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THE EXISTENCE OF EQUIVARIANT PURE FREE RESOLUTIONS

by David EISENBUD, Gunnar FLØYSTAD & Jerzy WEYMAN

Dedicated to Jürgen Herzog, on the occasion of his sixty-fifth birthday

ABSTRACT. — Let $A = K[x_1, \dots, x_m]$ be a polynomial ring in m variables and let $\mathbf{d} = (d_0 < \dots < d_m)$ be a strictly increasing sequence of $m + 1$ integers. Boij and Söderberg conjectured the existence of graded A -modules M of finite length having *pure free resolution* of type \mathbf{d} in the sense that for $i = 0, \dots, m$ the i -th syzygy module of M has generators only in degree d_i .

This paper provides a construction, in characteristic zero, of modules with this property that are also $GL(m)$ -equivariant. Moreover, the construction works over rings of the form $A \otimes_K B$ where A is a polynomial ring as above and B is an exterior algebra.

RÉSUMÉ. — Soit $A = K[x_1, \dots, x_m]$ un anneau polynomial à m variables et soit $\mathbf{d} = (d_0 < \dots < d_m)$ une suite strictement croissante de $m + 1$ nombres entiers. Boij et Söderberg ont conjecturé l'existence de A -modules gradués M de longueur finie ayant une *résolution pure et libre* de type \mathbf{d} dans le sens où pour $i = 0, \dots, m$ les générateurs du i -ème module de syzygies de M sont uniquement de degré d_i .

Cet article présente une construction, en caractéristique zéro, de modules avec cette propriété qui sont aussi $GL(m)$ -équivariants. La construction fonctionne aussi pour les anneaux de la forme $A \otimes_K B$ où A est un anneau polynomial comme ci-dessus et B est une algèbre extérieure.

Introduction

Let $\mathbf{d} = (d_0 < \dots < d_m)$ be a strictly increasing sequence of integers. In their remarkable paper [2], Boij and Söderberg conjectured the existence of a graded module M of finite length over every polynomial ring $A = K[x_1, \dots, x_m]$ whose minimal free resolution has the form

$$0 \rightarrow A^{\beta_m}(-d_m) \rightarrow \dots \rightarrow A^{\beta_0}(-d_0).$$

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Such a resolution is said to be *pure* of type \mathbf{d} . This paper provides a construction, in characteristic zero, that gives more: a $GL(m)$ -equivariant module with a pure resolution. In addition we give another construction of pure resolutions, of modules supported on determinantal varieties.

The constructions in this paper gave the first proof that the Boij-Söderberg existence conjecture was correct over any field. A subsequent paper, by Eisenbud-Schreyer [7] has verified the conjecture (along with the other conjectures in the Boij-Söderberg paper) over arbitrary fields, but with (generally) much worse bounds on the ranks of the modules constructed, and with no $GL(m)$ -equivariance. Also, Sam and Weyman [16] have provided a more direct proof that the modules we construct actually have pure resolutions. It is established in [10] that the equivariant resolution we construct, or rather its multigraded Betti diagram, has a canonical position among multigraded resolutions which become pure when considering total degrees.

One of the major questions left open by these papers is the nature of the semigroup of possible degrees of modules having pure resolutions with a given degree sequence. There is a unique minimum possibility, determined solely by integrality considerations, and it is easy to see that only integral multiples of this minimum can occur. It is known from many examples that not all occur. We make some conjectures in this direction, supported by the examples in this paper, in Section 6. In particular, Conjecture 6.1 asserts that every sufficiently high multiple that is integral *does* occur. The construction of Eisenbud-Schreyer [7] is complementary to this one, in that putting together the two equivariant examples produced here with the example produced there shows that this conjecture is true for many degree sequences.

We are able to make our constructions in a more general context than the polynomial rings that occur in [2] or [7]: we define resolutions over a free strictly commutative $\mathbf{Z}/2$ -graded algebra that specialize to the necessary pure resolutions in the polynomial algebra case, and also give pure resolutions over exterior algebras. This suggests that there may be a stronger version of the Boij-Söderberg conjectures/Eisenbud-Schreyer theory addressing resolutions over the exterior algebra.

Our constructions make use of Schur functors, $\mathbf{Z}/2$ -graded Schur functors and Bott's Theorem. This last is what limits our method to characteristic 0. It remains an open question whether such examples exist in characteristic $p > 0$.

By a participation $\lambda = (\lambda_1, \dots, \lambda_s, \dots)$ we shall mean an (infinite) weakly decreasing sequence of integers which is eventually zero.

Let $V = V_0 \oplus V_1$ be a $\mathbf{Z}/2$ -graded vector space with dimension vector (m, n) . We recall (see Section 1) that there exist $\mathbf{Z}/2$ -graded versions of Schur modules

$$\mathcal{S}_\lambda(V) = \bigoplus_{\mu \subset \lambda} S_\mu V_0 \otimes S_{\lambda'/\mu'} V_1.$$

Here λ' denotes the conjugate partition to λ . For example, the conjugate partition to (2) is $(1, 1)$.

We work over the free $\mathbf{Z}/2$ -graded algebra

$$R = \text{Sym}(V) = \bigoplus_{i \geq 0} \mathcal{S}_i(V) = \text{Sym}(V_0) \otimes \bigwedge^\bullet(V_1).$$

Define the positive integers e_i for $2 \leq i$, by setting

$$e_i = \lambda_{i-1} - \lambda_i + 1,$$

so that $e_i = 1$ for $i \gg 0$. For convenience we set $d_i = \sum_{j=1}^i e_j$, and $d_0 = 0$. We also set $\mathbf{d} = (d_0, d_1, \dots)$.

Next, define a sequence of partitions $\alpha(\mathbf{d}, i)$ for $i \geq 0$. We set $\alpha(\mathbf{d}, 0) = \lambda$ and, for $i \geq 1$,

$$\alpha(\mathbf{d}, i) = (\lambda_1 + e_1, \dots, \lambda_i + e_i, \lambda_{i+1}, \dots, \lambda_s, \dots).$$

The $\mathbf{Z}/2$ -graded endomorphisms of $V_0 \oplus V_1$ form a $\mathbf{Z}/2$ -graded Lie algebra $\mathfrak{gl}(V)$. We define a complex of free $\mathfrak{gl}(V)$ -equivariant R -modules, with the terms

$$\mathbf{F}(\mathbf{d})_0 = \mathcal{S}_\lambda V \otimes R,$$

$$\mathbf{F}(\mathbf{d})_i = \mathcal{S}_{\alpha(\mathbf{d}, i)} V \otimes R(-e_1 - \dots - e_i) \text{ for } i \geq 1.$$

Though this complex is in general infinite, it becomes finite of length at most $\dim V_0$ in the case when $V_1 = 0$. The differential

$$\partial_i : \mathbf{F}(\mathbf{d})_i \rightarrow \mathbf{F}(\mathbf{d})_{i-1}$$

is given on the generators by the $\mathbf{Z}/2$ -graded Pieri maps (see Section 1)

$$\mathcal{S}_{\alpha(\mathbf{d}, i)} V \rightarrow \mathcal{S}_{\alpha(\mathbf{d}, i-1)} V \otimes \mathcal{S}_{e_i} V = \mathcal{S}_{\alpha(\mathbf{d}, i-1)} V \otimes R_i.$$

We now state the main results of this paper.

THEOREM 0.1. — *The complex $\mathbf{F}(\mathbf{d})_\bullet$ is an acyclic complex of $\mathfrak{gl}(V)$ -equivariant, free R -modules. It is also pure, with the i -th differential of degree e_i .*

Remark. — When $n > 0$ the complexes $\mathbf{F}(\mathbf{d})_\bullet$ are infinite, but eventually linear. In fact, if the partition λ has s non-zero parts, the complex $\mathbf{F}(\mathbf{d})_\bullet$ becomes linear after s steps.

For the second construction we fix two $\mathbf{Z}/2$ -graded spaces V and U (with U of dimension vector (d, e)) and we work over the symmetric algebra $S = \text{Sym}(V \otimes U)$. Fix a sequence \mathbf{d} as above.

We define an infinite complex $\mathbf{H}(\mathbf{d})$ of free $\mathfrak{gl}(V) \times \mathfrak{gl}(U)$ -equivariant S -modules, with the terms

$$\begin{aligned} \mathbf{H}(\mathbf{d})_0 &= \mathcal{S}_\lambda V \otimes S, \\ \mathbf{H}(\mathbf{d})_i &= \mathcal{S}_{\alpha(\mathbf{d},i)} V \otimes \mathcal{S}_{d_i} U \otimes S(-e_1 - \dots - e_i) \text{ for } i \geq 1. \end{aligned}$$

The differential

$$\partial_i : \mathbf{H}(\mathbf{d})_i \rightarrow \mathbf{H}(\mathbf{d})_{i-1}$$

is given on the generators by the $\mathbf{Z}/2$ -graded Pieri maps (see Section 1)

$$\begin{array}{c} \mathcal{S}_{\alpha(\mathbf{d},i)}(V) \otimes \mathcal{S}_{d_i} U \\ \downarrow \\ \mathcal{S}_{\alpha(\mathbf{d},i-1)}(V) \otimes \mathcal{S}_{d_{i-1}} U \otimes \mathcal{S}_{e_i} V \otimes \mathcal{S}_{e_i} U \\ \cap \\ \mathcal{S}_{\alpha(\mathbf{d},i-1)} V \otimes \mathcal{S}_{d_{i-1}} U \otimes \mathcal{S}_{e_i}. \end{array}$$

THEOREM 0.2. — *The complex $\mathbf{H}(\mathbf{d})_\bullet$ is an acyclic complex of free S -modules that is $\mathfrak{gl}(V) \times \mathfrak{gl}(U)$ -equivariant. It is pure, and the i -th differential has degree e_i .*

The paper is organized as follows. Section 1 is devoted to recalling necessary notions from representation theory of $GL(m)$. In §2 we briefly review the material needed from representation theory of $\mathbf{Z}/2$ -graded Lie algebra $\mathfrak{gl}(V)$. In §3 and §4 we prove Theorems 0.1 and 0.2 in special cases. In §5 we deduce the general cases from the cases already treated. Further conjectures and open problems are discussed in §6.

We are grateful to F.-O. Schreyer for pointing out to us that certain known complexes constructed by multilinear algebra (see §2) give examples including all the pure resolutions whose Betti tables have just two rows with nonzero terms, and thus setting us on the idea of using Schur functors to construct pure resolutions.

1. Cohomology of Homogeneous Bundles on Projective Spaces

For the convenience of the reader we review a few necessary results from representation theory.

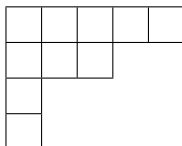
We work over a field K of characteristic zero. We denote by E a vector space of dimension m over K (or sometimes a vector bundle of rank m on an algebraic variety), and write $A = \text{Sym}(E)$ for the symmetric algebra. Here and in the sequel we use the language of vector bundles but always work with the associated locally free sheaves.

There is a one-to-one correspondence between the irreducible polynomial representations of the group $GL(E)$ and the set of *partitions* $\lambda = (\lambda_1, \dots, \lambda_m)$ with at most m parts. Here we allow this weakly decreasing sequence also to consist of negative integers. The representation corresponding to λ will be written $S_\lambda E$, and λ is called the *highest weight* for $S_\lambda E$. The construction of these representations is functorial; in characteristic 0, for example, one may view the representation $S_\lambda E$ as the image of $E \otimes \dots \otimes E$, the tensor product with $t = \sum_i \lambda_i$ factors, under the projection map defined by a *Young symmetrizer*, which is a certain element of the group algebra of the symmetric group on t letters that acts by permuting the factors of the tensor product (see [11] §4.1 and §6.1.) For this reason the construction extends to the case where E is a vector bundle on an arbitrary space, and the proof below will imply that the complexes of vector bundles we construct are resolutions because acyclicity can be proved fiberwise.

For example the d -th symmetric power of E is $S_{(d,0,\dots,0)}E$, which we will often denote by $S_d E$. Of course $A = \bigoplus_{d \geq 0} S_d E$. The one-dimensional representation $\wedge^m E$ corresponds to the weight $(1^m) := (1, 1, \dots, 1)$. For any λ and integer p we have

$$S_\lambda E \otimes (\wedge^m E)^p = S_{\lambda+(p^m)}$$

where $\lambda + (p^m) = (\lambda_1 + p, \dots, \lambda_m + p)$. Thus we may assume that all the λ_i are non-negative and that $\lambda_m = 0$ when this is convenient. It is useful to visualize λ as a *Young frame*, a diagram of boxes in which the i -th column of boxes extends down λ_i boxes from a given baseline; for example, the Young frame for $\lambda = (4, 2, 2, 1, 1)$ is



There is a general formula giving the decomposition—in characteristic 0—of the tensor product of two representations, called the Littlewood-Richardson Rule. Here we will only use the simple special case called the *Pieri Formula*, which gives the decomposition of $S_\lambda E \otimes S_d E$ for any λ and d . To express it, we define $|\lambda| := \sum_{i=1}^m \lambda_i$, and we write $\mu \supset \lambda$ if $\mu_i \geq \lambda_i$ for $i = 1, \dots, n$. We will say that μ/λ is a *vertical strip* if $\mu \supset \lambda$ and if $\mu_i \leq \lambda_{i-1}$ for $i = 2, \dots, n$. In the case where all the λ_i are non-negative, this means that the Young frame for λ fits into the upper left hand corner of the Young frame for μ , and that no two boxes of μ that are outside λ lie in the same row.

Since the decomposition is once again given by applying Young symmetrizers, it works for vector bundles as well.

THEOREM 1.1 (Pieri’s Formula). — *If E is a vector bundle defined on an algebraic variety of characteristic 0 then*

$$S_\lambda E \otimes S_i E = \bigoplus_\mu S_\mu E$$

where the sum is taken over all partitions $\mu \supset \lambda$ such that $|\mu| - |\lambda| = i$ and μ/λ is a vertical strip.

Proof. — See [17] (2.3.5) or [11], Appendix A, (A.7). See also [14], Chapter 1. □

The other result from representation theory that we need is a special case of Borel-Bott-Weyl theory. Let $\text{Grass}(1, E)$ denote the Grassmannian of 1-dimensional subspaces of E , which may also be viewed as a the projective space,

$$\text{Grass}(1, E) = \mathbf{P}(E^*) \cong \mathbf{P}^{m-1}.$$

Let \mathcal{R} denote the tautological rank one sub-bundle on $\text{Grass}(1, E)$, and let \mathcal{Q} the the quotient bundle, with *tautological exact sequence*

$$0 \rightarrow \mathcal{R} \rightarrow E \otimes \mathcal{O}_{\mathbf{P}^{m-1}} \rightarrow \mathcal{Q} \rightarrow 0.$$

For any sheaf \mathcal{G} on $\text{Grass}(1, E)$, let $H^i(\mathcal{G})$ denote the cohomology $H^i(\mathbf{P}^{m-1}, \mathcal{G})$. The result we need describes this cohomology in the case of an equivariant sheaf $\mathcal{G} = S_\alpha \mathcal{Q} \otimes S_u \mathcal{R}$. To express it we need two other pieces of notation. For any permutation σ we write $l(\sigma)$ for the *length* of σ , that is, the minimal number of transpositions necessary to express σ as a product of transpositions. We write ρ for the partition $\rho = (m - 1, m - 2, \dots, 1, 0)$.

THEOREM 1.2 (Bott’s Theorem in a special case). — *With notation as above, $H^i(S_\alpha \mathcal{Q} \otimes S_u \mathcal{R})$ is nonzero for at most one index i . More precisely, consider the sequence of integers $(\alpha, u) + \rho = (\alpha_1 + m - 1, \dots, \alpha_{m-1} + 1, u)$.*

- 1) If the sequence $(\alpha, u) + \rho$ has a repetition then the sheaf $S_\alpha \mathcal{Q} \otimes S_u \mathcal{R}$ has all cohomology equal to zero.
- 2) If the sequence $(\alpha, u) + \rho$ has no repetitions then there exists a unique permutation σ such that $\beta := \sigma((\alpha, u) + \rho) - \rho$ is non-increasing. In this case $S_\alpha \mathcal{Q} \otimes S_u \mathcal{R}$ has only one nonvanishing cohomology group, which is

$$H^{l(\sigma)}(S_\alpha \mathcal{Q} \otimes S_u \mathcal{R}) = S_\beta E.$$

Proof. — The dual form of this result is [17], (4.1.9). The version used here follows by the duality result given in Exercise 2.18b in Weyman [17]. A very short argument for Bott’s theorem may be found in [5]. \square

COROLLARY 1.3. — *Let $\lambda_1 \geq \dots \geq \lambda_{m-1}$ be a sequence of non-negative integers, and let $\mathcal{B} = \text{Sym}(\mathcal{Q})$.*

- a) *If $\lambda_{m-1} = 0$, then there is an equivariant isomorphism of graded $A := \text{Sym}(E)$ -modules*

$$H^0(S_{(\lambda_1, \dots, \lambda_{m-1})} \mathcal{Q} \otimes \mathcal{B}) \cong S_{(\lambda_1, \dots, \lambda_{m-1}, 0)} E \otimes A.$$

- b) *If $\lambda_{m-1} > 0$ then $H^0(S_{(\lambda_1, \dots, \lambda_{m-1})} \mathcal{Q} \otimes \mathcal{B})$ has an equivariant minimal resolution by free graded A -modules of the form*

$$0 \rightarrow S_{(\lambda_1, \dots, \lambda_{m-1}, 1)} E \otimes A(-1) \rightarrow S_{(\lambda_1, \dots, \lambda_{m-1}, 0)} E \otimes A$$

Proof. — From the tautological exact sequence above we derive a resolution of each $\text{Sym}_d(\mathcal{Q})$, and thus of the graded algebra \mathcal{B} , which takes the form

$$0 \rightarrow A(-1) \otimes \mathcal{R} \rightarrow A \otimes \mathcal{O}_{\mathbf{P}^{m-1}} \rightarrow \mathcal{B} \rightarrow 0.$$

We tensor this resolution with $S_{(\lambda_1, \dots, \lambda_{m-1})} \mathcal{Q}$ and form the long exact sequence in cohomology,

$$0 \rightarrow A(-1) \otimes H^0(S_{(\lambda_1, \dots, \lambda_{m-1})} \mathcal{Q} \otimes \mathcal{R}) \rightarrow A \otimes H^0(S_{(\lambda_1, \dots, \lambda_{m-1})} \mathcal{Q}) \rightarrow H^0(S_{(\lambda_1, \dots, \lambda_{m-1})} \mathcal{Q} \otimes \mathcal{B}) \rightarrow A(-1) \otimes H^1(S_{(\lambda_1, \dots, \lambda_{m-1})} \mathcal{Q} \otimes \mathcal{R}).$$

If $\lambda_{m-1} = 0$, Bott’s Theorem shows that all the cohomology of $\mathcal{R} \otimes S_{(\lambda_1, \dots, \lambda_{m-1})} \mathcal{Q}$ vanishes. By Bott’s Theorem, $H^0(S_\lambda \mathcal{Q}) = S_\lambda E$, so we get Part a). If, on the other hand, $\lambda_{m-1} > 0$ then Bott’s Theorem shows that the H^1 term is zero, and the resulting equivariant short exact sequence is the one given in Part b). \square

We remark that the use of the complex in b), which is the push-down of the complex $S_\lambda \mathcal{Q} \otimes \bigwedge^\bullet(\mathcal{R})$, is a simple example of the geometric technique described in [17].

2. $\mathbf{Z}/2$ -Graded Representation Theory

For the proof of Theorem 0.1 we will use the results of [1] giving the structure of R as a module over a $\mathbf{Z}/2$ -graded Lie algebra $\mathfrak{g} := \mathfrak{gl}(V)$. For the convenience of the reader we give a brief sketch of what is needed. Let $V = V_0 \oplus V_1$ be a $\mathbf{Z}/2$ -graded vector space of dimension (m, n) .

The $\mathbf{Z}/2$ -graded Lie algebra $\mathfrak{gl}(V)$ is the vector space of $\mathbf{Z}/2$ -graded endomorphisms of $V = V_0 \oplus V_1$. Thus

$$\mathfrak{gl}(V) = \mathfrak{gl}(V)_0 \oplus \mathfrak{gl}(V)_1,$$

where $\mathfrak{gl}(V)_0$ is the set of endomorphisms preserving the grading of V and $\mathfrak{gl}(V)_1$ is the set of endomorphisms of V shifting the grading by 1. Additively

$$\mathfrak{gl}(V)_0 = \text{End}_K(V_0) \oplus \text{End}_K(V_1),$$

$$\mathfrak{gl}(V)_1 = \text{Hom}_K(V_0, V_1) \oplus \text{Hom}_K(V_1, V_0)$$

The commutator of the pair of homogeneous elements $x, y \in \mathfrak{gl}(V)$ is defined by the formula

$$[x, y] = xy - (-1)^{\text{deg}(x)\text{deg}(y)}yx.$$

By a $\mathfrak{gl}(V)$ -module we mean a $\mathbf{Z}/2$ -graded vector space $M = M_0 \oplus M_1$ with a bilinear map of $\mathbf{Z}/2$ -graded vector spaces $\circ : \mathfrak{gl}(V) \times M \rightarrow M$ satisfying the identity

$$[x, y] \circ m = x \circ (y \circ m) - (-1)^{\text{deg}(x)\text{deg}(y)}y \circ (x \circ m)$$

for homogeneous elements $x, y \in \mathfrak{gl}(V), m \in M$.

In contrast to the classical theory, not every representation of the $\mathbf{Z}/2$ -graded Lie algebra $\mathfrak{gl}(V)$ is semisimple. For example its natural action on mixed tensors $V^{\otimes k} \otimes V^{*\otimes l}$ is in general not completely reducible. However, its action on $V^{\otimes t}$ decomposes just as in the ungraded case:

THEOREM 2.1 ([1]). — *The action of $\mathfrak{gl}(V)$ on $V^{\otimes t}$ is completely reducible for each t . More precisely, the analogue of Schur's double centralizer theorem holds and the irreducible $\mathfrak{gl}(V)$ -modules occurring in the decomposition of $V^{\otimes t}$ are in 1-1 correspondance with irreducible representations of the symmetric group Σ_t on t letters. These irreducibles are the ($\mathbf{Z}/2$ -graded) Schur functors*

$$\mathcal{S}_\lambda(V) = e(\lambda)V^{\otimes t}$$

where $e(\lambda)$ is a Young idempotent corresponding to a partition λ in the group ring of the symmetric group Σ_t .

(This result is also proven in Section 1 of [8].) The notation is consistent with the notation above in the sense that the d -th homogeneous component of the ring $\mathcal{S}(V)$ is $\mathcal{S}_d(V)$ where d represents the partition (d) with one part.

Here we use the symbol \mathcal{S}_λ to denote the $\mathbf{Z}/2$ -graded version of the Schur functor S_λ ; the latter acts on ungraded vector spaces. The partition (d) with only one part will be denoted simply d , so for example $\mathcal{S}_2(V) = S_2(V_0) \oplus (V_0 \otimes V_1) \oplus \wedge^2 V_1$ and similarly $\wedge^2 V = \mathcal{S}_{(1,1)} V = \wedge^2 V_0 \oplus V_0 \otimes V_1 \oplus S_2(V_1)$. In each case the decomposition is as representations of the subalgebra $\mathfrak{gl}(V_0) \times \mathfrak{gl}(V_1) \subset \mathfrak{gl}(V)$. Similar decompositions hold for all $\mathcal{S}_d V$ and $\wedge^d V$. The Pieri formula (and the Littlewood-Richardson rule) generalize verbatim to $\mathbf{Z}/2$ -graded Schur functors:

PROPOSITION 2.2 ($\mathbf{Z}/2$ -graded Pieri Formula). — *If V is a $\mathbf{Z}/2$ -graded vector space, and λ a partition, we have an isomorphism of $\mathfrak{gl}(V)$ -modules*

$$\mathcal{S}_\lambda V \otimes \mathcal{S}_i V = \bigoplus_{\mu} \mathcal{S}_\mu V$$

where the sum is taken over all partitions μ such that $|\mu| - |\lambda| = i$ and μ/λ is a vertical strip.

This follows from the results of [1].

3. First Construction of Pure Resolutions in the Even Case

Let E be an m -dimensional vector space, or more generally a rank m vector bundle on an algebraic variety, over a field of characteristic 0. Fix a strictly increasing sequence of integers $\mathbf{d} = (d_0, d_1, \dots, d_m)$. We will produce a pure acyclic equivariant complex of length m with terms in degrees d_0, \dots, d_m . To simplify notation we set

$$e_0 := d_0, \quad e_i := d_i - d_{i-1}, \text{ for } i = 1, \dots, m,$$

and we sometimes write $\mathbf{e} = (e_0, \dots, e_m)$ for the sequence corresponding to \mathbf{d} .

We will construct a complex

$$\mathbf{F}(\mathbf{d})_\bullet = \mathbf{F}(\mathbf{d})(E)_\bullet : 0 \rightarrow F(\mathbf{d})_m \rightarrow F(\mathbf{d})_{m-1} \rightarrow \dots \rightarrow F(\mathbf{d})_1 \rightarrow F(\mathbf{d})_0$$

where $F(\mathbf{d})_i$ is a free $A = \text{Sym}(E)$ -module generated in degree $d_i = -e_1 - \dots - e_i$. Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be the partition with parts $\lambda_i =$

$e_0 + \sum_{i+1}^m (e_j - 1)$ (so in particular $\lambda_m = e_0$). We define a sequence of partitions $\alpha(\mathbf{d}, i)$ for $0 \leq i \leq m$ by

$$\begin{aligned} \alpha(\mathbf{d}, 0) &= \lambda, \\ \alpha(\mathbf{d}, i) &= (\lambda_1 + e_1, \lambda_2 + e_2, \dots, \lambda_i + e_i, \lambda_{i+1}, \dots, \lambda_m), \end{aligned}$$

and set

$$F(\mathbf{d})_i := S_{\alpha(\mathbf{d}, i)} E \otimes A(-e_0 - e_1 - \dots - e_i).$$

We could of course have reduced to the case $d_0 = 0$; as defined below, the resolutions cases with $d_0 \neq 0$ are obtained from the ones with $d_0 = 0$ simply by tensoring with the 1-dimensional representation $(\wedge^m E)^{d_0}$. We will sometimes make the assumption $d_0 = 0$ for simplicity, but we will need the case with $d_0 > 0$ for induction.

To make it easier to think about these complexes, we give a pictorial representation. The following example contains all features of the general case.

Example 3.1. — Take $m = 4$ and $\mathbf{d} = (0, 4, 6, 9, 11)$, so that $\mathbf{e} = (0, 4, 2, 3, 2)$. Then the partition $\alpha(\mathbf{d}, i)$ (for $0 \leq i \leq 4$) is the subdiagram filled with numbers $\leq i$ in the Young diagram

0	0	0	4
0	0	3	4
0	0	3	
0	2	3	
1	2		
1			
1			
1			

Here $\alpha(\mathbf{d}, 0)$ is the partition in which the difference between the i -th and $i + 1$ 'st column equals $e_{i+1} - 1$. We get $\alpha(\mathbf{d}, i)$ from $\alpha(\mathbf{d}, i - 1)$ by adding e_i boxes to the i -th column. Notice that for each $i \geq 1$ there is exactly one row in the diagram above containing boxes numbered i and $i + 1$ — these are the highest box with the number i and the lowest box with the number $i + 1$. This is a general phenomenon that makes it possible for us to define a differential, there is really no choice about how to construct it because of the following observation:

Because $S_{\alpha(\mathbf{d}, i)} E$ is obtained from $S_{\alpha(\mathbf{d}, i-1)} E$ by adding e_i boxes in one column, the Pieri Formula implies that it occurs exactly once in the decomposition of $S_{\alpha(\mathbf{d}, i-1)} E \otimes S_{e_i} E$ into irreducible $GL(E)$ -modules. Thus there is a unique (up to scalar) nonzero equivariant map of A -modules

$$\phi(\mathbf{d}, i) : F(\mathbf{d})_i \rightarrow F(\mathbf{d})_{i-1},$$

and it has degree 0 in the grading coming from A since the generators of $F(\mathbf{d})_i$ have degree e_i more than those of $F(\mathbf{d})_{i-1}$.

For any $i \leq m - 2$, there is a row of $\alpha(\mathbf{d}, i + 2)$ containing two more boxes than are present in $\alpha(\mathbf{d}, i)$. The Pieri Formula thus implies that $S_{\alpha(\mathbf{d}, i)}E$ does not occur in $S_{\alpha(\mathbf{d}, i-2)}E \otimes S_{e_i+e_{i-1}}E$, so $\phi(\mathbf{d}, i - 1)\phi(\mathbf{d}, i) = 0$, so the maps $\phi(\mathbf{d}, i)$ make $\mathbf{F}(\mathbf{d})_\bullet$ into a complex. This argument actually shows that any equivariant maps of $\text{GL}(E)$ modules $\mathbf{F}(\mathbf{d})_i \rightarrow \mathbf{F}(\mathbf{d})_{i-1}$ would make $\mathbf{F}(\mathbf{d})_\bullet$ into a complex; and by the construction above, any nonzero equivariant maps make it into a complex isomorphic to $\mathbf{F}(\mathbf{d})_\bullet$. We will use this uniqueness in the proof below.

Here is our main result in the even case:

THEOREM 3.2. — *If E is a vector space of dimension m over a field of characteristic 0, and $\mathbf{d} = (d_0, \dots, d_m)$ is a strictly increasing sequence of integers, then*

- 1) *The complex*

$$\mathbf{F}(\mathbf{d})(E)_\bullet : 0 \rightarrow F(\mathbf{d})_m \xrightarrow{\phi(\mathbf{d}, m)} \dots \rightarrow F(\mathbf{d})_1 \xrightarrow{\phi(\mathbf{d}, 1)} F(\mathbf{d})_0 \rightarrow 0$$

is a minimal graded free resolution, and the generators of $F(\mathbf{d})_i$ have degree d_i .

- 2) *The module $M(\mathbf{d}) := \text{coker } \phi(\mathbf{d}, 1)$ resolved by $\mathbf{F}(\mathbf{d})(E)_\bullet$ is equivariant for $\text{GL}(E)$. As a representation, $M(\mathbf{d})$ is isomorphic to the direct sum of all the irreducible summands of $S_{\alpha(\mathbf{d}, 0)}E \otimes \text{Sym}(E)$ corresponding to the partitions that do not contain $\alpha(\mathbf{d}, 1)$. In particular $M(\mathbf{d})$ is finite dimensional as a vector space, and is zero in degrees $\geq \alpha(\mathbf{d}, 1)_1$.*

Remark. — If we simply think of each $\mathbf{F}(\mathbf{d})_i$ as a sum of representations, and define $M(\mathbf{d})$ as the sum of the representations in part 2), then in the augmented complex consisting of $\mathbf{F}(\mathbf{d})_\bullet$ and $M(\mathbf{d})$, each irreducible representation that occurs in one term occurs either in the term before or the term after, but not both. Moreover, in a given $\mathbf{F}(\mathbf{d})_i$ no representation occurs more than once. Thus we see that it is combinatorially possible that $\mathbf{F}(\mathbf{d})_\bullet$ is a resolution of $M(\mathbf{d})$. To make this into a proof of Theorem 1, one could first apply the Acyclicity Lemma of [15], which implies that it is enough to prove the acyclicity of $\mathbf{F}(\mathbf{d})_\bullet$ after replacing the variables (x_1, \dots, x_m) in $\text{Sym}(E) = K[x_1, \dots, x_m]$ by $(1, 0, \dots, 0)$. To finish the proof, one would need to show that the highest weight vector of each $\text{GL}(x_2, \dots, x_m)$ -representation contained in both $S_{\alpha(\mathbf{d}, i)}$ and $S_{\alpha(\mathbf{d}, i-1)}$ is mapped from the first module into a nonzero vector in the second. The

proof below shows that this must in fact be true! But we do not at present know how to prove it directly.

Proof of Theorem 3.2. — We use induction on m and (in the last part of the proof) on $d_m - d_0 = \sum_{i \geq 1} e_i$. If $m = 1$ then the complex has the form

$$\mathbf{F}(e_0, e_1)_\bullet : A(-e_1 - e_0) \rightarrow A(-e_0)$$

with the map being the multiplication by $x_1^{e_1}$, and the assertions are trivial. On the other hand, if $d_m - d_0 = m$, the smallest possible value, then all the e_i are 1 and the complex $\mathbf{F}(\mathbf{d})_\bullet$ is simply the Koszul complex on the variables in the polynomial ring A , so the theorem is true in this case as well.

We next show that part 1) of the theorem, for a given m , implies part 2) for that m . We use Pieri’s formula to understand the $F(\mathbf{d})_i$, and assume that $\mathbf{F}(\mathbf{d})_\bullet$ is a resolution of $M(\mathbf{d})$. Since no $S_\beta E$ occurs with multiplicity more than 1 in a term of the complex, a representation is present (with multiplicity 1) in $M(\mathbf{d})$ if it is present in $F(\mathbf{d})_0$ but not $F(\mathbf{d})_1$; and it is absent from $M(\mathbf{d})$ if it is either absent from $F(\mathbf{d})_0$ or present in $F(\mathbf{d})_0$ and also in $F(\mathbf{d})_1$ but not in $F(\mathbf{d})_2$.

First, if $\beta \not\supseteq \alpha(\mathbf{d}, 1)$ then S_β cannot occur in $F(\mathbf{d})_1 = A \otimes S_{(\mathbf{d},1)}E$, so if S_β is present in $F(\mathbf{d})_0 = A \otimes S_\lambda E$ then it is present in $M(\mathbf{d})$.

Next suppose that $\beta \supset \alpha(\mathbf{d}, 1)$ and S_β occurs in $A \otimes S_\lambda E$. It is clear from the Pieri formula that S_β also occurs in $A \otimes S_{\alpha(\mathbf{d},1)}E$. But since S_β occurs in $A \otimes S_\lambda E$, and $\beta_1 \geq \alpha(\mathbf{d}, 1) > \lambda_1$, we must have $\beta_2 \leq \lambda_1$. It follows that $\beta \not\supseteq \alpha(\mathbf{d}, 2)$. Thus S_β does not occur in $F(\mathbf{d})_2$, so it is in the image of $F(\mathbf{d})_1 \rightarrow F(\mathbf{d})_0$, and thus cannot occur in $M(\mathbf{d})$, completing the proof of part 2) based on part 1).

In the inductive step in the proof of part 1) it will be sufficient to verify that $\mathbf{F}(\mathbf{d})_\bullet$ is a resolution when $d_0 = 0$, which we now assume. Consider the sheaf of algebras $\mathcal{B} = \text{Sym}(\mathcal{Q})$ on $\text{Grass}(1, E) \cong \mathbf{P}^{m-1}$, and let $\mathbf{F}(\mathbf{d})(\mathcal{Q})$ be the corresponding complex of vector bundles on $\text{Grass}(1, E)$. The bundle \mathcal{Q} has rank $m - 1$, so applying our induction on the dimension of E to the fibers of the bundle \mathcal{Q} at each point, we see that the complex of vector bundles $\mathbf{F}(d_0, \dots, d_{m-1})(\mathcal{Q})_\bullet$, and with it the complex

$$\mathcal{F}_\bullet := \mathbf{F}(d_0, \dots, d_{m-1})(\mathcal{Q})_\bullet \otimes \left(\bigwedge^{m-1} \mathcal{Q} \right)^{\otimes e_{m-1}}$$

is acyclic. Its terms are the Schur functors on \mathcal{Q} with highest weights

$$\alpha'(\mathbf{d}, i) := \alpha(\mathbf{d}, i)_1, \dots, \alpha(\mathbf{d}, i)_{m-1}$$

for $0 \leq i \leq m - 1$ —the same as $\alpha'(\mathbf{d}, i)$ but with the last part $\alpha(\mathbf{d}, i)_m$, which is $e_0 = d_0 = 0$ omitted to make a partition of length $m - 1$. By

induction, \mathcal{F}_\bullet is a resolution of a \mathcal{B} -module that we may call $M_{\mathcal{Q}}(\mathbf{d})$, which is a direct sum of finitely many representations, each a Schur functor of \mathcal{Q} .

Next consider the complex obtained from \mathcal{F} by taking global sections,

$$H^0(\mathcal{F}_\bullet) : 0 \rightarrow H^0(\mathcal{F}_{m-1}) \rightarrow \cdots \rightarrow H^0(\mathcal{F}_0).$$

By Bott's Theorem, $H^j(\mathcal{F}_i) = 0$ for all i and all $j > 0$. Breaking the complex \mathcal{F} into short exact sequences, one sees by induction that this implies the acyclicity of the complex $H^0(\mathcal{F}_\bullet)$, and this is a resolution of the A -module $H^0(M_{\mathcal{Q}}(\mathbf{d}))$.

By the Corollary 1.3, each term

$$H^0(\mathcal{F}_i) = H^0\left(\mathbf{F}(d_0, \dots, d_{m-1})(\mathcal{Q})_i \otimes \left(\bigwedge^{m-1} \mathcal{Q}\right)^{\otimes e_{m-1}}\right)$$

of $H^0(\mathcal{F}_\bullet)$ is either free or has a free resolution of length 1. We distinguish these two cases. The reader will find an explicit example for each of these cases in Example 3.3 and Example 3.4 below, and it may be helpful to consider the pictures there while reading the following.

Case 1. $e_m = 1$. In this case each $\mathbf{F}(d_0, \dots, d_{m-1})(\mathcal{Q})_i \otimes (\bigwedge^{m-1} \mathcal{Q})^{\otimes e_{m-1}}$ for $i \leq m - 2$ satisfies the conditions of Part a) of the Corollary to Bott's Theorem. Thus the modules $H^0(\mathcal{F}_i)$ for $i < m - 1$ are free, and are the same as those of $\mathbf{F}(\mathbf{d})_\bullet$. By part b), on the other hand the last term $H^0(\mathcal{F}_{m-1})$ has homological dimension 1, and we see that the terms of its resolution furnish the remaining two terms of $\mathbf{F}(\mathbf{d})_\bullet$. By the uniqueness of the nonzero maps of the given degree between the terms of $\mathbf{F}(\mathbf{d})_\bullet$, we may identify $H^0(\mathcal{F}_\bullet)$ with this complex, proving acyclicity as required for part a).

Case 2. $e_m > 1$. In this case, Part b) of the Corollary to Bott's Theorem shows that each $H^0(\mathcal{F}_i)$ has an equivariant free resolution of length 1. From that Corollary we see moreover that the resolution takes the form

$$0 \rightarrow \mathbf{F}(\mathbf{d}')_i \rightarrow \mathbf{F}(\mathbf{d})_i \rightarrow H^0(\mathcal{F}_i) \rightarrow 0$$

for $i = 0, \dots, m - 1$, where \mathbf{d}' is given by

$$1, d_1 + 1, \dots, d_{m-1} + 1, d_m$$

corresponding to the sequence

$$\mathbf{e}' := (1, e_1, \dots, e_{m-1}, e_m - 1)$$

and we have simplified the notation by writing $\mathbf{F}(\mathbf{d}')$ and $\mathbf{F}(\mathbf{d})$ instead of $\mathbf{F}(\mathbf{d}')(E)$ and $\mathbf{F}(\mathbf{d})(E)$.

Because $\mathbf{F}(\mathbf{d})(E)_i$ is a free A -module generated by a representation, and everything splits as $\mathrm{GL}(E)$ -modules, we can lift the differential on $H^0\mathcal{F}_\bullet$ to get the following commutative diagram, where each column is exact.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow & \mathbf{F}(\mathbf{d}')_m & \rightarrow & \mathbf{F}(\mathbf{d}')_{m-1} & \rightarrow \cdots \rightarrow & \mathbf{F}(\mathbf{d}')_0 & \\
 & \downarrow & & \downarrow & & \cdots & \downarrow \\
 0 \rightarrow & \mathbf{F}(\mathbf{d}')_m = \mathbf{F}(\mathbf{d})_m & \rightarrow & \mathbf{F}(\mathbf{d})_{m-1} & \rightarrow \cdots \rightarrow & \mathbf{F}(\mathbf{d})_0 & \\
 & \downarrow & & \downarrow & & \cdots & \downarrow \\
 0 \rightarrow & 0 & \rightarrow & H^0\mathcal{F}_{m-1} & \rightarrow \cdots \rightarrow & H^0\mathcal{F}_0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

The horizontal maps are constructed simply to be $\mathrm{GL}(E)$ -equivariant and make the diagram commute, except that we take the upper left map $\mathbf{F}(\mathbf{d}')_m \rightarrow \mathbf{F}(\mathbf{d}')_{m-1}$ to be the map coming from the complex $\mathbf{F}(\mathbf{d}')_\bullet$, we take the upper left vertical map $\mathbf{F}(\mathbf{d}')_m \rightarrow \mathbf{F}(\mathbf{d}')_m = \mathbf{F}(\mathbf{d})_m$ to be the equality (so that the left-most column is also exact), and we take the horizontal map $\mathbf{F}(\mathbf{d})_m \rightarrow \mathbf{F}(\mathbf{d})_{m-1}$ to be the composite of the two maps above it, assuring the commutativity of the upper left-hand square of the diagram.

We will now prove that the two upper horizontal rows are in fact the complexes $\mathbf{F}(\mathbf{d}')_\bullet$ and $\mathbf{F}(\mathbf{d})_\bullet$. As defined, the left-hand map in the upper row is the right map, and the map below it in the middle row is at least nonzero. From the uniqueness statement in the definition of the differentials of our complexes, all the other horizontal maps will be the correct ones as soon as we know that they are all nonzero.

Let \mathcal{G}_\bullet be the total complex of the double complex made from the two upper rows of the diagram, so that \mathcal{G}_\bullet is a resolution of the same module $H^0(M_Q)$ as that resolved by $H^0(\mathcal{F}_\bullet)$. The last vertical map $\mathbf{F}(\mathbf{d}')_m \rightarrow \mathbf{F}(\mathbf{d})_m$ defines a quotient complex of \mathcal{G}_\bullet , and is an isomorphism. We may take the kernel of this quotient map, it has the same homology as \mathcal{G}_\bullet , arriving at a complex

$$\mathcal{G}'_\bullet : \quad 0 \rightarrow \mathbf{F}(\mathbf{d}')_{m-1} \rightarrow \cdots \rightarrow \mathbf{F}(\mathbf{d})_0$$

of length m that is, once again, a resolution of the module $H^0(M_Q)$.

First, we note that the complex \mathcal{G}'_\bullet is graded, with degree 0 differentials, if we give the generators of each $\mathbf{F}(\mathbf{d})_i$ the degree d_i as in the definition of $\mathbf{F}(\mathbf{d})_\bullet$, and similarly for the $\mathbf{F}(\mathbf{d}')_i$. This is because the unique occurrence of the representation $S_{\alpha(\mathbf{d},i)}$ that generates $\mathbf{F}(\mathbf{d})_i$, in $\mathbf{F}(\mathbf{d})_{i-1}$ is in $S_{e_i} \otimes S_{\alpha(\mathbf{d},i-1)}$, and similarly for the $\mathbf{F}(\mathbf{d}')_i$. It follows that all the maps

in the resolution \mathcal{G}'_\bullet are given by matrices of elements of positive degree in A ; that is, the resolution \mathcal{G}'_\bullet is minimal. From this minimality it follows that for each i the constructed map $\mathbf{F}(\mathbf{d})_i \rightarrow \mathbf{F}(\mathbf{d})_{i-1}$ is nonzero; for if it vanished then by exactness $\mathbf{F}(\mathbf{d})_i$ would be in the image of $\mathbf{F}(\mathbf{d})_{i+1} \oplus \mathbf{F}(\mathbf{d}'_i)$, which is impossible.

Since the $M_{\mathcal{Q}}(\mathbf{d})$ is the direct sum of finitely many Schur functors applied to \mathcal{Q} , Bott's Theorem tells us that the cohomology module $H^0(M_{\mathcal{Q}}(\mathbf{d}))$ is a direct sum of finitely many representations, each a Schur functor of E , and is thus finite-dimensional as a vector space. It follows that the dual of \mathcal{G}'_\bullet is also a minimal free resolution of an A -module of finite length. The dual argument to that just given shows that all the maps $\mathbf{F}(\mathbf{d}')_i \rightarrow \mathbf{F}(\mathbf{d}')_{i-1}$ are nonzero as well.

We have now proven the existence of a short exact sequence of complexes

$$0 \rightarrow \mathbf{F}(\mathbf{d}')_\bullet \rightarrow \mathbf{F}(\mathbf{d})_\bullet \rightarrow H^0(\mathcal{F}_\bullet) \rightarrow 0.$$

We know that the complex $H^0(\mathcal{F}_\bullet)$ is acyclic. Since $d'_m - d'_0 = d_m - d_0 - 1$, our second induction shows that the complex $\mathbf{F}(\mathbf{d}')_\bullet$ is acyclic as well. From the long exact sequence in homology associated to the short exact sequence of complexes, we see that $\mathbf{F}(\mathbf{d})_\bullet$ is acyclic too, and the proof is done. \square

Example 3.3. — Take $\mathbf{d} = (0, 2, 5, 7, 8)$ so that $\mathbf{e} = (0, 2, 3, 2, 1)$. Our Young diagram is

0	0	3	4
0	2	3	
0	2		
1	2		
1			

The complex $\mathbf{F}(\mathbf{d})_\bullet$ has terms

$$(3, 1, 0, 0) \leftarrow (5, 1, 0, 0) \leftarrow (5, 4, 0, 0) \leftarrow (5, 4, 2, 0) \leftarrow (5, 4, 2, 1)$$

where we write λ instead of $S_\lambda E \otimes A$. To get the acyclicity by induction we start with the complex of sheaves on the projective space with the terms

$$(3, 1, 0) \leftarrow (5, 1, 0) \leftarrow (5, 4, 0) \leftarrow (5, 4, 2)$$

where λ is the shorthand for $S_\lambda \mathcal{Q} \otimes \mathcal{B}$. Taking modules of sections we get A -modules with free resolutions (written as columns)

$$\begin{array}{ccccccc}
 (3, 1, 0, 0) & \leftarrow & (5, 1, 0, 0) & \leftarrow & (5, 4, 0, 0) & \leftarrow & (5, 4, 2, 0) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & (5, 4, 2, 1)
 \end{array}$$

where again we write λ instead of $S_\lambda E \otimes A$. So the mapping cone is the required complex.

Example 3.4. — Take $\mathbf{d} = (0, 2, 5, 6, 8)$, so that $\mathbf{e} = (0, 2, 3, 1, 2)$. Our Young diagram is

0	0	0	4
0	2	3	4
0	2		
1	2		
1			

The complex $\mathbf{F}(\mathbf{d})_\bullet$ has terms

$$(3, 1, 1, 0) \leftarrow (5, 1, 1, 0) \leftarrow (5, 4, 1, 0) \leftarrow (5, 4, 2, 0) \leftarrow (5, 4, 2, 2)$$

where we write λ instead of $S_\lambda E \otimes A$. To get the acyclicity by induction we start with the complex of sheaves on the projective space with the terms

$$(3, 1, 1) \leftarrow (5, 1, 1) \leftarrow (5, 4, 1) \leftarrow (5, 4, 2)$$

where λ is the shorthand for $S_\lambda \mathcal{Q} \otimes \mathcal{B}$. Taking A -modules of sections we get A -modules with free resolutions (written as columns)

$$\begin{array}{cccccccc} (3, 1, 1, 0) & \leftarrow & (5, 1, 1, 0) & \leftarrow & (5, 4, 1, 0) & \leftarrow & (5, 4, 2, 0) & \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ (3, 1, 1, 1) & \leftarrow & (5, 1, 1, 1) & \leftarrow & (5, 4, 1, 1) & \leftarrow & (5, 4, 2, 1) & \end{array}$$

where again we write λ instead of $S_\lambda E \otimes A$. The first row is the required complex $\mathbf{F}(\mathbf{d})_\bullet$ without the last term $(5, 4, 2, 2)$. But the second row is the complex $\mathbf{F}(1, 3, 6, 7, 8)_\bullet$ without the last term. It corresponds to the Young diagram

0	0	0	0
0	2	3	4
0	2		
1	2		
1			

where the row of zeros is added because $e_0 = 1$. Now we notice that the last missing term of this complex is also $(5, 4, 2, 2)$, which proves that the homology of the top row is isomorphic to this free A -module, and this concludes the proof.

Remarks. — The ranks of the modules in a pure resolution are easy to calculate from the Herzog-Kühl equations [12]; see Section 2.1 of [2]. In the case of complexes $\mathbf{F}(\mathbf{d})_\bullet$ these formulas are special cases of the Weyl

dimension formula. The multiplicative form of these formulas was one of the motivations for looking at Schur functors.

Similarly, it is a standard result that for a graded module of finite length over a polynomial ring, the dimension of the socle equals the Cohen-Macaulay type, that is, the rank of the last syzygy module. The representation in the highest degree of $M(\mathbf{d})$ corresponds to the partition we get from $\alpha(\mathbf{d}, 0)$ by adding one box to each of the first $\alpha(\mathbf{d}, 1)_1 - 1$ rows. It is amusing to see that this is the partition $\alpha(\mathbf{d}, m)$ with the first row of length m removed. So the socle of $M(\mathbf{d})$ is the representation in the highest degree, as it should since the last term in the resolution is pure.

We finish this section by analyzing some of the features of the complexes we constructed.

PROPOSITION 3.5. — *Let us assume the sequence \mathbf{e} is symmetric, i.e. $e_i = e_{m+1-i}$ for $i = 1, \dots, m$. Then the complex $\mathbf{F}(\mathbf{d})_\bullet$ is self-dual, i.e. $\mathbf{F}(\mathbf{d})_\bullet^* = \mathbf{F}(\mathbf{d})_\bullet$. This duality is $GL(E)$ -equivariant.*

Proof. — Assume that the sequence \mathbf{e} is symmetric. Let $\lambda = \alpha(\mathbf{d}, 0)$. Consider the rectangular partition $\mu := (\lambda_1 + e_1)^m$. It is clear that the partitions $\alpha(\mathbf{d}, i)$ and $\alpha(\mathbf{d}, m - i)$ are complementary with respect to this rectangle. This means we have a $GL(E)$ -equivariant nondegenerate pairing

$$S_{\alpha(\mathbf{d}, i)}E \otimes S_{\alpha(\mathbf{d}, m-i)}E \rightarrow \left(\bigwedge^m E\right)^{\otimes(\lambda_1 + e_1)}.$$

This can be extended to the isomorphism of complexes we claim. □

Example 3.6. — Let us take $m = 5$ and $\mathbf{e} = (2, 3, 1, 3, 2)$. The corresponding picture is

0	0	0	0	5
0	0	0	4	5
0	0	0	4	
0	2	3	4	
0	2			
1	2			
1				

The partitions $\alpha(\mathbf{d}, i)$ for $i = 0, \dots, 5$ correspond to the boxes with entries $\leq i$. Considering the boxes with entries $\geq i + 1$ (letting the empty boxes have entry 6), if we turn the rectangle 180° this is the partition $\alpha(\mathbf{d}, 5 - i)$.

4. Modules Supported on Determinantal Varieties

In this section we describe another way of constructing a Cohen-Macaulay module whose free resolution is pure, with given degree shifts \mathbf{d} . These modules are supported on the degeneracy locus of a generic map of free modules $G \rightarrow F$, and equivariant for $GL(F) \times GL(G)$. (Of course one can derive non-equivariant artinian modules from them by reducing modulo a general sequence of linear forms, at least in the case where the ground field is infinite.) This family of resolutions generalizes the ones described by Kirby [13] and Buchsbaum and Eisenbud [3] (see [6] Appendix A2.6 for an exposition) and re-interpreted by Weyman ([17] exercises 37-39, chapter 6), though in the special case treated by those authors the resolutions work in arbitrary characteristic, while the method used here to obtain the generalization depends on characteristic 0.

With notation as in Section 1, we fix the strictly increasing sequence $\mathbf{d} = (d_0, \dots, d_s)$ and its sequence of differences $\mathbf{e} = (e_0 = d_0, e_1 = d_1 - d_0, \dots)$. Take two vector spaces F, G , with $\dim(F) = 1 + \sum_{i=1}^s (e_i - 1)$ and $\dim(G) = \dim(F) + s - 1$. Let B be the polynomial ring $B = \text{Sym}(F \otimes G^*)$. Consider the Grassmannian $\text{Grass}(1, F)$ of lines in F (this is just a projective space), with tautological sequence

$$0 \rightarrow \mathcal{R} \rightarrow F \otimes \mathcal{O}_{\text{Grass}(1, F)} \rightarrow \mathcal{Q} \rightarrow 0,$$

so that \mathcal{Q} is a bundle of rank $\dim(F) - 1$.

Consider the incidence variety

$$Z = \{(\phi, \mathcal{R}) \in \text{Hom}(F, G) \times \text{Grass}(1, F) \mid \phi|_{\mathcal{R}} = 0\}.$$

This is one of the desingularizations of the determinantal variety defined by the maximal minors of the generic matrix ϕ , denoted in the Section 6.5 of [17] by $Z_{s-1}^{(1)}$.

Consider the partition $\lambda(\mathbf{d}) = ((s-1)^{e_s-1}, (s-2)^{e_{s-1}-1}, \dots, 0^{e_1-1})$, and let $\mathcal{N}(\mathbf{d})$ be the sheaf $\mathcal{N}(\mathbf{d}) = S_{\lambda(\mathbf{d})} \mathcal{Q} \otimes \mathcal{O}_Z$. To describe the second family of complexes we set

$$\gamma(\mathbf{d}, i) := ((s-1)^{e_s-1}, (s-2)^{e_{s-1}-1}, \dots, i^{e_{i+1}-1}, i^{e_i}, (i-1)^{e_{i-1}-1}, \dots, 1^{e_1-1}).$$

The partition $\gamma(\mathbf{d}, i)$ is conjugate to the partition $\alpha(\mathbf{d}, i)$ defined in the introduction.

THEOREM 4.1. — *$H^i(Z, \mathcal{N}(\mathbf{d})) = 0$ for $i > 0$, and $H^0(Z, \mathcal{N}(\mathbf{d}))$ has a pure $GL(F) \times GL(G)$ -equivariant minimal resolution $\mathbf{H}(\mathbf{d})_\bullet$ of type \mathbf{d} , with terms*

$$\mathbf{H}(\mathbf{d})_i = S_{\gamma(\mathbf{d}, i)} F \otimes \bigwedge^{d_i - d_0} G^* \otimes B(-d_i + d_0).$$

Proof. — Let $p : Z \rightarrow Grass(1, F)$ be the projection map. Because p is an affine map it suffices for the first statement to show that $H^i p_*(\mathcal{N}(\mathbf{d})) = 0$ for $i > 0$. However,

$$p_*(\mathcal{N}(\mathbf{d})) = S_{\lambda(\mathbf{d})}\mathcal{Q} \otimes Sym(\mathcal{Q} \otimes G^*).$$

Since this does not involve \mathcal{R} , it has no higher cohomology.

To prove the second statement, we apply the Basic Theorem (5.1.2) from [17] to the sheaf $\mathcal{N}(\mathbf{d})$. In the notation of that result, we set $\xi = \mathcal{R} \otimes G^*$, $\eta = \mathcal{Q} \otimes G^*$ and $\mathcal{V} = S_{\lambda(\mathbf{d})}\mathcal{Q}$. We get a complex $\mathbf{F}(S_{\lambda(\mathbf{d})}\mathcal{Q})_\bullet$ which is our $\mathbf{H}(\mathbf{d})_\bullet$, which is a resolution of $H^0 p_*(\mathcal{N}(\mathbf{d}))$ because the higher cohomology $H^i p_*(\mathcal{N}(\mathbf{d}))$ vanishes.

The direct calculation of the cohomology groups using Bott’s theorem ([17], (4.1.9)), dualized using exercise 18 b), p.83), gives the terms of our complex. More precisely, the calculation comes down to applying Bott’s Theorem to the weights

$$((s - 1)^{e_s - 1}, (s - 2)^{e_{s-1} - 1}, \dots, 0^{e_1 - 1}, u)$$

for $0 \leq u \leq dim(G)$. The partition $\gamma(\mathbf{d}, i)$ comes from the term with $u = d_i - d_1 + 1 + i$. □

Remark. — The above theorem is the specialization of Theorem 0.2 we get by setting $V = V_1 = F$ and $U = U_1 = G^*$.

5. Proofs of Theorems 0.1 and 0.2

Proof of Theorem 0.1. — The differential in $\mathbf{F}(\mathbf{d})_\bullet$ is $\mathfrak{gl}(V)$ -equivariant by definition. The $\mathbf{Z}/2$ -graded Pieri formula implies $\mathbf{F}(\mathbf{d})_\bullet$ is a complex. By Berele-Regev theory, the $\mathfrak{gl}(V)$ action on each homogeneous component of each module $\mathbf{F}(\mathbf{d})_i$ is semi-simple. Thus the homogeneous components of the homology are direct sums of Schur modules $\mathcal{S}_\mu V$.

Now assume that for some λ and some (m, n) the complex $\mathbf{F}(\mathbf{d})_\bullet$ is not acyclic, so some module $\mathcal{S}_\mu V$ consists of cycles that are not boundaries. But then, for a bigger dimension vector (m, n) , such that $\mathcal{S}_\mu(V_0) \neq 0$, the same Schur module $\mathcal{S}_\mu V$ is in the homology. If m is large enough, this Schur module contains elements in some weight that does not involve V_1 , that is, elements defined by tableaux that contain only basis elements from V_0 .

Since the differential of $\mathbf{F}(\mathbf{d})_\bullet$ preserves the weight space decomposition, this implies that the complex $\mathbf{F}(\mathbf{d})_\bullet$ for V_0 is not acyclic. This contradicts Theorem 3.2, proving the Theorem. □

Proof of Theorem 0.2. — The proof of Theorem 0.2 follows the same outline as the proof of Theorem 0.1, except that we use subspaces $U_1 \subset U$ and $V_1 \subset V$, and Theorem 4.1, where we set $F = V_1$ and $G^* = U_1$. \square

Remark. — Consider Theorem 0.1 in the odd case, that is where $m = 0$, and set $F = V_1$. Then $R = \bigwedge^\bullet(F)$ and the complex $\mathbf{F}(\mathbf{d})_\bullet$ has terms

$$\mathbf{F}(\mathbf{d})_i = S_{\alpha(\mathbf{d},i)} E \otimes R(-e_1 - \cdots - e_i).$$

In this way we get truncations of the Tate resolutions constructed by Fløystad [9]. An alternate proof of Theorem 0.1. could be obtained by reducing to the odd case. The proof in the current paper avoids Bernstein-Gel'fand-Gel'fand duality. One can also relate the even and odd cases to each other by means of Schur duality.

6. Some open problems and conjectures

In 6.1–6.5 below we work over the polynomial ring A .

CONJECTURE 6.1. — *Every sufficiently large integral point in the ray defining the possible pure Betti tables of graded modules of finite length over a polynomial ring, with a given degree sequence, is actually the Betti table of the free resolution of a Cohen-Macaulay module.*

The following examples are expressed in terms of the smallest integral multiple of the Betti number on a ray of pure resolutions, corresponding to the given degree sequence \mathbf{d} , which we call the primitive vector of Betti numbers. In all these examples $m = 3$.

Example 6.2. — $\mathbf{d} = (0, 3, 4, 7)$. The primitive vector of Betti numbers is $\beta = (1, 7, 7, 1)$. The construction of §3 gives Betti numbers 6β , the Eisenbud-Schreyer construction gives 15β and the construction from §4 gives 50β . Thus all three are needed in order to conclude the conjecture for this extremal ray. But in this case we know that the primitive vector is achieved by the minimal free resolution of the ideal of 6×6 Pfaffians of a 7×7 skew-symmetric matrix of linear forms; see [4].

Example 6.3. — $\mathbf{d} = (0, 4, 9, 13)$. The primitive vector of Betti numbers is $\beta = (5, 13, 13, 5)$. The construction of §3 gives Betti numbers 18β , the Eisenbud-Schreyer construction gives 380β and the construction of §4 gives 9075β . These three examples together imply the truth of the conjecture on this ray.

Example 6.4. — $\mathbf{d} = (0, 1, 4, 6)$. The primitive vector of Betti numbers is $\beta = (5, 8, 5, 2)$. All three constructions give 5β . This is the smallest sequence for $n = 3$ where we cannot conclude the conjecture using our three constructions.

The material presented in this paper raises some other interesting questions.

Problem 6.5. — Equivariant Boij-Söderberg conjectures. We use the notation of Section 3. Let \mathbf{F}_\bullet be a $GL(E)$ -equivariant acyclic complex of A -modules. Is it in the cone generated by the Betti tables of pure resolutions constructed in Section 3 in the sense that there exist $GL(E)$ -representations W and W_1, \dots, W_s and degree shifts $\mathbf{d}(i)$ (for $1 \leq i \leq s$) such that for each $j = 0, \dots, m$ we have isomorphisms of $GL(E)$ -modules

$$W \otimes \mathbf{F}_j = \bigoplus_{i=1}^s W_i \otimes \mathbf{F}(\mathbf{d}(i))_j?$$

In particular, assuming \mathbf{F}_\bullet is pure with degree shifts \mathbf{d} , does it mean that $W \otimes \mathbf{F}_\bullet = W' \otimes \mathbf{F}(\mathbf{d})_\bullet$ for some $GL(E)$ -modules W, W' ?

Problem 6.6. — $\mathbf{Z}/2$ -graded Betti tables. Let $R = Sym(V_0) \otimes \bigwedge^\bullet(V_1)$. Is the Betti table of every acyclic complex of free R -modules in the cone generated by the Betti tables of pure acyclic complexes? What are the facets of the cone generated by Betti tables of pure acyclic complexes of free modules, is this cone self-dual in some sense?

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