for $k \neq 0$ easily follows. Moreover, the assertion (i) for $k = 0$ is reduced to

$$F_{p_0} M \subset V^0 M. \tag{2.1.1}$$

(Indeed, this implies $y_1 F_{p_0} M \subset V^{>0} M$, and hence $F_{p_0} \operatorname{Gr}_V^0 M$ is a quotient of $\mathcal{O}_X \bigotimes_{\mathcal{O}_Y} F_{p_0} M$.) Then (2.1.1) follows from the strict surjectivity of

$$\partial/\partial y_1 : \operatorname{Gr}_V^{\alpha+1}(M, F[1]) \to \operatorname{Gr}_V^{\alpha}(M, F) \quad \text{for} \quad \alpha < 0,$$

see [22], 3.2.1.3, where $(F[1])_p = F_{p-1}$ and $V_\alpha = V^{-\alpha}$.

For the assertion (ii) it is enough to show $F_{p_0} V^{>0} M = 0$ under the condition $\operatorname{supp} F_{p_0} M \subset X$. By the definition of the filtration V, we have the injectivity of

$$y_1 : V^\alpha M \to V^{\alpha+1} M \quad \text{for} \quad \alpha > 0.$$

If $F_{p_0} V^{>0} M \neq 0$, then its support cannot be contained in $X = \{y_1 = 0\}$ by the above injectivity of y_1. So the assertion follows. This completes the proof of Prop. 2.1. □

2.2. Proof of Theorem 1. — Set $(M,F)=\mathrm{Sp}_X(\mathcal{O}_Y,F)$. Applying Prop. 2.1 (i) to the pull-back by

$$\tilde{i}_x : \{x\}\times\mathbb{C}^r \to \widetilde{Y},$$

we get the assertion (i), because the pull-back by

$$i'_\xi : \{\xi\} \to \Lambda$$

is a non-characteristic pull-back (since $\xi\in\Lambda$ is sufficiently general), and is defined by the pull-back as $\mathcal{O}$-modules, see Remark 1.3 (ii). (Note that the pull-back by $\Lambda\to\{x\}\times\mathbb{C}^r$ is the inverse of the direct image by the closed embedding if we restrict to a neighborhood of ξ where Λ is smooth. So this is essentially trivial.)

If x is a sufficiently general point of $Z_{\beta,E}$, we may assume $Z_{\beta,E}$ is a point by [9], Th. 5.3 taking a sufficiently general member of a family of transversal slices to $Z_{\beta,E}$. Here $\dim Y$ and $d_{\beta,E}$ are replaced respectively by $\dim Y - d_{\beta,E}$ and 0. Thus the assertion (iii) is reduced to (ii), and the latter follows from Prop. 2.1 (ii). This completes the proof of Theorem 1.

2.3. Proof of Theorem 2. — The first assertion is clear since the restriction of $\mathcal{H}^\bullet\,\mathrm{Sp}_X\,\mathbb{C}_Y$ to S'_j underlies a variation of mixed Hodge structure.

For the second assertion, note that the stratification $\{S'_j\}$ of N_XY gives a stratification satisfying Thom's (a_f)-condition for the function t in (1.1.2) since $Y\times\mathbb{C}^*$ is smooth, see *e.g.* [9], Prop. 2.17 and the references there. So the assertions follow from the same argument as in the proofs of [9], Theorems 1.2 and 5.3.

2.4. Proof of Theorem 3. — The first assertion on the roots of b-function follows from [4], Cor. 2.8, and the second assertion on the jumping coefficients follows from Theorem 1 (i). Indeed, if the second assertion does not hold, then the degree i part of $\mathcal{M}(\beta,x)$ vanishes for $i\gg 0$ (since the graded algebra $\overline{\mathcal{A}}$ is generated by the degree 1 part). But this implies that $\mathrm{supp}\,\mathcal{M}(\beta,x)\subset\{0\}$ in $\mathit{Spec}\,\overline{\mathcal{A}}(x)=(T_XY)_x$, and it contradicts Theorem 1 (i). This completes the proof of Theorem 3.

2.5. Remark. — Let $E_\Lambda=\{\alpha\mid m_{\Lambda,\alpha}\neq 0\}$, and $R_{f,x}$ be the set of the roots of $b_{f,x}(-s)$, where $b_{f,s}(s)$ is defined for the germ (X,x), see [4]. Then the first assertion of Theorem 3 is equivalent to

$$\bigcup_\Lambda \exp(-2\pi i E_\Lambda)\subset\exp(-2\pi i R_{f,x}),$$

where Λ runs over the irreducible components of $(N_XY)_x$. However, the equality does not always hold (*e.g.* if $f=x^2y$) unless we take the union over the irreducible components Λ of $(N_XY)_y$ for any $y \in Z$ sufficiently near x.

3. Product formula

In this section we show a product formula which implies the well-definedness of the spectrum.

3.1. Cartesian product. — For $a = 1, 2$, let Y_a be a smooth complex algebraic variety or a complex manifold, and X_a be a closed subvariety of Y_a. Let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$ with the projection pr_a to the a-th factor. Let $\mathcal{I}_a$ denote the ideal of $X_a \subset Y_a$. Then $\mathcal{I}_X = \mathrm{pr}_1^* \mathcal{I}_1 + \mathrm{pr}_2^* \mathcal{I}_2$, and

$$\mathcal{I}_X^i = \sum_{p+q=i} \mathcal{I}_1^p \boxtimes \mathcal{I}^q = \sum_{p+q=i} \mathrm{pr}_1^* \mathcal{I}_1^p \cap \mathrm{pr}_2^* \mathcal{I}_2^q, \tag{3.1.1}$$

i.e. the filtration $\{\mathcal{I}_X^i\}_{i\in\mathbb{N}}$ is the convolution of the filtrations $\{\mathrm{pr}_1^* \mathcal{I}_1^p\}_{p\in\mathbb{N}}$ and $\{\mathrm{pr}_2^* \mathcal{I}_2^q\}_{q\in\mathbb{N}}$. Here the last isomorphism of 3.1.1 is shown by taking the exterior product of the exact sequences

$$0 \to \mathcal{I}_1^p \to \mathcal{O}_{Y_1} \to \mathcal{O}_{Y_1}/\mathcal{I}_1^p \to 0,$$
$$0 \to \mathcal{I}_2^q \to \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_2}/\mathcal{I}_2^q \to 0,$$

which gives a diagram of short exact sequences by the exactness of exterior product. By a similar argument, we get then (see also [1], 3.1.2.9)

$$\mathcal{I}_X^i/\mathcal{I}_X^{i+1} = \bigoplus_{p+q=i} \big(\mathcal{I}_1^p/\mathcal{I}_1^{p+1} \boxtimes \mathcal{I}_2^q/\mathcal{I}_2^{q+1}\big), \tag{3.1.2}$$

i.e.

$$\bigoplus_{i\in\mathbb{N}} \mathcal{I}_X^i/\mathcal{I}_X^{i+1} = \Big(\bigoplus_{p\in\mathbb{N}} \mathcal{I}_1^p/\mathcal{I}_1^{p+1}\Big) \boxtimes \Big(\bigoplus_{q\in\mathbb{N}} \mathcal{I}_2^q/\mathcal{I}_2^{q+1}\Big),$$

and hence

$$N_XY = N_{X_1}Y_1 \times N_{X_2}Y_2. \tag{3.1.3}$$

3.2. Proposition. — *With the above notation we have a canonical isomorphism*

$$\mathrm{Sp}_X(\mathcal{O}_Y, F) = \mathrm{Sp}_{X_1}(\mathcal{O}_{Y_1}, F) \boxtimes \mathrm{Sp}_{X_2}(\mathcal{O}_{Y_2}, F), \tag{3.2.1}$$

where the action of T_s on the left-hand side corresponds to $T_s \boxtimes T_s$ on the right-side hand.

Proof. — Let M_a be the direct image of $\mathcal{O}_{Y_a}$ by the graph embedding as in 1.1. Let

$$(M,F)=(M_1,F)\boxtimes(M_2,F).$$

These have the filtration V of Kashiwara [14] and Malgrange [18]. Using the same argument as in the proof of (3.1.1), we see that the filtration V on M is the convolution of pr_1^*V and pr_2^*V, *i.e.*

$$V^{\alpha}M=\sum_{\alpha_1+\alpha_2=\alpha}V^{\alpha_1}M_1\boxtimes V^{\alpha_2}M_2.$$

(Indeed, the filtration defined by the right-hand side satisfies the conditions of the filtration V.) Then we have

$$\mathrm{Gr}_V^{\alpha}(M,F)=\bigoplus_{\alpha_1+\alpha_2=\alpha}\mathrm{Gr}_V^{\alpha_1}(M_1,F)\boxtimes\mathrm{Gr}_V^{\alpha_2}(M_2,F).$$

So the assertion follows. □

3.3. Corollary. — *With the above notation, write* $\widehat{\mathrm{Sp}}(X_a,\Lambda_a)=\sum_{\alpha}m_{a,\Lambda_a,\alpha}t^{\alpha}$ *for an irreducible component* Λ_a *of* $(N_{X_a}Y_a)_{x_a}$*, and* $\widehat{\mathrm{Sp}}(X,\Lambda)=\sum_{\alpha}m_{\Lambda,\alpha}t^{\alpha}$ *for* $\Lambda=\Lambda_1\times\Lambda_2$ *under the isomorphism* (3.1.3). *Then*

$$m_{\Lambda,\alpha}=\sum_{\alpha_1\,\widetilde{+}\,\alpha_2=\alpha}m_{1,\Lambda_1,\alpha_1}m_{2,\Lambda_2,\alpha_2}. \tag{3.3.1}$$

where $\alpha_1\,\widetilde{+}\,\alpha_2$ *is defined to be* $\alpha_1+\alpha_2$ *if* $\lceil\alpha_1\rceil-\alpha_1+\lceil\alpha_2\rceil-\alpha_2\geqslant 1$, *and* $\alpha_1+\alpha_2-1$ *otherwise. Here* $\lceil\alpha\rceil$ *is the smallest integer which is greater than or equal to* α.

Proof. — Let $\xi=(\xi_1,\xi_2)\in N_XY=N_{X_1}Y_1\times N_{X_2}Y_2$. By Prop. 3.2 we have

$$i_{\xi}^*\,\mathrm{Sp}_X(\mathcal{O}_Y,F)=i_{\xi_1}^*\,\mathrm{Sp}_{X_1}(\mathcal{O}_{Y_1},F)\boxtimes i_{\xi_2}^*\,\mathrm{Sp}_{X_2}(\mathcal{O}_{Y_2},F).$$

So the assertion follows. □

3.4. Corollary. — *The spectrum* $\widehat{\mathrm{Sp}}(X,\Lambda)$ *is essentially independent of* Y *using the isomorphism 3.1.3 for* $X_2=pt$.

Proof. — Since the spectrum is defined analytically locally, it is enough to compare the embedding $X\to Y$ with

$$X\to Y=Y\times\{0\}\to Y\times\mathbb{C}.$$

So the assertion follows from 3.3, since $\widehat{\mathrm{Sp}}(X_2,\Lambda_2)=t$ in the case X_2 is a (reduced) point, $\dim Y_2=1$, and $\Lambda_2=N_{X_2}Y_2$. □

4. Monomial ideal case

In this section, we treat the monomial ideal case, and prove Th. 4.4.

4.1. Notation. — Assume $Y = \mathbb{C}^n$, and X is defined by a monomial ideal $\mathfrak{a}$ of $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$. We denote by x^ν the monomial corresponding to $\nu \in \mathbb{N}^n$. Let $\Gamma_{\mathfrak{a}} \subset \mathbb{N}^n$ be the semigroup corresponding to $\mathfrak{a}$, *i.e.*

$$\Gamma_{\mathfrak{a}} = \big\{\nu \in \mathbb{N}^n \mid x^\nu \in \mathfrak{a}\big\}.$$

Let $P_{\mathfrak{a}}$ be the convex hull of $\Gamma_{\mathfrak{a}}$ in $\mathbb{R}^n_{\geqslant 0}$ which is called the Newton polyhedron of $\mathfrak{a}$. Let $J(Y, \alpha X) \subset \mathbb{C}[x]$ denote the multiplier ideals of $\mathfrak{a}$. Set $\mathbf{1} = (1, \dots, 1)$, and

$$U(\alpha) = \big\{\nu \in \mathbb{N}^n \mid \nu + \mathbf{1} \in (\alpha + \varepsilon) P_{\mathfrak{a}} \quad \text{with} \quad 0 < \varepsilon \ll 1\big\}.$$

Then we have by Howald [13]

$$J(Y, \alpha X) = \sum_{\nu \in U(\alpha)} \mathbb{C}\, x^\nu. \tag{4.1.1}$$

See [5] for a combinatorial description of the roots of the b-function.

For an $(n-1)$-dimensional compact face σ of $P_{\mathfrak{a}}$, let L_σ be the linear function such that $L_\sigma^{-1}(1) \supset \sigma$. Let c_σ be the smallest positive integer such that $c_\sigma L_\sigma$ has integral coefficients. Let

$$G'_\sigma = \mathbb{Z}^n \cap L_\sigma^{-1}(0),$$

and G_σ be the subgroup generated by $\nu - \nu'$ for $\nu, \nu' \in \Gamma_{\mathfrak{a}} \cap \sigma$. Set

$$e_\sigma = \big|G'_\sigma / G_\sigma\big|.$$

For a face σ of $P_{\mathfrak{a}}$, let $\overline{B}_\sigma \subset \mathbb{C}[x]$ be the $\mathbb{C}$-subalgebra generated by x^ν for $\nu \in \sigma \cap \Gamma_{\mathfrak{a}}$. Let

$$\overline{B} = \sum_\sigma \overline{B}_\sigma \subset \mathbb{C}[x],$$

where the multiplication of x^ν and $x^{\nu'}$ in $\overline{B}$ is given by $x^{\nu+\nu'}$ if $x^\nu, x^{\nu'} \in \overline{B}_\sigma$ for some σ, and it vanishes otherwise. Set

$$\overline{A} = \bigoplus_{i \in \mathbb{N}} \mathfrak{a}^i / \mathfrak{a}^{i+1}.$$

With the above notation, we have the following:

4.2. Proposition. — *$\overline{A}_{\mathrm{red}} = \overline{B}$.*

Proof. — For $\nu \in \mathbb{N}^n$, set

$$v(\nu) = \min\{L_\sigma(\nu)\},$$

where σ runs over the $(n-1)$-dimensional faces of $P_{\mathfrak{a}}$. Let $\widehat{\sigma}$ be the cone generated by σ in the real vector space $\mathbb{R}^n$. Then

$$\mathbb{R}^n_{>0} \subset \bigcup_\sigma \widehat{\sigma} \subset \mathbb{R}^n_{\geqslant 0},$$

and

$$(4.2.1) \qquad v(\nu) = \begin{cases} L_\sigma(\nu) & \text{if} \quad \nu \in \widehat{\sigma}, \\ 0 & \text{if} \quad \nu \notin \bigcup_\sigma \widehat{\sigma}. \end{cases}$$

So we get for $\nu \in \mathbb{R}^n_{>0}$

$$(4.2.2) \qquad \nu \in \alpha P_{\mathfrak{a}} \Longleftrightarrow v(\nu) \geqslant \alpha.$$

Note that

$$(4.2.3) \qquad v(\nu + \nu') = L_{\sigma''}(\nu + \nu') \geqslant v(\nu) + v(\nu') \quad \text{if} \quad \nu + \nu' \in \widehat{\sigma}''.$$

Let C be a positive number such that for any $\nu \in \mathbb{N}^n$ and $k \in \mathbb{N}$

$$(4.2.4) \qquad x^\nu \in \mathfrak{a}^k \quad \text{if} \quad v(\nu) \geqslant k + C.$$

For the existence of such C, it is enough to show

$$(4.2.5) \qquad \sum^k \Gamma_{\mathfrak{a}} + \mathbb{R}^n_{\geqslant 0} \supset (k+C)P_{\mathfrak{a}},$$

where $\sum^k S := \bigl\{\sum_{i=1}^k v_i \mid v_i \in S\bigr\}$ for $S \subset \mathbb{R}^n$. Then the assertion is reduced to

$$(4.2.6) \qquad \sum^k (\sigma \cap \Gamma_{\mathfrak{a}}) + \mathbb{R}^n_{\geqslant 0} \supset \widehat{\sigma} \cap (k+C)P_{\mathfrak{a}} \quad \text{for any} \quad \sigma.$$

If σ is not compact, it is the union of $\sigma' + \mathbb{R}^I_{\geqslant 0}$ for compact faces σ' of σ, where I is the subset of $\{1, \dots, n\}$ such that σ is stable by adding the i-th unit vector for $i \in I$. So the assertion is reduced to the case σ compact. By increasing induction on k, it is further reduced to

$$(4.2.7) \qquad (\sigma \cap \Gamma_{\mathfrak{a}}) + \widehat{\sigma} \supset \widehat{\sigma} \cap \alpha P_{\mathfrak{a}} \quad \text{if} \quad \alpha \gg 0,$$

since $v(\nu) = 1$ for $\nu \in \sigma \cap \Gamma_{\mathfrak{a}}$. Then the assertion is proved by replacing σ with a simplex σ' defined by vertices of σ.

Set

$$N = \bigl[(C+1)/\varepsilon\bigr] + 1 \quad \text{with} \quad \varepsilon = \min\bigl\{c_\sigma^{-1}\bigr\},$$

where σ runs over the $(n-1)$-dimensional faces of $P_{\mathfrak{a}}$. Note that

$$(4.2.8) \qquad v(\nu) \geqslant k + \varepsilon \quad \text{if} \quad v(\nu) > k \quad \text{with} \quad \nu \in \mathbb{N}^n.$$

Let I_k be the ideal of $\mathbb{C}[x]$ generated by $x^\nu \in \mathfrak{a}^k$ with $v(\nu) > k$. Set

$$\overline{I} = \bigoplus_{k\in\mathbb{N}} I_k/\mathfrak{a}^{k+1}.$$

Then $\overline{I}$ is an ideal of $\overline{A}$ such that $\overline{I}^N = 0$ and $\overline{A}/\overline{I} = \overline{B}$ by the above arguments. So the assertion follows. □

4.3. Corollary.

(i) *The irreducible components Λ of N_XY are given by Spec $\overline{B}_\sigma$ for $(n-1)$-dimensional faces σ of $P_\mathfrak{a}$, and we have for any faces σ, σ' of $P_\mathfrak{a}$*

$$\mathit{Spec}\,\overline{B}_\sigma \cap \mathit{Spec}\,\overline{B}_{\sigma'} = \mathit{Spec}\,\overline{B}_{\sigma\cap\sigma'}.$$

(ii) *The irreducible components Λ of $(N_XY)_0$ are given by Spec $\overline{B}_\sigma$ for $(n-1)$-dimensional compact faces σ of $P_\mathfrak{a}$.*

Proof. — The first assertion of (i) is clear by Prop. 4.2. Let

$$J_\sigma = \mathrm{Ker}(\overline{B} \to \overline{B}_\sigma).$$

This is generated over $\mathbb{C}$ by x^ν for $\nu \in \mathbb{N}^n$ such that $x^\nu \in \overline{B}$ and $\nu \notin \widehat{\sigma}$. So we get

$$J_{\sigma\cap\sigma'} = J_\sigma + J_{\sigma'},$$

and the last assertion of (i) follows. For the assertion (ii), note that the image of *Spec* $\overline{B}_\sigma$ in Y is a point if and only if σ is compact. Then the assertion (ii) follows from (i). □

4.4. Theorem. — *For Λ = Spec $\overline{B}_\sigma$ with σ compact, we have in the notation of 4.1*

$$\widehat{\mathrm{Sp}}(X,\Lambda) = \sum_{i=1}^{c_\sigma} e_\sigma t^{i/c_\sigma}.$$

Proof. — Let $\overline{B}'_\sigma$ be the localization of $\overline{B}_\sigma$ by the monomials x^ν in $\overline{B}_\sigma$. Then

$$\mathrm{rank}_{\overline{B}'_\sigma}\, \mathcal{M}(\beta) \bigotimes_{\overline{A}} \overline{B}'_\sigma = \begin{cases} e_\sigma & \text{if} \quad \beta \in c_\sigma^{-1}\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.1}$$

So the assertion follows from Theorem 1 together with Prop. 1.4, 4.2 and Cor. 4.3. □

4.5. Comparison. — With the notation of 4.1, let $f = \sum_\nu c_\nu x^\nu \in \mathbb{C}\{\{x\}\}$. Assume f is non-degenerate with respect to the Newton boundary $\partial P_\mathfrak{a}$, see [15]. Let $D = f^{-1}(0) \subset (Y,0) = (\mathbb{C}^n, 0)$. Assume

$$X_{\mathrm{red}} = \{0\} \quad \text{so that} \quad \mathrm{Sing}\, D = \{0\}.$$

Then we may assume $f \in \mathbb{C}[x]$ by the finite determination property as is well-known. Let $\mathrm{JC}(Y,X)$, $\mathrm{JC}(Y,D)$ denote the sets of jumping coefficients. With the notation in the proof of Prop. 4.2 we have by Howald [13] (see 4.1.1 above)

$$\mathrm{JC}(Y,X) = \bigl\{v(\nu) \mid \nu \in \mathbb{Z}^n_{>0}\bigr\}. \tag{4.5.1}$$

Note that $\mathrm{JC}(Y,D)$ contains always 1, and we have the periodicity

$$\mathrm{JC}(Y,D) = \bigl(\mathrm{JC}(Y,D) \cap (0,1]\bigr) + \mathbb{N}. \tag{4.5.2}$$

Write $\mathrm{Sp}(f,0) = \sum_\alpha m_{f,\alpha} t^\alpha$ (the spectrum of f). Define the set of exponents by

$$E(D,0) = \{\alpha \mid m_{f,\alpha} \neq 0\}, \quad E(X,\Lambda) = \{\alpha \mid m_{\Lambda,\alpha} \neq 0\}.$$

Then we have

$$\begin{array}{ccc} \mathrm{JC}(Y,D) \cap (0,1) & \overset{(1)}{=\!=} & \mathrm{JC}(Y,X) \cap (0,1) \\ {\scriptstyle (2)}\big\| & & \cap{\scriptstyle (3)} \\ E(D,0) \cap (0,1) & \underset{(4)}{\subset} & \bigcup_\Lambda E(X,\Lambda) \cap (0,1) \end{array} \tag{4.5.3}$$

Indeed, we have the equality (1) by Howald [12], and (2) by Budur [3]. The inclusion (3) follows from Theorem 1 and (4.4.1) or Theorem 4.4 and (4.5.1). The spectrum of f is calculated by Steenbrink [26] (see also [21], [30]). This implies that the composition of (1) and (2) is the equality, since the exponents at most 1 are given by restricting the right-hand side of (4.5.1). Combined with Theorem 4.4, the last assertion also implies (4). Note that equality does not necessarily hold for (3) and (4), see Ex. 4.6–4.7 below.

As for the b-function, it is well known that the roots of the b-function are not determined by the Newton boundary even if f is non-degenerate in the sense of [15]. For example, the roots of the b-function of $f_\lambda = x^5+y^4+\lambda x^3y^2$ depend on whether $\lambda = 0$ or not. Indeed, for $\lambda \neq 0$, $\frac{11}{20}$ is a root of $b_{f_\lambda}(-s)$ and $\frac{31}{20}$ is not. For $\lambda = 0$, see (4.7.1) below.

4.6. Example. — Assume $Y = \mathbb{C}^n$ and $f = (f_1, \ldots, f_n)$ with $f_i = x_i^{m_i}$, where $x_1, \ldots, x_n$ are the coordinates of $\mathbb{C}^n$ and the m_i are positive integers. Let σ be the unique $(n-1)$-dimensional compact face of $P_{\mathfrak{a}}$, *i.e.* the convex hull of $\{m_1\mathbf{e}_1, \ldots, m_n\mathbf{e}_n\}$, where $\mathbf{e}_i$ is the i-th unit vector of $\mathbb{R}^n$. Then $L_\sigma = \sum_i x_i/m_i$, and

$$c_\sigma = \mathrm{LCM}(m_1, \ldots, m_n), \quad e_\sigma = m_1 \cdots m_n / c_\sigma. \tag{4.6.1}$$

Indeed, let $\overline{G}_\sigma$ and $\overline{G}'_\sigma$ denote respectively the injective image of G_σ and G'_σ by the projection $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$ to the first $n-1$ factors. Set

$$b = \mathrm{LCM}(m_1, \ldots, m_{n-1}), \quad d = \mathrm{GCD}(b, m_n).$$

Then $b = b'd$, $m_n = m'_n d$ with $b', m'_n \in \mathbb{N}$. This implies $c_\sigma = m_n b'$, and

$$|\mathbb{Z}^{n-1}/\overline{G}_\sigma| = m_1 \cdots m_{n-1}, \quad |\mathbb{Z}^{n-1}/\overline{G}'_\sigma| = b'.$$

So the assertion for e_σ follows. The assertion for c_σ is clear.

Set

$$E = \Big\{(a_1, \dots, a_n) \in \mathbb{N}^n \mid a_i \in [1, m_i]\Big\}.$$

Then

$$\tag{4.6.2} \begin{aligned} \mathrm{JC}(Y, X) &= \Big\{ \sum_{i=1}^{n} \frac{a_i}{m_i} \mid a_i \in \mathbb{Z}_{>0} \Big\}, \\ \widehat{\mathrm{Sp}}(X, 0) &= \sum_{i=1}^{c_\sigma} e_\sigma t^{i/c_\sigma}, \\ b_f(s) &= \Big[\prod_{(a_1,\dots,a_n)\in E} \Big(s + \sum_{i=1}^{n} \frac{a_i}{m_i} \Big) \Big]_{\mathrm{red}}. \end{aligned}$$

Here $[\prod_j (s + \beta_j)^{n_j}]_{\mathrm{red}} = \prod_j (s + \beta_j)$ if the β_j are mutually different and $n_j \in \mathbb{Z}_{>0}$. The assertion on the spectrum follows from Th. 4.4 or Cor. 3.3 using (4.6.1). The other assertions follow from (4.5.1) and [4], Th. 5. This example shows that it is not necessarily easy to determine i and j_0 in Theorem 3 in general.

4.7. Example. — Let $f = \sum_i x_i^{m_i}$ and $D = f^{-1}(0) \subset Y = \mathbb{C}^n$. Set

$$\widetilde{E} = \Big\{(a_1, \dots, a_n) \in \mathbb{N}^n \mid a_i \in [1, m_i - 1]\Big\}.$$

Then

$$\tag{4.7.1} \begin{aligned} \mathrm{JC}(Y, D) \cap (0, 1] &= \Big\{ \sum_{i=1}^{n} \frac{a_i}{m_i} \mid a_i \in \mathbb{Z}_{>0} \Big\} \cap (0, 1], \\ \mathrm{Sp}(D, 0) &= \prod_{i=1}^{n} (t - t^{1/m_i})/(t^{1/m_i} - 1), \\ \tilde{b}_f(s) &= \Big[\prod_{(a_1,\dots,a_n)\in \widetilde{E}} \Big(s + \sum_{i=1}^{n} \frac{a_i}{m_i} \Big) \Big]_{\mathrm{red}}, \end{aligned}$$

where $\tilde{b}_f(s) = b_f(s)/(s+1)$. The assertions on $\mathrm{JC}(Y, D)$ and $\mathrm{Sp}(D, 0)$ follow from [12] and [29]. The assertion on $\tilde{b}_f(s)$ is an unpublished result of Kashiwara asserting that in the isolated weighted homogeneous singularity case, the roots of $\tilde{b}_f(-s)$ coincide with the exponents and have multiplicity 1. (This also follows from [17].) Note that the formula for $\mathrm{Sp}(D, 0)$ holds in the isolated weighted homogeneous singularity case if we replace $1/m_i$ by the weights w_i.

4.8. Example. — In the monomial ideal case, j_0 in Theorem 3 is bounded by $n-1$. Indeed, with the notation of the proof of Prop. 4.2 we have for $\beta \in \mathbb{Q} \cap (0,1]$ and a face σ of $P_{\mathfrak{a}}$

$$\min\Big\{L_\sigma(\nu) \mid \nu \in \mathbb{Z}^n \cap \widehat{\sigma},\ L_\sigma(\nu) - \beta \in \mathbb{Z}\Big\} \leqslant n. \tag{4.8.1}$$

To show this, we may replace $\widehat{\sigma}$ with $\widehat{\sigma}' + \sum_{i \in I} \mathbb{R}^I_{\geqslant 0}$ where σ' is a simplex defined by vertices $\{v_i\}$ of a compact face of σ, and I is as in the proof of Prop. 4.2. Set

$$D_{\sigma'} = \operatorname{Int} \widehat{\sigma}' \smallsetminus \bigcup_i (\operatorname{Int} \widehat{\sigma}' + v_i),$$

where $\operatorname{Int} \widehat{\sigma}'$ is the interior of $\widehat{\sigma}'$. Then

$$\operatorname{Int} \widehat{\sigma}' = \bigcup_{\nu \in \mathbb{N}} \Big(D_\sigma + \sum_i \nu_i v_i\Big),$$

and (4.8.1) follows. By a similar argument, $\mathrm{JC}(Y, X)$ is stable by adding any positive integers in the monomial ideal case. Note that $j_0 = n-1$ if the m_i in 4.6 are mutually prime. In general it is unclear whether j_0 is always bounded by $n-1$.

Acknowledgments. — This work was partially supported by ANR-08-BLAN-0317-02 (SEDIGA) and Kakenhi 19540023. Additionally, A. Dimca is grateful to ASSMS, Government College University, Lahore, Pakistan, where part of the work on this paper was done.

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Manuscrit reçu le 11 décembre 2009,
accepté le 10 mars 2010.

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