

ANNALES DE L'INSTITUT FOURIER

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On entire functions represented by Dirichlet series. IV

Annales de l'institut Fourier, tome 16, n° 2 (1966), p. 209-223

<http://www.numdam.org/item?id=AIF_1966__16_2_209_0>

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ON ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES (IV)

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1. Let

$$f(s) = \sum_{n=1}^{\infty} \alpha_n e^{s\lambda_n}, \quad s = \sigma + it$$

represent an entire function, where

$$(1.1) \quad \overline{\lim}_{n \rightarrow \infty} n/\lambda_n = D < \infty;$$

$$(1.2) \quad \underline{\lim}_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = h > 0,$$

such that ([10], p. 201) $hD \leq 1$, and

$$(1.3) \quad 0 = \lambda_0 < \lambda_1 < \dots < \lambda_n \rightarrow \infty$$

as $n \rightarrow \infty$. Now $f(s)$ represents an entire function and so its abscissa of absolute convergence must be infinite, that is

$$(1.3') \quad \overline{\lim}_{n \rightarrow \infty} \log |\alpha_n|/\lambda_n = -\infty.$$

Let us define χ_n as follows:

$$\chi_n = \frac{\log |\alpha_{n-1}/\alpha_n|}{\lambda_n - \lambda_{n-1}}.$$

Then χ_n is a non-decreasing function of n (see [1]) and $\rightarrow \infty$ as $n \rightarrow \infty$. The fact is similar to what G. Valiron describes about rectified ratio in his book ([12], p. 32). So we have:

$$0 \leq \chi_1 \leq \chi_2 \leq \dots \leq \chi_n \leq \dots; \quad \chi_n \rightarrow \infty, \quad n \rightarrow \infty.$$

Let $\mu(\sigma)$ be the maximum term in the representation of $\sum |\alpha_n| e^{\sigma \lambda_n}$ and call it as the maximum term of $f(S)$. Let $\lambda_{\nu(\sigma)}$

be that value of λ_n which makes $|a_n|e^{\sigma\lambda_n}$ the maximum term and call $\lambda_{\nu(\sigma)}$ as the rank of $\mu(\sigma)$. Let us similarly correspond $\mu_{(m)}(\sigma)$ and $\lambda_{\nu(m)(\sigma)}$ to $f^{(m)}(S)$, the m -th derivative of $f(S)$ as we have done about $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$ connecting them with $f(S)$, where $\mu_{(0)}(\sigma) \equiv \mu(\sigma)$, $\lambda_{\nu(0)(\sigma)} \equiv \lambda_{\nu(\sigma)}$. It is well-known that ([13]; [4], pp. 1-2)

$$(1.4) \quad \log \mu(\sigma) = \int_1^\sigma \lambda_{\nu(x)} dx.$$

We define the order $(R)\rho$ and lower order $(R)\lambda$ of $f(s)$ as follows :

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log M(\sigma)}{\sigma} = \frac{\rho}{\lambda};$$

where $M(\sigma) = \text{l.u.b.}_{-\infty < t < \infty} |f(s)|$.

According to Mandelbrojt ([10], p. 216) we call ρ as the Ritt order (to be written as order $(R)\rho$) of $f(s)$. We, therefore, naturally call the lower limit in $\log \log M(\sigma)/\sigma$ as $\sigma \rightarrow \infty$ to be the lower order $(R)\lambda$. However, I shall drop the word (R) in the sequel. The results starting after Theorem C and onwards are expected to be new; Theorems A and B have already appeared but the secretary wishes them to incorporate here. This paper is to be considered as a sequel to my previous papers [6; 7; 8 et 9]. For the sake of completeness I start with the following result ([4], Th. 1).

2. THEOREM A. — For an entire function $f(s) = \sum_{n=1}^{\infty} a_n e^{\sigma\lambda_n}$ where $\{\lambda_n\}$ satisfies (1.2), then

$$(2.1) \quad \mu(\sigma) \leq M(\sigma) < \mu(\sigma) \left[\left(1 + \frac{1}{L\sigma} \right) \lambda_{\nu(\sigma + \sigma/\lambda_{\nu(\sigma)})} + 1 \right],$$

where $L = h - \varepsilon$, ε being an arbitrarily taken small positive number.

We now proceed to prove it. The left-hand inequality in (2.1) is obvious in view of Ritt's inequality :

$$|a_n|e^{\sigma\lambda_n} \leq M(\sigma).$$

Let

$$W(\sigma) = \sum_{n=1}^{\infty} e^{-G_n + \sigma\lambda_n}, \quad G_n = -\log |a_n|.$$

Suppose p is a positive integer $> \lambda_{\nu(\sigma)}$, such that $\chi_p > \sigma$. Let $q \geq p$. Now

$$\begin{aligned} e^{-G_q} e^{\sigma \lambda_q} &< e^{-G_{p-1}} e^{\sigma \lambda_{p-1}} \exp\{(\sigma - \chi_p)(\lambda_q - \lambda_{p-1})\} \\ &\leq \mu(\sigma) \exp\{(\sigma - \chi_p)(\lambda_q - \lambda_{p-1})\}. \end{aligned}$$

Hence

$$W(\sigma) < \mu(\sigma) \left[p + \sum_{q=p}^{\infty} \left(\frac{e^{\sigma}}{e^{\chi_p}} \right)^{\lambda_q - \lambda_{p-1}} \right].$$

Hence in view of (1.2), if we write $x = \exp(\chi_p - \sigma)$, then $\chi > 1$ and so

$$\sum_{q=p}^{\infty} \left(\frac{e^{\sigma}}{e^{\chi_p}} \right)^{\lambda_q - \lambda_{p-1}} < \chi^{-L} + \chi^{-2L} + \dots = \frac{1}{x^L - 1}.$$

Therefore

$$W(\sigma) < \mu(\sigma) \left[p + \frac{e^{L\sigma}}{e^{L\chi_p} - e^{L\sigma}} \right].$$

Let

$$p = \lambda_{\nu(\sigma + \sigma/\lambda_{\nu(\sigma)})} + 1,$$

we find that

$$e^{L\chi_p} - e^{L\sigma} > e^{L\sigma} \{ e^{L\sigma/\lambda_{\nu(\sigma)}} - 1 \}$$

and therefore the right-hand part in (2.1) follows.

Making use of Theorem A, we prove ([4], Th. 2, p. 5):

THEOREM B. — *Let $f(s)$ be an entire function of order ρ and lower order λ ; λ_n satisfies (1.2) in the expansion of $f(s)$. Then*

$$(2.2) \quad \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \frac{\rho}{\lambda}; \quad (0 \leq \rho \leq \infty; 0 \leq \lambda \leq \infty).$$

As regards the proof, the upper limit is similar to a result proved by Valiron ([12], p. 33), care is only to be taken that during the course of proof, we use the fact that $\log \mu(\sigma)$ is a convex function of σ [2]. From the previous theorem and the fact that if ρ is finite, we notice that

$$\log M(\sigma) \sim \log \mu(\sigma), \quad \sigma \rightarrow \infty.$$

Let

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \rho < \infty,$$

so that from (1.4), for $\sigma \geq \sigma_0$

$$\log \mu(\sigma) < K + \frac{e^{(\rho+\varepsilon)\sigma}}{\rho + \varepsilon}.$$

Therefore

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log M(\sigma)}{\sigma} \leq \rho.$$

Let us suppose now

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log M(\sigma)}{\sigma} = \rho_1 (\leq \rho).$$

Therefore from (1.4) and the relation $\mu(\sigma) \leq M(\sigma)$, we find that

$$2\lambda_{\nu(\sigma)} \leq \int_{\sigma}^{\sigma+2} \lambda_{\nu(x)} dx < (1 + \varepsilon)e^{(\sigma+2)\chi_{\rho+\varepsilon}},$$

and so we find that

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} \leq \rho_1.$$

Therefore $\rho = \rho_1$. Therefore the ratios $\log \log M(\sigma)/\sigma$ and $\log \lambda_{\nu(\sigma)}/\sigma$ have the same upper limit. To prove that

$$\underline{\lim}_{\sigma \rightarrow \infty} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \lambda,$$

we proceed in some other way. Let

$$\underline{\lim}_{\sigma \rightarrow \infty} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \alpha.$$

With the help of (1.4) and $\mu(\sigma) \leq M(\sigma)$, one easily finds that for any constant $C > 0$.

$$C\lambda_{\nu(\sigma)} \leq \log \mu(\sigma + c) \leq \log M(\sigma + c) < e^{(\lambda+\varepsilon)(\sigma+c)},$$

for an arbitrarily large value of σ . This implies $\alpha \leq \lambda$. If $\lambda = 0$, then $\alpha = 0$ and there is nothing to prove. Let $0 \leq \alpha < \infty$. Choose β and γ , such that $\alpha < \beta$ and $\alpha/\beta < \gamma < 1$. Hence

$$(2.3) \quad \lambda_{\nu(\sigma)} < e^{\beta\sigma}, \quad (\gamma\sigma_n \leq \sigma \leq \sigma_n)$$

where $\{\sigma_n\}$ is a sequence of σ , such that $\sigma_n \rightarrow \infty$ as $n \rightarrow \infty$. We shall show that

$$\frac{\log M(\sigma)}{\log \mu(\sigma)} \rightarrow 1,$$

as $\sigma \rightarrow \infty$ through the sequence for which (2.3) holds (it is not assumed that ρ is finite: if ρ is finite we cannot claim necessarily that $\log M(\sigma) \sim \log \mu(\sigma)$).

Let δ and ϵ' be two positive numbers such that

$$\gamma < \delta < 1; \quad \gamma/\delta < \epsilon' < 1.$$

Put $\delta\sigma_n = \xi_n$. Then for $n \geq n_0$, $\gamma\sigma_n < \epsilon'\xi_n < \xi_n < \sigma_n - \frac{1}{2}$.

Further, let $\mu(0) = 1$, which we may without loss of generality. Then from (1.4)

$$\log \mu(\xi_n) = \log \mu(\xi_n \epsilon') + \int_{\epsilon'\xi_n}^{\xi_n} \lambda_{\nu(x)} dx.$$

But $\log \mu(\epsilon'\xi_n) < \epsilon'\xi_n \lambda_{\nu(\epsilon'\xi_n)}$, so

$$\begin{aligned} \log \mu(\xi_n) &> \log \mu(\epsilon'\xi_n) + (1 - \epsilon')\xi_n \lambda_{\nu(\epsilon'\xi_n)} \\ &> \frac{1}{\epsilon'} \log \mu(\epsilon'\xi_n). \end{aligned}$$

Hence

$$(2.4) \quad \begin{aligned} (1 - \epsilon') \log \mu(\xi_n) &< \int_{\epsilon'\xi_n}^{\xi_n} \lambda_{\nu(x)} dx \\ &< \frac{1}{\beta} [e^{\beta\xi_n} - e^{\beta\epsilon'\xi_n}], \end{aligned}$$

for all $n \geq n_0$. But from Theorem A

$$\begin{aligned} \log M(\xi_n) &< \log \mu(\xi_n) + \log \lambda_{\nu(\xi_n + \xi_n/\lambda_{\nu(\xi_n)})} + 0(1) \\ &< \log \mu(\xi_n) + \log \lambda_{\nu(2\xi_n)} + 0(1) \\ &< \log \mu(\xi_n) + 2\beta\xi_n + 0(1). \end{aligned}$$

Hence we get for all $n \geq n_0$.

$$\begin{aligned} \log \log M(\xi_n) &< (1 + 0(1)) \log \log \mu(\xi_n) \\ &< (1 + 0(1))\beta\xi_n, \end{aligned}$$

from (2.4). Consequently $\lambda \leq \beta$ and as $(\beta - \alpha)$ can be made arbitrarily small we see that $\lambda \leq \alpha$; and this, when combined with the already established inequality: $\lambda \geq \alpha$, gives the required result.

Next, I give the following result ([5], p. 45).

THEOREM C. — Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

be an entire function, where $\{\lambda_n\}$ satisfies (1.2), of order ρ and lower order λ ($0 < \rho \leq \infty$; $0 \leq \lambda < \infty$). Then

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \leq 1 - \frac{\lambda}{\rho}.$$

Proof. — We have

$$\begin{aligned} \log \mu(\sigma) &= \sum_{\lambda_n \leq \sigma} (\lambda_n - \lambda_{n-1})(\sigma - \lambda_n) \\ &= \sigma \lambda_{\nu(\sigma)} - \sum_{\lambda_n \leq \sigma} (\lambda_n - \lambda_{n-1}) \lambda_n. \end{aligned}$$

But for all $n \geq n_0$ (from Th. B)

$$\log \lambda_n < (\rho + \varepsilon) \lambda_n.$$

So we find

$$\sum_{\lambda_n \leq \sigma} (\lambda_n - \lambda_{n-1}) \lambda_n > \sum_{\lambda_n \leq \sigma, n \geq n_0} (\lambda_n - \lambda_{n-1}) \frac{\log \lambda_n}{\rho + \varepsilon}.$$

Let N be the largest integer such that $\lambda_N \leq \sigma$, then

$$\begin{aligned} \sum_{\lambda_n \leq \sigma} (\lambda_n - \lambda_{n-1}) \lambda_n &> \frac{1}{\rho + \varepsilon} \{ \lambda_N \log \lambda_N + O(\lambda_n) \} \\ &= \frac{1}{\rho + \varepsilon} \{ \lambda_{\nu(\sigma)} \log \lambda_{\nu(\sigma)} \} + O(\lambda_{\nu(\sigma)}). \end{aligned}$$

So that for $\sigma \geq \sigma_0$

$$\log \mu(\sigma) < \sigma \lambda_{\nu(\sigma)} \left\{ 1 - \frac{\lambda - \varepsilon}{\rho + \varepsilon} + O(1) \right\}$$

and the result follows.

3. Below I construct an example to exhibit that the result of Th. C is best possible in view of the fact that if $\lambda < \infty$, $\rho = \infty$, then

$$(3.1) \quad \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} = 1.$$

Example 1. — Let

$$f(s) = \sum_{n=N}^{\infty} \left\{ \frac{e^s}{I(\lambda_n)} \right\}^{\lambda_n},$$

where $\lambda_{n+1} = \lambda_n$; N is a positive integer, such that $I(\lambda_N) \geq e$ and that

$$\log I(x) = \int_{x_0}^x \frac{dt}{t\theta(t) \log t} \rightarrow \infty,$$

as $x \rightarrow \infty$, where further.

(i) $\theta(x)$ is a positive, continuous and non-decreasing function for $x \geq x_0$ and $\rightarrow \infty$ with x , and has a derivative;

$$(ii) \quad \frac{x\theta'(x)}{\theta(x)} \leq \frac{1}{\log x \log \log x \log \log \log x}, \quad x \geq x_0.$$

Demonstration of the aim. — According to a result ([10], p. 217, eq. (94)) we see that the order ρ of $f(s)$ is

$$\begin{aligned} &= \overline{\lim}_{n \rightarrow \infty} \frac{\lambda_n \log \lambda_n}{\lambda_n \log I(\lambda_n)} \\ &\geq \overline{\lim}_{n \rightarrow \infty} \frac{\log \lambda_n}{A \log \log \lambda_n}, \end{aligned}$$

from (ii) and the integral representation of $I(x)$, A being a finite number. Therefore the order ρ of $f(s)$ is infinite. Let

$$\chi_n = \log \left\{ \frac{I(\lambda_n)^{\lambda_n}}{I(\lambda_{n-1})^{\lambda_{n-1}}} / (\lambda_n - \lambda_{n-1}) \right\},$$

then it is easily found that $\chi_{n+1} > \chi_n (n > n_0)$ and that $\chi_n \rightarrow \infty$, as $n \rightarrow \infty$. Hence for $\chi_n \leq \sigma < \chi_{n+1}$,

$$\log \mu(\sigma) = \{\sigma - \log I(\lambda_n)\} \lambda_n, \quad \lambda_n = \lambda_{\nu(\sigma)}.$$

Therefore

$$\frac{\log \mu(\chi_{n+1})}{\chi_{n+1} \lambda_{\nu(\chi_{n+1})}} = 1 - \frac{(1 + o(1)) \log I(\lambda_n)}{\log I(\lambda_{n+1}) + o(\log I(\lambda_n))}.$$

Further

$$\log I(\lambda_{n+1}) - \log I(\lambda_n) > (1 + o(1)) \frac{l_2 \lambda_{n+1}}{l_3 \lambda_{n+1}},$$

and as $\log I(\lambda_n) < A l_2 \lambda_n$, $A = a$ constant, we find that

$$\frac{\log I(\lambda_{n+1})}{\log I(\lambda_n)} \rightarrow \infty, \quad (n \rightarrow \infty)$$

and so

$$\frac{\log \mu(\chi_{n+1})}{\chi_{n+1} \lambda_{\nu(\chi_{n+1})}} \rightarrow 1, \quad (n \rightarrow \infty)$$

and hence

$$(3.2) \quad \overline{\lim}_{n \rightarrow \infty} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \geq 1.$$

Further

$$\begin{aligned} \log \mu(\chi_{n+1}) &= \frac{\lambda_n \lambda_{n+1} \log \{I(\lambda_{n+1})/I(\lambda_n)\}}{\lambda_{n+1} - \lambda_n} \\ &= (1 + o(1)) \lambda_n \log I(\lambda_{n+1}), \end{aligned}$$

and therefore

$$\log \log \mu(\chi_{n+1}) \sim \log \log I(\lambda_{n+1}) + \log \lambda_n,$$

and as $\chi_{n+1} \sim \log I(\lambda_{n+1})$, it follows that

$$\lambda = \underline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log \mu(\sigma)}{\sigma} = 0.$$

Hence from Theorem C

$$(3.3) \quad \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \leq 1.$$

Inequalities (3.2) and (3.3) provide the demonstration of our aim.

Example 2. — Let us consider the function defined by (see Theorem 6 [3], p. 22 where I put $\beta = 1$)

$$f(s) = \sum_{n=1}^{\infty} \left(\frac{e^s}{\lambda_n} \right)^{\lambda_n}, \quad \lambda_{n+1} = \alpha^{\lambda_n}; \quad \alpha \geq e; \quad \lambda_1 = \alpha.$$

The function $f(s)$ is certainly an entire function on account of (1.3)'. The order ρ of $f(s)$ is in this case

$$= \overline{\lim}_{n \rightarrow \infty} \frac{\lambda_n \log \lambda_n}{\lambda_n \log \lambda_n} = 1.$$

Also

$$\mu(\sigma) = \{e^\sigma / \lambda_n\}^{\lambda_n}; \quad \lambda_n = \lambda_{\nu(\sigma)},$$

for $\chi_n \leq \sigma < \chi_{n+1}$, where

$$\chi_n = \frac{\lambda_n \log \lambda_n - \lambda_{n-1} \log \lambda_{n-1}}{\lambda_n - \lambda_{n-1}}.$$

Then

$$\begin{aligned}
 \log \mu(\chi_n) &= \lambda_n(\lambda_n - \log \lambda_n) \\
 &= \frac{\lambda_n \lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \log (\lambda_n / \lambda_{n-1}) \\
 (3.4) \quad &= (1 + 0(1)) \lambda_{n-1} \log \lambda_n; \\
 \log \log \mu(\chi_n) &= (1 + 0(1)) + \log \lambda_{n-1} + \log \log \lambda_n.
 \end{aligned}$$

Also $\chi_n \rightarrow \infty$ as $n \rightarrow \infty$, we see that

$$(3.5) \quad \frac{\log \log \mu(\chi_n)}{\chi_n} = 0(1) + \frac{1}{\chi_n} (\log \lambda_{n-1} + \log \log \lambda_n).$$

Now

$$\begin{aligned}
 \frac{\log \lambda_{n-1}}{\chi_n} &= \frac{\log \lambda_{n-1} (\lambda_n - \lambda_{n-1})}{\lambda_n \log \lambda_n - \lambda_{n-1} \log \lambda_{n-1}} \\
 &= \frac{\lambda_n \log \lambda_{n-1} + 0(\lambda_n)}{\lambda_n \lambda_{n-1} \log \alpha + 0(\lambda_n)} \\
 (3.6) \quad &= (1 + 0(1)) \frac{\log \lambda_{n-1}}{\lambda_{n-1} \log \alpha} \rightarrow 0 \quad (n \rightarrow \infty).
 \end{aligned}$$

Also $\log \log \lambda_n = (1 + 0(1)) \log \lambda_{n-1}$ and so the right-hand term in (3.5) $\rightarrow 0$ as $n \rightarrow \infty$ in view of (3.6). Therefore the lower order λ of $f(s)$ is zero on account of (3.5). Hence from Theorem C

$$(3.7) \quad \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \leq 1.$$

Also

$$\begin{aligned}
 \frac{\log \mu(\chi_{n+1})}{\chi_{n+1} \lambda_{\nu(\chi_{n+1})}} &= 1 - \frac{\log \lambda_n}{\chi_{n+1}} \\
 &= 1 - \frac{(\lambda_{n+1} - \lambda_n) \log \lambda_n}{\lambda_{n+1} \log \lambda_{n+1} - \lambda_n \log \lambda_n} \rightarrow 1 \quad (n \rightarrow \infty),
 \end{aligned}$$

for the above solution see the technique used in getting (3.6). Hence

$$(3.8) \quad \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \geq 1.$$

Therefore from (3.7) and (3.8) one gets

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} = 1,$$

giving thereby again a best possible nature of Theorem C in case $\lambda = 0$ and $\rho < \infty$.

4. Results involving derivatives of $f(s)$:

I have already spoken in the article 1 about $\mu_{(m)}(\sigma)$ and $\lambda_{\nu(m)\chi(\sigma)}$. I first prove:

THEOREM D. — *For all $\sigma \geq \sigma_0$ (σ_0 is a fixed large number) one should have:*

$$\mu_{(m)}(\sigma) > \mu(\sigma) \left[\frac{\log \mu(\sigma)}{\sigma} \right]^m,$$

m is an integer ≥ 0 . This result I stated in a previous paper ([6], p. 235) without proof.

Proof. — We have:

$$(4.1) \quad \lambda_{\nu(m)\chi(\sigma)} \leq \frac{\mu_{(m+1)}(\sigma)}{\mu_{(m)}(\sigma)} \leq \lambda_{\nu(m+1)\chi(\sigma)}, \quad m = 0, 1, \dots$$

When $m = 0$ in (4.1), it reduces to a result which I have proved in ([3], p., Theorem 2) as follows

$$\begin{aligned} \mu_{(1)}(\sigma) &= |a_{\nu(1)\chi(\sigma)}| \lambda_{\nu(1)\chi(\sigma)} \exp(\sigma \lambda_{\nu(1)\chi(\sigma)}) \leq \lambda_{\nu(1)\chi(\sigma)} \mu(\sigma); \\ \mu_{(1)}(\sigma) &= |a_{\nu(1)\chi(\sigma)}| \lambda_{\nu(1)\chi(\sigma)} \exp(\sigma \lambda_{\nu(1)\chi(\sigma)}) \geq |a_{\nu(\sigma)}| \lambda_{\nu(\sigma)} \exp(\sigma \lambda_{\nu(\sigma)}) \\ &= \lambda_{\nu(\sigma)} \mu(\sigma). \end{aligned}$$

The case $m \geq 1$ can also be treated by simple definitions, for let

$$f^{(m)}(S) = \sum A_n e^{s\lambda_n}, \quad \lambda_{\nu(m)\chi(\sigma)} = \lambda_N; \quad \lambda_{\nu(m+1)\chi(\sigma)} = \lambda_{N_1},$$

then

$$\mu_{(m+1)}(\sigma) = \lambda_{N_1} |A_{N_1}| \exp(\sigma \lambda_{N_1}) \leq \lambda_{N_1} \mu_{(m)}(\sigma),$$

and

$$\mu_{(m)}(\sigma) = \frac{1}{\lambda_N} (\lambda_N |A_N| \exp(\sigma \lambda_N)) \leq \frac{\mu_{(m+1)}(\sigma)}{\lambda_{\nu(m)\chi(\sigma)}},$$

and so these two inequalities complete (4.1) and from which we have:

$$\begin{aligned} \lambda_{\nu(\sigma)} \leq \frac{\mu_{(1)}(\sigma)}{\mu(\sigma)} \leq \lambda_{\nu(1)\chi(\sigma)} \leq \frac{\mu_{(2)}(\sigma)}{\mu_{(1)}(\sigma)} \leq \dots \leq \lambda_{\nu(m-1)\chi(\sigma)} \\ \leq \frac{\mu_{(m)}(\sigma)}{\mu_{(m-1)}(\sigma)} \leq \lambda_{\nu(m)\chi(\sigma)}. \end{aligned}$$

Multiplying the ratios involving these μ 's one finds that

$$(4.2) \quad \frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} \geq \lambda_{\nu(m-1)\chi(\sigma)} \dots \lambda_{\nu(\sigma)} \geq (\lambda_{\nu(\sigma)})^m.$$

Now from (1.3)' we get, for K to be sufficiently large,

$$(4.3) \quad \begin{aligned} \log |a_{\nu(\sigma)}| &< -K\lambda_{\nu(\sigma)}; \quad \sigma \geq \sigma_0 \\ |a_{\nu(\sigma)}| &< \exp(-k\lambda_{\nu(\sigma)}) < 1, \quad \sigma \geq \sigma_0. \end{aligned}$$

Again

$$(4.4) \quad \begin{aligned} \log \mu(\sigma) &= \log |a_{\nu(\sigma)}| + \sigma\lambda_{\nu(\sigma)} \\ &< \sigma\lambda_{\nu(\sigma)}, \quad \sigma \geq \sigma_0 \end{aligned}$$

from (4.3). The inequalities (4.2) and (4.4) result in for $\sigma \geq \sigma_0$

$$\frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} > \left(\frac{\log \mu(\sigma)}{\sigma} \right)^m.$$

The above theorem is useful in deducing the following interesting.

THEOREM E. — *One has (with the terms involved in to be known):*

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log (\mu_{(m)}(\sigma)/\mu(\sigma))^{1/m}}{\sigma} = \frac{\rho}{\lambda};$$

Proof. — We have:

$$\begin{aligned} \frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} &\leq \lambda_{\nu(1)\chi(\sigma)} \dots \lambda_{\nu(m)\chi(\sigma)} \\ &\leq (\lambda_{\nu(m)\chi(\sigma)})^m. \end{aligned}$$

Now $f^{(m)}(s)$ also posses the same order ρ and lower order λ as $f(s)$ has, and so (cf. Theorem B)

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \lambda_{\nu(m)\chi(\sigma)}}{\sigma} = \frac{\rho}{\lambda};$$

consequently

$$(4.5) \quad \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log (\mu_{(m)}(\sigma)/\mu(\sigma))^{1/m}}{\sigma} \leq \frac{\rho}{\lambda};$$

But Theorem D provides us the inequality (to be deduced with the help of Theorem B and (1.4)) ⁽¹⁾

$$(4.6) \quad \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log (\mu_{(m)}(\sigma) / \mu(\sigma))^{1/m}}{\sigma} \geq \frac{\rho}{\lambda};$$

The inequalities (4.5) and (4.6) yield the desired result.

Remark. — Theorem D has been stated without any proof by Srivastav ([11], p. 89 (i)) and that too under the restrictive condition that $\lambda > 0$. The proof of Theorem D removes this superfluous restriction which Srivastav asserts. Secondly, Srivastav claims to prove Theorem E but to the best my surprise there is no clue available to its proof in his paper wherever he mentions it. I wish to add that I have stated Theorem D without proof in a recent paper of mine ([6], Theorem 1).

5. Towards the end of this paper, I would like to add a new result on the mean values of entire Dirichlet functions. To the best of my knowledge I introduced these means and discovered their properties relating to the order and lower order of $f(S)$ in a recent paper [9]. I do here a little more. I define

$$A_k(\sigma) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(S)|^k dt,$$

where the sequence $\{\lambda_n\}$ satisfies (1.1)-(1.3); $0 < k < \infty$.

THEOREM F. — *If $f(S)$ satisfies the conditions stated in § 5, then we have:*

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log A_k(\sigma)}{\sigma} = \frac{\rho}{\lambda};$$

⁽¹⁾ From (1.4), (i)

$$\log \mu(\sigma) \leq (1 + 0(1))\sigma\lambda_{(\sigma)} \quad \text{and so} \quad \log \log \mu(\sigma)/\sigma \leq 0(1) + \log \lambda_{(\sigma)}/\sigma;$$

and (ii) for $k > 0$, $\log \mu(\sigma + k) \geq k\lambda_{(\sigma)}$ and so

$$\log \log \mu(\sigma + k)/(1 + 0(1)) (\sigma + k) \geq 0(1) + \log \lambda_{(\sigma)}/\sigma.$$

From (i) et (ii) one deduces that

$$\overline{\lim}_{\sigma \rightarrow \infty} \log \log \mu(\sigma)/\sigma = \overline{\lim}_{\sigma \rightarrow \infty} \log \lambda_{(\sigma)}/\sigma.$$

Remark. — If $k = 2$, I have got the above result in a recent paper ([9], Theorem 1) where I supposed further that χ_n was non-decreasing. Here we need not, as one will soon find, make this supposition.

Proof of Theorem F. — One does have

$$A_k(\sigma) \leq \{M_s(\sigma)\}^k,$$

where

$$M_s(\sigma) = \max_{|t| \leq T} |f(\sigma + it)|.$$

But (see for references [9] and also [10])

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log M_s(\sigma)}{\sigma} = \frac{\rho}{\lambda};$$

So we find that

$$(5.1) \quad \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log A_k(\sigma)}{\sigma} \leq \frac{\rho}{\lambda};$$

To get the other part, it is sufficient to consider $f(S)$ in the representation given by:

$$f(S) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}.$$

Then, if $S' = \Delta + ix$; $a_n = \alpha_n + i\beta_n$, we have

$$\begin{aligned} & f(\Delta + ix) \\ &= \sum_{n=0}^{\infty} [(\alpha_n \cos \lambda_n x - \beta_n \sin \lambda_n x) + i(\alpha_n \sin \lambda_n x + \beta_n \cos \lambda_n x)] e^{\Delta \lambda_n}; \\ & \operatorname{Rl}\{f(\Delta + ix)\} = \sum_{n=0}^{\infty} (\alpha_n \cos \lambda_n x - \beta_n \sin \lambda_n x) e^{\Delta \lambda_n}. \end{aligned}$$

Therefore

$$(*) \quad \alpha_m e^{\Delta \lambda_m} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T \operatorname{Rl}\{f(\Delta + ix)\} \cos \lambda_m x \, dx, \quad m > 0.$$

$$\begin{aligned} (**) \quad -\beta_m e^{\Delta \lambda_m} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T \operatorname{Rl}\{f(\Delta + ix)\} \sin \lambda_m x \, dx, \quad m > 0. \\ \alpha_0 &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T \operatorname{Rl}\{f(\Delta + ix)\} \, dx. \end{aligned}$$

Therefore from (*) and (**)

$$\begin{aligned}
 \text{Re}\{f(\sigma + it)\} &= \sum_{n=0}^{\infty} (\alpha_n \cos \lambda_n t - \beta_n \sin \lambda_n t) e^{\sigma \lambda_n} \\
 (5.2) \quad &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T \text{Re}\{f(\Delta + ix)\} \left\{ 1 + \sum_{n=1}^{\infty} \cos \{(x-t)\lambda_n\} e^{(\sigma-\Delta)\lambda_n} \right\} dx.
 \end{aligned}$$

We can treat (5.2) as an analogue to Poisson’s formula in power series. Therefore, if we start our series for $f(s)$ from $n = 1$ to ∞ , then

$$|f(s)| \leq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(\Delta + ix)| 2 \sum_{n=1}^{\infty} \exp\{(\sigma - \Delta)\lambda_n\} dx,$$

and since the right-hand side is independent of t , one finds that

$$\begin{aligned}
 (5.3) \quad M(\sigma) &\leq 2A(\Delta) \left(\sum_{n=1}^{n_0-1} + \sum_{n=n_0}^{\infty} \right) \exp\{(\sigma - \Delta)\lambda_n\} \\
 &< 2A(\Delta) \left[(n_0 - 1) \exp\{(\sigma - \Delta)\lambda_1\} + \sum_{n=n_0}^{\infty} \exp\{(\sigma - \Delta)\lambda_n\} \right].
 \end{aligned}$$

But

$$\begin{aligned}
 \sum_{n=n_0}^{\infty} \exp\{(\sigma - \Delta)\lambda_n\} &< \exp\{(\sigma - \Delta)\lambda_1\} \\
 &\quad \{ 1 + \exp(\sigma - \Delta)L + \exp(\sigma - \Delta)2L + \dots \}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 M(\sigma) &< 2A(\Delta) \\
 &\quad \left[(n_0 - 1) \exp(\sigma - \Delta)\lambda_1 + \frac{\exp\{(\sigma - \Delta)\lambda_1\} \exp(\Delta L)}{\exp(\Delta L) - \exp(\sigma L)} \right].
 \end{aligned}$$

Let $\Delta = \sigma + \eta$, $\eta > 0$. Then on simplifications, one gets

$$(5.4) \quad M(\sigma) < O(1) A(\sigma + \eta).$$

Similarly taking $\{f(s)\}^k$ instead of $f(s)$, one can prove that

$$(5.5) \quad (M(\sigma))^k < O(1) A_k(\sigma + \eta),$$

where the constants $O(1)$ in (5.4) and (5.5) might not be the same, and so

$$(5.6) \quad \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log A_k(\sigma)}{\sigma} \geq \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log M(\sigma)}{\sigma} = \frac{\rho}{\lambda};$$

The inequalities (5.1) and (5.6) yield the required result. I might like to discuss further results on the means defined by $A_k(\sigma)$ in a next sequel of my work.

Before I close up the discussion, I would like to express my warm thanks to the University Grants Commission, India about its partial support for the project undertaken by me.

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Manuscrit reçu le 19 octobre 1965.

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