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COALGEBRAIC APPROACH TO THE LODAY INFINITY CATEGORY, STEM DIFFERENTIAL FOR $2n$-ARY GRADED AND HOMOTOPY ALGEBRAS

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ABSTRACT. — We define a graded twisted-coassociative coproduct on the tensor algebra the desuspension space of a graded vector space V. The coderivations (resp. quadratic "degree 1" codifferentials, arbitrary odd codifferentials) of this coalgebra are 1-to-1 with sequences of multilinear maps on V (resp. graded Loday structures on V, sequences that we call Loday infinity structures on V). We prove a minimal model theorem for Loday infinity algebras and observe that the Lod$_\infty$ category contains the L$_\infty$ category as a subcategory. Moreover, the graded Lie bracket of coderivations gives rise to a graded Lie "stem" bracket on the cochain spaces of graded Loday, Loday infinity, and $2n$-ary graded Loday algebras. This stem bracket restricts to the graded Nijenhuis-Richardson and Grabowski-Marmo brackets, and it encodes, beyond the already mentioned cohomologies, those of graded Lie, graded Poisson, graded Jacobi, Lie infinity, as well as that of $2n$-ary graded Lie algebras.

RÉSUMÉ. — Nous définissons un coproduit gradué et coassociatif tordu sur l'algèbre tensorielle d'un espace vectoriel gradué V. Les codérivations (resp. codifférentielles quadratiques de "degré 1", codifférentielles impaires quelconques) de cette co-algèbre sont en correspondance biunivoque avec les suites d'applications multilinéaires sur V (resp. structures graduées de Loday sur V, suites que nous appelons structures de Loday infinies sur V). Nous prouvons un théorème du modèle minimal pour les algèbres infinies de Loday et observons que la catégorie Lod$_\infty$ contient la catégorie L$_\infty$ comme sous-catégorie. En plus, le crochet de Lie gradué des codérivations conduit à un crochet de Lie gradué "souche" sur les espaces des cochaînes des algèbres de Loday graduées, de Loday infinies et de Loday graduées $2n$-aires. Le crochet souche se restreint aux crochets gradués de Nijenhuis-Richardson et de Grabowski-Marmo, et il encode, au-delà des cohomologies déjà mentionnées, celles des algèbres de Lie graduées, de Poisson graduées, de Jacobi graduées, Lie infinies, ainsi que celle des algèbres de Lie graduées $2n$-aires.

1. Introduction

The prevailing approach to cohomology consists in associating in a canonical way a differential module to the investigated algebraic structure. For instance, in [21] and [15] (resp. in [33]) the author(s) define(s) Lichnerowicz-Jacobi and Nambu-Poisson cohomologies (resp. the cohomology of graded Leibniz algebras with trivial coefficients) essentially as the cohomology of a Lie algebroid (resp. as the homology of a differential graded dual Leibniz algebra) that is associated with any Jacobi or Nambu-Poisson manifold [it suffices to solve the Lie algebra homomorphism equation for the natural anchor map] (resp. that is induced by the considered Leibniz algebra [via dualization]).

The most natural way to obtain a differential space from an algebraic structure is the square-zero or canonical element method that was initiated by De Wilde and Lecomte in [8]. They look for a graded Lie algebra, such that there is a 1-to-1 correspondence between the studied algebraic structures and the degree 1 elements of the Lie algebra space that square to zero with respect to the Lie bracket. The adjoint action by the given structure then provides a differential graded Lie algebra, the cohomology of which has the usual properties with respect to formal deformations of the algebraic structure.

Two efficient tools allow finding the mentioned graded Lie algebra:

— The coalgebraic technique consists in the identification of the cochain space of the investigated algebra with certain coderivations, in such a way that the algebraic structure can be viewed as an odd codifferential. These identifications enable constructing a graded Lie bracket on the space of cochains by transfer of the commutator bracket of coderivations; the considered algebraic structures are then canonical elements of this transferred bracket. The procedure was applied by Stasheff [39] in the case of associative and Lie algebras and by Penkava [34] for strongly homotopy associative and Lie algebras.

— The operadic theory, as developed in [28], extends a work by Balavoine [6] and shows that Stasheff's approach is applicable for any quadratic operad $\mathcal{P}$. The authors associate to $\mathcal{P}$ a cofree coalgebra over the dual operad, whose quadratic codifferentials correspond to the $\mathcal{P}-$algebra structures on a graded vector space V. This construction yields the homology and cohomology theories of $\mathcal{P}-$algebras on V and allows defining strongly homotopy $\mathcal{P}-$algebra structures on V as arbitrary codifferentials.

In the present communication we provide a tensor coalgebra that induces the proper concepts of Loday infinity algebras and morphisms, and define

the cohomologies of $\mathbb{Z}^n$-graded Loday, Loday infinity, and $2p$-ary graded Loday algebras. This leads to a graded Lie "stem" bracket, in which are encrypted, in addition to the preceding cohomologies, the coboundary operators of graded Lie, graded Poisson, graded Jacobi, Lie infinity, and $2p$-ary graded Lie algebras [30]. Our approach is self-contained, results are explicit, and they suggest that the operadic theory is not confined to the quadratic case.

The paper is organized as follows.

In Section 2, we study the properties of cohomology algebras, which are implemented by square-zero elements in a graded Lie algebra, with respect to formal deformations of these elements. Our upshots extend similar properties for the adjoint Hochschild (resp. Chevalley-Eilenberg, Leibniz) cohomology and deformations of associative (resp. Lie, Loday) structures, which were proved in [11] (resp. [32], [5]).

Section 3 contains the definition of a Zinbiel coalgebra structure Δ on the tensor algebra $T(W)$ of a $\mathbb{Z}^n$-graded vector space W. We provide explicit formulæ for the reconstruction of coderivations and cohomomorphisms from their corestriction maps.

In Section 4, we transfer the $\mathbb{Z}^n$-graded Lie bracket of coderivations of the mentioned tensor coalgebra $T(W)$ of the desuspension space $W := \downarrow V$ of an underlying $\mathbb{Z}^n$-graded vector space V, and get a $\mathbb{Z}^{n+1}$-graded (resp. $\mathbb{Z}^n$-graded) Lie bracket on the $\mathbb{Z}^{n+1}$-graded vector space of weighted multilinear maps on V (resp. on the $\mathbb{Z}^n$-graded vector space of sequences of shifted weighted multilinear maps on V). We determine the explicit form of this pullback "stem" bracket and show that its $\mathbb{Z}^{n+1}$-graded version coincides in the case of a nongraded underlying space V (resp. of graded skew-symmetric multilinear mappings on V) with Rotkiewicz's bracket [38] pertaining to left Loday structures [and corresponds to Balavoine's bracket [6] concerning right Loday structures] (resp. with the graded Nijenhuis-Richardson bracket [20]).

Codifferentials of our dual Leibniz coalgebra $T(W)$ are characterized in Section 5. We prove that $\mathbb{Z}^n$-graded Loday structures on V can be viewed as (resp. we define strongly homotopy Loday structures on V as) degree $e_1 := (1, 0, \ldots, 0) \in \mathbb{Z}^n$ quadratic (resp. odd degree) codifferentials of $(T(\downarrow V), \Delta)$. Loday infinity structures and Loday infinity morphisms are described in terms of sequences of weighted multilinear maps that satisfy explicitly depicted sequences of constraints: Our Lod_∞ algebras are really differential graded Loday algebras up to homotopy and the Lod_∞ category contains the L_∞ category as a subcategory.

Loday infinity (quasi)-isomorphisms are investigated. In Section 6, we prove a minimal model theorem for strongly homotopy Loday algebras, and deduce that any Loday infinity quasi-isomorphism has a quasi-inverse — a theorem whose Lie infinity counterpart plays a key-role in Deformation Quantization.

In Section 7, we deal with graded and strongly homotopy cohomologies. $\mathbb{Z}^n$-graded Loday [resp. strongly homotopy Loday] structures are canonical for the $\mathbb{Z}^{n+1}$-graded [resp. $\mathbb{Z}^n$-graded] stem bracket, so that we obtain a natural cohomology theory and an explicit coboundary operator. In the nongraded (resp. the antisymmetric) [resp. the Lie infinity] case, our $\mathbb{Z}^n$-graded Loday [resp. Loday infinity] cohomology operator coincides with the Loday (resp. graded Chevalley-Eilenberg) [resp. Lie infinity] differential given in [7] and [6] (resp. in [20]) [resp. in [34] and [9]].

Further, graded Poisson and Jacobi cohomologies were defined purely algebraically by Grabowski and Marmo in [14]. The authors prove existence and uniqueness of a $\mathbb{Z}^{n+1}$-graded Jacobi (resp. Poisson) bracket on the algebra of antisymmetric graded first order polydifferential operators (resp. of graded polyderivations). We compute this "Grabowski-Marmo" bracket explicitly and explain how the corresponding cohomologies are induced by our stem bracket.

Finally, essentially two p-ary extensions of the Jacobi identity were investigated during the last decades. The first, see e.g. [10], leads to the Nambu-Lie structure, see [31], the second, see [30], [40], [41], will in this text be referred to as p-ary Lie structure. We define analogously p-ary (p even) $\mathbb{Z}^n$-graded Loday structures and their cohomology. These graded p-ary Loday algebras are special strongly homotopy Loday algebras, so that we have to prove that the two stem bracket induced cohomologies coincide.

2. Canonical elements of graded Lie algebras

Unless otherwise stated, all vector spaces that we consider in this text are spaces over a field $\mathbb{K}$ of characteristic 0, and all graded vector spaces are $\mathbb{Z}^n$-graded, $n \in \mathbb{N}^*$. The $\mathbb{Z}^n$-degree $\deg(v)$ of a vector v or the $\mathbb{Z}^n$-weight $\deg(f)$ of a graded linear map f are often denoted by the same symbol v or f. If $v, f \in \mathbb{Z}^n$ are two such degrees, we set $\langle v, f \rangle = \sum_i v_i f_i$. A homogeneous vector or graded linear map w is termed *odd*, if $\langle w, w \rangle \in \mathbb{Z}$ is an odd number. Eventually, again except differently stipulated, all graded brackets are of weight 0.

DEFINITION 2.1. — *We call* canonical element *of a graded Lie algebra (GLA)* $(V, \{-, -\})$, *any odd element* $\pi \in V$ *that verifies* $\{\pi, \pi\} = 0$.

It follows from the Jacobi identity that a GLA $(V, \{-, -\})$ endowed with a canonical element π is a differential graded Lie algebra (DGLA) $(V, \{-, -\}, \partial_\pi)$ for $\partial_\pi := \{\pi, -\}$. Of course the cohomology space of any DGLA is canonically a GLA. We denote the cohomology GLA of the preceding DGLA by $H_\pi(V)$.

Let us mention two basic examples. If V is a $\mathbb{Z}^n$-graded vector space, we set $M(V) = \bigoplus_{(B,b) \in \mathbb{Z}^n \times \mathbb{Z}} M^{(B,b)}(V)$, where $M^{(B,b)}(V) = 0$ for all $b \leqslant -2$, $M^{(B,-1)}(V) = V^B$, and where for each $b \geqslant 0$, $M^{(B,b)}(V)$ is the space of all $(b+1)$-multilinear maps $B \colon V^{\times(b+1)} \to V$ that have weight B. Further, we denote by $A(V)$ the graded vector subspace of $M(V)$, which is made up by the $\mathbb{Z}^n$-graded skew-symmetric maps.

Example 2.2. — The pair $(M(V), [-, -]^{\mathrm{G}})$ (resp. $(A(V), [-, -]^{\mathrm{NR}})$), where $[-, -]^{\mathrm{G}}$ (resp. $[-, -]^{\mathrm{NR}}$) denotes the graded Gerstenhaber (resp. Nijenhuis-Richardson) bracket, is a $\mathbb{Z}^{n+1}$-graded Lie algebra. Its canonical elements of degree $(0, 1) \in \mathbb{Z}^n \times \mathbb{Z}$ are the associative graded (resp. graded Lie) algebra structures on V. If π is such an element, $[\pi, \pi]^{\mathrm{G}} = 0$ (resp. $[\pi, \pi]^{\mathrm{NR}} = 0$) exactly means that π is associative (resp. verifies the graded Jacobi identity). The cohomology $H_\pi(M(V))$ (resp. $H_\pi(A(V))$) coincides with the adjoint Hochschild (resp. Chevalley-Eilenberg) cohomology of (V, π).

For details pertaining to these examples, we refer the reader to [20].

Next we show that cohomology algebras $H_\pi(V)$ that are implemented by canonical elements π are good tools to study formal deformations of π.

We set $V[[\nu]] = \bigoplus_{\alpha \in \mathbb{Z}^n} V^\alpha[[\nu]]$, where $V^\alpha[[\nu]]$ is the space of formal power series in a formal parameter ν with coefficients in the degree α term V^α of V. A formal power series $\pi_\nu = \sum_{i=0}^\infty \nu^i \pi_i \in V^{\deg(\pi)}[[\nu]]$ with first term $\pi_0 = \pi$ is a *formal deformation* of π, if it squares to zero w.r.t. the natural extension of the bracket $\{-, -\}$ to a bilinear map of the space $V[[\nu]]$, i.e. if $\{\pi_\nu, \pi_\nu\} = \sum_{p=0}^\infty \nu^p \sum_{i+j=p} \{\pi_i, \pi_j\} = 0$. A *formal deformation of order* q is a formal series π_ν that is truncated at order q in ν and satisfies the condition

$$\text{(1)} \qquad \sum_{i+j=p} \{\pi_i, \pi_j\} = 0, \quad 1 \leqslant p \leqslant q.$$

We refer to formal deformations of order 1 as *infinitesimal deformations*.

PROPOSITION 2.3. — *The cohomology space $H_\pi^{2\deg(\pi)}(V)$ contains the obstructions to extension of formal deformations of order at least 1 to higher order deformations.*

Proof. — Assume that π admits an order q, $q \geqslant 1$, deformation π_ν and define E_p in such a way that condition (1) reads $E_p = -2\partial_\pi(\pi_p)$, $1 \leqslant p \leqslant q$. In view of (1), we then have

$$\partial_\pi(E_{q+1}) = - \sum_{\substack{k+l+j=q+1 \\ k,l,j \neq 0}} \{\{\pi_k, \pi_l\}, \pi_j\},$$

which vanishes due to Jacobi's identity. In order to extend deformation π_ν to order $q+1$, see (1), cocycle $E_{q+1} \in V^{2\deg(\pi)}$ must be a coboundary. □

LEMMA 2.4. — *Let π be a canonical element of a GLA $(V, \{-,-\})$ and consider a series $\chi_\nu = \sum_{i=1}^\infty \nu^i \chi_i \in V^0[[\nu]]$. If π_ν is a formal deformation of π, then $\exp(\mathrm{ad}\,\chi_\nu)\,\pi_\nu$ is a formal deformation of π as well.*

Proof. — Let us mention that exp denotes the exponential series and stress that

$$(2) \qquad (\mathrm{ad}\,\chi_\nu)^k \pi_\nu = \big\{\chi_\nu\{\chi_\nu \ldots \underbrace{\{\chi_\nu, \pi_\nu\} \ldots\}\big\}}_{k}$$

$$= \sum_{p=0}^\infty \nu^p \sum_{i_1+\ldots+i_k+j=p} \big\{\chi_{i_1}\{\chi_{i_2} \ldots \{\chi_{i_k}, \pi_j\} \ldots\}\big\}.$$

It follows that the coefficient of ν^p in the exponential series over k is made up by a finite number of terms in $V^{\deg(\pi)}$; indeed, if $k \geqslant p+1$, at least one of the χ_{i_ℓ} vanishes. Moreover, the coefficient of ν^0 contains only the term $k = 0$, and thus $(\exp(\mathrm{ad}\,\chi_\nu)\,\pi_\nu)_0 = \pi$. Eventually, as $\mathrm{ad}\,\chi_\nu$ is a derivation of the bracket $\{-,-\}$, we have

$$(\mathrm{ad}\,\chi_\nu)^k \{\pi_\nu, \pi_\nu\} = \sum_{\substack{r+s=k \\ r,s \geqslant 0}} C_k^r \{(\mathrm{ad}\,\chi_\nu)^r \pi_\nu, (\mathrm{ad}\,\chi_\nu)^s \pi_\nu\},$$

where C_k^r is the binomial coefficient. Hence,

$$\exp(\mathrm{ad}\,\chi_\nu)\{\pi_\nu, \pi_\nu\} = \big\{\exp(\mathrm{ad}\,\chi_\nu)\pi_\nu, \exp(\mathrm{ad}\,\chi_\nu)\pi_\nu\big\},$$

which completes the proof of the lemma. □

Recall that two formal deformations π_ν and π'_ν of π are said to be *equivalent* (resp. *equivalent up to order q, $q \geqslant 1$*), if there is a series χ_ν of the type specified in Lemma 2.4, such that $\exp(\mathrm{ad}\,\chi_\nu)\,\pi_\nu = \pi'_\nu$ (resp. $\exp(\mathrm{ad}\,\chi_\nu)\,\pi_\nu = \pi'_\nu + \mathcal{O}(\nu^{q+1})$). A deformation π_ν of π is called *trivial* (resp. *trivial up to order q, $q \geqslant 1$*), if π_ν is equivalent to π (resp. equivalent to π up to order q).

PROPOSITION 2.5. — *If $H_\pi^{\deg(\pi)}(V) = 0$, any formal deformation of π is trivial.*

Proof. — Let $\pi_\nu := \pi + \sum_{i=1}^\infty \nu^i \pi_i$ be a formal deformation of π. We first prove that π_ν is trivial up to order 1, then we proceed by induction. Condition (1) and the assumption imply that there exists $\chi_1 \in V^0$, such that $\pi_1 = \partial_\pi(\chi_1)$. When setting $\chi_\nu^{(1)} = \nu\chi_1$, we get $\exp(\operatorname{ad}\chi_\nu^{(1)})\pi_\nu = \pi + \mathcal{O}(\nu^2)$. Suppose now that π_ν is trivial up to order q ($q \geqslant 1$), or, equivalently, that there is a series $\chi_\nu^{(q)}$, such that $\pi'_\nu := \exp(\operatorname{ad}\chi_\nu^{(q)})\pi_\nu = \pi + \nu^{q+1}\pi'_{q+1} + \mathcal{O}(\nu^{q+2})$. As above, since π'_ν is a deformation of π, there exists $\chi'_{q+1} \in V^0$ that verifies $\pi'_{q+1} = \partial_\pi(\chi'_{q+1})$. Set now $\chi_\nu^{(q+1)} := \chi_\nu^{(q)} + \nu^{q+1}\chi'_{q+1}$ and $\pi''_\nu := \exp(\operatorname{ad}\chi_\nu^{(q+1)})\pi_\nu$. It follows from equation (2) that $\pi''_\nu - \pi'_\nu = -\nu^{q+1}\partial_\pi(\chi'_{q+1}) + \mathcal{O}(\nu^{q+2})$. Hence, $\pi''_\nu = \pi + \mathcal{O}(\nu^{q+2})$, which completes the proof. $\qquad\square$

Concerning infinitesimal deformations, it is easily seen from the above explanations that

PROPOSITION 2.6. — *Infinitesimal deformations of a canonical element π of a GLA $(V, \{-, -\})$ are classified up to first order equivalence by $H_\pi^{\deg(\pi)}(V)$.*

3. Zinbiel tensor coalgebra

Let us briefly recall some well-known facts. A *graded coalgebra* (C, Δ) is a graded vector space $C = \bigoplus_{\alpha \in \mathbb{Z}^n} C^\alpha$ together with a *coproduct* Δ, *i.e.* a linear map $\Delta \colon C \to C \otimes C$ that verifies $\Delta(C^\alpha) \subset \bigoplus_{\beta + \gamma = \alpha} C^\beta \otimes C^\gamma$. A *homomorphism* from (C, Δ) to a graded coalgebra (C', Δ') is a weight 0 linear map $\mathcal{F} \colon C \to C'$, such that $\Delta'\mathcal{F} = (\mathcal{F} \otimes \mathcal{F})\Delta$. In this text, the tensor product of linear maps is defined by $(f \otimes g)(v_1 \otimes v_2) = (-1)^{\langle g, v_1 \rangle} f(v_1) \otimes g(v_2)$, where we used the Koszul-sign rule. Further, a homogeneous *coderivation* of (C, Δ) is a linear map $Q \colon C \to C$ of weight $\deg(Q)$ that satisfies the *co-Leibniz identity* $\Delta Q = (Q \otimes \operatorname{id} + \operatorname{id} \otimes Q)\Delta$, where id is the identity map of C. Weight α coderivations form a vector space $\operatorname{CoDer}^\alpha(C)$, and the space $\operatorname{CoDer}(C) = \bigoplus_{\alpha \in \mathbb{Z}^n} \operatorname{CoDer}^\alpha(C)$ of all coderivations carries a natural $\mathbb{Z}^n$-graded Lie algebra structure provided by the graded commutator bracket.

To any $\mathbb{Z}^n$-graded vector space V, we associate the (reduced) associative tensor algebra $T(V) = \bigoplus_{p=1}^\infty V^{\otimes p}$ (the full tensor algebra includes

the term $V^{\otimes 0} = \mathbb{K}$ as well), which carries two natural gradings, the $\mathbb{Z}$-gradation $T(V) = \bigoplus_{p=1}^{\infty} T^p V$, $T^p V := V^{\otimes p}$, and the $\mathbb{Z}^n$-gradation $T(V) = \bigoplus_{\alpha \in \mathbb{Z}^n} T(V)^{\alpha}$, $T(V)^{\alpha} := \bigoplus_{p=1}^{\infty} (T^p V)^{\alpha}$, $(T^p V)^{\alpha} = \bigoplus_{\beta_1 + \ldots + \beta_p = \alpha} V^{\beta_1} \otimes \ldots \otimes V^{\beta_p}$. In the following, unless differently stated, we view $T(V)$ as $\mathbb{Z}^n$-graded vector space.

In order to define and study a coproduct on $T(V)$ that is adapted to Loday structures, we need additional notations. For any p-tuple $N^{(p)} := (1, \ldots, p)$, $p \in \mathbb{N}^*$, an *unshuffle* $I = (i_1, \ldots, i_k)$, $1 \leqslant k \leqslant p$, of $N^{(p)}$ is a naturally ordered subset of $N^{(p)}$. The length of I is denoted by $|I|$. If I and J are two nonintersecting unshuffles, we set $(I; J) = (i_1, \ldots, i_{|I|}; j_1, \ldots, j_{|J|})$, and $I \cup J$ is the unique unshuffle that coincides with $(I; J)$ as a set. Similarly, $V_{(I;J)} = (v_{i_1}, \ldots, v_{i_{|I|}}; v_{j_1}, \ldots, v_{j_{|J|}})$, $v_\ell \in V^{v_\ell}$. The sign $(-1)^{(I;J)}$ is the signature of the permutation $(I; J) \to I \cup J$, whereas $\varepsilon_V(I; J)$ is the Koszul-sign implemented by $V_{(I;J)} \to V_{I \cup J}$.

PROPOSITION 3.7. — *Let V be a $\mathbb{Z}^n$−graded vector space. The coproduct $\Delta \colon T(V) \to T(V) \bigotimes T(V)$, defined by*

(3)
$$\Delta(v_1 \otimes \ldots \otimes v_p) = \sum_{\substack{I \cup J = N^{(p-1)} \\ I \neq \emptyset}} \varepsilon_V(I; J)\, V_I \bigotimes V_J \otimes v_p \quad (v_\ell \in V^{v_\ell}, p \geqslant 1),$$

provides a Zinbiel (or graded dual Leibniz) coalgebra structure on $T(V)$, i.e. a graded coalgebra structure that verifies

(4)
$$\left(\mathrm{id} \bigotimes \Delta\right) \Delta = \left(\Delta \bigotimes \mathrm{id}\right) \Delta + \left(T \bigotimes \mathrm{id}\right) \left(\Delta \bigotimes \mathrm{id}\right) \Delta,$$

where $T \colon T(V) \bigotimes T(V) \to T(V) \bigotimes T(V)$ is the twisting map, which exchanges two elements of the $\mathbb{Z}^n$-graded space $T(V)$ modulo the corresponding Koszul-sign.

Proof. — The proposition is a consequence of the fact that the free Zinbiel algebra on V is $T(V)$ endowed with the half-shuffle product, see [27]. For a direct proof, compute the images of $v_1 \otimes \ldots \otimes v_p$ by the three terms of (4). The first (resp. second; third) one is equal to

(5)
$$\sum_{\substack{I \cup K \cup L = N^{(p-1)} \\ I, K \neq \emptyset}} \varepsilon_V(I; K; L) V_I \bigotimes V_K \bigotimes V_L \otimes v_p$$

(resp. to the same sum, but confined to those unshuffles that verify $i_{|I|} < k_{|K|}$; $k_{|K|} < i_{|I|}$, since $\varepsilon_V(K; I; L)(-1)^{\langle V_I, V_K \rangle} = \varepsilon_V(I; K; L)$). Hence, the result. $\qquad \square$

THEOREM 3.8. — *The mapping*

$$(6) \quad \psi_Q^V \colon \mathrm{CoDer}^Q(T(V)) \ni Q \to (Q_1, Q_2, \ldots) =: \sum_p Q_p \in \prod_{p \geqslant 1} M^{(Q,p-1)}(V),$$

which assigns to any (weight Q) coderivation Q its (weight Q) corestriction maps $Q_p \colon T^p V \hookrightarrow T(V) \xrightarrow{Q} T(V) \xrightarrow{\mathrm{pr}} V$, where pr denotes the canonical projection, is a vector space isomorphism, the inverse of which associates to any sequence $(Q_1, Q_2, \ldots)$ the coderivation Q that is defined by

$$(7) \quad Q(v_1 \otimes \ldots \otimes v_p) = \sum_{\substack{I \cup J \cup K = N^{(p)} \\ I, J < K}} \varepsilon_V(I; J)(-1)^{\langle Q, V_I \rangle} V_I \otimes Q_{|J|+1}$$
$$\cdot (V_J \otimes v_{k_1}) \otimes V_{K \smallsetminus k_1},$$

where $I < K$ means that $i_{|I|} < k_1$ and where $v_\ell \in V^{v_\ell}$.

Remark 3.9. — The isomorphisms ψ_Q^V, $Q \in \mathbb{Z}^n$, (for inverses, see equation (7)) induce an isomorphism ψ^V between $\mathrm{CoDer}(T(V))$ and the corresponding direct sum of direct products. Further, if we denote by $\mathrm{CoDer}_p^Q(T(V))$, $Q \in \mathbb{Z}^n, p \in \mathbb{N}^*$, the image by $(\psi_Q^V)^{-1}$ of $M^{(Q,p-1)}(V)$, isomorphism ψ_Q^V restricts to an isomorphism $\psi_{(Q,p)}^V$ between these spaces. If no confusion arises, we write ψ instead of ψ^V, ψ_Q^V, or $\psi_{(Q,p)}^V$.

Proof. — Since the Zinbiel coalgebra is cofree, its coderivations are completely determined by their corestrictions. It thus suffices to show that equation (7) actually defines a coderivation. Although the coderivation condition is quite easily checked for $p \leqslant 3$, the general proof is a little technical: It will not be reproduced here, but can be found in [2]. □

Remark 3.10. — Many upshots of this note are valid in any linear symmetric monoidal category. However, to facilitate following comparisons with results of other papers, we continue working in the category of $\mathbb{Z}^n$-graded vector spaces.

Like coderivations, cohomomorphisms from $(T(V), \Delta)$ to $(T(V'), \Delta)$ are characterized by their corestriction maps.

Remark 3.11. — Let V and V' be two $\mathbb{Z}^n$-graded vector spaces. A coalgebra cohomomorphism $\mathcal{F} \colon (T(V), \Delta) \longrightarrow (T(V'), \Delta)$ is uniquely defined by its (weight 0) corestriction maps $\mathcal{F}_p \colon T^p V \to V'$, $p \geqslant 1$, via the equation

$$(8) \quad \mathcal{F}(v_1 \otimes \cdots \otimes v_p) = \sum_{s=1}^{p} \sum_{\substack{I^1 \cup \cdots \cup I^s = N^{(p)} \\ I^1, \ldots, I^s \neq \emptyset \\ i^1_{|I^1|} < \cdots < i^s_{|I^s|}}} \varepsilon_V(I^1; \cdots; I^s) \mathcal{F}_{|I^1|}$$
$$\cdot (V_{I^1}) \otimes \cdots \otimes \mathcal{F}_{|I^s|}(V_{I^s}),$$

where $v_\ell \in V^{v_\ell}$.

Set now $e_1 = (1, 0, \ldots, 0) \in \mathbb{Z}^n$ and consider the desuspension operator $\downarrow : V \to \downarrow V$, where $\downarrow V$ is the same space as V up to the shift $(\downarrow V)^\alpha = V^{\alpha + e_1}$ of gradation. The inverse map of $\downarrow$ is denoted by $\uparrow$. The mapping $\downarrow^{\otimes p} := \downarrow \otimes \ldots \otimes \downarrow$, p factors, *i.e.* the mapping

$$(9) \quad \downarrow^{\otimes p} : V^{\otimes p} \ni v_1 \otimes \cdots \otimes v_p \to (-1)^{\sum_{s=1}^{p} \langle (p-s)e_1, v_s \rangle} \downarrow v_1 \otimes \cdots$$
$$\otimes \downarrow v_p \in (\downarrow V)^{\otimes p},$$

is an isomorphism of weight $-p\, e_1$, whose inverse is $(-1)^{p(p-1)/2} \uparrow^{\otimes p}$.

Remark 3.12. — The isomorphisms

$$(10) \quad \sigma^{\downarrow V}_{(Q,p)} : M^{(Q,p-1)}(\downarrow V) \ni Q_p \to \pi_p$$
$$:= \uparrow \circ Q_p \circ \downarrow^{\otimes p} \in M^{(Q+(1-p)e_1, p-1)}(V), \quad Q \in \mathbb{Z}^n, \ p \in \mathbb{N}^*,$$

(their inverses are defined by $(-1)^{p(p-1)/2} \downarrow \circ \pi_p \circ \uparrow^{\otimes p}$) generate isomorphisms $\sigma^{\downarrow V}_Q$ and $\sigma^{\downarrow V}$ between the corresponding direct products and direct sums of direct products. If no confusion is possible, we omit super- and subscripts and denote these isomorphisms simply by σ. Isomorphisms (10) extend of course to multilinear maps on $\downarrow V$ valued in $\downarrow V'$.

Remark 3.13. — Theorem 3.8 and Remarks 3.9, 3.11 and 3.12 show that weight Q coderivations $Q : (T(\downarrow V), \Delta) \to (T(\downarrow V), \Delta)$ can be viewed as sequences $\pi = (\pi_1, \pi_2, \ldots) = \sum_p \pi_p$ of weight $Q + (1-p)e_1$ multilinear maps $\pi_p : V^{\times p} \to V$, and that cohomomorphisms $\mathcal{F} : (T(\downarrow V), \Delta) \to (T(\downarrow V'), \Delta)$ (which by definition have weight 0) can be seen as sequences $f = (f_1, f_2, \ldots) = \sum_p f_p$ of weight $(1-p)e_1$ multilinear maps $f_p : V^{\times p} \to V'$.

4. Stem bracket

When combining the isomorphisms σ^{-1} and ψ^{-1}, we get, for $A \in \mathbb{Z}^n$, $a \in \mathbb{N}$, a vector space isomorphism

(11) $\phi_{(A,a)} \colon M^{(A,a)}(V) \ni A \to Q^A = (0,\dots,0,Q^A_{a+1},0,\dots)$

$$\in \mathrm{CoDer}^{A+ae_1}_{a+1}(T(\downarrow V)),$$

where $Q^A_{a+1} = (-1)^{a(a+1)/2} \downarrow \circ A \circ \uparrow^{\otimes(a+1)}$ and where Q^A is the coderivation that is obtained from Q^A_{a+1} via extension equation (7).

Let now $M_r(V)$ be the vector space $M(V)$, except for the terms V^B, $B \in \mathbb{Z}^n$, which are omitted.

THEOREM 4.14. — *The $\mathbb{Z}^{n+1}$-graded vector space $M_r(V)$ is a $\mathbb{Z}^{n+1}$-graded Lie algebra, when endowed with the bracket*

(12) $[A,B]^\otimes := (-1)^{1+\langle ae_1, be_1+B \rangle} \phi^{-1}_{(A+B,a+b)}\Big(\big[\phi_{(A,a)}(A), \phi_{(B,b)}(B) \big] \Big),$

$A \in M^{(A,a)}(V)$, $B \in M^{(B,b)}(V)$, where $[-,-]$ denotes the $\mathbb{Z}^n$-graded Lie bracket of the space of coderivations of $(T(\downarrow V), \Delta)$.

Proof. — It follows from equation (7) that the p-th corestriction map $[\phi_{(A,a)}(A), \phi_{(B,b)}(B)]_p$ vanishes if $p \neq a+b+1$, so that the bracket $[\phi_{(A,a)}(A), \phi_{(B,b)}(B)]$ is really a coderivation in $\mathrm{CoDer}^{A+B+(a+b)e_1}_{a+b+1}(T(\downarrow V))$. The sign $(-1)^{1+\langle ae_1, be_1+B \rangle}$ ensures that the $\mathbb{Z}^n$-graded Lie bracket of coderivations induces a $\mathbb{Z}^{n+1}$-graded Lie bracket $[-,-]^\otimes$. □

As the map $\phi = \psi^{-1} \circ \sigma^{-1}$ is also a (weight 0) $\mathbb{Z}^n$-graded vector space isomorphism

(13) $\phi \colon C(V) := \bigoplus_{Q \in \mathbb{Z}^n} \prod_{p \geqslant 1} M^{(Q+(1-p)e_1, p-1)}(V) \to \mathrm{CoDer}(T(\downarrow V))$

$$= \bigoplus_{Q \in \mathbb{Z}^n} \mathrm{CoDer}^Q(T(\downarrow V)),$$

the next proposition is obvious.

PROPOSITION 4.15. — *The $\mathbb{Z}^n$-graded vector space $C(V)$ is a $\mathbb{Z}^n$-graded Lie algebra for the bracket*

(14) $[\pi, \rho]^{\bar\otimes} = \phi^{-1}[\phi\pi, \phi\rho] = \sum_{q \geqslant 1} \sum_{s+t=q+1} (-1)^{1+(s-1)\langle e_1, \rho \rangle} [\pi_s, \rho_t]^\otimes,$

where $\pi = \sum_s \pi_s \in C^\pi(V)$ and $\rho = \sum_t \rho_t \in C^\rho(V)$ are two homogeneous $C(V)$-elements of $\mathbb{Z}^n$-degree π and ρ respectively.

Remark 4.16. — In the following, we refer to $[-,-]^{\bar\otimes}$ (resp. $[-,-]^\otimes$) as the $\mathbb{Z}^n$-graded (resp. $\mathbb{Z}^{n+1}$-graded) stem bracket.

THEOREM 4.17. — *The $\mathbb{Z}^{n+1}$-graded stem bracket on $M_r(V)$ explicitly reads*

$$(15) \qquad [A,B]^{\otimes} = j_A B - (-1)^{\langle (A,a),(B,b)\rangle} j_B A,$$

where

$$(16) \quad (j_B A)(v_1 \otimes \cdots \otimes v_{a+b+1}) = (-1)^{\langle A,B\rangle} \sum_{\substack{I\cup J\cup K = N^{(a+b+1)} \\ I,J<K,\ |J|=b}}$$

$$\cdot (-1)^{\langle B,V_I\rangle + b|I|}(-1)^{(I;J)}\varepsilon_V(I;J)A(V_I \otimes B(V_J \otimes v_{k_1}) \otimes V_{K\smallsetminus k_1}),$$

for any $A \in M^{(A,a)}(V)$, $B \in M^{(B,b)}(V)$, $a,b \geqslant 0$, and $v_\ell \in V^{v_\ell}$.

Proof. — It follows from equation (7) that the $(a+b+1)$-th corestriction of $[\phi_{(A,a)}(A), \phi_{(B,b)}(B)] = [Q^A, Q^B]$ is obtained by just restricting the involved composite maps to $(\downarrow V)^{\otimes(a+b+1)}$. The description of the isomorphisms $\phi^{-1}_{(A,a)}$ then shows that

$$(17) \quad [A,B]^{\otimes} = -(-1)^{\langle ae_1, be_1 + B\rangle}\left(\uparrow \circ Q^A \circ Q^B \circ \downarrow^{\otimes(a+b+1)}\right.$$

$$\left. -(-1)^{\langle A+ae_1, B+be_1\rangle} \uparrow \circ Q^B \circ Q^A \circ \downarrow^{\otimes(a+b+1)}\right).$$

If $V_{N^{(a+b+1)}} = v_1 \otimes \cdots \otimes v_{a+b+1}$, with $v_\ell \in V^{v_\ell}$, we get

$$\left(\uparrow \circ Q^A \circ Q^B \circ \downarrow^{\otimes(a+b+1)}\right)(V_{N^{(a+b+1)}}) = (-1)^{\beta_1}\left(\uparrow \circ Q^A \circ Q^B\right)(\downarrow V_{N^{(a+b+1)}}),$$

$$\beta_1 = \langle e_1, \sum_{s\geqslant 1}(a+b+1-s)v_s\rangle,$$

where $\downarrow V_{N^{(a+b+1)}} = \downarrow v_1 \otimes \cdots \otimes \downarrow v_{a+b+1}$. Formula (7) yields

$$(18) \quad Q^B(\downarrow V_{N^{(a+b+1)}}) = \sum_{\substack{I\cup J\cup K = N^{(a+b+1)} \\ I,J<K,\ |J|=b}} (-1)^{\beta_2}\varepsilon_{\downarrow V}(I;J)\downarrow V_I \otimes Q^B_{b+1}$$

$$\cdot (\downarrow V_J \otimes \downarrow v_{k_1}) \otimes \downarrow V_{K\smallsetminus k_1},$$

$$\beta_2 = \Big\langle B + be_1, V_I + |I|e_1 \Big\rangle.$$

Moreover,

$$Q^B_{b+1}(\downarrow V_J \otimes \downarrow v_{k_1}) = (-1)^{\beta_3} \downarrow B(V_J \otimes v_{k_1}), \beta_3$$

$$= \Big\langle e_1, \sum_{s\geqslant 1}(b+1-s)v_{j_s}\Big\rangle,$$

as the sign $(-1)^{b(b+1)/2}$ inside Q^B_{b+1} and the sign due to the shift of the $\mathbb{Z}^n$-gradation cancel each other out. When noticing that Q^A evaluated on

an element of $(\downarrow V)^{\otimes(a+1)}$ is nothing but Q_{a+1}^A, we obtain

$$(19) \quad \left(\uparrow \circ Q^A \circ Q^B \circ \downarrow^{\otimes(a+b+1)}\right)(V_{N^{(a+b+1)}})$$

$$= \sum_{\substack{I \cup J \cup K = N^{(a+b+1)}I, \\ J < K, \ |J| = b}} (-1)^\ell \varepsilon_{\downarrow V}(I;J) A\left(V_I \otimes B(V_J \otimes v_{k_1}) \otimes V_{K \setminus k_1}\right),$$

with $\ell = \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6$, where

$$\beta_4 = \left\langle e_1, \sum_{s \geqslant 1} (a+1-s) v_{i_s} \right\rangle, \ \beta_5 = (a - |I|) \langle e_1, B + V_J + v_{k_1} \rangle,$$

$$\text{and } \beta_6 = \left\langle e_1, \sum_{s \geqslant 2} (a - |I| - s + 1) v_{k_s} \right\rangle$$

are generated by $\uparrow^{\otimes(a+1)}$ and where, again, the sign inside Q_{a+1}^A and the sign due to the shift neutralize.

We will prove that

$$(20) \quad (-1)^\ell \varepsilon_{\downarrow V}(I;J) = (-1)^{\langle B, ae_1 \rangle + \langle B, V_I \rangle + b|I|}(-1)^{(I;J)} \varepsilon_V(I;J).$$

Observe first that an appropriate regrouping of terms yields

$$\ell = \ell_1 + \ell_2 := (\langle B, ae_1 \rangle + \langle B, V_I \rangle + b|I|) + \left(\left\langle e_1, \sum_{s=1}^{a+b+1} (a+b+1-s) v_s \right\rangle\right.$$

$$+ \left\langle e_1, \sum_{s=1}^{|I|} (a+b+1-s) v_{i_s} \right\rangle + \left\langle e_1, \sum_{s=|I|+1}^{|I|+|J|} (a+b+1-s) v_{j_{s-|I|}} \right\rangle$$

$$+ \left.\left\langle e_1, \sum_{s=|I|+|J|+1}^{a+b+1} (a+b+1-s) v_{k_{s-|I|-|J|}} \right\rangle\right),$$

where, since in view of the conditions $I, J < K$ the concatenation $(I;J)$ is a permutation of $1, \ldots, |I| + |J|$, the term ℓ_2 reads (modulo even terms)

$$\ell_2 = \left\langle e_1, \sum_{s=1}^{|I|} (s + i_s) v_{i_s} \right\rangle + \left\langle e_1, \sum_{s=|I|+1}^{|I|+|J|} (s + j_{s-|I|}) v_{j_{s-|I|}} \right\rangle.$$

If permutation $(I;J)$ is a transposition $(I;J) = (1, \ldots, q-1, q+1, q, q+2, \ldots, |I| + |J|)$, then $\ell_2 = \langle e_1, v_q + v_{q+1} \rangle$. It is now easily checked that for a transposition

$$(21) \quad (-1)^{\ell_2} \varepsilon_{\downarrow V}(I;J) = (-1)^{(I;J)} \varepsilon_V(I;J)$$

and that for a composition of transpositions all the factors of this last equation are the products of the corresponding factors induced by the involved

transpositions. It follows that equations (21) and (20) hold true for any permutation $(I; J)$.

Eventually equation (19) may be written

$$\left(\uparrow \circ Q^A \circ Q^B \circ \downarrow^{\otimes(a+b+1)}\right)\left(V_{N^{(a+b+1)}}\right)$$
$$= \sum_{\substack{I \cup J \cup K = N^{(a+b+1)} \\ I,J<K, \; |J|=b}} (-1)^m (-1)^{(I;J)} \varepsilon_V(I; J) A\left(V_I \otimes B(V_J \otimes v_{k_1}) \otimes V_{K \smallsetminus k_1}\right),$$

where $m = \langle B, ae_1 \rangle + \langle B, V_I \rangle + b|I|$.

If we define the insertion operator

$$j_B A = (-1)^{\langle ae_1 + A, B \rangle} \uparrow \circ Q^A \circ Q^B \circ \downarrow^{\otimes(a+b+1)},$$

$A \in M^{(A,a)}(V)$, $B \in M^{(B,b)}(V)$, $a, b \geqslant 0$, we finally get the announced result. $\qquad\square$

Remark 4.18. —   It is easily checked that the restriction of the stem bracket $[-,-]^\otimes$ to the subspace $A_r(V)$ of $M_r(V)$, made up by graded skew-symmetric multilinear maps on V, coincides with the graded Nijenhuis-Richardson bracket $[-,-]^{\mathrm{NR}}$ that was introduced in [20].

5. Graded and strongly homotopy Loday structures

Let us recall that a *codifferential* of a coalgebra is a coderivation that squares to 0.

PROPOSITION 5.19. — *A homogenous odd weight coderivation Q of the coalgebra $(T(V), \Delta)$ is a codifferential if and only if, for any $p \geqslant 1$, the following equation holds identically:*
(22)
$$\sum_{\substack{I \cup J \cup K = N^{(p)} \\ I,J<K}} \varepsilon_V(I; J)(-1)^{\langle Q, V_I \rangle} Q_{|I|+|K|}\left(V_I \otimes Q_{|J|+1}(V_J \otimes v_{k_1}) \otimes V_{K \smallsetminus k_1}\right) = 0.$$

Proof. — As Q is an odd weight coderivation, $2Q^2 = [Q, Q]$ and so Q^2 is also a coderivation. Thus, according to Theorem 3.8, the condition $Q^2 = 0$ is satisfied if and only if the corestriction maps Q_p^2, $p \geqslant 1$, of Q^2 vanish. It is easily seen that $Q_p^2(V_{N^{(p)}})$ is exactly the LHS of (22). $\qquad\square$

DEFINITION 5.20. — *A graded Loday algebra $(V, \{-,-\})$ (GLodA for short) is made up by a $\mathbb{Z}^n$-graded vector space V and a weight 0 bilinear bracket $\{-,-\}$ that satisfies the graded Jacobi identity*
(23)
$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{\langle a, b \rangle}\{b, \{a, c\}\},$$

for any homogeneous $a, b, c \in V$.

Nongraded Loday algebras were introduced by Loday in [26] and are also known as *Leibniz algebras*. For further information on graded Loday algebras, we refer the reader to [18] or [1].

THEOREM 5.21. — *Let* $\mathrm{Lod}(V)$ *be the set of* $\mathbb{Z}^n$-*graded Loday structures on a* $\mathbb{Z}^n$-*graded vector space* V, *and denote by* $\mathrm{CoDiff}_p^Q(T(\downarrow V))$, $Q \in \mathbb{Z}^n$, $p \in \mathbb{N}^*$, *the set of codifferentials* Q *of* $(T(\downarrow V), \Delta)$, *which have weight* Q *and whose corestriction maps all vanish except* Q_p. *Then,*

1. *The restriction of* $\phi_{(0,1)}$ *to* $\mathrm{Lod}(V)$ *is a bijection and*

(24) $$\mathrm{Lod}(V) \simeq \mathrm{CoDiff}_2^{e_1}(T(\downarrow V)),$$

2. *The graded Loday structures on* V *are exactly the canonical elements of weight* $(0, 1)$ *of the graded Lie algebra* $(M_r(V), [-, -]^\otimes)$.

Proof. — Since $\phi_{(0,1)}$ is a bijection between $M^{(0,1)}(V)$ and $\mathrm{CoDer}_2^{e_1}(T(\downarrow V))$, it suffices, in order to account for point 1, to prove that

$$\phi_{(0,1)}(\mathrm{Lod}(V)) = \mathrm{CoDiff}_2^{e_1}(T(\downarrow V)).$$

If $\pi \in \mathrm{Lod}(V)$, its image $\phi_{(0,1)}(\pi) = Q^\pi = (0, Q_2^\pi, 0, \ldots) \in \mathrm{CoDer}_2^{e_1}(T(\downarrow V))$ is a codifferential, if

(25) $$Q_2^\pi(Q_2^\pi(\downarrow v_1 \otimes \downarrow v_2) \otimes \downarrow v_3) + (-1)^{\langle e_1, \downarrow v_1 \rangle} Q_2^\pi(\downarrow v_1 \otimes Q_2^\pi(\downarrow v_2 \otimes \downarrow v_3))$$
$$+ (-1)^{\langle e_1 + \downarrow v_1, \downarrow v_2 \rangle} Q_2^\pi(\downarrow v_2 \otimes Q_2^\pi(\downarrow v_1 \otimes \downarrow v_3)) = 0,$$

see Proposition 5.19 and note that condition (22) is trivial for $p \neq 3$. When remembering that $Q_2^\pi = - \downarrow \circ \pi \circ (\uparrow \otimes \uparrow)$, we easily check that condition (25) reads

$$(-1)^{\langle v_2, e_1 \rangle} \downarrow [\pi(\pi(v_1 \otimes v_2) \otimes v_3) - \pi(v_1 \otimes \pi(v_2 \otimes v_3))$$
$$+ (-1)^{\langle v_1, v_2 \rangle} \pi(v_2 \otimes \pi(v_1 \otimes v_3))] = 0.$$

As π verifies the graded Jacobi identity, the last requirement is fulfilled.

Conversely, if $Q = (0, Q_2, 0, \ldots) \in \mathrm{CoDiff}_2^{e_1}(T(\downarrow V))$, then $\pi := \phi_{(0,1)}^{-1}(Q)$ is a graded Loday structure.

As regards point 2, note that any graded Loday structure π is odd. Furthermore,

$$[\pi, \pi]^\otimes = -(-1)^{\langle e_1, e_1 \rangle} \phi_{(0,2)}^{-1}([\phi_{(0,1)}(\pi), \phi_{(0,1)}(\pi)]) = 2\phi_{(0,2)}^{-1}(\phi_{(0,1)}^2(\pi)),$$

so that the graded Loday structures are exactly the canonical elements of weight $(0, 1)$. $\qquad\square$

There are two ways to make an algebraic structure more flexible, categorification and homotopyfication. In the sequel, we extend graded Loday structures, see equation (24), and investigate the homotopy version of Loday algebras.

DEFINITION 5.22. — *A strongly homotopy Loday algebra or a Loday infinity algebra is a $\mathbb{Z}^n$-graded vector space V endowed with a codifferential of odd weight of the tensor coalgebra $(T(\downarrow V), \Delta)$.*

We denote by $\text{Lod}_\infty^Q(V)$, $Q \in \mathbb{Z}^n$, $\langle Q, Q \rangle$ odd, the set

$$\text{Lod}_\infty^Q(V) \simeq \text{CoDiff}^Q(T(\downarrow V))$$

of weight Q Loday infinity (Lod_∞^Q for short) structures on V, whereas $\text{Lod}_\infty(V)$ denotes the set $\text{Lod}_\infty^{e_1}(V)$ — as most infinity structures considered below have weight e_1. Since Q is odd, $Q \in \text{Lod}_\infty^Q(V)$ if and only if $Q \in \text{CoDer}^Q(T(\downarrow V))$ and $[Q, Q] = 2Q^2 = 0$. Hence, in view of Remark 3.13 and Proposition 4.15, the sequence "definition" of Loday infinity algebras:

PROPOSITION 5.23. — *A Loday infinity algebra is a $\mathbb{Z}^n$-graded vector space V together with a sequence of structure maps*

$$\pi = (\pi_1, \pi_2, \dots) = \sum_p \pi_p \in C^Q(V) = \prod_{p \geqslant 1} M^{(Q+(1-p)e_1, p-1)}(V)$$

of odd degree Q, such that

$$(26) \qquad \sum_{s+t=p} (-1)^{1+(s-1)\langle e_1, Q \rangle} [\pi_s, \pi_t]^\otimes = 0, \ \forall p \geqslant 2.$$

In the usual case of Lod_∞ structures π on V, the first three conditions (26) mean that (V, π_1) is a chain complex, that π_1 is a $\mathbb{Z}^n$-graded derivation of the bilinear map π_2, and that π_2 is a $\mathbb{Z}^n$-graded Loday structure modulo homotopy π_3.

Example 5.24. — If the structure maps of a Lod_∞ algebra (V, π) all vanish, except π_1 (resp. except π_2, except π_1 and π_2), (V, π) is a chain complex (resp. a $\mathbb{Z}^n$-graded Loday algebra (GLodA), a differential graded Loday algebra($DGLodA$)).

Example 5.25. — Let (V, π) and (V', π') be two Lod_∞ algebras. Their direct sum $(V \oplus V', \pi \oplus \pi')$ is a Lod_∞ algebra, where the structure maps are defined by

$$(\pi \oplus \pi')_p(v_1 + v'_1, \dots, v_p + v'_p) := \pi_p(v_1, \dots, v_p) + \pi'_p(v'_1, \dots, v'_p),$$

for any $p \geqslant 1$.

It is obvious from the structure of the explicit form of the stem bracket $[-, -]^{\otimes}$, see Theorem 4.17, that if π and π' verify the Lod$_\infty$ conditions (26) in V and V' respectively, then $\pi \oplus \pi'$ verifies the same conditions in $V \oplus V'$.

As a Lod$_\infty$ structure on V is a weight e_1 codifferential Q of the coalgebra $(T(\downarrow V), \Delta)$, it is natural to give the following coalgebraic

DEFINITION 5.26. — *Let (V, Q) and (V', Q') be two Lod$_\infty$ algebras. A Lod$_\infty$ morphism from (V, Q) to (V', Q') is a coalgebra cohomomorphism $\mathcal{F}: (T(\downarrow V), \Delta) \longrightarrow (T(\downarrow V'), \Delta)$, such that $Q'\mathcal{F} = \mathcal{F}Q$.*

Since the composition of Lod$_\infty$ morphisms is again a Lod$_\infty$ morphism, and as, for any Lod$_\infty$ algebra, the identity map is a morphism (say $\mathcal{I}$, with corestriction maps $\mathcal{I}_1 = \text{id}$ and $\mathcal{I}_p = 0$, for $p \geqslant 2$), Lod$_\infty$ algebras and their morphisms form a category, which we denote by **Lod$_\infty$**.

Next we give the sequence "definition" of Lod$_\infty$ morphisms, see Remark 3.13.

PROPOSITION 5.27. — *Let (V, π) and (V', π') be two Lod$_\infty$ algebras. A Lod$_\infty$ morphism $f: (V, \pi) \to (V', \pi')$ is a sequence of weight $(1-p)e_1$ multilinear maps $f_p: V^{\times p} \to V'$, $p \geqslant 1$, which satisfy the condition*

$$(27) \quad \sum_{s=1}^{p} \sum_{\substack{I^1 \cup \cdots \cup I^s = N^{(p)} \\ I^1, \ldots, I^s \neq \emptyset \\ i^1_{|I^1|} < \cdots < i^s_{|I^s|}}} (-1)^\omega (-1)^{(I^1; \cdots ; I^s)} \varepsilon_V(I^1; \cdots ; I^s)$$

$$\cdot \pi'_s \left(f_{|I^1|}(V_{I^1}), \ldots, f_{|I^s|}(V_{I^s}) \right)$$

$$= \sum_{\substack{I \cup J \cup K = N^{(p)} \\ I, J < K}} (-1)^\lambda (-1)^{(J;I)} \varepsilon_V(I; J) f_{|I|+|K|} \left(V_I, \pi_{|J|+1}(V_J, v_{k_1}), V_{K \smallsetminus k_1} \right),$$

where

$$(28) \quad \omega = \frac{s(s-1)}{2} + \sum_{1 \leqslant r \leqslant s} (s-r)|I^r| + \sum_{2 \leqslant r \leqslant s} \left\langle (|I^r|+1)e_1, V_{|I^1|} + \cdots + V_{|I^{r-1}|} \right\rangle$$

and

$$(29) \quad \lambda = \left\langle (1 + |J|)e_1, V_I + (p+1)e_1 \right\rangle,$$

for all $p \geqslant 1$.

Proof. — When using (7) and (8), we see that $Q'\mathcal{F}$ and $\mathcal{F}Q$ are similarly composed of the respective corestriction maps, so that they coincide, if their corestrictions do. This leads to a reformulation of the intertwining condition $Q'\mathcal{F} = \mathcal{F}Q$ in terms of $Q'_p, \mathcal{F}_p, Q_p$, $p \geqslant 1$. The translation of

this reformulation by means of the sequences π', f, π, and in particular the above signs, are obtained by a direct computation that is based upon similar arguments than those used in the proof of Theorem 4.17. We refrain from providing the details. $\qquad\square$

Remark 5.28. — The first constraint (27) states that f_1 is a chain map between (V, π_1) and (V', π'_1), whereas the second means that f_2 measures the deviation from f_1 being a $(V, \pi_2)-(V', \pi'_2)$ homomorphism. If (V, π) and (V', π') are DGLodAs, map f_1 is a DGLodA morphism. It can be shown that the category $\mathbf{L}_\infty$ of L_∞ algebras and morphisms is a subcategory of $\mathbf{Lod}_\infty$.

DEFINITION 5.29. — A Lod_∞ quasi-isomorphism from a Lod_∞ algebra (V, π) to a Lod_∞ algebra (V', π') is a Lod_∞ morphism $f \colon (V, \pi) \to (V', \pi')$, such that the chain map $f_1 \colon (V, \pi_1) \to (V', \pi'_1)$ induces an isomorphism $f_{1\,\sharp} \colon H(V, \pi_1) \to H(V', \pi'_1)$ in cohomology. In particular, f is called a Lod_∞ isomorphism, if $f_1 \colon V \to V'$ is an isomorphism.

If $\mathcal{F} \simeq f$ and $\mathcal{G} \simeq g$ are two composable Lod_∞ morphisms, we denote by $g \circ f$ the sequence of multilinear maps that corresponds to the Lod_∞ morphism $\mathcal{GF}$. Similarly, $\pi' \circ f$ and $f \circ \pi$ are the sequences that represent $Q'\mathcal{F}$ and $\mathcal{F}Q$. The Lod_∞ morphism condition (27) then reads $\pi' \circ f = f \circ \pi$. We use these and analogous notations below.

The next upshots will be needed in the following.

PROPOSITION 5.30.

1. Any coalgebra cohomomorphism $f \colon (T(\downarrow V), \Delta) \to (T(\downarrow V'), \Delta)$, which corresponds to a sequence $f = (f_1, f_2, \ldots)$, whose first element f_1 is bijective, is invertible, i.e. there is a coalgebra cohomomorphism $f^{-1} \colon (T(\downarrow V'), \Delta) \to (T(\downarrow V), \Delta)$, such that $f \circ f^{-1} = \mathrm{id}$ and $f^{-1} \circ f = \mathrm{id}$, where id is the unit cohomomorphism $\mathrm{id} = (\mathrm{id}, 0, 0, \ldots)$.

2. If (V, π) denotes a Lod_∞ algebra, any sequence $f = (f_1, f_2, \ldots)$ of weight $(1 - p)e_1$ multilinear maps $f_p \colon V^{\times p} \to V$, whose first element f_1 is the identity map of V, induces a new Lod_∞ structure $f \circ \pi \circ f^{-1}$ on V and f is a Lod_∞ isomorphism between (V, π) and $(V, f \circ \pi \circ f^{-1})$.

3. Any Lod_∞ isomorphism $f \colon (V, \pi) \to (V', \pi')$ admits an inverse f^{-1} that is a Lod_∞ isomorphism as well.

Proof.

1. Let $\mathcal{F} \colon (T(\downarrow V), \Delta) \to (T(\downarrow V'), \Delta)$ be a coalgebra cohomomorphism, whose first corestriction $\mathcal{F}_1 \colon \downarrow V \to \downarrow V'$ is bijective. If there is an inverse cohomomorphism $\mathcal{G} \colon (T(\downarrow V'), \Delta) \to (T(\downarrow V), \Delta)$, it follows from the

condition $\mathcal{F}\mathcal{G} = \mathcal{I}$ and from equation (8) that $\mathcal{G}_1 = \mathcal{F}_1^{-1}$ and that, for any $p \geqslant 2$,

$$\mathcal{G}_p(\downarrow v_1', \ldots, \downarrow v_p') = -\sum_{s=2}^{p} \sum_{\substack{I^1\cup\cdots\cup I^s = N^{(p)} \\ I^1,\ldots,I^s \neq \emptyset \\ i^1_{|I^1|} < \cdots < i^s_{|I^s|}}}$$

$$\cdot\, \varepsilon_{\downarrow V'}(I^1; \cdots; I^s)\mathcal{F}_1^{-1}\mathcal{F}_s\Big(\mathcal{G}_{|I^1|}(\downarrow V_{I^1}') \otimes \cdots \otimes \mathcal{G}_{|I^s|}(\downarrow V_{I^s}')\Big).$$

The last equation provides inductively the corestriction maps of a cohomomorphism $\mathcal{G}$. One can check that $\mathcal{G}$ not only verifies $\mathcal{F}\mathcal{G} = \mathcal{I}$, but also $\mathcal{G}\mathcal{F} = \mathcal{I}$.

2. Take a Lod_∞ structure Q on V and a cohomomorphism $\mathcal{F} \colon (T(\downarrow V), \Delta) \to (T(\downarrow V), \Delta)$, such that $\mathcal{F}_1 = \mathrm{id}$. Since $(f \otimes g) \circ (h \otimes k) = (-1)^{\langle g, h\rangle}(f \circ h) \otimes (g \circ k)$, with self-explaining notations, it is easily seen that $\mathcal{F}Q\mathcal{F}^{-1}$, where $\mathcal{F}^{-1}$ is the inverse cohomomorphism $\mathcal{G}$ given by Item 1, is a weight e_1 codifferential of $T(\downarrow V)$, i.e. a Lod_∞ structure on V. Eventually, $\mathcal{F}$ is obviously a Lod_∞ morphism, and, in view of the assumption $\mathcal{F}_1 = \mathrm{id}$, even a Lod_∞ isomorphism.

3. Consider two Lod_∞ algebras (V, Q), (V', Q') and a Lod_∞ isomorphism $\mathcal{F}$, i.e. a cohomomorphism, such that $\mathcal{F}_1$ is bijective and $Q'\mathcal{F} = \mathcal{F}Q$ $(\star)$. It then follows from Item 1 that there is an inverse cohomomorphism $\mathcal{F}^{-1}$, such that $(\mathcal{F}^{-1})_1 = (\mathcal{F}_1)^{-1}$, and from equation $(\star)$ that $Q\mathcal{F}^{-1} = \mathcal{F}^{-1}Q'$. $\square$

The following key-theorem generalizes the last item of Proposition 5.30.

THEOREM 5.31. — *If $f \colon (V, \pi) \to (V', \pi')$ is a Lod_∞ quasi-isomorphism, it admits a quasi-inverse, i.e. there exists a Lod_∞ quasi-isomorphism $g \colon (V', \pi') \to (V, \pi)$, which induces the inverse isomorphism in cohomology, i.e. $g_{1\sharp} = (f_{1\sharp})^{-1}$.*

We prove this theorem, which does not hold true in the category of DGLodAs, in the next section.

6. Minimal model theorem for Loday infinity algebras

DEFINITION 6.32. — *A Lod_∞ algebra (V, π) is minimal, if $\pi_1 = 0$. It is contractible, if $\pi_p = 0$, for $p \geqslant 2$, and if in addition $H(V, \pi_1) = 0$.*

THEOREM 6.33. — *Each Lod_∞ algebra is Lod_∞ isomorphic to the direct sum of a minimal Lod_∞ algebra and a contractible Lod_∞ algebra.*

Theorem 6.33 was proved for A_∞ (resp. L_∞) algebras in [16] (resp. in [17] and e.g. [4]). In the sequel, we provide a proof in the Lod_∞ case.

Proof. — Let (V, π) be a Lod_∞ algebra. For any $\alpha \in \mathbb{Z}^n$, denote by Z^α and B^α the trace on V^α of the kernel and the image of π_1. Consider a supplementary vector subspace V_m^α of B^α in Z^α and a supplementary subspace W^α of Z^α in V^α. Let Z, B, V_m, and W be the corresponding graded spaces. Then, the complex (V, π_1) decomposes into the direct sum of the complex $(V_m, 0)$, with vanishing differential, and the complex $(V_c := B \oplus W, \pi_1)$, with trivial cohomology. It follows that the sequence $f^{(1)} := (id, 0, \ldots)$ is a Lod_∞ isomorphism from (V, π) to the Lod_∞ algebra $L_1 := (V_m \oplus V_c, 0 \oplus \pi_1, \pi_2, \pi_3, \ldots)$. We will transform inductively the maps π_p, $p \geqslant 2$, via Lod_∞ isomorphisms, into mappings of the form $\pi_p^m \oplus 0$, such that $\pi^m := (0, \pi_2^m, \pi_3^m, \ldots)$ be a minimal Lod_∞ structure on V_m.

LEMMA 6.34. — *Consider the operator $\delta \colon V \to V$ that is defined, for any $v \in V_m \oplus W$, by $\delta(v) = 0$, and, for any $v \in B$, by $\delta(v) = w$, where w is the unique element $w \in W$, such that $\pi_1(w) = v$. Let P be the projection onto V_m with respect to the decomposition $V = V_m \oplus V_c$. Then, δ is a homotopy operator between the complex endomorphisms P and id of (V, π_1), i.e. $\pi_1 \delta + \delta \pi_1 = id - P$.*

Proof. — Obvious.   □

Let us construct π_2^m. Consider a sequence $f^{(2)} := (id, f_2, 0, 0, \ldots)$, where f_2 is a weight $-e_1$ bilinear map on V. According to item 2 of Proposition 5.30, $f^{(2)}$ defines a Lod_∞ isomorphism

$$L_1 \to (V_m \oplus V_c, \pi_1^{(2)}, \pi_2^{(2)}, \pi_3^{(2)}, \ldots) =: (V_m \oplus V_c, \pi^{(2)}),$$

and $\pi^{(2)}$ is a Lod_∞ structure on $V_m \oplus V_c$, if and only if $\pi^{(2)} \circ f^{(2)} = f^{(2)} \circ \pi$. In view of equation (27), this condition implies that (take $p = 1$) $\pi_1^{(2)} = \pi_1 = 0 \oplus \pi_1$, that (take $p = 2$), for $v_1 \in V^{v_1}, v_2 \in V$,

$$(30) \quad \pi_2^{(2)}(v_1, v_2) = -\pi_1 f_2(v_1, v_2) + \pi_2(v_1, v_2) - f_2(\pi_1 v_1, v_2)$$
$$- (-1)^{\langle e_1, v_1 \rangle} f_2(v_1, \pi_1 v_2),$$

and (take $p \geqslant 3$) it provides the $\pi_p^{(2)}$, $p \geqslant 3$, in terms of f_2.

It suffices to find a weight $-e_1$ bilinear map f_2, such that the resulting $\pi_2^{(2)}$ maps $V_m \times V_m$ to V_m and vanishes elsewhere. Indeed, the restriction π_2^m of $\pi_2^{(2)}$ to $V_m \times V_m$ is then a weight 0 bilinear map on V_m and $\pi_2^{(2)} = \pi_2^m \oplus 0$. If we choose the $\pi_p^{(2)}$, $p \geqslant 3$, given by f_2, intertwining condition $\pi^{(2)} \circ f^{(2)} = f^{(2)} \circ \pi$ is satisfied and $f^{(2)}$ is a Lod_∞ isomorphism between L_1 and $(V_m \oplus V_c, 0 \oplus \pi_1, \pi_2^m \oplus 0, \pi_3^{(2)}, \pi_4^{(2)}, \ldots)$. When continuing step by step,

we finally get a Lod_∞ isomorphism $\ldots f^{(3)} \circ f^{(2)} \circ f^{(1)}$ between (V, π) and $(V_m \oplus V_c, 0 \oplus \pi_1, \pi_2^m \oplus 0, \pi_3^m \oplus 0, \ldots)$. It eventually follows from the explicit form of the stem bracket, see Example 5.25 and subsequent explanation, that, since $(0 \oplus \pi_1, \pi_2^m \oplus 0, \pi_3^m \oplus 0, \ldots)$ verifies the Lod_∞ structure condition on $V_m \oplus V_c$, the two terms of this direct sum verify the same condition on V_m and V_c respectively.

Let us define f_2 as follows:

(31)
$$f_2(v_1, v_2) = \begin{cases} \delta\pi_2(v_1, v_2) + P\pi_2(w, v_2), & \text{if } (v_1, v_2) \in B^\alpha \times Z^\beta, \\ & \quad v_1 = \pi_1(w), \\ \delta\pi_2(v_1, v_2) + \frac{1}{2}P\pi_2(w, v_2), & \text{if } (v_1, v_2) \in B^\alpha \times W^\beta, \\ & \quad v_1 = \pi_1(w), \\ \delta\pi_2(v_1, v_2) + (-1)^{\langle e_1, \alpha \rangle} P\pi_2(v_1, w'), & \text{if } (v_1, v_2) \in Z^\alpha \times B^\beta, \\ & \quad v_2 = \pi_1(w'), \\ \delta\pi_2(v_1, v_2) + (-1)^{\langle e_1, \alpha \rangle}\frac{1}{2}P\pi_2(v_1, w'), & \text{if } (v_1, v_2) \in W^\alpha \times B^\beta, \\ & \quad v_2 = \pi_1(w'), \\ \delta\pi_2(v_1, v_2), & \text{otherwise.} \end{cases}$$

Map f_2 is well-defined, *i.e.* the two definitions on $B^\alpha \times B^\beta \subset (B^\alpha \times Z^\beta) \cap (Z^\alpha \times B^\beta)$ coincide, as the Lod_∞ structure condition (26) implies

$$0 = P\pi_1\pi_2(v_1, v_2) = P\pi_2(\pi_1 v_1, v_2) + (-1)^{\langle e_1, \alpha \rangle} P\pi_2(v_1, \pi_1 v_2),$$

for any $v_1 \in V^\alpha, v_2 \in V$. Indeed, when writing this upshot for $w \in W^{\alpha - e_1}$ and w', see equation (31), we get the announced result.

It remains to show that $\pi_2^{(2)}$ sends $V_m \times V_m$ to V_m and vanishes elsewhere.

If $(v_1, v_2) \in Z \times Z$, equation (30) and Lemma 6.34 yield $\pi_2^{(2)}(v_1, v_2) = \delta\pi_1\pi_2(v_1, v_2) + P\pi_2(v_1, v_2)$. But, since the Lod_∞ condition entails that $\pi_1\pi_2(v_1, v_2) = 0$, we get $\pi_2^{(2)}(v_1, v_2) = P\pi_2(v_1, v_2)$. Furthermore, for any $(w, v_2) \in W \times Z$, condition (26) implies that $P\pi_2(\pi_1 w, v_2) = 0$, whereas for any $(v_1, w') \in Z \times W$, we obtain $P\pi_2(v_1, \pi_1 w') = 0$. Therefore,

$$\pi_2^{(2)}(v_1, v_2) = \begin{cases} P\pi_2(v_1, v_2) =: \pi_2^m(v_1, v_2) \in V_m, & \text{if } (v_1, v_2) \in V_m \times V_m, \\ 0, & \text{if } (v_1, v_2) \in B \times V_m \text{ or } (v_1, v_2) \in V_m \times B \text{ or } (v_1, v_2) \in B \times B. \end{cases}$$

If $(v_1, v_2) \in (W \times Z) \cup (Z \times W) \cup (W \times W)$, equation (30), Lemma 6.34, and condition (26) allow checking that $\pi_2^{(2)}(v_1, v_2) = 0$.

Hence,

$$\pi_2^{(2)}(v_1, v_2) = \begin{cases} \pi_2^m(v_1, v_2) = P\pi_2(v_1, v_2) \in V_m, & \text{if } (v_1, v_2) \in V_m \times V_m, \\ 0, & \text{otherwise,} \end{cases}$$

so that it suffices to continue by induction. $\qquad \square$

We are now prepared to prove Theorem 5.31.

Proof. — Let $f\colon (V,\pi) \to (V',\pi')$ be a Lod_∞ quasi-isomorphism. According to the minimal model theorem, there is a Lod_∞ isomorphism h (resp. h') that identifies the Lod_∞ algebra (V,π) (resp. (V',π')) to a direct sum $(V_m \oplus V_c, \pi^m \oplus \pi^c)$ (resp. $(V'_m \oplus V'_c, \pi'^m \oplus \pi'^c)$). Furthermore, since the inclusion $\mathfrak{i} := (i,0,0,\ldots)\colon (V_m, \pi^m) \to (V_m \oplus V_c, \pi^m \oplus \pi^c)$ (resp. the projection $\mathfrak{p} := (P',0,0,\ldots)\colon (V'_m \oplus V'_c, \pi'^m \oplus \pi'^c) \to (V'_m, \pi'^m))$ is a Lod_∞ quasi-isomorphism, the sequence $h^{\mathfrak{i}} := h^{-1} \circ \mathfrak{i}$ (resp. $h'^{\mathfrak{p}} := \mathfrak{p} \circ h'$) is a Lod_∞ quasi-isomorphism from (V_m, π^m) (resp. (V',π')) to (V,π) (resp. (V'_m, π'^m)). Therefore, the map $f^m := h'^{\mathfrak{p}} \circ f \circ h^{\mathfrak{i}}$ is a Lod_∞ quasi-isomorphism between (V_m, π^m) and (V'_m, π'^m). But, as $H(V_m, \pi_1^m) = V_m$ and $H(V'_m, \pi_1'^m) = V'_m$, the map $(f_1^m)_\sharp = f_1^m\colon V_m \to V'_m$ is an isomorphism, and so f^m has a Lod_∞ isomorphism inverse $(f^m)^{-1}$, with $(f^m)_1^{-1} = (f_1^m)^{-1}$, see Proposition 5.30. Consequently, the sequence $g := h^{\mathfrak{i}} \circ (f^m)^{-1} \circ h'^{\mathfrak{p}}$ is a Lod_∞ quasi-isomorphism from (V',π') to (V,π). Moreover, $g_{1\sharp} = (f_{1\sharp})^{-1}$. Indeed, observe first that $g_1 = i \circ (f_1^m)^{-1} \circ P'$ and $f_1^m = P' \circ f_1 \circ i$. For any $[v'] \in H(V',\pi_1')$, we thus get $g_{1\sharp}[v'] = [(f_1^m)^{-1}P'v'] \in H(V,\pi_1)$. On the other hand, $(f_{1\sharp})^{-1}[v'] =: [v] \in H(V,\pi_1)$, hence $f_1 v = v' + \pi_1'\mathfrak{v}'$, $\mathfrak{v}' \in V'$. It now suffices to show that there is $\mathfrak{v} \in V$, such that $(f_1^m)^{-1}P'v' = v + \pi_1\mathfrak{v}$, i.e.

$$P'v' = f_1^m(v + \pi_1\mathfrak{v}) = P'f_1 v + P'f_1\pi_1\mathfrak{v} = P'(v' + \pi_1'\mathfrak{v}') + P'\pi_1'f_1\mathfrak{v}.$$

This condition is satisfied for any $\mathfrak{v} \in V$, since $P'\pi_1' = 0$. □

In view of the preceding proof, we have the following

COROLLARY 6.35. — *Each Lod_∞ algebra is Lod_∞ quasi-isomorphic to a minimal one.*

7. Graded and strongly homotopy algebra cohomologies

7.1. Graded Loday and Chevalley-Eilenberg cohomologies

Let $\pi \in \mathrm{Lod}(V)$ be a $\mathbb{Z}^n$-graded Loday structure on V. As π is canonical for the $\mathbb{Z}^{n+1}$-GLA $(M_r(V), [-,-]^\otimes)$, see Theorem 5.21, it is clear that the cohomology of the induced DGLA, with differential $\partial_\pi = [\pi,-]^\otimes$, should roughly be the cohomology of the considered Loday algebra.

PROPOSITION 7.36. — *The graded Loday cohomology operator ∂_π of a Loday structure $\pi = \{-,-\}$ on a vector space V, reads, for any $B \in$*

$M^{(B,b)}(V)$, $b \geqslant -1$, and any homogeneous $v_1, \ldots, v_{b+2} \in V$,

$$(\partial_\pi B)(v_1, \ldots, v_{b+2}) = (-1)^{b+1}\{B(v_1, \ldots, v_{b+1}), v_{b+2}\}$$

$$- \sum_{i=1}^{b+1}(-1)^{i-1}(-1)^{\langle B+v_1+\cdots+v_{i-1}, v_i\rangle}\{v_i, B(v_1, \ldots, \widehat{v_i}, \ldots, v_{b+1}, v_{b+2})\}$$

(32)

$$+ \sum_{i=1}^{b+1}\sum_{j=1}^{i}(-1)^{j+1}(-1)^{\langle v_j, v_{j+1}+\cdots+v_i\rangle}$$
$$\cdot B(v_1, \ldots, \widehat{v_j}, \ldots, v_i, \{v_j, v_{i+1}\}, v_{i+2}, \ldots, v_{b+2}).$$

Proof. — For $b \geqslant 0$, the explicit form of ∂_π is a consequence of Theorem 4.17. Equation (32) suggests extending ∂_π to $M^{-1}(V) = V$ by $(\partial_\pi v)(w) := \pi(v, w) = \{v, w\}$, for any $v, w \in V$. The extended operator ∂_π is of course still a cohomology operator. See also [26]. $\qquad\square$

DEFINITION 7.37. — *The graded Loday cohomology of a $\mathbb{Z}^n$-graded Loday algebra (V, π) is the cohomology of the complex $(M(V), \partial_\pi)$, where ∂_π is given by equation (32).*

Remark 7.38. — In the non-graded case, Operator (32) coincides with the (non-graded) Loday coboundary operator, see [7], and in the antisymmetric situation, it is (the opposite of) the graded Chevalley-Eilenberg differential, see [20]. For an explicit computation of a graded Chevalley-Eilenberg cohomology, see [36] and [37].

7.2. Graded Poisson and Jacobi cohomologies

In the following we prove that the stem bracket $[-, -]^\otimes$ not only restricts to the Nijenhuis-Richardson bracket, see Remark 4.18, but also to the Grabowski-Marmo bracket, which was defined in [14], and in particular to the Schouten-Jacobi and Schouten brackets.

7.2.1. Definition

It is well-known that the triplet made up by the space of multivector fields of a manifold, the wedge product, and the Schouten-Nijenhuis bracket, is a graded Poisson algebra of weight $\alpha = -1$. Let us recall that, more generally, a *graded Poisson algebra* of weight $\alpha \in \mathbb{Z}^n$ is an associative $\mathbb{Z}^n$-graded commutative algebra $\mathcal{A}$, endowed with a bilinear bracket $\{-, -\}$ of weight α, which is α-*graded antisymmetric*, *i.e.* verifies

$\{u,v\} = -(-1)^{\langle u+\alpha, v+\alpha \rangle}\{v,u\}$, and satisfies the graded Jacobi identity, as well as the graded Leibniz rule

$$(33) \qquad \{u, vw\} = \{u,v\}w + (-1)^{\langle u+\alpha, v\rangle}v\{u,w\},$$

for any homogeneous $u, v, w \in \mathcal{A}$. The concept of *graded Jacobi algebra* of weight α is defined similarly, except that the associative algebra must have a unit 1 and that condition (33) is replaced by

$$(34) \qquad \{u, vw\} = \{u,v\}w + (-1)^{\langle u+\alpha, v\rangle}v\{u,w\} - \{u,1\}vw.$$

Note that we cannot confine our study to graded Jacobi (resp. Poisson) structures of weight $\alpha = 0$, as weight α does not disappear via an α-shift in the grading of V.

Equation (34) (resp. equation (33)) means that a graded Jacobi (resp. Poisson) structure of weight α is an α-antisymmetric graded first order bidifferential operator (resp. graded biderivation) of $\mathcal{A}$. It follows that the appropriate graded Jacobi (resp. Poisson) cochain space is the space $\mathrm{Diff}_1(\mathcal{A})$ (resp. $\mathrm{Der}(\mathcal{A})$) of α-antisymmetric graded first order polydifferential operators (resp. graded polyderivations) of $\mathcal{A}$.

In [14], the authors investigated these operators using a variant of Krasil'shchik's calculus, see [19], which is based upon a particular bidegree of the spaces $M(\mathcal{A})$, $A(\mathcal{A})$, or $\mathrm{Diff}_1(\mathcal{A})$ (resp. $\mathrm{Der}(\mathcal{A})$). More precisely, let us fix $\alpha \in \mathbb{Z}^n$ (to avoid overcrowded notations, the dependence on α of the below constructed objects will not be specified). We then denote by $\mathfrak{M}(\mathcal{A})$ the usual vector space $M(\mathcal{A})$ with the $\mathbb{Z}^{n+1}$-gradation

$$(35) \qquad \mathfrak{M}(\mathcal{A}) = \bigoplus_{(A,a)\in\mathbb{Z}^n\times\mathbb{Z}} \mathfrak{M}^{(A+\alpha a, a)}(\mathcal{A}),$$

where the space $\mathfrak{M}^{(A+\alpha a, a)}(\mathcal{A})$ of bidegree $(A + \alpha a, a)$ vanishes for $a < -1$, coincides with $\mathcal{A}^A$ for $a = -1$, and is, for $a \geqslant 0$, the space of $(a+1)$-linear maps on $\mathcal{A}$ that have weight A. The subspaces of α-antisymmetric multilinear mappings will be denoted by $\mathfrak{A}^{(A+\alpha a, a)}(\mathcal{A})$. Moreover, the subspaces $\mathfrak{Diff}_1^{(*,a)}(\mathcal{A})$ (resp. $\mathfrak{Der}^{(*,a)}(\mathcal{A})$) and the associative graded commutative dot-product "·" on these subspaces are described inductively w.r.t. a, see [14] and [19]. We then get the following

PROPOSITION 7.39. — *Let $\mathcal{A}$ be an associative $\mathbb{Z}^n$-graded commutative unital algebra and let $\alpha \in \mathbb{Z}^n$. With respect to the bidegree (35), the pair $(\mathfrak{Diff}_1(\mathcal{A}), \cdot)$ is an associative $\mathbb{Z}^{n+1}$-graded commutative unital algebra, · has weight $(\alpha, 1)$, and the pair $(\mathfrak{Der}(\mathcal{A}), \cdot)$ is a $\mathbb{Z}^{n+1}$-graded subalgebra.*

As the objective is to define the graded Jacobi (resp. Poisson) cohomology, our task is to find on the cochain algebra $(\mathfrak{Diff}_1(\mathcal{A}), \cdot)$ (resp.

$(\mathfrak{Der}(\mathcal{A}), \cdot))$ a $\mathbb{Z}^{n+1}$-graded Lie structure (of weight $(0,0) \in \mathbb{Z}^n \times \mathbb{Z}$) or even a graded Jacobi (resp. Poisson) structure. If such a bracket $[-,-]^{\mathrm{GM}}$ exists and satisfies

(36) $$[v,w]^{\mathrm{GM}} = 0, [A,v]^{\mathrm{GM}} = (-1)^{1-a}A(v),$$

it necessarily also verifies

(37) $$[v,A]^{\mathrm{GM}} = -(-1)^{\langle v-\alpha, A+\alpha a\rangle + a}[A,v]^{\mathrm{GM}}$$

and

$$
\begin{aligned}
[A,B]^{\mathrm{GM}}(v) &= (-1)^{1-a-b}[[A,B]^{\mathrm{GM}},v]^{\mathrm{GM}} \\
\text{(38)} \quad\quad &= (-1)^{1-a-b}[A,[B,v]^{\mathrm{GM}}]^{\mathrm{GM}} \\
&\quad + (-1)^{1+\langle B+\alpha b, v-\alpha\rangle - a}[[A,v]^{\mathrm{GM}},B]^{\mathrm{GM}} \\
&= (-1)^a[A,B(v)]^{\mathrm{GM}} + (-1)^{\langle B+\alpha b, v-\alpha\rangle}[A(v),B]^{\mathrm{GM}},
\end{aligned}
$$

where v,w (resp. A,B) are homogeneous elements of $\mathcal{A} \subset \mathfrak{Diff}_1(\mathcal{A})$ (resp. of $\mathfrak{Diff}_1(\mathcal{A})$). The bracket that is defined inductively by equations (36), (37), and (38) actually fits, see [14] and [19]:

THEOREM 7.40. — *If $\mathcal{A}$ is an associative $\mathbb{Z}^n$-graded commutative unital algebra and $\alpha \in \mathbb{Z}^n$, there is a unique $\mathbb{Z}^{n+1}$-graded Jacobi bracket $[-,-]^{\mathrm{GM}}$ of degree $(0,0) \in \mathbb{Z}^n \times \mathbb{Z}$ on the associative $\mathbb{Z}^{n+1}$-graded commutative algebra $(\mathfrak{Diff}_1(\mathcal{A}), \cdot)$ that verifies $[A,v]^{\mathrm{GM}} = (-1)^{1-a}A(v)$, $A \in \mathfrak{Diff}_1^{(*,a)}(\mathcal{A})$, $v \in \mathcal{A}$. Moreover, $(\mathfrak{Der}(\mathcal{A}), [-,-]^{\mathrm{GM}}, \cdot)$ is a $\mathbb{Z}^{n+1}$-graded Poisson algebra. Furthermore, the graded Jacobi (resp. Poisson) structures of weight α on $\mathcal{A}$, are exactly the canonical elements of degree $(2\alpha, 1)$ of this graded Jacobi (resp. Poisson) algebra.*

This result immediately leads to the

DEFINITION 7.41. — *If π denotes a graded Jacobi (resp. Poisson) structure of weight $\alpha \in \mathbb{Z}^n$ on the (usual) algebra $\mathcal{A}$, we refer to the cohomology of the DGLA $(\mathfrak{Diff}_1(\mathcal{A}), [-,-]^{\mathrm{GM}}, [\pi,-]^{\mathrm{GM}})$ (resp. $(\mathfrak{Der}(\mathcal{A}), [-,-]^{\mathrm{GM}}, [\pi,-]^{\mathrm{GM}}))$ as the graded Jacobi (resp. Poisson) cohomology of $(\mathcal{A}, \pi)$.*

Of course, in the (ungraded) geometric case $\mathcal{A} = C^\infty(M)$ (usual notations), the graded Jacobi (resp. Poisson) cohomology coincides with the standard Lichnerowicz-Jacobi (resp. Lichnerowicz-Poisson) cohomology, see [25] (resp. [24]). More detailed information on Jacobi (resp. Poisson) cohomology and its computation can be found in [23], [21], and [22] (resp. [35], [29], and [3]).

7.2.2. Link with the stem bracket

PROPOSITION 7.42. — *For any fixed $\alpha \in \mathbb{Z}^n$, any $A \in \mathfrak{Diff}_1^{(A+\alpha a,a)}(\mathcal{A})$ and any $B \in \mathfrak{Diff}_1^{(B+\alpha b,b)}(\mathcal{A})$, the Grabowski-Marmo bracket is given by*

$$(39) \qquad [A,B]^{\mathrm{GM}} = A \square B - (-1)^{\langle A+\alpha a, B+\alpha b\rangle + ab} B \square A,$$

where $A \square B$ is defined inductively by

$$v \square w := 0, \quad A \square v := (-1)^{1-a} A(v), \ a \geqslant 0, \quad v \square A := 0, \ a \geqslant 0,$$

$$(40) \quad (A \square B)(v) := (-1)^a A \square B(v) + (-1)^{\langle B+\alpha b, v-\alpha\rangle} A(v) \square B, \ a, b \geqslant 0,$$

for any $v \in \mathcal{A}^v$, $w \in \mathcal{A}$. Further, the explicit form of the square-product is

$$(41) \quad (A \square B)(v_1, \ldots, v_{a+b+1})$$
$$= (-1)^{1+ab} \sum_{\substack{I \cup J = N^{(a+b+1)} \\ |I|=b+1, \ |J|=a}} (-1)^{(I;J)} \varepsilon_{\downarrow V}(I;J) A(B(V_I), V_J),$$

where $v_i \in \mathcal{A}^{v_i}$, $\downarrow V = \downarrow v_1 \otimes \cdots \otimes \downarrow v_{a+b+1}$, and $\downarrow v_i \in \mathcal{A}^{v_i - \alpha}$.

Proof. — The proof of equation (39) is by induction on $a+b$. If $a+b \leqslant -1$, the claim is obviously true, see equations (36) and (37). Equations (38) and (40) allow checking the rest. In order to determine the explicit form (41), we proceed similarly. The announced upshot is clear for $a = -1$ or $b = -1$, hence, in particular for $a + b \leqslant -1$. Let us now assume that it is valid for $a+b \leqslant k-1$, $k \geqslant 0$, and examine the case $a+b = k$. As already pointed out, if $a = -1$ or $b = -1$, our conjecture is verified. Otherwise, $a, b \geqslant 0$, and, if we set $V'' = v_2 \otimes \cdots \otimes v_{a+b+1}$, if we use equation (40) and the induction assumption, and if we exchange v_1 and $B(V_I'')$ in the second term of the result produced in this way, we obtain

$$(A \square B)(v_1, v_2, \ldots, v_{a+b+1})$$

$$= (-1)^{1+ab} \left[\sum_{\substack{I \cup J = \{2,\ldots,a+b+1\} \\ |I|=b, \ |J|=a}} (-1)^{(I;J)} \varepsilon_{\downarrow V''}(I;J) A(B(v_1, V_I''), V_J'') \right.$$

$$+ (-1)^{1+\langle V_I'' - \alpha(b+1), v_1 - \alpha\rangle + b} \sum_{\substack{I \cup J = \{2,\ldots,a+b+1\} \\ |I|= b+1, |J|=a-1}}$$

$$\left. \cdot (-1)^{(I;J)} \varepsilon_{\downarrow V''}(I;J) A(B(V_I''), v_1, V_J'') \right].$$

In the final upshot for which we look, $\mathcal{I}$ and $\mathcal{J}$ are unshuffles that form a partition of $N^{(a+b+1)}$, so that 1 is either the first element of $\mathcal{I}$ or of $\mathcal{J}$. The first term $\sum \cdots$ (resp. second term $(-1)^{\cdots} \sum \cdots$) of the RHS of the last equation, corresponds exactly to the first (resp. second) possibility. Indeed,

$$(-1)^{(I;J)} \varepsilon_{\downarrow V''}(I; J) = (-1)^{(\mathcal{I};\mathcal{J})} \varepsilon_{\downarrow V}(\mathcal{I}; \mathcal{J})$$

$$\left(\text{resp. } (-1)^{1+\langle V_I'' - \alpha(b+1), v_1 - \alpha \rangle + b} (-1)^{(I;J)} \varepsilon_{\downarrow V''}(I; J) = (-1)^{(\mathcal{I};\mathcal{J})} \varepsilon_{\downarrow V}(\mathcal{I}; \mathcal{J}) \right),$$

where $V = v_1 \otimes \cdots \otimes v_{a+b+1}$. $\qquad \square$

In the following, we denote by $\downarrow \mathcal{A}$ the vector space $\mathcal{A}$ endowed with the gradation $(\downarrow \mathcal{A})^\gamma = \mathcal{A}^{\gamma + \alpha}$. Set now

(42) $$\mathfrak{A}(\mathcal{A}) = \bigoplus_{(A,a) \in \mathbb{Z}^n \times \mathbb{Z}} \mathfrak{A}^{(A + \alpha a, a)}(\mathcal{A})$$

$$\left(\text{resp. } A(\downarrow \mathcal{A}) = \bigoplus_{(A,a) \in \mathbb{Z}^n \times \mathbb{Z}} A^{(A + \alpha a, a)}(\downarrow \mathcal{A}) \right),$$

where $\mathfrak{A}^{(A + \alpha a, a)}(\mathcal{A})$ (resp. $A^{(A + \alpha a, a)}(\downarrow \mathcal{A})$) is the space of multilinear maps $A \colon \mathcal{A}^{\times(a+1)} \to \mathcal{A}$ (resp. $\widetilde{A} \colon (\downarrow \mathcal{A})^{\times(a+1)} \to \downarrow \mathcal{A}$) that have weight A (resp. $A + \alpha a$), and that are further α-graded antisymmetric (resp. graded antisymmetric). Obviously, the map $\sim \colon A \to \widetilde{A}$, with $\widetilde{A}$ defined by $\widetilde{A}(\tilde{v}_1, \ldots, \tilde{v}_{a+1}) = \downarrow A(\uparrow \tilde{v}_1, \ldots, \uparrow \tilde{v}_{a+1})$, is a $\mathbb{Z}^{n+1}$-graded vector space isomorphism, the inverse of which is $\smile \colon \widetilde{A} \to A$, where $A(v_1, \ldots, v_{a+1}) = \uparrow \widetilde{A}(\downarrow v_1, \ldots, \downarrow v_{a+1})$. The isomorphism $\sim$ pulls of course the Nijenhuis-Richardson graded Lie bracket $[-, -]^{\mathrm{NR}}$ on the usual space $A(\downarrow \mathcal{A})$, associated with the $\mathbb{Z}^n$-graded vector space $\downarrow \mathcal{A}$, back to a graded Lie bracket

$$\smile [\sim -, \sim -]^{\mathrm{NR}} =: \sim^*[-, -]^{\mathrm{NR}} = \sim^*[-, -]^{\otimes}|_{A(\downarrow \mathcal{A})}$$

on $\mathcal{A}(\mathcal{A})$.

PROPOSITION 7.43. — *The Grabowski-Marmo bracket $[-, -]^{\mathrm{GM}}$ is the restriction of $\sim^*[-, -]^{\otimes}|_{A(\downarrow \mathcal{A})}$ to $\mathfrak{Diff}_1(\mathcal{A})$.*

Proof. — The explicit form of the restriction of the stem bracket, see Theorem 4.17, to skew-symmetric mappings, *i.e.* the explicit expression of the Nijenhuis-Richardson bracket, as well as the forms (41) and (39) of the Grabowski-Marmo bracket imply that, for $A \in \mathfrak{Diff}_1^{(A + \alpha a, a)}(\mathcal{A})$ and $B \in \mathfrak{Diff}_1^{(B + \alpha b, b)}(\mathcal{A})$, the preceding pullback reads

$$\left(\sim^*[-, -]^{\otimes}|_{A(\downarrow \mathcal{A})} \right)(A, B) = \smile i_{\widetilde{A}} \widetilde{B} - (-1)^{\langle A + \alpha a, B + \alpha b \rangle + ab} \smile i_{\widetilde{B}} \widetilde{A},$$

where $\smile i_{\widetilde{B}} \widetilde{A} = (-1)^{1 + \langle A + \alpha a, B + \alpha b \rangle + ab} A \square B$, so that the result follows. $\qquad \square$

Finally:

COROLLARY 7.44. — *The stem bracket is (up to reading through a canonical isomorphism) a graded Lie bracket on the spaces of graded Loday, graded Lie, graded Poisson, and graded Jacobi cochains, for which the corresponding algebraic structures are canonical elements, and that thus encodes the graded cohomologies of all these structures.*

7.3. Strongly homotopy and graded p-ary Loday cohomologies

Since Lod_∞^Q structures on a $\mathbb{Z}^n$-graded vector space V, $Q \in \mathbb{Z}^n$, $\langle Q, Q \rangle$ odd, are the degree Q canonical elements π of the $\mathbb{Z}^n$-graded Lie algebra $(C(V), [-,-]^{\bar\otimes})$, we have the natural

DEFINITION 7.45. — *The cohomology of a Loday infinity algebra (V, π) is the cohomology of the DGLA $(C(V), [-,-]^{\bar\otimes}, [\pi, -]^{\bar\otimes})$, where the coboundary operator is, for any $\rho \in C^p(V)$, explicitly given by*

$$[\pi, \rho]^{\bar\otimes} = \sum_{q \geqslant 1} \sum_{s+t=q+1} (-1)^{1+(s-1)\langle e_1, \rho \rangle} [\pi_s, \rho_t]^{\otimes}$$

(and equations (15) and (16)).

1. We first examine the case $\pi = \pi_p$, $p \in \{2, 4, \ldots\}$. If the odd degree $Q \in \mathbb{Z}^n$ of π is chosen to be equal to $(p-1)e_1$, we have

(43) $\pi = \pi_p \in M^{(0,p-1)}(V)$, $[\pi_p, \pi_p]^{\otimes} = 0$, p even.

Essentially two p-ary extensions of the Jacobi identity were investigated during the last decades. If $[-,-,\ldots,-]$ denotes a p-linear bracket on V, the first is the generalization, which requires that the adjoint action $[v_1, v_2, \ldots, v_{p-1}, -]$ be a derivation for the p-ary bracket $[w_1, w_2, \ldots, w_p]$, see e.g. [10], and leads to *Nambu-Lie* or, in the nonantisymmetric context, to *Nambu-Loday* structures, see [31]. The second was suggested by P. Michor and A. Vinogradov, see [30], and further studied in [40] and in [41]. We refer to this last p-ary extension as *p-ary Lie* or *p-ary Loday* structure. A *p-ary $\mathbb{Z}^n$-graded Lie structure* on a $\mathbb{Z}^n$-graded vector space V is a map $P_p \in A^{(0,p-1)}(V)$, such that $i_{P_p} P_p = 0$ (where i is the interior product used in the definition of the graded Nijenhuis-Richardson bracket) or, if p is even, equivalently, such that $[P_p, P_p]^{\mathrm{NR}} = 0$. Moreover, the cohomology of a p-ary Lie algebra (V, P_p) is the cohomology of the DGLA $(A(V), [-,-]^{\mathrm{NR}}, [P_p, -]^{\mathrm{NR}})$, see [30].

Analogously, a map $\pi_p \in M^{(0,p-1)}(V)$, $p \in \{2, 4, \ldots\}$, that verifies $[\pi_p, \pi_p]^{\otimes} = 0$, is a *$p$-ary $\mathbb{Z}^n$-graded Loday structure* on V. The cohomology

of such an algebra (V, π_p) should be defined (and was defined in the case $p = 2$ roughly) as the cohomology of the DGLA $(M_r(V), [-, -]^\otimes, [\pi_p, -]^\otimes)$.

In the following, we explain why

Remark 7.46. — The cohomology space of $\pi = \pi_p$, see equation (43), viewed as degree $(p-1)e_1$ strongly homotopy Loday structure, coincides with the preceding cohomology space of $\pi = \pi_p$, viewed as p-ary $\mathbb{Z}^n$-graded Loday structure.

Let us first mention that for noninfinity algebras, it is conventional to substitute in cochain space $C(V)$ a direct sum for the direct product, so that

$$C(V) = \bigoplus_{s \in \mathbb{N}^*} \bigoplus_{Q \in \mathbb{Z}^n} M^{(Q+(1-s)e_1, s-1)}(V) = \bigoplus_{s \in \mathbb{N}^*} M^{s-1}(V) = M_r(V).$$

Hence, $C(V)$ carries a bigrading that is shifted with respect to the usual bigradation

$$M_r(V) = \bigoplus_{s \in \mathbb{N}^*} \bigoplus_{Q \in \mathbb{Z}^n} M^{(Q, s-1)}(V)$$

of $M_r(V)$. We emphasize that in the first bigrading the space of bidegree (Q, s) is, unlike in Section 7.2, but just as above that section, the space $M^{(Q+(1-s)e_1, s-1)}(V)$ — whereas no ambiguity is possible for the second grading.

It is now easy to see that the cohomology spaces of $(C(V), [-, -]^{\bar\otimes}, [\pi_p, -]^{\bar\otimes})$ and $(M_r(V), [-, -]^\otimes, [\pi_p, -]^\otimes)$ coincide. Indeed, if π is a Lod_∞^Q structure on V and if $\rho_t \in M^{(\rho+(1-t)e_1, t-1)}(V)$, we have

$$(44) \qquad [\pi, \rho_t]^{\bar\otimes} = \sum_{q \geq 1} (-1)^{1+(q-t)\langle e_1, \rho \rangle} [\pi_{q-t+1}, \rho_t]^\otimes$$

$$\in \prod_{q \geq 1} M^{(Q+\rho+(1-q)e_1, q-1)}(V),$$

so that, in the case $\pi = \pi_p$, $Q = (p-1)e_1$, where necessarily $q = t + p - 1$, the weight of cohomology operator $[\pi_p, -]^{\bar\otimes}$ with respect to the first mentioned bigrading is $((p-1)e_1, p-1)$. On the other hand, since $\pi_p \in M^{(0, p-1)}(V)$, it is clear that the weight of operator $[\pi_p, -]^\otimes$ with respect to the second bigradation is $(0, p-1)$. It follows that the cohomology spaces of $(C(V), [\pi_p, -]^{\bar\otimes})$ and $(M_r(V), [\pi_p, -]^\otimes)$, say $\bar{H}$ and H, are both $\mathbb{Z}^{n+1}$-graded. Space $\bar{H}^{(\rho+(t-1)e_1, t-1)}$ (resp. $H^{(\rho, t-1)}$) is encoded in the cocycle equation $[\pi_p, \rho_t]^{\bar\otimes} = 0$, i.e. $[\pi_p, \rho_t]^\otimes = 0$, where $\rho_t \in M^{(\rho, t-1)}(V)$ (resp. $[\pi_p, \rho_t]^\otimes = 0$, $\rho_t \in M^{(\rho, t-1)}(V)$), and in the coboundary equation, which

reads $\rho_t = [\pi_p, \tau]^{\bar{\otimes}} = [\pi_p, \pm\tau]^{\otimes}$, with $\tau \in M^{(\rho, t-p)}(V)$ (resp. $\rho_t = [\pi_p, \tau]^{\otimes}$, $\tau \in M^{(\rho, t-p)}(V)$), for any cocycle ρ_t. This observation entails that

$$\bar{H}^{(\rho+(t-1)e_1, t-1)} = H^{(\rho, t-1)}.$$

Hence, the cohomology spaces $\bar{H}$ and H coincide, but their natural GLA structures, which are induced by $[-,-]^{\bar{\otimes}}$ and $[-,-]^{\otimes}$, are $\mathbb{Z}^n$- and $\mathbb{Z}^{n+1}$-graded respectively.

For $p = 2$, we of course recover the abovedefined cohomology space of a graded Loday algebra.

2. If π is not a sequence of all but one vanishing elements, equation (44) implies that the Loday infinity coboundary operator $[\pi, -]^{\bar{\otimes}}$ maps $M^{(\rho+(1-t)e_1, t-1)}(V)$ into $\prod_{q \geqslant 1} M^{(Q+\rho+(1-q)e_1, q-1)}(V)$. Hence, the Loday infinity cohomology is not $\mathbb{Z}$-graded. Nevertheless, if we consider the (decreasing) filtration

$$C_k(V) = \bigoplus_{R \in \mathbb{Z}^n} \prod_{s \geqslant k} M^{(R+(1-s)e_1, s-1)}(V), \ k \geqslant 1,$$

and if $\rho = \sum_R \sum_{t \geqslant k} \rho_{R,t} \in C_k(V)$, we get

$$[\pi, \rho]^{\bar{\otimes}} = \sum_R \sum_{q \geqslant 1} \sum_{s+t=q+1} (-1)^{1+(s-1)\langle e_1, R \rangle} [\pi_s, \rho_{R,t}]^{\otimes}$$

$$\in \bigoplus_{R \in \mathbb{Z}^n} \prod_{q \geqslant k} M^{(Q+R+(1-q)e_1, q-1)}(V).$$

Indeed, as $k \leqslant t$, we have $q \leqslant k-1 \Rightarrow q \leqslant t-1 \Leftrightarrow s = q-t+1 \leqslant 0$, so that the sum over s, t vanishes for these q. The observation yields

$$[\pi, C_k(V)]^{\bar{\otimes}} \subset C_k(V).$$

Eventually,

Remark 7.47. — The pair $(C(V), [\pi, -]^{\bar{\otimes}})$ is a differential filtered space and admits a spectral sequence.

3. Let $\mathcal{C}(V)$ be the $\mathbb{Z}^n$-graded vector subspace

$$\mathcal{C}(V) = \bigoplus_{Q \in \mathbb{Z}^n} \mathcal{C}^Q(V) = \bigoplus_{Q \in \mathbb{Z}^n} \prod_{s \geqslant 1} A^{(Q+(1-s)e_1, s-1)}(V)$$

of $C(V)$. Note that $\mathcal{C}(V)$ is closed for the bracket $[-,-]^{\bar{\otimes}}$ and that Lie infinity structures are canonical elements of the graded Lie subalgebra $(\mathcal{C}(V), [-,-]^{\bar{\otimes}})$. Hence, the *cohomology of a Lie infinity algebra* (V, π) is the cohomology of the DGLA $(\mathcal{C}(V), [-,-]^{\bar{\otimes}}, [\pi, -]^{\bar{\otimes}})$. If the L_∞ structure

has the degree e_1, the restriction of the coboundary operator $[\pi, -]^{\bar{\otimes}}$ to $\mathcal{C}(V)$ coincides with the differential studied in [34] and used in [9].

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