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ON MICROLOCAL ANALYTICITY OF SOLUTIONS OF FIRST-ORDER NONLINEAR PDE

by Shif BERHANU (*)

ABSTRACT. — We study the microlocal analyticity of solutions u of the nonlinear equation

$$u_t = f(x,t,u,u_x)$$

where $f(x,t,\zeta_0,\zeta)$ is complex-valued, real analytic in all its arguments and holomorphic in (ζ_0,ζ). We show that if the function u is a C^2 solution, $\sigma \in \operatorname{Char} L^u$ and $\frac{1}{i}\sigma([L^u,\overline{L^u}]) < 0$ or if u is a C^3 solution, $\sigma \in \operatorname{Char} L^u$, $\sigma([L^u,\overline{L^u}]) = 0$, and $\sigma([L^u,[L^u,\overline{L^u}]]) \neq 0$, then $\sigma \notin WF_a u$. Here $WF_a u$ denotes the analytic wave-front set of u and $\operatorname{Char} L^u$ is the characteristic set of the linearized operator. When $m = 1$, we prove a more general result involving the repeated brackets of L^u and $\overline{L^u}$ of any order.

RÉSUMÉ. — Nous étudions l'analyticité microlocale des solutions de l'équation non linéaire

$$u_t = f(x,t,u,u_x)$$

où $f(x,t,\zeta_0,\zeta)$ est une fonction analytique réelle, à valeurs complexes, et holomorphe en (ζ_0,ζ). Nous montrons que si u est une solution de classe C^2, $\sigma \in \operatorname{Char} L^u$ et $\frac{1}{i}\sigma([L^u,\overline{L^u}]) < 0$, ou si u est une solution de classe C^3, $\sigma \in \operatorname{Char} L^u$, $\sigma([L^u,\overline{L^u}]) = 0$ et $\sigma([L^u,[L^u,\overline{L^u}]]) \neq 0$, alors $\sigma \notin WF_a(u)$. Ici, $WF_a(u)$ désigne le front d'onde analytique de u et $\operatorname{Char} L^u$ l'ensemble caractéristique de l'opérateur linéarisé. Quand $m = 1$, nous démontrons un résultat plus général faisant intervenir les crochets des opérateurs L^u et $\overline{L^u}$ de tout ordre.

1. Introduction

This paper studies the local and microlocal analyticity of solutions of the nonlinear PDE

$$u_t = f(x,t,u,u_x) \tag{1.1}$$

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where u is always assumed to be at least C^2, $f(x,t,\zeta_0,\zeta)$ is complex-valued, real analytic in all its arguments and holomorphic in (ζ_0,ζ). The variable x varies in an open subset of $\mathbb{R}^m$, t in an interval in $\mathbb{R}$, and (ζ_0,ζ) varies in an open subset of $\mathbb{C}^{m+1}$.

When u is a C^2 solution of (1.1), it was proved in [7] that the analytic wave-front set of u is contained in the characteristic set of the linearized operator

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)\frac{\partial}{\partial x_j}. \tag{1.2}$$

For the analogous result in the C^∞ case see [5] and [1]. Here we prove that if u is a C^2 solution of (1.1), $\sigma \in \operatorname{Char} L^u$ (= the characteristic set of L^u) and $\frac{1}{i}\sigma([L^u,\overline{L^u}]) < 0$ or u is a C^3 solution of (1.1), $\sigma \in \operatorname{Char} L^u$, $\sigma([L^u,\overline{L^u}]) = 0$, and $\sigma([L^u,[L^u,\overline{L^u}]]) \neq 0$, then $\sigma \notin WF_a u$ where $WF_a u$ denotes the analytic wave-front set of u. In the linear case, for a real analytic vector field with no singularities, these results are due to H. Lewy and C. H. Chang [4] respectively. Chang's result was generalized to a real analytic, linear partial differential operator of principal type in the works [8] and [9]. In this paper we follow the approach of [7] which requires that we prove the corresponding regularity results for a nonanalytic vector field L which has only C^1 coefficients when u is C^2 and C^2 coefficients when u is C^3. Since the known linear results require one more derivative for the first integrals of L, we give here a self contained proof. Actually, to prove Chang's result when the vector field L has lower regularity (Lemma 3.2), we also assume that the vector field satisfies an additional condition (see condition (3.22)) which involves the existence of first integrals satisfying convenient Cauchy conditions on each noncharacteristic hyperplane through the origin. Fortunately, this additional condition is satisfied by the linearized operator L^u. With this additional condition, we are able to use the ideas in the more recent article [6] to prove Chang's result for L of lower regularity. Observe that the brackets $[L^u,\overline{L^u}]$ and $[L^u,[L^u,\overline{L^u}]]$ are defined when u is C^2 and C^3 respectively. When $m = 1$ and u is a C^k solution of (1.1), we prove a microlocal analyticity result that involves assumptions on brackets of L^u and $\overline{L^u}$ up to length k.

Complex-valued solutions of first order nonlinear pdes arise in numerous applications. For example, the initial value problem for the complex inviscid Burger's equation

$$u_t + u\,u_x = 0, \quad u(x,0) = f(x)$$

has complex-valued solutions of physical significance (see [3]). This complex Burger's equation also arises in geometrical problems (see for example [11] and [10]).

The article is organized as follows. In section 2 we state the main results and present some examples. Section 3 is devoted to results for linear vector fields. Section 4 applies the results in section 3 to the nonlinear pde (1.1).

2. Statement of results and examples

In the sequel $f(z, w, \zeta_0, \zeta)$ will denote a holomorphic function in a neighborhood $\Omega \times \mathcal{N}$ of $((0,0),(a,\omega))$ in $\mathbb{C}^{m+1} \times \mathbb{C}^{m+1}$. We assume $U \subset \Omega \cap \mathbb{R}^{m+1}$ is a neighborhood of $(0,0) \in \mathbb{R}^{m+1}$ and we will consider a solution $u \in C^2(U)$ of

$$u_t = f(x,t,u,u_x)$$

under the assumption that

$$u(0,0) = a,\ u_x(0,0) = \omega, \quad \text{and } (u(x,t), u_x(x,t)) \in \mathcal{N} \text{ for all } (x,t) \in U.$$

Let

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)\frac{\partial}{\partial x_j}.$$

Theorem 2.1. — *Suppose $u \in C^2(U)$ is a solution of the nonlinear pde*

$$u_t = f(x,t,u,u_x).$$

If $\sigma \in \operatorname{Char} L^u$ and $\frac{1}{i}\sigma([L^u, \overline{L^u}]) < 0$, then $\sigma \notin WF_a u$.

Theorem 2.2. — *Suppose $u \in C^3(U)$ is a solution of*

$$u_t = f(x,t,u,u_x).$$

If $\sigma \in \operatorname{Char} L^u$, $\sigma([L^u, \overline{L^u}]) = 0$, and $\sigma([L^u,[L^u, \overline{L^u}]]) \neq 0$, then $\sigma \notin WF_a u$.

In Theorem 2.3 below we will consider brackets of the planar vector fields $L = \dfrac{\partial}{\partial t} + c(x,t)\dfrac{\partial}{\partial x}$ and $\overline{L}$ of order k. By a bracket of order 1 we mean L or $\overline{L}$, order 2 will mean $[L, \overline{L}]$, while a bracket of order 3 by definition is either $[L,[L,\overline{L}]]$ or $[\overline{L},[L,\overline{L}]]$. Continuing this way, a bracket of order j by definition has the form $[L, M_{j-1}]$ or $[\overline{L}, M_{j-1}]$ where M_{j-1} is a bracket of order $j-1$.

Theorem 2.3. — *Assume $m = 1$ and $u \in C^k(U)$ is a solution of*

$$u_t = f(x, t, u, u_x).$$

Assume that $\sigma(L) = 0$ whenever L is a repeated bracket of L^u and $\overline{L^u}$ of length $< k$ and $\sigma(M) \neq 0$ for some repeated bracket of L^u and $\overline{L^u}$ of length k. If k is even and $\frac{1}{i}\sigma(M) < 0$, then $\sigma \notin WF_a u$, and if k is odd, $\sigma \notin WF_a u$.

Example 2.4. — Let u be a C^3 solution of the equation

$$u_t + u\, u_x = \lambda(x, t).$$

where $\lambda(x, t)$ is a real analytic function in a neighborhood of the origin in $\mathbb{R}^2$. If $\operatorname{Im} u(0) \neq 0$, then the linearized operator L^u is elliptic and so by the main result of [7], u is real analytic near the origin. We assume that $\operatorname{Im} u(0) = 0$. Using this and the equation that u satisfies, we get

$$[L^u, L^{\overline{u}}](0) = (\overline{\lambda(0)} - \lambda(0))\frac{\partial}{\partial x}.$$

Hence by Theorem 2.1, if $\sigma = (0, 0; \xi^0, \tau^0) \in \operatorname{Char} L^u$, and $\operatorname{Im} \lambda(0)\xi^0 > 0$, then $\sigma \notin WF_a(u)$, while if $\operatorname{Im} \lambda(0)\xi^0 < 0$, then $-\sigma \notin WF_a(u)$. Next assume that $\operatorname{Im} \lambda(0) = 0$. Then we have

$$[L^u, [L^u, L^{\overline{u}}]](0) = \left(\overline{\lambda_t(0)} - \lambda_t(0) + u(0)(\overline{\lambda_x(0)} - \lambda_x(0))\right)\frac{\partial}{\partial x}.$$

By Theorem 2.2, we conclude that if $\operatorname{Im} \lambda(0) = 0$, $\operatorname{Im} \lambda_x(0) = 0$, and $\operatorname{Im} \lambda_t(0) \neq 0$, then u is real analytic near the origin.

Example 2.5. — Consider next the semilinear equation

$$\frac{\partial u}{\partial t} + it^{2k} a(x, t)\,\frac{\partial u}{\partial x} = f(x, t, u)$$

where $a(x, t)$ is real analytic near the origin in $\mathbb{R}^2$, $f(x, t, \zeta_0)$ is real analytic in all variables, and holomorphic in ζ_0. If k is a nonnegative integer and $\Re a(0, 0) \neq 0$, then by Theorem 2.3 and the result in [7], any solution u is real analytic near the origin.

3. Some lemmas on first-order linear pdes

Let

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} c_j(x, t)\frac{\partial}{\partial x_j} \tag{3.1}$$

be a complex vector field in an open neighborhood Ω of the origin in $\mathbb{R}^{m+1}$. In Lemma 3.1 below we will assume that the coefficients $c_j \in C^1(\Omega)$. We will assume that there are m complex-valued functions Ψ_i $(1 \leqslant i \leqslant m)$ which are C^1 in Lemma 3.1 and C^2 in Lemma 3.2 such that

$$Z_i(x,t) = x_i + t\Psi_i(x,t)$$

solve

$$LZ_i = 0, \quad 1 \leqslant i \leqslant m. \tag{3.2}$$

We will write $\Psi = (\Psi_1, \dots, \Psi_m)$, and $Z = (Z_1, \dots, Z_m)$. Observe that at a point $(x,0)$ near the origin, the characteristic set of L is given by

$$\operatorname{Char} L|_{(x,0)} = \{(x,0;\xi,\tau) : \operatorname{Im}\Psi(x,0)\cdot\xi = 0,\ \tau = \Re\Psi(x,0)\cdot\xi,\ (\xi,\tau) \neq (0,0)\}. \tag{3.3}$$

The latter follows from the equations,

$$c(x,t) = -Z_x^{-1} \cdot Z_t,\ Z_x = I + t\Psi_x, \quad \text{and } Z_t = \Psi + t\Psi_t.$$

Lemma 3.1. — *Suppose L has C^1 coefficients and the $\Psi_j \in C^1(\Omega)$. Let $h \in C^1(\Omega)$ be a solution of $Lh = 0$. If $\sigma = (0,0;\xi^0,\tau^0) \in \operatorname{Char} L$ and $\frac{1}{2i}\sigma([L,\overline{L}]) < 0$, then $(0,\xi^0) \notin WF_a h(x,0)$.*

Proof. — By adding variables as in [4], we may assume that L is a CR vector field near the origin. This means that for some j, $\operatorname{Im}\Psi_j(0) \neq 0$. Without loss of generality assume that

$$\operatorname{Im}\Psi_1(0) \neq 0. \tag{3.4}$$

Observe next that the linear change of coordinates

$$x_l' = x_l + t\Re\Psi_l(0),\ t' = t$$

allow us to assume, after dropping the primes, that

$$\Re\Psi_j(0) = 0, \text{ for all } j = 1, ..., m. \tag{3.5}$$

We can use (3.4) and (3.5) to replace $Z_2, \dots, Z_m$ by a linear combination of $Z_1, \dots, Z_m$ and apply a linear change of coordinates to get

$$Z_j = x_j + t\Psi_j,\ 1 \leqslant j \leqslant m, \text{ and } \Psi_1(0) = i, \Psi_j(0) = 0, \text{ for } 2 \leqslant j \leqslant m. \tag{3.6}$$

The equation $LZ_l = 0$ implies that

$$\Psi_l + t\frac{\partial \Psi_l}{\partial t} + c_l + \sum_{j=1}^{m} c_j t \frac{\partial \Psi_l}{\partial x_j} = 0 \tag{3.7}$$

and so from (3.6) and (3.7),

$$c_1(0) = -i \text{ and } c_j(0) = 0 \text{ for } j \geqslant 2. \tag{3.8}$$

The condition that $(0,0;\xi^0,\tau^0)\in \mathrm{Char}\, L$ therefore means that $\tau^0=0=\xi_1^0$ and $\xi_j^0\neq 0$ for some $j\geqslant 2$. In particular, $\xi^0\neq 0$ and

$$\xi^0\cdot \mathrm{Im}\,\Psi(0)=0. \tag{3.9}$$

We may assume that

$$\xi^0=(0,1,0,\ldots,0). \tag{3.10}$$

We have

$$[L,\overline{L}]=\sum_{l=1}^m A_l(x,t)\frac{\partial}{\partial x_l}$$

where

$$A_l(x,t)=\frac{\partial \overline{c_l}}{\partial t}-\frac{\partial c_l}{\partial t}+\sum_{j=1}^m c_j\frac{\partial \overline{c_l}}{\partial x_j}-\sum_{j=1}^m \overline{c_j}\frac{\partial c_l}{\partial x_j}. \tag{3.11}$$

We will express $A_l(0,0)$ using the Ψ_j. From (3.7) we have

$$\Psi_l(x,0)+c_l(x,0)=0. \tag{3.12}$$

Subtract (3.12) from (3.7), divide by t, and let $t\to 0$ to arrive at (recalling that Ψ and L are C^1):

$$2\frac{\partial \Psi_l}{\partial t}(x,0)+\frac{\partial c_l}{\partial t}(x,0)+\sum_{j=1}^m c_j(x,0)\frac{\partial \Psi_l}{\partial x_j}(x,0)=0. \tag{3.13}$$

From (3.12) and (3.13), we get:

$$\frac{\partial c_l}{\partial t}(x,0)=-2\frac{\partial \Psi_l}{\partial t}(x,0)+\sum_{j=1}^m \Psi_j(x,0)\frac{\partial \Psi_l}{\partial x_j}(x,0). \tag{3.14}$$

Thus from (3.6), (3.11), (3.12) and (3.14), we conclude

$$A_l(0,0)=4i\frac{\partial\,\mathrm{Im}\,\Psi_l}{\partial t}(0).$$

Thus, the assumption that $\frac{1}{2i}\sigma([L,\overline{L}])<0$ implies that

$$\frac{\partial\,\mathrm{Im}\,\Psi_2}{\partial t}(0)=\frac{\partial\,\mathrm{Im}\,\Psi}{\partial t}(0)\cdot\xi^0<0. \tag{3.15}$$

Next, we show that coordinates (x,t) and first integrals $Z_l=x_l+t\Psi_l$ can be chosen so that (3.6), (3.10) and (3.15) still hold and in addition,

$$\frac{\partial\,\mathrm{Im}\,\Psi_l}{\partial x_j}(0)=0\quad\text{for all } l,j.$$

Define

$$\widetilde{Z}_l(x,t)=Z_l+\sum_{k=1}^m a_{lk}Z_1Z_k\ \text{l = 1, ..., m,}$$

where

$$a_{l,k} = \begin{cases} -\frac{1}{2}\dfrac{\partial \operatorname{Im}\Psi_l}{\partial x_k}(0), & k=1 \\ \\ -\dfrac{\partial \operatorname{Im}\Psi_l}{\partial x_k}(0), & 2 \leqslant k \leqslant m. \end{cases}$$

Note that

$$\widetilde{Z}_l(x,t) = x_l + \sum_{k=1}^{m} a_{lk} x_1 x_k + t\widetilde{\Psi_l}(x,t),$$

where

$$\widetilde{\Psi_l}(x,t) = \Psi_l + \sum_k a_{l,k}(x_1\Psi_k + x_k\Psi_1 + t\Psi_1\Psi_k).$$

By the choice of the a_{lk} and the fact that $\Psi_1(0) = i$, we have

$$\frac{\partial \widetilde{\operatorname{Im}\Psi_l}}{\partial x_j}(0) = 0 \text{ for all } l,j.$$

Introduce new coordinates

$$\tilde{x}_l = x_l + \sum_{k=1} a_{lk} x_1 x_k, \quad \tilde{t} = t,\ 1 \leqslant l \leqslant m.$$

These change of coordinates are smooth and hence L is still C^1 in these coordinates. After dropping the tildes both in the new coordinates and the first integrals, we have:

$$Z_j = x_j + t\Psi_j \quad \text{with} \quad \frac{\partial \operatorname{Im}\Psi_l}{\partial x_j}(0) = 0 \quad \text{for all} \quad l,j \tag{3.16}$$

and (3.6), (3.10) and (3.15) still hold. Moreover, the new coordinates preserve the set $\{t=0\}$ and so L still has the form

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} c_j(x,t)\frac{\partial}{\partial x_j}.$$

Let $\eta(x) \in C_0^\infty(B_r(0))$, where $B_r(0)$ is a ball of small radius r centered at $0 \in \mathbb{R}^m$ and $\eta(x) \equiv 1$ when $|x| \leqslant r/2$. We will be using the FBI transform

$$F_\kappa(t,z,\zeta) = \int_{\mathbb{R}^m} e^{i\zeta\cdot(z-Z(x,t)) - \kappa\langle\zeta\rangle[z-Z(x,t)]^2}\eta(x)h(x,t)\,dZ$$

where for $z \in \mathbb{C}^m$, we write $[z]^2 = \sum_{j=1}^m z_j^2$, $\langle\zeta\rangle = (\zeta\cdot\zeta)^{1/2}$ is the main branch of the square root, $dZ = dZ_1 \wedge \ldots \wedge dZ_m = \det Z_x(x,t)dx_1 \wedge \ldots \wedge dx_m$, and $\kappa > 0$ is a parameter which will be chosen later.

To prove that $(0,\xi^0) \notin WF_a(h(x,0))$, we need to show that for some $\kappa > 0$ and constants $C_1, C_2 > 0$,

$$\begin{aligned} |F_\kappa(0,z,\zeta)| &= \left| \int e^{i\zeta.(z-x)-\kappa\langle\zeta\rangle[z-x]^2} \eta(x)h(x,0)\,dx \right| \\ &\leqslant C_1 e^{-c_2|\zeta|} \end{aligned} \tag{3.17}$$

for z near 0 in $\mathbb{C}^m$ and ζ in a conic neighborhood of ξ^0 in $\mathbb{C}^m$. Let $U = B_r(0) \times (0,\delta)$ for some δ small. Since h and the Z_j are solutions, the form

$$\omega = e^{i\zeta.(z-Z(x,t))-\kappa\langle\zeta\rangle[z-Z(x,t)]^2} h(x,t) dZ_1 \wedge dZ_2 \wedge \dots \wedge dZ_m$$

is a closed form. This is well known when the Z_j are C^2 and when they are only C^1 as in our case, one can prove that ω is closed by approximating the Z_j by smoother functions. By Stokes' theorem, we therefore have

$$F_\kappa(0,z,\zeta) = \int_{\{t=0\}} \eta\omega = \int_{t=\delta} \eta\omega - \iint_U d\eta \wedge \omega. \tag{3.18}$$

We will show that κ, δ and $r > 0$ can be chosen so that each of the two integrals on the right side of (3.18) satisfies an estimate of the form (3.17). Set

$$Q(z,\zeta,x,t) = \frac{\Re(i\zeta \cdot (z - Z(x,t)) - \kappa\langle\zeta\rangle[z - Z(x,t)]^2)}{|\zeta|}.$$

Observe that it is sufficient to show that there is $C > 0$ so that $Q(0,\xi^0,x,t) \leqslant -C$ for $(x,t) \in (\operatorname{supp}\eta \times \{\delta\}) \cup (\operatorname{supp} d\eta \times [0,\delta])$. For then, $Q(z,\zeta,x,t) \leqslant -C/2$ for the same (x,t), z near 0 in $\mathbb{C}^m$, and ζ in a conic neighborhood of ξ^0 in $\mathbb{C}^m$. We recall that $\xi^0 = (0,1,\dots,0)$, and so $|\xi^0| = 1$. We have:

$$\begin{aligned} Q(0,\xi^0,x,t) &= \Re(-i\xi^0 \cdot (x + t\Psi) - \kappa[x + t\Psi]^2) \\ &= t\xi^0 \cdot \operatorname{Im}\Psi(x,t) - \kappa[|x|^2 + t^2|\Re\Psi|^2 + 2t\langle x, \Re\Psi\rangle - t^2|\operatorname{Im}\Psi|^2]. \end{aligned} \tag{3.19}$$

Since Ψ is C^1, using (3.6), (3.15), and (3.16),

$$t(\xi^0 \cdot \operatorname{Im}\Psi(x,t)) = -C_1 t^2 + o(|x|t + t^2) \tag{3.20}$$

where $C_1 = -\dfrac{\partial \operatorname{Im}\Psi}{\partial t}(0) \cdot \xi^0 > 0$. Let $C \geqslant |\operatorname{Im}\Psi|^2 + 1$ on U, and set $\alpha = \frac{C_1}{8C}$. Note that (3.20) allows us to choose r and δ small enough so that on U,

$$t(\xi^0 \cdot \operatorname{Im}\Psi(x,t)) \leqslant -\frac{C_1}{2}t^2 + \alpha|x|^2. \tag{3.21}$$

From (3.19) and (3.21), we get:

$$Q(0,\xi^0,x,t) \leqslant -\frac{C_1}{2}t^2 + \alpha|x|^2 - \kappa[|x|^2 - 2t|x||\Re\Psi| - t^2|\operatorname{Im}\Psi|^2].$$

Since $\Re\Psi(0) = 0$, we may assume r and δ are small enough so that

$$2t|x||\Re\Psi| \leqslant t^2 + |x|^2/2$$

and hence

$$Q(0,\xi^0,x,t) \leqslant -\frac{C_1}{2}t^2 + \alpha|x|^2 - \kappa|x|^2 - \kappa|x|^2/2 + \kappa C t^2.$$

Choose $\kappa = \frac{3C_1}{8C}$. Recalling that $\alpha = \frac{C_1}{8C}$, we get:

$$Q(0,\xi^0,x,t) \leqslant -\frac{C_1}{8}t^2 - \frac{C_1}{16C}|x|^2$$

and so on $\operatorname{supp}\eta \times \{\delta\} \cup (\operatorname{supp}(d\eta) \times [0,\delta])$, $Q(0,\xi^0,x,t) \leqslant -C$ for some $C > 0$. This proves the Lemma. □

In the following lemma we will assume that the vector field

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^m c_j(x,t)\frac{\partial}{\partial x_j}$$

satisfies the following condition:

(3.22)

for each hyperplane Σ *in* $\mathbb{R}^{m+1}$ *of the form* $\Sigma = \Big\{(x,t) : t = \sum_{j=1}^m a_j x_j\Big\}$,

there exist C^2 *functions* Z_j^Σ *near* 0 *such that* $LZ_j^\Sigma = 0$ *for* $j = 1,\dots,m$

and $Z_j^\Sigma(x,t) = x_j + \Big(t - \sum_{k=1}^m a_k x_k\Big)\Psi_j^\Sigma(x,t)$ *for some* C^2 *functions* Ψ_j^Σ.

Lemma 3.2. — *Suppose* $L = \frac{\partial}{\partial t} + \sum_{j=1}^m c_j(x,t)\frac{\partial}{\partial x_j}$ *is* C^2 *and* CR *at* 0. *Assume that* L *satisfies condition (3.22) above and* $h \in C^1(\Omega)$ *is a solution of* $Lh = 0$. *If* $\sigma = (0,0;\xi^0,\tau^0) \in \operatorname{Char} L$, $\sigma([L,\overline{L}]) = 0$, *and* $\sigma([L,[L,\overline{L}]]) \neq 0$, *then* $(0,\xi^0) \notin WF_a h(x,0)$.

Remark 3.3. — If L has first integrals that are of class C^3, then Lemma 3.2 would follow from the results in [4] (or [6]) and condition (3.22) is not needed.

Proof. — Since $\sigma([L,[L,\overline{L}]]) \neq 0$, we can find $\theta \in (0,2\pi)$, $\theta \notin \{\pi/2, 3\pi/2, \pi\}$ such that if $L' = e^{i\theta}L$, then

$$\Re\sigma([L',[L',\overline{L'}]]) > \sqrt{3}|\operatorname{Im}\sigma([L',[L',\overline{L'}]])|. \tag{3.23}$$

Using the fact that $L' = X + iY$ is CR near 0, we can find a hyperplane $\Sigma = \{(x,t) : t = \sum_{j=1}^m a_j x_j\}$ such that

$$X(0) \in T_0\Sigma \quad \text{and} \quad Y(0) \notin T_0\Sigma. \tag{3.24}$$

The hypotheses on L tell us that we can find C^2 functions

$$Z_j^\Sigma(x,t) = x_j + \left(t - \sum_{k=1}^m a_k x_k\right) \Psi_j^\Sigma(x,t) \quad 1 \leqslant j \leqslant m$$

with $\Psi_j^\Sigma(x,t)$ C^2 such that $LZ_j^\Sigma = 0$. Consider the change of coordinates $F(x,t) = (x, t - \sum_{j=1}^m a_j x_j) = (x', t')$. Observe that

$$F^*\tilde{\sigma} = \sigma, \quad \text{where} \quad \tilde{\sigma} = \sum_{j=1}^m (\xi_j^0 + a_j\tau^0)dx'_j + \tau^0 dt'. \tag{3.25}$$

If $\Sigma_0 = \{(x', t') : t' = -\sum_{j=1}^m a_j x'_j\}$, we need to show that

$$i_{\Sigma_0}^* \tilde{\sigma} \notin WF_a(h \circ F^{-1}|_{\Sigma_0}).$$

Observe next that this is equivalent to showing that

$$i_M^* \tilde{\sigma} = \sum_{j=1}^m (\xi_j^0 + a_j\tau^0) dx'_j \notin WF_a(\tilde{h}(x', 0)) \tag{3.26}$$

where $\tilde{h} = h \circ F^{-1}$ and $M = \{(x', 0)\}$. Indeed, according to Theorem 4.1 of [2], if θ^0 is a characteristic covector at the origin, X is a real analytic, maximally real submanifold through the origin, $\pi_X(\theta^0)$ denotes the pull-back of θ^0 to X, and h_X is the trace of h on X, then $\pi_X(\theta^0) \notin WF_a(h_X)$ if and only if $\pi_Y(\theta^0) \notin WF_a(h_Y)$ for any other Y like X. Using (3.24), we have

$$L' = \sum_{j=1}^m c_j e^{i\theta} \frac{\partial}{\partial x'_j} + ib \frac{\partial}{\partial t'}$$

where $b(0)$ is real and nonzero. Moreover, by replacing θ with $\theta + \frac{\pi}{2}$ if necessary, we may assume that $b(0) > 0$. Dividing by b will not affect the condition that

$$\Re\tilde{\sigma}([L', [L', \overline{L'}]]) > \sqrt{3}|\operatorname{Im} \tilde{\sigma}([L', [L', \overline{L'}]])|.$$

Therefore, we may assume that

$$L' = \sum_{j=1}^m a_j \frac{\partial}{\partial x'_j} + i \frac{\partial}{\partial t'}$$

where the a_j are C^2. Expressing the Z_j^Σ in (x', t') coordinates we have first integrals

$$Z_j(x', t') = x'_j + t'_j \Psi_j(x', t') \quad 1 \leqslant j \leqslant m,$$

with the Ψ_j C^2. We can now drop the primes and assume

$$L = \sum_{j=1}^{m} a_j \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial t} \tag{3.27}$$

is a C^2 vector field with C^2 first integrals

$$Z_j(x,t) = x_j + t\Psi_j(x,t) \quad 1 \leqslant j \leqslant m,$$

$\sigma = (\xi^0, \tau^0) \in \operatorname{Char} L, \sigma([L,\overline{L}]) = 0$, and

$$\Re\sigma([L,[L,\overline{L}]]) > \sqrt{3}|\operatorname{Im}\sigma([L,[L,\overline{L}]])|. \tag{3.28}$$

The function $h(x,t)$ is a C^1 solution near the origin and we need to show that $(0,\xi^0) \notin WF_a(h(x,0))$. Since L is CR near 0, $\Re a_j(0) \neq 0$ for some j, and so we may assume that $\Re a_1(0) \neq 0$. By stretching the x_1 coordinate, we may also assume that $\Re a_1(0) = 1$ and thus after dividing by 2,

$$L = \frac{\partial}{\partial \overline{z}} + \sum_{j=1}^{m} a_j \frac{\partial}{\partial x_j}, \quad \Re a_1(0) = 0. \tag{3.29}$$

where $z = x_1 + it$. The form (3.29) implies that $\operatorname{Im}\Psi_1(0) = 1$ and hence we can use linear change of coordinates and a substitution of $Z_2, ..., Z_m$ by a linear combination of $Z_1, ..., Z_m$ as in the proof of Lemma 3.1 to get

$$Z_j = x_j + t\Psi_j, \quad 1 \leqslant j \leqslant m \quad \text{and } \Psi(0) = (i, 0, ..., 0). \tag{3.30}$$

These changes will not affect the validity of (3.28) and we still have

$$L = \frac{\partial}{\partial \overline{z}} + \sum_{j=1}^{m} a_j \frac{\partial}{\partial x_j} \quad \text{for some } a_j \text{ that are } C^2.$$

We next proceed as in the proof of Lemma 3.1 to find coordinates (x,t) and first integrals

$$Z_j = x_j + t\Psi_j, \quad 1 \leqslant j \leqslant m$$

such that

$$\frac{\partial \operatorname{Im}\Psi_l}{\partial x_j}(0) = 0, \quad \text{for all } l, j. \tag{3.31}$$

Note that (3.30) still holds and the form of L is still the same. The equations $LZ_l = 0$ become

$$\frac{1}{2}\delta_{1l} + \frac{i}{2}\Psi_l + t(\Psi_l)_{\overline{z}} + a_l + \sum_{j=1}^{m} a_j t\frac{\partial \Psi_l}{\partial x_j} = 0 \tag{3.32}$$

which imply

$$\frac{1}{2} + \frac{i}{2}\Psi_1(x,0) + a_1(x,0) = 0, \quad \frac{i}{2}\Psi_j(x,0) + a_j(x,0) = 0 \quad \text{for } j \geqslant 2. \tag{3.33}$$

From (3.30), (3.31), and (3.33), we get

$$a_l(0) = 0, \quad \text{and } \frac{\partial \Re a_l}{\partial x_j}(0) = 0 \text{ for all } l, j. \tag{3.34}$$

The condition that $\sigma = (0, 0, \xi^0, \tau^0) \in \operatorname{Char} L$ now means that $\tau^0 = 0 = \xi_1^0$, and $\xi_j^0 \neq 0$ for some $j \geqslant 2$. We may assume that

$$\xi^0 = (0, 1, 0, .., 0). \tag{3.35}$$

We have

$$[L, \overline{L}] = \sum_{l=1}^{m} \left(\frac{\partial \overline{a_l}}{\partial \overline{z}} - \frac{\partial a_l}{\partial z} \right) \frac{\partial}{\partial x_l} + \sum_l \sum_j \left(a_j \frac{\partial \overline{a_l}}{\partial x_j} - \overline{a_j} \frac{\partial a_l}{\partial x_j} \right) \frac{\partial}{\partial x_l} \tag{3.36}$$

and hence using (3.34) and (3.35) we get:

$$\sigma([L, \overline{L}]) = \frac{\partial \overline{a_2}}{\partial \overline{z}}(0) - \frac{\partial a_2}{\partial z}(0) \tag{3.37}$$

Differentiating (3.32) with respect to z, we get

$$i \frac{\partial \Psi_l}{\partial z}(0) - i \frac{\partial \Psi_l}{\partial \overline{z}}(0) + 2 \frac{\partial a_l}{\partial z}(0) = 0.$$

This latter equation, (3.37), and the assumption that $\sigma([L, \overline{L}]) = 0$ lead to

$$\frac{\partial \operatorname{Im} \Psi_2}{\partial t}(0) = 0. \tag{3.38}$$

Next using (3.36) and (3.34), we get

$$\begin{aligned} [L, [L, \overline{L}]](0) = & \sum_{l=1}^{m} \left(\frac{\partial^2 \overline{a_l}}{\partial \overline{z}^2}(0) - \frac{\partial^2 a_l}{\partial \overline{z} \partial z}(0) \right) \frac{\partial}{\partial x_l} \\ & + \sum_{l=1}^{m} \sum_{j=1}^{m} \left(\frac{\partial a_j}{\partial \overline{z}} \frac{\partial \overline{a_l}}{\partial x_j} - \frac{\partial \overline{a_j}}{\partial \overline{z}} \frac{\partial a_l}{\partial x_j} \right)(0) \frac{\partial}{\partial x_l} \\ & - \sum_{l=1}^{m} \sum_{j=1}^{m} \left(\frac{\partial \overline{a_l}}{\partial \overline{z}} - \frac{\partial a_l}{\partial z} \right)(0) \frac{\partial a_j}{\partial x_l}(0) \frac{\partial}{\partial x_j} \end{aligned}$$

and hence

$$\begin{aligned} \sigma([L, [L, \overline{L}]]) = & \frac{\partial^2 \overline{a_2}}{\partial \overline{z}^2}(0) - \frac{\partial^2 a_2}{\partial \overline{z} \partial z}(0) \\ & - i \sum_{j=1}^{m} \left(\frac{\partial a_j}{\partial \overline{z}} + 2 \frac{\partial \overline{a_j}}{\partial \overline{z}} - \frac{\partial a_j}{\partial z} \right)(0) \frac{\partial \operatorname{Im} a_2}{\partial x_j}(0). \end{aligned} \tag{3.39}$$

We will express (3.39) using Ψ_2. Since Ψ is C^2, we can differentiate (3.32) to get:

$$\begin{aligned} i\frac{\partial\Psi_2}{\partial\overline{z}}+t\frac{\partial^2\Psi_2}{\partial\overline{z}^2}+\frac{\partial a_2}{\partial\overline{z}}+\sum_{j=1}^{m}\frac{\partial a_j}{\partial\overline{z}}t\frac{\partial\Psi_2}{\partial x_j}& \\ +\frac{i}{2}\sum_{j=1}^{m}a_j\frac{\partial\Psi_2}{\partial x_j}+\sum_{j=1}^{m}a_jt\frac{\partial^2\Psi_2}{\partial\overline{z}\partial x_j}&=0. \end{aligned}\tag{3.40}$$

At $t=0$, we have :

$$i\frac{\partial\Psi_2}{\partial\overline{z}}(x,0)+\frac{\partial a_2}{\partial\overline{z}}(x,0)+\frac{i}{2}\sum_{j=1}^{m}a_j(x,0)\frac{\partial\Psi_2}{\partial x_j}(x,0)=0.\tag{3.41}$$

Subtract (3.41) from (3.40), divide by t, let $t\to 0$ and evaluate at $x=0$ to get:

$$\begin{aligned} i\partial_t\partial_{\overline{z}}\Psi_2(0)+\partial_{\overline{z}}^2\Psi_2(0)+\partial_t\partial_{\overline{z}}a_2(0)+\sum_{j=1}^{m}\frac{\partial a_j}{\partial\overline{z}}(0)\frac{\partial\Psi_2}{\partial x_j}(0)& \\ +\frac{i}{2}\sum_{j=1}^{m}\frac{\partial a_j}{\partial t}(0)\frac{\partial\Psi_2}{\partial x_j}(0)&=0. \end{aligned}\tag{3.42}$$

Next differentiate (3.41) with respect to x_1 which leads to:

$$i\partial_{x_1}\partial_{\overline{z}}\Psi_2(0)+\partial_{x_1}\partial_{\overline{z}}a_2(0)+\frac{i}{2}\sum_{j=1}^{m}\frac{\partial a_j}{\partial x_1}(0)\frac{\partial\Psi_2}{\partial x_j}(0)=0.\tag{3.43}$$

From (3.42) and (3.43) we conclude that

$$\begin{aligned} \frac{\partial^2 a_2}{\partial z\partial\overline{z}}(0)=-i\frac{\partial^2\Psi_2}{\partial z\partial\overline{z}}(0)+\frac{i}{2}\frac{\partial^2\Psi_2}{\partial\overline{z}\partial\overline{z}(0)}& \\ -\frac{i}{2}\sum_{j=1}^{m}\frac{\partial a_j}{\partial z}(0)\frac{\partial\Psi_2}{\partial x_j}(0)+\frac{i}{2}\sum_{j=1}^{m}\frac{\partial a_j}{\partial\overline{z}}(0)\frac{\partial\Psi_2}{\partial x_j}(0).& \end{aligned}\tag{3.44}$$

By a similar reasoning, we also get

$$\frac{\partial^2\overline{a_2}}{\partial\overline{z}\partial\overline{z}}(0)=\frac{i}{2}\frac{\partial^2\overline{\Psi_2}}{\partial\overline{z}\partial\overline{z}}(0)-i\frac{\partial^2\Psi_2}{\partial z\partial\overline{z}}(0)-i\sum_{j=1}^{m}\frac{\partial\overline{a_j}}{\partial\overline{z}}(0)\frac{\partial\overline{\Psi_2}}{\partial x_j}(0)$$

and hence (3.39) can be written as

$$\sigma([L,[L,\overline{L}]])=\frac{\partial^2\operatorname{Im}\Psi_2}{\partial\overline{z}\partial\overline{z}}(0)-2\frac{\partial^2\operatorname{Im}\Psi_2}{\partial\overline{z}\partial z}(0).\tag{3.45}$$

From (3.30), (3.31) and (3.38), we know that the linear part of $\operatorname{Im}\Psi_2$ at 0 is 0 and so since it is C^2, we have:
(3.46)
$$\operatorname{Im}\Psi_2(x_1,x',t) = at^2+3bx_1t+3cx_1^2+O(|x'|t+|x'|^2+|x'||x_1|)+o(x_1^2+t^2).$$
From (3.45) and (3.46), it follows that
$$\sigma([L,[L,\overline{L}]]) = \frac{3}{2}(-a-c+ib)$$
and hence (3.28) implies that
$$-a-c > \sqrt{3}|b|.$$
We proceed now as in [6], Lemma III.5. From (3.46) we have:
$$\begin{aligned} t\operatorname{Im}\Psi_2(x_1,x',t) &= at^3+3bx_1t^2+3cx_1^2t \\ &\qquad + O(|x'|t^2+|x'|^2|t|+|x'||x_1||t|)+o(tx_1^2+t^3). \end{aligned} \tag{3.47}$$
The error terms in (3.47) are not the same as the ones in (15) of [6]. However, we will show that the arguments in [6] will still work. Let $\tilde{Z_2} = Z_2+\mu Z_1^3$ for some $\mu\in\mathbb{R}$ to be determined. Set $\tilde{x_2} = \Re\tilde{Z_2}$ and leave t and the other x_k and Z_k unchanged. In these new coordinates, in (3.47), a is replaced by $a-\mu$ and c by $c+\mu$. Observe that these changes of coordinates are C^2 and so in the new coordinates, the vector field L is C^1. However, this will be of no consequence in what follows. As observed in [6], the inequality
$$-a-c > \sqrt{3}|b|$$
allows us to choose μ so that the quadratic form
$$(a-\mu)t^2+bx_1t^2+3(c+\mu)x_1^2 \quad \text{is negative definite.}$$
Hence there exist $\alpha>0$ and $C>0$ such that for $t\geqslant 0$,
$$t\operatorname{Im}\Psi_2(x_1,x',t) \leqslant -\alpha(t^3+x_1^2t)+C(|x'|t^2+|x'|^2t+|x'||x_1|t). \tag{3.48}$$
Next for $0<\lambda\leqslant\lambda_0$ where λ_0 is small, change the coordinates and first integrals as follows:
$$\tilde{x}_1 = \frac{x_1}{\lambda},\ \tilde{t} = \frac{t}{\lambda},\ \tilde{x}_2 = \frac{x_2}{\lambda^3},\ \tilde{x}_k = \frac{x_k}{\lambda^2} \text{ for } k\geqslant 3$$
and
$$\widetilde{Z_1} = \frac{Z_1}{\lambda},\ \widetilde{Z_2} = \frac{Z_2}{\lambda^3},\ \widetilde{Z_k} = \frac{Z_k}{\lambda^2} \text{ for } k\geqslant 3.$$
Removing the tildes, we have:
$$t\operatorname{Im}\Psi_2(x_1,x',t) \leqslant -\alpha t^3+C\lambda(|x|^3+t^3), \tag{3.49}$$
where we may assume that $0<\alpha<1$ (the left hand side depends on λ but we are suppressing this dependence in the notation). We are now ready

to estimate the FBI transform. For some $\delta > 0$ to be chosen small, let $U = \{(x,t) : |x| < 6\delta,\ 0 < t < \delta\}$. Let $\eta(x) \in C_0^\infty(\mathbb{R}^m)$ such that $\operatorname{supp}\eta \subset \{x : |x| \leqslant 5\delta\}$, and $\eta \equiv 1$ for $|x| \leqslant 4\delta$. Since $Z_j(x,0) = x_j,\ 1 \leqslant j \leqslant m$, Proposition II.6 in [6] allows us to choose $\kappa = \frac{\alpha\delta}{6}$. Choose λ and δ small enough so that for $(x,t) \in U$,

$$C\lambda(|x|^3 + t^3) \leqslant \frac{\alpha}{4}\delta^3. \tag{3.50}$$

Since $\Psi(0) = (i, 0, ..., 0)$, we may assume that on U,

$$|\operatorname{Im}\Psi(x,t)|^2 \leqslant 2, \text{ and } 2t|x||\Re\Psi(x,t)| \leqslant t^2 + \frac{|x|^2}{2}. \tag{3.51}$$

We use (3.49), (3.50), and (3.51) to estimate

$$\begin{aligned} Q(0,\xi^0,x,t) &= t\operatorname{Im}\Psi(x,t) - \kappa[|x|^2 + t^2|\Re\Psi|^2 + 2t\langle x, \Re\Psi\rangle - t^2|\operatorname{Im}\Psi|^2] \\ &\leqslant -\alpha t^3 + \frac{\alpha}{4}\delta^3 - \kappa\frac{|x|^2}{2} + 3\kappa t^2 \\ &= -\alpha t^3 + \frac{\alpha}{4}\delta^3 - \frac{\alpha\delta}{12}|x|^2 + \frac{\alpha\delta}{2}t^2. \end{aligned}$$

Therefore, if $|x| \leqslant 6\delta$ and $t = \delta$, then

$$Q(0,\xi^0,x,t) \leqslant -\frac{3\alpha}{4}\delta^3 + \frac{\alpha}{2}\delta^3 = -\frac{\alpha}{4}\delta^3,$$

while if $0 \leqslant t \leqslant \delta$ and $x \in \operatorname{supp} d\eta$,

$$Q(0,\xi^0,x,t) \leqslant \frac{\alpha}{4}\delta^3 - \frac{16\alpha}{12}\delta^3 + \frac{\alpha}{2}\delta^3 = -\frac{7\alpha}{12}\delta^3.$$

Thus in any case, the FBI transform has the required exponential decay which proves the Lemma. □

Lemma 3.4. — *Let $L = \dfrac{\partial}{\partial t} + c(x,t)\dfrac{\partial}{\partial x}$ be a C^{k-1} vector field on a neighborhood of 0 in $\mathbb{R}^2$ with a C^{k-1} first integral $Z(x,t) = x + t\Psi(x,t)$. Let $\sigma = (0,0;\xi^0,\tau^0) \in \operatorname{Char} L$. Assume that $\sigma(M) = 0$ whenever M is a bracket of L and $\overline{L}$ of length less than k and $\sigma(M_k) \neq 0$ for some bracket of length k. Let h be a C^1 solution of $Lh = 0$ near the origin. If k is even and $\frac{1}{i}\sigma(M_k) < 0$, then $(0,\xi^0) \notin WF_a h(x,0)$ and if k is odd, $(0,\xi^0) \notin WF_a h(x,0)$.*

Proof. — We may assume that $k \geqslant 3$ since $k = 2$ is contained in Lemma 3.1 and $k = 1$ is Lemma 1.3 in [7]. Write $\Re\Psi(x,t) = p(x,t) + g(x,t)$ where $p(x,t)$ is a polynomial of degree $k-1$ and $g \in C^{k-1}$ with $D^\alpha g(0) = 0$ for $|\alpha| \leqslant k-1$. By introducing the coordinates $x' = x + tp(x,t)$, $t' = t$, we may assume that our first integral $Z(x,t) = x + t\Psi(x,t)$ satisfies

$$D^\alpha \Re\Psi(x,t) = 0 \quad \text{for } |\alpha| \leqslant k-1. \tag{3.52}$$

Let $M_n = f_n(x,t)\dfrac{\partial}{\partial x}$ be a repeated bracket of L and $\overline{L}$ of length n, $2 \leqslant n \leqslant k$. We will show that f_n has the form

(3.53)
$$f_n = -2i\partial_t^{n-1}\Im c(t) + \partial_t^{n-2}\left(c\frac{\partial \overline{c}}{\partial x} - \overline{c}\frac{\partial c}{\partial x}\right) + \sum_{j=1}^{n-3}\partial_t^j\left(e_j\frac{\partial f_{n-j-1}}{\partial x} - f_{n-j-1}\frac{\partial e_j}{\partial x}\right) + e_{n-1}\frac{\partial f_{n-1}}{\partial x} - f_{n-1}\frac{\partial e_{n-1}}{\partial x}$$

where $e_j(x,t) = c(x,t)$ or $\overline{c}(x,t)$ and $f_l\dfrac{\partial}{\partial x}$ is some bracket of length l for $1 \leqslant l \leqslant n-1$. Indeed, (3.53) holds for $n = 2$ since

$$[L,\overline{L}] = -2i\partial_t \operatorname{Im} c(x,t)\frac{\partial}{\partial x} + \left(c\frac{\partial \overline{c}}{\partial x} - \overline{c}\frac{\partial c}{\partial x}\right)\frac{\partial}{\partial x}.$$

Assume it also holds for all brackets of length $\leqslant n$. Let M be a bracket of length $n+1$. By definition, either $M = [L, M_n]$ or $M = [\overline{L}, M_n]$ where M_n is a bracket of length n and hence $M_n = f_n\dfrac{\partial}{\partial x}$ with f_n as in (3.53). We have

(3.54)
$$\begin{aligned}[L, M_n] &= \frac{\partial f_n}{\partial t}\frac{\partial}{\partial x} + \left(c\frac{\partial f_n}{\partial x} - f_n\frac{\partial c}{\partial x}\right)\frac{\partial}{\partial x} \\ &= \Big\{-2i\partial_t^n \operatorname{Im} c(t) + \partial_t^{n-1}\Big(c\frac{\partial \overline{c}}{\partial x} - \overline{c}\frac{\partial c}{\partial x}\Big) + \sum_{j=1}^{n-2}\partial_t^j\Big(e_j\frac{\partial f_{n-j}}{\partial x} \\ &\qquad - f_{n-j}\frac{\partial e_j}{\partial x}\Big)\Big\}\frac{\partial}{\partial x} + \Big(c\frac{\partial f_n}{\partial x} - f_n\frac{\partial c}{\partial x}\Big)\frac{\partial}{\partial x}\end{aligned}$$

and

$$[\overline{L}, M_n] = \frac{\partial f_n}{\partial t}\frac{\partial}{\partial x} + \left(\overline{c}\frac{\partial f_n}{\partial x} - f_n\frac{\partial \overline{c}}{\partial x}\right)\frac{\partial}{\partial x},$$

and so it follows that (3.53) holds for all n. Suppose now $\sigma(M) = 0$ whenever M is a bracket of L and $\overline{L}$ of length $\leqslant n$. We want to show that

$$(3.55)\qquad \begin{aligned}&(1)\ \partial_t^j c(0) = 0, \qquad \forall j \leqslant n-1 \text{ and} \\ &(2)\ \partial_t^j f_{n-l}(0) = 0,\ \ \forall j \leqslant l-1,\ 2 \leqslant l \leqslant n-2\end{aligned}$$

whenever $f_s\dfrac{\partial}{\partial x}$ is a bracket of length s, $1 \leqslant s \leqslant n-1$. Because of (3.52) and the assumption that $\sigma \in \operatorname{Char} L$, (3.55) clearly holds for $n = 2$. Suppose it holds for $n-1$. Then

$$(3.56)\quad \partial_t^j c(0) = 0,\ \forall j \leqslant n-2 \text{ and } \partial_t^j f_{n-l}(0) = 0,\ \forall j \leqslant l-1,\ 2 \leqslant l \leqslant n-3$$

whenever $f_s \dfrac{\partial}{\partial x}$ is a bracket of length s, $1 \leqslant s \leqslant n-2$. We will first prove part (2) of (3.55) by induction on l. For $l = 2$, suppose $f_{n-2} \dfrac{\partial}{\partial x}$ is a bracket of length $n-2$. Then

$$f_{n-1} \frac{\partial}{\partial x} = \left[L, f_{n-2} \frac{\partial}{\partial x} \right]$$

is a bracket of length $n-1$ where

$$f_{n-1} = \frac{\partial f_{n-2}}{\partial t} - c \frac{\partial f_{n-2}}{\partial t} + f_{n-2} \frac{\partial c}{\partial x}.$$

Since $f_{n-1}(0) = f_{n-2}(0) = c(0) = 0$, it follows that $\dfrac{\partial f_{n-2}}{\partial t}(0) = 0$ and so (3.55) holds for $l = 2$. Assume it holds for some $2 < l < n-2$. We want to prove that if $f_{n-l-1} \dfrac{\partial}{\partial x}$ is a bracket of length $n-l-1$, then $\partial_t^j f_{n-l-1}(0) = 0$ for $j \leqslant l$. By (3.56), we only need to show this for $j = l$. Observe that

$$f_{n-l} \frac{\partial}{\partial x} = \left[L, f_{n-l-1} \frac{\partial}{\partial x} \right]$$

is of length $n-l$ where

$$f_{n-l} = \frac{\partial f_{n-l-1}}{\partial t} - f_{n-l-1} \frac{\partial c}{\partial x} + c \frac{\partial f_{n-l-1}}{\partial x}. \tag{3.57}$$

Apply ∂_t^{l-1} to (3.57) and use the fact that $\partial_t^{l-1} f_{n-l}(0) = 0$ (since (3.55) holds for l) and

$$\partial_t^j f_{n-l-1}(0) = \partial_t^j c(0) = 0 \quad \text{for } j \leqslant l-1$$

by (3.56). We conclude that $\partial_t^l f_{n-l-1}(0) = 0$ and hence (3.55) holds for all l. To prove (1), in view of (3.56), we only need to show that $\partial_t^{n-1} c(0) = 0$. From $f_n(0) = 0$, equation (3.53) for f_n and application of (2) and (1) for $j \leqslant n-1$, we conclude that

$$\partial_t^{n-1} \operatorname{Im} c(0) = 0.$$

Next from $LZ = 0$, we have $t\Psi_t + \Psi + c(1 + t\Psi_x) = 0$ and hence hence since $\partial_t^j c(0) = 0$ for $j = 0, ..., n-2$, we get

$$n \partial_t^{n-1} \Psi(0) + \partial_t^{n-1} c(0) = 0 \quad n \leqslant k-1. \tag{3.58}$$

Since $\partial_t^{n-1} \Re \Psi(0) = 0$ by (3.52), (3.58) implies that

$$\partial_t^{n-1} c(0) = \partial_t^{n-1} \operatorname{Im} c(0) = 0.$$

Hence (1) and (2) hold for all $n \leqslant k-1$. We can now use the hypotheses of the Lemma, (3.52), (3.53), and (3.58) to conclude that

(3.59)
$$\partial_t^j \operatorname{Im}\Psi(0) = 0 \text{ for } j < k-1 \text{ and } \quad \partial_t^j \operatorname{Im}\Psi(0) \neq 0 \ (< 0 \text{ when } k \text{ is even}).$$

We will now estimate the FBI transform. We assume without loss of generality that k is even and $\xi^0 = 1$. For some $\delta > 0$ small and m a positive integer, let

$$U = \{(x,t) : |x| < (m+1)\delta^k,\ 0 < t < \delta\}.$$

Let $\eta(x) \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\eta \subset \{x : |x| < (m+\frac{1}{2})\delta^k\}$, and $\eta \equiv 1$ for $|x| \leqslant (\frac{m+1}{2})\delta^k$. Since $\Psi \in C^{k-1}$, by (3.52) we can find $C_2 > 0$ such that

$$2|\langle x, t\Re\Psi(x,t)\rangle| \leqslant C_2|x|(t^k + |x|^{k-1}).$$

Using (3.59) we also have for $t \geqslant 0$,

$$\operatorname{Im}\Psi(x,t) \leqslant -Ct^{k-1} + C_1|x| \quad \text{for some } C,\ C_1 > 0,$$

and

$$t^2\,|\operatorname{Im}\Psi(x,t)|^2 \leqslant C_3\,t^2(t^{2k-2} + |x|^2)$$

for some $C_3 > 0$. Choose $\delta < \frac{C}{4C_1}$. Choose $\kappa = \frac{\alpha}{\delta^k}$ where $\alpha = \frac{C}{2(m+1)(C_2+C_3)}$. We then have (for $t \geqslant 0$):

$$\begin{aligned}\Re Q(x,t,0,\xi^0) &= t\ \operatorname{Im}\Psi(x,t) - \kappa[|x|^2 + 2\langle x, t\Re\Psi\rangle + t^2|\Re\Psi|^2 - t^2|\operatorname{Im}\Psi|^2] \\ &\leqslant -Ct^k + C_1|x|t - \kappa[|x|^2 + 2\langle x, t\Re\Psi\rangle - t^2|\operatorname{Im}\Psi|^2] \\ &\quad\ \text{which for } \delta \text{ small since } k \geqslant 3 \\ &\leqslant -Ct^k + C_1|x|t - \kappa[\frac{|x|^2}{2} - C_2|x|t^k - C_3t^{2k}].\end{aligned}$$

Hence when $t = \delta$, we get

$$\begin{aligned}\Re Q(x,t,0,\xi^0) &\leqslant -C\delta^k + C_1(m+1)\delta^{k+1} + C_2(m+1)\alpha\delta^k + C_3\alpha\delta^k \\ &< 0 \ \text{ for } \delta \text{ small enough since } \alpha = \frac{C}{2(m+1)(C_2+C_3)}.\end{aligned}$$

When $0 < t < \delta$ and $x \in \operatorname{supp} d\eta$,

$$\Re Q(x,t,0,\xi^0) \leqslant C_1(m+1)\delta^{k+1} - \frac{\alpha(m+1)^2}{4}\delta^k + C_2\alpha(m+1)\delta^k + 2C_3\alpha\delta^k < 0$$

for m chosen so that $(m+1)^2 > 4C_2(m+1) + 8C_3$ and δ is small enough. This proves Lemma 3.4. $\square$

We will next work in (x, t, ξ, τ) space and apply the results of our lemmas by using a trick from [7]. Let

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} c_j(x,t)\frac{\partial}{\partial x_j}$$

be a vector field in a neighborhood Ω of the origin in $\mathbb{R}^{m+1}$. Introduce an additional variable $s \in \mathbb{R}$, a parameter $\theta \in [0, 2\pi)$ and define

$$L_\theta = \frac{\partial}{\partial s} - e^{-i\theta}L.$$

Observe that if $h \in C^1(\Omega)$ is a solution of $Lh = 0$, it is also a solution of $L_\theta h = 0$ in $\Omega \times \mathbb{R}$. We will say that L_θ satisfies the integrability condition (3.2) if there are m functions $\Psi_i^\theta \in C^1(\Omega \times I)$ (I an open interval in $\mathbb{R}$ centered at 0) such that, if

$$Z_i^\theta = x_i + s\Psi_i^\theta(x,t,s),$$

then $L_\theta Z_i^\theta = 0$. Note that $Z_{m+1}^\theta = t + e^{-i\theta}s$ is also a solution whose value at $s = 0$ equals t. For the proof of Theorem 2.2, We will also be interested in the following stronger integrability condition for L_θ:

$$\text{(3.60)}\quad \begin{aligned} &\text{for each hyperplane } \Sigma \text{ in } \mathbb{R}^{m+2} \text{ of the form}\\ &\Sigma = \{(x,t,s) : s = \sum_{j=1}^{m} a_j x_j + a_{m+1}t\}, \text{ there exist } C^2 \text{ functions}\\ &Z_j^\Sigma\ (1 \leqslant j \leqslant m+1) \text{ near } 0 \text{ such that } L_\theta Z_j^\Sigma = 0 \text{ and}\\ &Z_j^\Sigma(x,t,s) = x_j + (s - \sum_{k=1}^{m} a_k x_k - a_{m+1}t)\Psi_j^\Sigma(x,t,s),\ 1 \leqslant j \leqslant m,\\ &Z_{m+1}^\Sigma(x,t,s) = t + (s - \sum_{k=1}^{m} a_k x_k - a_{m+1}t)\Psi_{m+1}^\Sigma(x,t,s))\\ &\text{for some } C^2 \text{ functions } \Psi_k^\Sigma(x,t,s),\ 1 \leqslant k \leqslant m+1. \end{aligned}$$

Consider the FBI transform in the variables (x, t) in $\mathbb{R}^{m+1}$:

$$\widetilde{F}_\kappa h(z,w,\zeta,\tau) = \int_{\mathbb{R}^{m+1}} e^{i[\zeta\cdot(z-x')+\tau(w-t')]-\kappa\langle\zeta,\tau\rangle[(z-x')^2+(w-t')^2]} \eta\, h(x',t')dx'dt'$$

where $\eta(x', t')$ is a smooth cut-off function supported near the origin in $\mathbb{R}^{m+1}$. Note that this is the same as the FBI transform considered before but in (x, t, s) space and computed on the hyperplane $s = 0$. An application of Lemmas 3.1, 3.2 and 3.4 leads to the following theorem which is a result on the microlocal analyticity of $h(x,t)$ as opposed to that of the trace $h(x,0)$.

Theorem 3.5. — *Suppose $\sigma = (0,0;\xi^0,\tau^0) \in \operatorname{Char} L$, and for some θ, the vector field L_θ satisfies (3.2) with C^1 first integrals. Let $h \in C^1(\Omega)$ be a solution of $Lh = 0$.*

(i) *If $\frac{1}{2i}\sigma([L,\overline{L}]) < 0$, then $\sigma \notin WF_a h$.*
(ii) *If L is C^2, L_θ satisfies condition (3.60) for some θ, $\sigma([L,\overline{L}]) = 0$, and $\sigma([L,[L,\overline{L}]]) \neq 0$, then $\sigma \notin WF_a h$.*
(iii) *Suppose $m = 1$, L is C^{k-1}, and for some θ, L_θ satisfies (3.2) with C^{k-1} first integrals. Assume that $\sigma(M) = 0$ whenever M is a bracket of L and $\overline{L}$ of length less than k and $\sigma(M_k) \neq 0$ for some bracket of length k. If k is even and $\frac{1}{i}\sigma(M_k) < 0$, then $\sigma \notin WF_a h$ and if k is odd, $\sigma \notin WF_a h$.*

Proof. — We will only prove (i) since (ii) and (iii) follow in a similar fashion. Let $\tilde{\sigma} = (0,0,0;\xi^0,\tau^0,0)$ which is a co-vector in (x,t,s) space. Observe that $\tilde{\sigma} \in \operatorname{Char} L_\theta$ and

$$\frac{1}{2i}\tilde{\sigma}([L_\theta,\overline{L_\theta}]) = \frac{1}{2i}\sigma([L,\overline{L}]) < 0.$$

Since $L_\theta h = 0$ and L_θ satisfies (3.2), by Lemma 3.1, for some $\kappa > 0$, we can find an open neighborhood $\mathcal{O}$ of 0 in $\mathbb{C}^{m+1}$, a conic neighborhood $\mathcal{C}$ of $\sigma = (0,0;\xi^0,\tau^0)$ in $\mathbb{C}^{m+1}$ and constants $C_1, C_2 > 0$ such that

$$|\tilde{F}_\kappa h(z,w,\zeta,\tau)| \leqslant C_1 e^{-C_2|\langle\zeta,\tau\rangle|}$$

for all $z \in \mathcal{O}$, and $\zeta \in \mathcal{C}$. It follows that $\sigma \notin WF_a h$. $\square$

Finally we shall need the following result which is Lemma 1.5 in [7]:

Lemma 3.6. — *Suppose $h(x,t,\lambda)$ is C^1 in all variables and depends analytically on λ. Assume that for each λ fixed, $(0,0;\xi^0,\tau^0) \notin WF_a h(x,t,\lambda)$. Then $(0,0;\xi^0,\tau^0) \notin WF_a h(x,t,t)$.*

4. Application to a nonlinear pde

In this section we will apply the results of section 3 by following [7] closely with some modifications that are needed for the proof of Theorem 2.2. Let $f(z,w,\zeta^0,\zeta)$ be a holomorphic function in a neighborhood $\widetilde{\Omega} \times \mathcal{N}$ of $((0,0),(a,\omega))$ in $\mathbb{C}^{m+1} \times \mathbb{C}^{m+1}$. Assume $U \subset \widetilde{\Omega} \times \mathbb{R}^{m+1}$ and consider a solution $u \in C^2(U)$ of the nonlinear pde

$$u_t = f(x,t,u,u_x) \tag{4.1}$$

satisfying

$$u(0,0)=a, u_x(0,0)=\omega \text{ and } (u(x,t),u_x(x,t))\in\mathcal{N}\ \forall(x,t)\in U. \tag{4.2}$$

Let

$$\mathcal{L}=\frac{\partial}{\partial t}-\sum_{j=1}^{m}\frac{\partial f}{\partial\zeta_j}(x,t,\zeta_0,\zeta)\frac{\partial}{\partial x_j}. \tag{4.3}$$

$\mathcal{L}$ is a vector field in Ω depending on the parameters $(\zeta_0,\zeta)\in\mathcal{N}$. Set

$$L^u=\frac{\partial}{\partial t}-\sum_{j=1}^{m}\frac{\partial f}{\partial\zeta_j}(x,t,u,u_x)\frac{\partial}{\partial x_j}.$$

Note that the vector field L^u has C^1 coefficients in U. Let $v=(u,u_x)$. It follows from (4.1) that (see [7])

$$L^u v=g(x,t,v) \tag{4.4}$$

where

$$g_0(x,t,\zeta_0,\zeta)=f(x,t,\zeta_0,\zeta)-\sum_{j=1}^{m}\zeta_j\frac{\partial f}{\partial\zeta_j}(x,t,\zeta_0,\zeta),$$

$$g_i(x,t,\zeta_0,\zeta)=f_{x_i}(x,t,\zeta_0,\zeta)+\zeta\frac{\partial f}{\partial\zeta_0}(x,t,\zeta_0,\zeta).$$

Consider the principal part of the holomorphic Hamiltonian of the system (4.4):

$$H=\mathcal{L}+g_0\frac{\partial}{\partial\zeta_0}+\sum_{j=1}^{m}g_j\frac{\partial}{\partial\zeta_j}.$$

If $\Psi(x,t,\zeta_0,\zeta)$ is a C^1 function holomorphic in (ζ_0,ζ), set

$$\Psi^v(x,t)=\Psi(x,t,v(x,t))$$

and let $\mathcal{L}^v$ be the vector field obtained from $\mathcal{L}$ by substituting $v(x,t)$ for (ζ_0,ζ) in each coefficient of $\mathcal{L}$ (recall that $v=(u,u_x)$). Thus $\mathcal{L}^v=L^u$. Equation (4.4) implies (see [7]) that

$$\mathcal{L}^v\Psi^v=(H\Psi)^v. \tag{4.5}$$

Let Z_i $(1\leqslant i\leqslant m)$ and Ξ_j $(0\leqslant j\leqslant m)$ be holomorphic solutions in $\widetilde{\Omega}\times\mathcal{N}$ (after contracting $\widetilde{\Omega}\times\mathcal{N}$) of the Cauchy problems

$$HZ_i=0,\quad Z_i|_{t=0}=x_i,\ 1\leqslant i\leqslant m \tag{4.6}$$

$$H\Xi_j=0,\quad \Xi_j|_{t=0}=\zeta_j,\ 0\leqslant j\leqslant m. \tag{4.7}$$

Using (4.5) we can see that $Z_i^v(x,t)$ $(1 \leqslant i \leqslant m)$ and $\Xi_j^v(x,t)$ $(0 \leqslant j \leqslant m)$ are C^1 solutions of the Cauchy problems

$$\mathcal{L}^v Z_i^v = 0, \quad Z_i^v(x,0) = x_i,\ 1 \leqslant i \leqslant m, \tag{4.8}$$

$$\mathcal{L}^v \Xi_j^v = 0, \quad \Xi_j^v(x,0) = v(x,0),\ 0 \leqslant j \leqslant m. \tag{4.9}$$

Consider next the map

$$F(z,w,\zeta_0,\zeta) = (Z(z,w,\zeta_0,\zeta), w, \Xi(z,w,\zeta_0,\zeta))$$

which is biholomorphic near $(0,0,a,\omega)$ and $F(0,0,a,\omega) = (0,0,a,\omega)$. Let

$$G(z',w',\zeta_0',\zeta') = (P(z',w',\zeta_0',\zeta'), w', Q(z',w',\zeta_0',\zeta'))$$

denote its inverse. Then Q is holomorphic and

$$Q(Z(z,w,\zeta_0,\zeta), w, \Xi(z,w,\zeta_0,\zeta)) = (\zeta_0,\zeta).$$

In particular,

$$v(x,t) = Q(Z^v(x,t), t, \Xi^v(x,t)).$$

Now $u(x,t)$ is also a solution of the equation

$$u_s = e^{-i\theta}(u_t - f(x,t,u,u_x))$$

which is of the same kind as (4.1), and the associated vector field $\mathcal{L}^\theta$ as in (4.3) is

$$\mathcal{L}^\theta = \frac{\partial}{\partial s} - e^{-i\theta}\mathcal{L}$$

where $\mathcal{L}$ is as in (4.3). Therefore, conclusion (4.8) applies, that is the vector field

$$(\mathcal{L}^\theta)^v = \frac{\partial}{\partial s} - e^{-i\theta}\mathcal{L}^v$$

has first integrals in $U \times \mathbb{R}$ as in (3.2). Observe that

$$(\mathcal{L}^\theta)^v = (\mathcal{L}^v)_\theta \tag{4.10}$$

where we recall that for a vector field M in (x,t) space such as $\mathcal{L}^v$,

$$M_\theta = \frac{\partial}{\partial s} - e^{-i\theta}\, M.$$

For each t', the function $Q(Z^v(x,t), t', \Xi^v(x,t))$ is a C^1 solution of $\mathcal{L}^v h = 0$, and is analytic with respect to t'. We are now ready to prove Theorems 2.1, 2.2 and 2.3. Since the arguments from here on are similar, we will only present the details for the proof of Theorem 2.2.

Proof of Theorem 2.2. — We are given $u \in C^3(U)$ is a solution of (4.1), $\sigma \in \operatorname{Char} L^u$, $\sigma([L^u, \overline{L^u}]) = 0$, and $\sigma([L^u, [L^u, \overline{L^u}]]) \neq 0$. In order to apply Theorem 3.5 (ii), we need to show that $(\mathcal{L}^v)_\theta$ satisfies condition (3.60). Consider the equation

$$w_s = f^\theta(x, t, w, w_x, w_t) \tag{4.11}$$

where

$$f^\theta(x, t, \zeta_0, \zeta_1, ..., \zeta_{m+1}) = e^{-i\theta}(\zeta_{m+1} - f(x, t, \zeta_0, ...\zeta_m)).$$

The function $w(x, t, s) = u(x, t)$ is a solution of equation (4.11). Equation (4.11) leads to the Hamiltonian

$$H^\theta = \mathcal{L}^\theta + g_0^\theta \frac{\partial}{\partial \zeta_0} + \sum_{j=1}^{m+1} g_j^\theta \frac{\partial}{\partial \zeta_j}$$

where

$$g_0^\theta(x, t, \zeta, \zeta_1, ..., \zeta_{m+1}) = f^\theta - \sum_{j=1}^{m+1} \zeta_j \frac{\partial f^\theta}{\partial \zeta_j},$$

$$g_i^\theta = f_{x_i}^\theta + \zeta_i \Big(\frac{\partial f^\theta}{\partial \zeta_0}\Big) \ 1 \leqslant i \leqslant m \quad \text{and} \quad g_{m+1}^\theta = f_t^\theta + \zeta_{m+1} \Big(\frac{\partial f^\theta}{\partial \zeta_0}\Big).$$

Since any hyperplane Σ of the form $s = \sum_{j=1}^m a_j x_j + a_{m+1} t$ is non-characteristic for H^θ, we can find holomorphic solutions $\widetilde{Z}_j(x, t, s, \zeta_0, \zeta_1, ..., \zeta_{m+1})$ $(1 \leqslant j \leqslant m+1)$ for the Cauchy problems

$$H^\theta \widetilde{Z}_j = 0, \ \widetilde{Z}_j|_\Sigma = x_j, \ 1 \leqslant j \leqslant m$$

and

$$H^\theta \widetilde{Z}_{m+1} = 0, \ \widetilde{Z}_{m+1}|_\Sigma = t.$$

Set $\tilde{Z}_j^v(x, t, s) = \tilde{Z}_j(x, t, s, u, u_x, u_t), \ 1 \leqslant j \leqslant m+1$. Then just as in (4.8), we have:

$$(\mathcal{L}^\theta)^v \tilde{Z}_j^v(x, t, s) = 0, \quad \tilde{Z}_j^v|_\Sigma = x_j$$

for $1 \leqslant j \leqslant m$, and $\tilde{Z}_{m+1}^v|_\Sigma = t$. Thus $(\mathcal{L}^v)_\theta = (\mathcal{L}^\theta)^v$ satisfies condition (3.60). By Theorem 3.5 (ii) applied to the vector field $\mathcal{L}^v$, $\sigma \notin WF_a h$ whenever $h(x, t) = Q(Z^v(x, t), t', \Xi^v(x, t))$ for some fixed t'. Finally, by Lemma 3.6, $\sigma \notin WF_a u$ since

$$v(x, t) = (u(x, t), u_x(x, t)) = Q(Z^v(x, t), t, \Xi^v(x, t)).$$

□

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