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FATOU-NAIM-DOOB LIMIT THEOREMS IN THE AXIOMATIC SYSTEM OF BRELOT

by Kohur GOWRISANKARAN

1. Introduction.

Let Ω be a locally compact Hausdorff space which is connected and has a countable base. Let \mathfrak{H} be a class of real valued continuous functions, called harmonic functions, on open subsets of Ω such that for each open set $W \subset \Omega$, the set \mathfrak{H}_W , consisting of all functions in \mathfrak{H} defined on W , is a real vector space. Let this class \mathfrak{H} satisfy the axioms 1, 2 and 3 of M. Brelot [1]. Let, moreover, there exist a potential > 0 on Ω .

The classical Fatou-Naim-Doob limit theorems were extended to the axiomatic system of M. Brelot in [2]. But, besides the above mentioned axioms, we had assumed the validity of axioms D and \mathfrak{R}_u [2]. The object of this paper is to show that the Fatou-Naim-Doob limit theorems (cf. Theorem 8) hold good in the axiomatic set up without these supplementary axioms (viz. D and \mathfrak{R}_u). The method consists in proving first, the limit theorems for a special class of superharmonic functions (cf. Theorem 4), and using it systematically to prove the general result. A novel feature in our proof is the consideration of a modified Dirichlet problem. We shall mostly follow the notation of [1, 2, 3].

Notation.

S^+ : The set of all non-negative superharmonic functions on Ω .

H^+ : The set of all non-negative harmonic functions on Ω .

Λ : A compact base for S^+ (compact in the T-topology [3]).

Δ_1 : The set of minimal harmonic functions contained in Λ .

For any $E \subset \Omega$ and $\nu \in S^+$,

$$R_\nu^E = \inf\{\omega : \omega \in S^+ \text{ and } \omega \geq \nu \text{ on } E\}.$$

For any $h \in \Delta_1$, $\mathcal{F}_h = \{E \subset \Omega : R_h^E \not\equiv h\}$ [2]. A set E is thin at $h \in \Delta_1$ if $R_h^E \equiv h$ (i.e. if \mathcal{F}_h leaves no trace on E).

The limits of any function f following \mathcal{F}_h , for any $h \in \Delta_1$, are called the fine limits of $f(x)$, as x tends to h . To every harmonic function $\omega \in H^+$ corresponds a unique measure μ_ω on Δ_1 , called the canonical measure corresponding to ω , such that $\omega = \int h \mu_\omega(dh)$. For any regular domain $\delta \subset \Omega$, and $x \in \delta$, $d\rho_x^\delta$ is the measure on $\partial\delta$ which associates to a finite continuous function f on $\partial\delta$ the integral $H_f(x)$. For the considerations below, let us fix a $u \in H^+$ with $u > 0$. Let μ_u be the canonical measure on Δ_1 , corresponding to u . A function ν on Ω is said to be super- u -harmonic (respectively- u -harmonic) if $u\nu$ is superharmonic (resp. harmonic) on Ω .

2. Fine limits of bounded u -harmonic functions.

LEMMA 1. — *Let $V \subset \Omega$ be an open set. Then, for every $x \in \Omega$, the mapping $h \rightarrow R_h^V(x)$ of $H^+ \rightarrow \mathbf{R}^+$ is lower semi-continuous.*

Proof. — Let $h_n \in H^+$ converge to $h \in H^+$. Let $\nu_n = R_{h_n}^V$. Then, ν_n is a non-negative superharmonic function on Ω and $\nu_n = h_n$ on V . Let $\nu = \liminf_{n \rightarrow \infty} \nu_n$. Let ω be a regular domain of Ω . Then,

$$\nu_n(y) \geq \int \nu_n(z) \rho_y^\omega(dz) \quad \text{for all } y \in \omega.$$

Hence,

$$\begin{aligned} \nu(y) = \liminf_{n \rightarrow \infty} \nu_n(y) &\geq \liminf_{n \rightarrow \infty} \int \nu_n(z) \rho_y^\omega(dz) \\ &\geq \int \nu(z) \rho_y^\omega(dz) \quad (\text{Fatou's Lemma}). \end{aligned}$$

(Note here that ν is a ρ_x^ω -measurable function.) Since ν is also non-negative, it follows that ν is an $S_{\mathcal{B}}$ -function, where \mathcal{B} is the class of all regular domains of Ω [1]. Hence, $\hat{\nu}$, the lower semi-continuous regularisation of ν , is a superharmonic function. But $\nu(y) = h(y)$, for all $y \in V$, and hence $\hat{\nu} = h$ on V .

It follows that $\nu \geq \hat{\nu} \geq R_h^\nu$ on Ω . This gives the required lower semi-continuity.

COROLLARY. — *For any regular domain δ of Ω and all $x \in \delta$, the function $h \rightarrow \int R_h^\nu(z) \rho_\delta^\nu(dz)$ is lower semi-continuous on H^+ .*

The corollary follows from the lemma by the use of Fatou's lemma.

LEMMA 2. — *The set \mathcal{E}_V of points of Δ_1 , where an open set $V \subset \Omega$ is thin, is a borel subset of Δ_1 .*

Proof. — Let $\{\delta_n\}$ be a countable covering of Ω by regular domains. Let, for each n , $x_n \in \delta_n$. Define,

$$F'_n = \{h \in \Lambda \cap H^+ : \int R_h^\nu(y) \rho_{x_n}^{\delta_n}(dy) < h(x_n)\}.$$

In view of the above lemma and its corollary, F'_n is a borel subset of Λ (in fact, a K_σ — set). Hence, $F'_n = F'_n \cap \Delta_1$ is a borel subset of Δ_1 . It can be proved as in [2], that $\bigcup_{n=1}^{\infty} F'_n$ is precisely the set \mathcal{E}_V . The lemma is proved.

THEOREM 1. — *Let $V \subset \Omega$ be any open set. Then $R_u^\nu \equiv u$ if and only if $\mu_u(\mathcal{E}_V) = 0$.*

Proof. — Let $\mu_u(\mathcal{E}_V) = 0$. For any $x \in \Omega$, we have,

$$R_u^\nu(x) = \int R_h^\nu(x) \mu_u(dh) \quad (\text{Th. 22.3, [3]}).$$

Since $R_h^\nu(x) = h(x)$, for all $h \in \Delta_1 - \mathcal{E}_V$, and $\mu_u(\mathcal{E}_V) = 0$, we get,

$$R_u^\nu(x) = \int h(x) \mu_u(dh) = u(x).$$

This is true whatever be $x \in \Omega$.

Conversely, suppose that $R_u^\nu \equiv u$. Let $\{\delta_n\}$ be a sequence covering Ω , each δ_n being a regular domain, and consider the sets $F'_n \subset \Delta_1$, as defined in the above lemma.

Let ν_k be the swept-out measure corresponding to the measure $d\rho_{x_k}^{\delta_k}$ relative to the sweeping out on V . (Th. 10. 1, [3]). (Note that $d\rho_{x_k}^{\delta_k}$ is with the compact support $\partial\delta_k$). The measure ν_k is such that, for any $\nu \in S^+$,

$$\int \nu(y) \nu_k(dy) = \int R_\nu^\nu(y) \rho_{x_k}^{\delta_k}(dy).$$

We have,

$$\begin{aligned} \int R_u^V(y) \rho_{x_k}^{\delta_k}(dy) &= \int u(z) \nu_k(dz) = \int \nu_k(dz) \int h(z) \mu_u(dh) \\ &= \int \mu_u(dh) \int h(z) \nu_k(dz) = \int \mu_u(dh) \int R_h^V(y) \rho_{x_k}^{\delta_k}(dy) \dots (1) \end{aligned}$$

(Lebesgue-Fubini Theorem).

Now,

$$\begin{aligned} \int h(x_k) \mu_u(dh) &= u(x_k) = \int R_u^V(y) \rho_{x_k}^{\delta_k}(dy) \quad (\text{hypothesis}) \\ &= \int \mu_u(dh) \int R_h^V(y) \rho_{x_k}^{\delta_k}(dy) \quad (\text{from (1)}). \end{aligned}$$

It follows that,

$$\int [h(x_k) - \int R_h^V(y) \rho_{x_k}^{\delta_k}(dy)] \mu_u(dh) = 0 \dots (2).$$

Since the integrand in the above equation is always ≥ 0 , we get, $h(x_k) = \int R_h^V(y) \rho_{x_k}^{\delta_k}(dy)$, for all $h \in \Delta_1$, except for a set of μ_u -measure zero. But the exceptional set where the inequality does not hold good is precisely F_k . Hence,

$$\mu_u(F_k) = 0.$$

It follows, from the above lemma, that $\mu_u(\mathcal{E}_V) = 0$. The theorem is proved.

COROLLARY. — *The greatest harmonic minorant of R_u^V is the function $\int h \mu_u^V(dh)$ where μ_u^V is the restriction of μ_u to $\Delta_1 - \mathcal{E}_V$. Hence, R_u^V is a potential if V is thin μ_u -almost everywhere on Δ_1 .*

The proof of the corollary is exactly as in (Cor. Th. II. 2, [2]).

THEOREM 2. — *Let $\omega > 0$ be a potential on Ω . Then $\frac{\omega}{u}$ has the fine limit zero, at μ_u -almost every element of Δ_1 .*

Proof. — It is enough to show that, for every rational number $r > 0$, the set $V_r = \left\{ x \in \Omega : \frac{\omega(x)}{u(x)} > r \right\}$ is thin μ_u -almost everywhere. But, since $\frac{\omega}{u}$ is a lower semi-continuous function, V_r is an open subset of Ω . Further, $R_u^V \leq \frac{\omega}{r}$. Hence R_u^V is a potential and it follows (Cor. to Theorem 1) that V_r is

thin at μ_u -almost every element of Δ_1 . This is true for every $r > 0$. The proof is completed easily.

The following result is an important corollary of the above theorem.

THEOREM 3. — *Let ν and ω be two non-negative harmonic functions on Ω such that their canonical measures μ_ν and μ_ω on Δ_1 are singular relative to each other. If, $\nu > 0$ on Ω , then, $\frac{\omega}{\nu}$ has the fine limit zero, at μ_ν -almost every element of Δ_1 .*

Proof. — Let $\nu' = \inf(\nu, \omega)$. Then it is clear that ν' is a potential on Ω . By the above theorem, we can find a set $E \subset \Delta_1$ of μ_u -measure zero such that, for every $h \in \Delta_1 - E$, $\text{fine lim}_{x \rightarrow h} \frac{\nu'(x)}{\nu(x)} = 0$. From this we easily deduce that, the $\text{fine lim}_{x \rightarrow h} \frac{\omega(x)}{\nu(x)} = 0$, for every $h \in \Delta_1 - E$. This completes the proof.

THEOREM 4. — *Let ω be a bounded u -harmonic function on Ω . Then, ω has a fine limit at μ_u -almost every element of Δ_1 .*

Proof. — Define, for a μ_u -summable function f on Δ_1 ,

$$\sigma_f = \int f(h) \cdot \frac{h}{u} \mu_u(dh).$$

For the characteristic function χ_E of a μ_u -measurable set $E \subset \Delta_1$, let us denote by σ_E the function σ_{χ_E} and σ'_E the function $\sigma_{\chi_E'}$. σ_f is a u -harmonic function, for every such f . Now, for a μ_u -measurable set $E \subset \Delta_1$, since $\sigma_E \leq 1$ on Ω ,

$$\text{fine lim sup}_{x \rightarrow h} \sigma_E(x) \leq 1 \quad \text{for all } h \in \Delta_1.$$

If either $\mu_u(E)$ or $\mu_u(\bigcap E)$ is zero, then $\sigma'_E = 1$ (or respectively $\sigma_E = 1$), and the fine limits of σ_E and σ'_E , exist at all points of Δ_1 . On the other hand, suppose $\mu_u(E) \neq 0$ and also

$$\mu_u(\bigcap E) \neq 0.$$

Then, $u\sigma_E$ and $u\sigma'_E$ are two harmonic functions > 0 on Ω and their canonical measures on Δ_1 (viz. μ_u restricted to E and $\int E$) are singular relative to each other. Hence, by the Theorem 3, σ_E/σ'_E has the fine limit zero at μ_u -almost every element of $\Delta_1 - E$. It follows then that,

$$\text{fine lim sup}_{x \rightarrow h} \sigma_E(x) \leq \text{fine lim}_{x \rightarrow h} \frac{\sigma_E(x)}{\sigma'_E(x)} = 0$$

for μ_u -almost every element of $\Delta_1 - E$, as $\sigma'_E(x) \leq 1$. Hence,

$$\text{fine lim sup}_{x \rightarrow h} \sigma_E(x) \leq \chi_E(h)$$

for μ_u -almost every $h \in \Delta_1 \dots (3)$.

In particular, the inequality (3) is valid for the complement of E and we deduce that,

$$\text{fine lim inf}_{x \rightarrow h} \sigma_E(x) \geq \chi_E(h) \quad \text{for} \quad \mu_u\text{-almost every } h \in \Delta_1.$$

In any case we get, for the characteristic function χ_E of a μ_u -measurable set E contained in Δ_1 ,

$$\text{fine lim}_{x \rightarrow h} \sigma_E(x) = \chi_E(h) \quad \text{for} \quad \mu_u\text{-almost every } h \in \Delta_1 \dots (4).$$

Suppose, now, $f \geq 0$ is a μ_u -measurable function on Δ_1 . Then, there exists an increasing sequence of non-negative simple functions s_n such that $\lim_{n \rightarrow \infty} s_n = f$. We deduce easily from (4) that

$$\text{fine lim}_{x \rightarrow h} \sigma_{s_n}(x) = s_n(h) \quad \text{for} \quad \mu_u\text{-almost every } h \in \Delta_1.$$

Hence, σ_f satisfies,

$$\text{fine lim inf}_{x \rightarrow h} \sigma_f(x) \geq s_n(h) \quad \text{for} \quad \mu_u\text{-almost every } h \in \Delta_1.$$

Now, it is easily seen that,

$$\text{fine lim inf}_{x \rightarrow h} \sigma_f(x) \geq f(h) \quad \text{for} \quad \mu_u\text{-almost every } h \in \Delta_1 \dots (5).$$

Let us now consider a bounded μ_u -measurable function g on Δ_1 (say $|g| \leq M$). Then, applying the inequality (5) to the

two functions $\sigma_{(M \pm g)}$, and noting that, $\sigma_{M \pm g} = M \pm \sigma_g$, we get that

$$\text{fine lim}_{x \rightarrow h} \sigma_g(x) = g(h) \quad \text{for} \quad \mu_u\text{-almost every } h \in \Delta_1.$$

Now, the proof of the theorem is completed by noting that any bounded u -harmonic function ω is equal to $u\sigma_g$, for some bounded μ_u -measurable function g on Δ_1 ; this g is unique (depending on ω) upto a set of μ_u -measure zero.

Remark 1. — In the course of the proof of the theorem, we have shown that, for any $f \geq 0$, which is μ_u -measurable,

$$\text{fine lim inf}_{x \rightarrow h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) \geq f(h_0),$$

for μ_u -almost every $h_0 \in \Delta_1$ (viz. the inequality (5)).

Remark 2. — For any bounded u -harmonic function ω on Ω , if $g(h) = \text{fine lim}_{x \rightarrow h} \omega(x)$, (the function g is defined upto a set of μ_u -measure zero), then g is μ_u -measurable and

$$\omega(x) = \int g(h) \frac{h(x)}{u(x)} \mu_u(dh).$$

In particular, if the fine limit is ≥ 0 for μ_u -almost every element of Δ_1 , then ω is non-negative.

Remark 3. — For any bounded super- u -harmonic function ν on Ω , the fine lim $\nu(x)$ exists for μ_u -almost every $h \in \Delta_1$.

THEOREM 5. — (*The Minimum Principle*). Let ν be a lower bounded super- u -harmonic function on Ω . Suppose that, for every $h \in \Delta_1 - E$, $\text{fine lim}_{x \rightarrow h} \sup \nu(x) \geq 0$, where E is a set with $\mu_u^*(E) = 0$. Then, ν is ≥ 0 on Ω .

Proof. — Let $\alpha > 0$ be such that $\nu \geq -\alpha$. Consider

$$\nu' = \text{Inf}(\nu, 1).$$

Then ν' is a super- u -harmonic function such that $\nu' \geq -\alpha$. The theorem would be proved if we show that $\nu' \geq 0$ on Ω .

Now, it is easily seen that $\text{fine } \limsup_{x \rightarrow h} \nu'(x) \geq 0$, for all $h \in \Delta_1 - E$. But, we know, (by the Remark 3 following the Theorem 4) that, the limit of ν' exists, following \mathcal{F}_h , for μ_u -almost every $h \in \Delta_1$; and this fine limit is precisely the fine limit of u_1 , where u_1 is the greatest u -harmonic minorant of ν' . Hence, we have that the fine limit of ν' is ≥ 0 at μ_u -almost every element of Δ_1 . It follows that $u_1 \geq 0$ (from the Remark 2, Theorem 4). A fortiori, $\nu' \geq 0$. This completes the proof of the theorem.

3. A Dirichlet problem.

Let Σ be the set of all lower bounded super- u -harmonic functions on Ω . Corresponding to any extended real valued function f on Δ_1 , define,

$$\Sigma_f = \left\{ \nu \in \Sigma : \exists \text{ a set } E_\nu \subset \Delta_1 \text{ of } \mu_u\text{-measure zero such that for} \right. \\ \left. \text{all } h \in \Delta_1 - E_\nu, \text{ fine } \liminf_{x \rightarrow h} \nu(x) \geq f(h) \right\}$$

$$\tilde{\Sigma}_f = \left\{ \nu \in \Sigma : \exists \text{ a set } F_\nu \subset \Delta_1 \text{ of } \mu_u\text{-measure zero such that for} \right. \\ \left. \text{all } h \in \Delta_1 - F_\nu, \text{ fine } \limsup_{x \rightarrow h} \nu(x) \geq f(h) \right\}$$

DEFINITION. — Corresponding to any extended real valued function f on Δ_1 , define, for all $x \in \Omega$,

$$\mathcal{H}_{f,u}(x) = \inf \{ \nu(x) : \nu \in \Sigma_f \}$$

$$\underline{\mathcal{H}}_{f,u}(x) = - \overline{\mathcal{H}}_{-f,u}(x)$$

and

$$\overline{\mathcal{D}}_{f,u}(x) = \inf \{ \nu(x) : \nu \in \tilde{\Sigma}_f \}.$$

It is easy to see that Σ_f is a saturated family of super- u -harmonic functions [1]. Hence $\mathcal{H}_{f,u}$ is either identically $\pm \infty$ or it is a u -harmonic function. Moreover, from the minimum principle, we deduce that $\mathcal{H}_{f,u} \geq \underline{\mathcal{H}}_{f,u}$ on Ω .

Also $\mathcal{H}_{f,u} \geq \overline{\mathcal{D}}_{f,u}$.

DEFINITION 2. — Let $u(\mathcal{R})$ be the class of extended real valued functions f on Δ_1 such that, $\mathcal{H}_{f,u} = \underline{\mathcal{H}}_{f,u}$ and this function u -harmonic on Ω . For functions $f \in u(\mathcal{R})$, we denote $\mathcal{H}_{f,u} = \overline{\mathcal{H}}_{f,u} = \underline{\mathcal{H}}_{f,u}$.

LEMMA 3. — *Every bounded μ_u -measurable function f on Δ_1 belongs to $u(\mathcal{R})$ and moreover*

$$\mathcal{H}_{f,u} = \int f(h) \frac{h}{u} \mu_u(dh).$$

Proof. — The u -harmonic function $\sigma_f = \int f(h) \frac{h}{u} \mu_u(dh)$ satisfies,

$$\text{fine lim}_{x \rightarrow h} \sigma_f(x) = f(h) \quad \text{for} \quad \mu_u\text{-almost every } h \in \Delta_1$$

(Theorem 4). Hence, $\overline{\mathcal{H}}_{f,u} \leq \sigma_f \leq \underline{\mathcal{H}}_{f,u}$. This completes the proof.

PROPOSITION 1. — *Let $\{f_n\}$ be an increasing sequence of extended real functions such that $\overline{\mathcal{H}}_{f_n,u} > -\infty$. Then,*

$$\lim \overline{\mathcal{H}}_{f_n,u} = \overline{\mathcal{H}}_{f,u}.$$

Proof. — Since $\overline{\mathcal{H}}_{f_n,u} \leq \overline{\mathcal{H}}_{f,u}$, for every n , it is enough to show that $\overline{\mathcal{H}}_{f,u} \leq \lim_{n \rightarrow \infty} \overline{\mathcal{H}}_{f_n,u}$, when the limit is not $+\infty$. Let $x_0 \in \Omega$. Given $\varepsilon > 0$, choose for every n , an element $\nu_n \in \Sigma_{f_n}$ such that

$$\overline{\mathcal{H}}_{f_n,u}(x_0) \geq \nu_n(x_0) - \frac{\varepsilon}{2^n}.$$

Consider $\omega = \lim \overline{\mathcal{H}}_{f_n,u} + \sum_{n=1}^{\infty} (\nu_n - \overline{\mathcal{H}}_{f_n,u})$. It is easily seen that ω is a super- u -harmonic function. Moreover $\omega \geq \nu_n$, for every n . Hence ω is lower bounded on Ω . Also, if E_{ν_n} is the set contained in Δ_1 such that $\mu_u(E_{\nu_n}) = 0$ and for all $h \in \Delta_1 - E_{\nu_n}$, $\text{fine lim inf}_{x \rightarrow h} \nu_n(x) \geq f_n(h)$, then,

$$\text{fine lim inf}_{x \rightarrow h} \omega(x) \geq f(h),$$

for all $h \in \Delta_1 - \bigcup_{n=1}^{\infty} E_{\nu_n}$. It follows that $\omega \in \Sigma_f$. Hence $\omega \geq \overline{\mathcal{H}}_{f,u}$. But,

$$\overline{\mathcal{H}}_{f,u}(x_0) \leq \omega(x_0) \leq \lim \overline{\mathcal{H}}_{f_n,u}(x_0) + \varepsilon.$$

The proof is now completed easily.

The following proposition is proved easily.

PROPOSITION 2. — $u(\mathcal{R})$ is a real vector space. Moreover, for $f, g \in u(\mathcal{R})$, $\mathcal{H}_{f,u} + \mathcal{H}_{g,u} = \mathcal{H}_{f+g,u}$.

LEMMA 4. — For any non-negative extended real valued function f on Δ_1 , $\mathcal{H}_{f,u} = 0$ is equivalent to the fact that $f = 0$ μ_u -almost everywhere.

Proof. — Suppose $f = 0$ except on a set of μ_u -measure zero. Let $\nu \in \Sigma_f$. Then clearly $\frac{1}{n} \nu \in \Sigma_f$ for all positive integers n . Hence $\mathcal{H}_{f,u} = 0$.

Conversely, suppose $\mathcal{H}_{f,u} = 0$. Let $A_n = \left\{ h : f(h) > \frac{1}{n} \right\}$. Then the characteristic function χ_n of $A_n \subset \Delta_1$ has the property that $\mathcal{H}_{\chi_n,u} = 0$. The lemma would be proved if we show that for any set $A \subset \Delta_1$, $\mathcal{H}_{\chi_A,u} = 0$ implies that $\mu_u^*(A) = 0$.

Let $\nu \in \Sigma_{\chi_A}$. That is, there exists a set E_ν of μ_u -measure zero such that fine $\liminf_{x \rightarrow h} \nu(x) \geq \chi_A(h)$, for all $h \in \Delta_1 - E_\nu$. Given $\varepsilon > 0$, let $V_\varepsilon = \{x \in \Omega : \nu(x) > 1 - \varepsilon\}$. Then, V_ε is an open set and V_ε is not thin at any point of $h \in A - E_\nu$. Now,

$$\frac{u\nu}{1 - \varepsilon} \geq R_u^{V_\varepsilon} \geq \int h \chi_{A-E_\nu}(h) \mu_u(dh) = \int h \chi_A(h) \mu_u(dh).$$

This inequality is true for all $\varepsilon > 0$. Hence

$$\nu \geq \int \frac{h}{u} \chi_A(h) \mu_u(dh).$$

In turn, this inequality is true for all $\nu \in \Sigma_{\chi_A}$, and we deduce,

$$\mathcal{H}_{\chi_A,u} \geq \frac{1}{u} \int h \chi_A(h) \mu_u(dh).$$

Hence, if $\mathcal{H}_{\chi_A,u} = 0$, then $\int h \chi_A(h) \mu_u(dh) = 0$. Now, we deduce easily that $\mu_u^*(A) = 0$. This completes the proof.

THEOREM 6. — Every μ_u -summable function f on Δ_1 belongs to $u(\mathcal{R})$ and moreover, $\mathcal{H}_{f,u}(x) = \int f(h) \frac{h(x)}{u(x)} \mu_u(dh)$ on Ω .

Proof. — Suppose f is a non-negative μ_u -summable function on Δ_1 . For each positive integer n , if, $f_n = \inf(f, n)$, then

$f_n \in u(\mathcal{R})$ and $\mathcal{H}_{f_n, u} = \int f_n(h) \frac{h}{u} \lambda_u(dh)$. (Lemma 3). Hence, we have,

$$\begin{aligned}\overline{\mathcal{H}}_{f, u} &= \lim_{n \rightarrow \infty} \mathcal{H}_{f_n, u} \quad (\text{Proposition 1}) \\ &= \lim_{n \rightarrow \infty} \int f_n(h) \frac{h}{u} \mu_u(dh) \\ &= \int f(h) \frac{h}{u} \mu_u(dh).\end{aligned}$$

Also,

$$\int f(h) \frac{h}{u} \mu_u(dh) = \lim \mathcal{H}_{f_n, u} \leq \underline{\mathcal{H}}_{f, u}.$$

It follows that

$$f \in u(\mathcal{R}) \quad \text{and} \quad \mathcal{H}_{f, u} = \int f(h) \frac{h}{u} \mu_u(dh).$$

Now the proof is completed easily.

Remark. — It can be proved that any function $f \in u(\mathcal{R})$ is necessarily equal μ_u -almost everywhere to a μ_u -summable function and that $\mathcal{H}_{f, u}$ is precisely $\int f(h) \frac{h}{u} \mu_u(dh)$.

4. The Main Result.

THEOREM 7. — *Let $f \geq 0$ be an extended real valued function on Δ_1 . Then, $\overline{\mathcal{D}}_{f, u} = \overline{\mathcal{H}}_{f, u}$.*

Proof. — It is enough to show that $\overline{\mathcal{D}}_{f, u} \geq \overline{\mathcal{H}}_{f, u}$.

First of all consider a function $f \geq 0$ which is bounded, say $f \leq M$. Consider $\tilde{\Sigma}_f^M = \{\nu \in \tilde{\Sigma}_f : \nu \leq M\}$. We assert that $\overline{\mathcal{D}}_{f, u} = \inf \{\nu : \nu \in \tilde{\Sigma}_f^M\}$. For, suppose $\nu \in \tilde{\Sigma}_f$. Then $\nu_M = \inf(\nu, M)$ is a super- u -harmonic function and satisfies

$$\limsup_{x \rightarrow h} \nu_M(x) \geq f(h),$$

for μ_u -almost every $h \in \Delta_1$. Hence, $\nu \geq \nu_M \geq \inf \{\nu : \nu \in \tilde{\Sigma}_f^M\}$. Hence $\overline{\mathcal{D}}_{f, u} \geq \inf \{\nu : \nu \in \tilde{\Sigma}_f^M\}$. The opposite inequality is obvious.

Now, let $\nu \in \tilde{\Sigma}_f^M$. Then, by Theorem 4, Remark 3, the

fine limit $\nu(x)$ exists for all $h \in \Delta_1 - E'_v$, where $\mu_u(E'_v) = 0$. But, by the defining property of $\nu \in \tilde{\Sigma}_f^M$, fine $\limsup_{x \rightarrow h} \nu(x) \geq f(h)$ for all $h \in \Delta_1 - F_v$, where $\mu_u(F_v) = 0$. It follows that,

$$\text{fine } \liminf_{x \rightarrow h} \nu(x) \geq f(h),$$

for all $h \in \Delta_1 - (E'_v \cup F_v)$. Hence, $\nu \geq \overline{\mathcal{H}}_{f,u}$. This is true for all $\nu \in \tilde{\Sigma}_f^M$ and we get that $\overline{\mathcal{D}}_{f,u} \geq \overline{\mathcal{H}}_{f,u}$.

Let us now consider any $f \geq 0$. Let, for every positive integer n , $f_n = \inf(f, n)$. Then, we have,

$$\overline{\mathcal{D}}_{f,u} \geq \lim \overline{\mathcal{D}}_{f_n,u} = \lim \overline{\mathcal{H}}_{f_n,u} = \overline{\mathcal{H}}_{f,u}.$$

This completes the proof of the theorem.

THEOREM 7. — *For every μ_u -summable function f on Δ_1 ,*

$$\text{fine } \lim_{x \rightarrow h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) = f(h_0),$$

for μ_u almost every $h_0 \in \Delta_1$.

Proof. — It is enough to prove the theorem assuming that $f \geq 0$. Define, for every $h_0 \in \Delta_1$,

$$\varphi'(h_0) = \text{fine } \limsup_{x \rightarrow h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh).$$

Let $\varphi = \sup(\varphi', f)$ and $\nu \in \tilde{\Sigma}_f$. Then, $\nu \geq \int f(h) \frac{h}{u} \mu_u(dh)$ and we see easily that the fine $\limsup_{x \rightarrow h} \nu(x) \geq \varphi(h)$, for μ_u -almost every $h \in \Delta_1$. It follows that $\nu \in \tilde{\Sigma}_\varphi$. This is true for all $\nu \in \tilde{\Sigma}_f$. Hence, $\overline{\mathcal{H}}_{f,u} \geq \overline{\mathcal{H}}_{\varphi,u}$. But $\overline{\mathcal{H}}_{f,u} = \overline{\mathcal{H}}_{f,u} \leq \overline{\mathcal{H}}_{\varphi,u}$. This implies that $\varphi \in u(\mathcal{R})$ and $\overline{\mathcal{H}}_{\varphi,u} = \overline{\mathcal{H}}_{f,u}$. Again, $\varphi - f \geq 0$ and $\overline{\mathcal{H}}_{\varphi-f,u} = 0$. We get, from the Lemma 4, that, $\varphi = f$, μ_u -almost everywhere. Hence,

$$\text{fine } \limsup_{x \rightarrow h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) \leq f(h_0)$$

for μ_u -almost every $h_0 \in \Delta_1$. But we have already proved that

the fine \liminf is $\geq f(h_0)$ for μ_u -almost every $h_0 \in \Delta_1$. Hence, we get,

$$\text{fine } \lim_{x \rightarrow h_0} \int f(h) \frac{h(x)}{u(x)} \mu_u(dh) = f(h_0)$$

for μ_u -almost every $h_0 \in \Delta_1$, completing the proof of the theorem.

THEOREM 8. (Fatou-Naïm-Doob). *For any $\nu \in S^+$, $\frac{\nu}{u}$ has a finite limit at μ_u -almost every element of Δ_1 .*

Proof. — Let ν be the canonical measure on Δ_1 corresponding to the greatest harmonic minorant of ν . Let ν_1 (respectively ν_2) be the absolutely continuous (resp. singular) part of ν relative to μ_u . Let f be the Radon-Nikodym derivative of ν_1 relative to μ_u (f is defined upto a set of μ_u measure zero). Then

$$\nu = \nu_1 + \nu_2 + \nu_3$$

where ν_3 is a potential, $\nu_2 = \int h \nu_2(dh)$ and $\nu_1 = \int f(h) h \mu_u(dh)$.

Now, $\frac{\nu_1}{u}$ has the fine limit f (note that f is finite μ_u almost everywhere), for μ_u -almost every element of Δ_1 . Also, $\frac{\nu_2 + \nu_3}{u}$ has the fine limit zero at μ_u -almost every element of Δ_1 . This completes the proof of the theorem.

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