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CONVERGENCE IN CAPACITY

by Yang XING

ABSTRACT. — We study the relationship between convergence in capacities of plurisubharmonic functions and the convergence of the corresponding complex Monge-Ampère measures. We find one type of convergence of complex Monge-Ampère measures which is essentially equivalent to convergence in the capacity C_n of functions. We also prove that weak convergence of complex Monge-Ampère measures is equivalent to convergence in the capacity C_{n-1} of functions in some case. As applications we give certain stability theorems of solutions of Monge-Ampère equations.

RÉSUMÉ. — Nous étudions la relation entre la convergence en capacité des fonctions pluri sous-harmoniques et la convergence des mesures de Monge-Ampère complexes correspondantes. Nous trouvons un type de convergence des mesures de Monge-Ampère complexe qui est essentiellement équivalent à la convergence en capacité C_n des fonctions. Nous montrons aussi que la convergence faible des mesures de Monge-Ampère complexes est équivalente à la convergence en capacité C_{n-1} des fonctions dans certains cas. Comme application nous donnons des théorèmes de stabilité des solutions des équations de Monge-Ampère.

1. Introduction

Let $PSH(\Omega)$ be the set of plurisubharmonic (psh) functions in a bounded domain Ω in \mathbb{C}^n . Write $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$. The inner capacity C_s of a subset E in Ω is defined by

$$C_s(E) = C_s(E, \Omega) = \sup \{C_s(F); F \text{ is a compact subset of } E\},$$

where

$$C_s(F) = \sup \left\{ \int_F (dd^c u)^s \wedge (dd^c |z|^2)^{n-s}; u \in PSH(\Omega), 0 < u < 1 \right\}.$$

Clearly, the capacity C_{n-1} is locally dominated by a constant multiple of C_n . The capacity C_n was introduced by Bedford and Taylor in their

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fundamental paper [3]. The capacities play a great role in pluripotential theory. They are very effective tools in the study of psh functions and complex Monge-Ampère operators, see [11] [6] [13] [15]. In particular, one can use capacities to deal with continuity of the Monge-Ampère operator $(dd^c)^n$. By examples of Cegrell [5] and Lelong [14] there exist uniformly bounded psh functions u_j which converge weakly to a psh functions u , but the Monge-Ampère measures $(dd^c u_j)^n$ do not converge weakly to $(dd^c u)^n$. It is also known that the Monge-Ampère operator is continuous under convergence of functions in capacity C_s with $s \geq n - 1$. In [17] we obtained the following convergence theorem.

THEOREM A. — [17] *Let $u \in PSH(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$ and u_j be locally uniformly bounded psh functions in Ω . If $u_j \rightarrow u$ in C_{n-1} on each $E \subset\subset \Omega$ then $(dd^c u_j)^n \rightarrow (dd^c u)^n$ weakly in Ω .*

Theorem A is useful in many applications. We shall give a stronger version of the theorem in terms of the capacity C_n . Our main goal in this paper is to study the converse problem: which convergence of the Monge-Ampère measures $(dd^c u_j)^n$ can imply convergence of the functions u_j in capacity? Unfortunately, weak convergence of measures $(dd^c u_j)^n$ to $(dd^c u)^n$ generally cannot imply weak convergence of the functions u_j to u (hence, there is no convergence of the u_j to u in capacity either.), even in the case that all the u_j as well as u coincide on the boundary of the domain. This is shown in a simple example: the functions $u_j(z) = \max(j \ln |z|, -1)$ converge to $u(z) = 0$ nowhere in the unit ball, whereas the measures $(dd^c u_j)^n$ converge weakly to zero. Therefore, additional assumptions on functions near the boundary are needed to guarantee a positive result. We shall give certain conditions such that weak convergence of $(dd^c u_j)^n$ to $(dd^c u)^n$ is equivalent to convergence of the u_j to u in capacity. Using the capacity C_{n-1} we obtain the following converse theorem.

THEOREM 1.1. — *Let Ω be a bounded pseudoconvex domain and $u \in PSH(\Omega)$. Suppose that $\{u_j\}$ is a sequence of locally uniformly bounded functions in $PSH \cap L^\infty(\Omega)$ such that*

- i) $u_j \rightarrow u$ weakly in Ω ;
- ii) $\liminf_{z \rightarrow \partial\Omega} (u_j - u) \geq 0$ uniformly for all j ;
- iii) $(dd^c u_j)^n \rightarrow (dd^c u)^n$ weakly in Ω .

Then $u_j \rightarrow u$ in C_{n-1} on each $E \subset\subset \Omega$.

We have an example to show that under the assumptions of Theorem 1.1 one cannot expect convergence in C_n of the u_j to u . Some conditions

stronger than weak convergence of Monge-Ampère measures are needed to ensure convergence in C_n of the functions. We obtain several results in this direction. Our results enable us to find solutions of the Monge-Ampère equations and furthermore to deal with stability of solutions. Recently, Cegrell and Kolodziej proved the following stability theorem.

STABILITY THEOREM. — [10] *Let Ω be a strictly pseudoconvex domain. Suppose that $d\mu = (dd^c v)^n$ for some $v \in PSH \cap L^\infty(\Omega)$ with $\lim_{\zeta \rightarrow z} v(\zeta) = 0$ for $z \in \partial\Omega$ and $\int_\Omega (dd^c v)^n < \infty$, and suppose that f_j are μ -measurable functions with $0 \leq f_j \leq 1$ such that the measures $f_j d\mu$ converge weakly to $f d\mu$. For a continuous function ϕ in $\partial\Omega$, denote by u_j the unique solution of the Dirichlet problem*

$$\begin{cases} u \in PSH \cap L^\infty(\Omega); \\ (dd^c u)^n = f_j d\mu; \\ \lim_{\zeta \rightarrow z} u(\zeta) = \phi(z) \quad \text{for } z \in \partial\Omega. \end{cases}$$

Then there exists $u \in PSH \cap L^\infty(\Omega)$ such that $(dd^c u)^n = f d\mu$ and $u_j \rightarrow u$ in C_n on Ω .

The stability Theorem was proved first in the special case when ϕ is smooth, f_j is continuous and μ has a compactly support, and then in general by an approximation argument. We give here a more general stability theorem.

COROLLARY 1.2. — *Let Ω be a bounded domain. Suppose that $\{u_j\}$ is a sequence of locally uniformly bounded functions in $PSH \cap L^\infty(\Omega)$ such that*

- i) $\limsup_{z \rightarrow \partial\Omega} |u_j - u_i| = 0$ uniformly for all j and i ;
- ii) $(dd^c u_j)^n$ converges weakly to a positive measure $d\mu$ in Ω ;
- iii) there exists a positive measure $d\nu$ vanishing on all pluripolar sets in Ω such that $(dd^c u_j)^n \leq d\nu$ in Ω for all j .

Then there exists $u \in PSH \cap L^\infty_{\text{loc}}(\Omega)$ such that $(dd^c u)^n = d\mu$ and $u_j \rightarrow u$ in C_n on Ω .

Moreover, without assuming that all the Monge-Ampère measures are dominated by some fixed measure vanishing on all pluripolar subsets, we have

COROLLARY 1.3. — *Let Ω be a bounded domain. Suppose that $\{u_j\}$ is a sequence of locally uniformly bounded functions in $PSH \cap L^\infty(\Omega)$ such that*

- i) $\limsup_{z \rightarrow \partial\Omega} |u_j - u_i| = 0$ uniformly for all j and i ;
 ii) there exists a positive measure $d\mu$ in Ω such that $g(dd^c u_j)^n$ converges weakly to $g d\mu$ in Ω uniformly for all $g \in PSH(\Omega)$ with $0 \leq g \leq 1$.

Then there exists $u \in PSH \cap L_{\text{loc}}^\infty(\Omega)$ such that $(dd^c u)^n = d\mu$ and $u_j \rightarrow u$ in C_n on Ω .

The assumption ii) of Corollary 1.3 is in fact a necessary condition as shown in Theorem 2.1 of this paper. So such a convergence of complex Monge-Ampère measures is essentially equivalent to the convergence in C_n of functions.

In the second part of the paper we extend our results to Cegrell's class $\mathcal{F}(\Omega)$ of unbounded psh functions on which the Monge-Ampère operator is well defined, where Ω is a hyperconvex domain. The subclass $\mathcal{F}^a(\Omega)$ consists of those functions u from $\mathcal{F}(\Omega)$ for which the Monge-Ampère measure has zero mass on any pluripolar subset, see [9]. Theorem A has been generalized to some unbounded functions in [8] and [16]. For functions in $\mathcal{F}(\Omega)$ Cegrell proved

THEOREM B. — [8] *Suppose that $u_j \in \mathcal{F}(\Omega)$, $j = 0, 1, \dots$ and $u_j \geq u_0$. If $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$ then $(dd^c u_j)^n \rightarrow (dd^c u)^n$ weakly in Ω .*

Under the hypotheses in Theorem B we prove that for each fixed $g \in PSH \cap L^\infty(\Omega)$ the currents $g(dd^c u_j)^n$ converge weakly to $g(dd^c u)^n$ in Ω . Moreover, in the case of $u_0 \in \mathcal{F}^a(\Omega)$ we obtain a stronger convergence of Monge-Ampère measures in the following sense.

THEOREM 1.4. — *Suppose that $u_0 \in \mathcal{F}^a(\Omega)$ and that $u_j \in PSH(\Omega)$ satisfy $0 \geq u_j \geq u_0$ in Ω . If $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$ then $g(dd^c u_j)^n$ converges weakly to $g(dd^c u)^n$ in Ω uniformly for all psh functions g with $0 \leq g \leq 1$.*

We also show that the assumption of convergence in capacity of Theorem 1.4 is sharp by giving a converse result, which is a generalization of Corollary 1.3 for functions in $\mathcal{F}^a(\phi, \Omega)$.

It is a great pleasure for me to thank Urban Cegrell for many fruitful comments.

2. Convergence for Bounded Plurisubharmonic Functions

In this section we consider sequences of locally uniformly bounded psh functions in Ω . First we prove a convergence theorem. Next we prove that

weak convergence of Monge-Ampère measures is equivalent to convergence in the capacity C_{n-1} of functions if all the functions converge weakly and have uniformly the same boundary values. Then in terms of the capacity C_n we obtain several converse results. Finally, we give different types of stability theorems for solutions of the complex Monge-Ampère equations.

Recall that a sequence of functions u_j is said to be convergent to a function u in C_s on a set E if for each constant $\delta > 0$ we have that $C_s\{z \in E; |u_j(z) - u(z)| > \delta\} \rightarrow 0$ as $j \rightarrow \infty$. We begin with a convergence theorem.

THEOREM 2.1. — *Let $u \in PSH(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ and let u_j be locally uniformly bounded psh functions in Ω . Then for any family B of locally uniformly bounded psh functions in Ω the following assertions hold.*

- (a) *If $u_j \rightarrow u$ in C_{n-1} on each $E \subset\subset \Omega$ then $g(dd^c u_j)^n$ converges weakly to $g(dd^c u)^n$ for each g in B .*
- (b) *If $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$ then $g(dd^c u_j)^n$ converges weakly to $g(dd^c u)^n$ in Ω uniformly for all $g \in B$, that is, for each given $\phi \in C^\infty_0(\Omega)$ we have that $\int_\Omega \phi g(dd^c u_j)^n \rightarrow \int_\Omega \phi g(dd^c u)^n$ uniformly for all g in B .*
- (c) *If $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$ and $g_j \in B$ converges weakly to $g \in B$, then $g_j(dd^c u_j)^n$ converges weakly to $g(dd^c u)^n$ in Ω .*

Proof. — Assertion (a) follows directly from Theorem A and the quasi-continuity of the psh function g , see [3].

To prove (b), by Theorem A we have that $(dd^c u_j)^n \rightarrow (dd^c u)^n$ weakly in Ω and hence we can assume that $B = \{g \in PSH(\Omega); 0 < g < 1\}$. On the other hand, for each given $\phi \in C^\infty_0(\Omega)$, by changing values of functions near the boundary $\partial\Omega$, we assume also that all the u_j coincide with u in $\Omega \setminus E$ for some subset $E \supset\supset \text{supp } \phi$. Hence for any $\varepsilon > 0$ and all $g \in B$, an integration by parts yields

$$\begin{aligned} & \int_\Omega \phi g((dd^c u_j)^n - (dd^c u)^n) \\ &= \int_{E \cap \{|u_j - u| < \varepsilon\}} (u_j - u) dd^c(\phi g) \wedge \left(\sum_{k=0}^{n-1} (dd^c u_j)^k \wedge (dd^c u)^{n-1-k} \right) \\ &+ \int_{E \cap \{|u_j - u| \geq \varepsilon\}} (u_j - u) dd^c(\phi g) \wedge \left(\sum_{k=0}^{n-1} (dd^c u_j)^k \wedge (dd^c u)^{n-1-k} \right) \\ &:= A_{\varepsilon,j}(\phi) + B_{\varepsilon,j}(\phi). \end{aligned}$$

Given ϕ in $C_0^\infty(\Omega)$, take a sufficiently large constant M such that $\phi = (\phi + M|z|^2) - M|z|^2 := \phi_1 - \phi_2$, where $0 \leq \phi_1, \phi_2 \in PSH \cap L^\infty(\Omega)$. For both $k = 1$ and $k = 2$ we have that $2dd^c(\phi_k g) = dd^c((g + \phi_k)^2) - dd^c(g^2) - dd^c(\phi_k^2)$, where all dd^c on the right-hand side act on the bounded psh functions in Ω . So there exists a constant D independent of ε and j such that $|A_{\varepsilon,j}(\phi)| \leq |A_{\varepsilon,j}(\phi_1)| + |A_{\varepsilon,j}(\phi_2)| \leq DC_n(E)\varepsilon$ and $|B_{\varepsilon,j}(\phi)| \leq |B_{\varepsilon,j}(\phi_1)| + |B_{\varepsilon,j}(\phi_2)| \leq DC_n(E \cap \{|u_j - u| > \varepsilon\}) \rightarrow 0$ as $j \rightarrow \infty$. This implies that $\int_\Omega \phi g (dd^c u_j)^n \rightarrow \int_\Omega \phi g (dd^c u)^n$ as $j \rightarrow \infty$ uniformly in B .

Finally, to prove (c) we write $g_j (dd^c u_j)^n - g (dd^c u)^n = g_j ((dd^c u_j)^n - (dd^c u)^n) + (g_j - g)((dd^c u)^n - (dd^c v_s)^n) + (g_j - g)(dd^c v_s)^n$, where v_s are smooth psh functions decreasing to u . By (b) the second term on the right-hand side of the last equality tends weakly to zero as $s \rightarrow \infty$ uniformly for all j . Then, for a fixed sufficiently large s , both the first and third term converge weakly to zero as $j \rightarrow \infty$. Hence we have proved (c) and the proof of Theorem 2.1 is complete. \square

As a consequence of Theorem 2.1 we obtain that weak convergence of psh functions g_j (with respect to the Lebesgue measure) implies weak convergence of the g_j with respect to any measure vanishing on all pluripolar sets.

COROLLARY 2.2. — *Suppose that a locally finite positive measure $d\mu$ vanishes on all pluripolar sets in Ω and suppose that $g_0 \in PSH(\Omega)$ is locally integrable with respect to the $d\mu$. If $g_j \in PSH(\Omega)$ converges weakly to a psh function g and $|g_j| \leq |g_0|$ in Ω for all j then $\int_E |g_j - g| d\mu \rightarrow 0$ as $j \rightarrow \infty$ on any $E \subset\subset \Omega$.*

Remark 2.3. — In the case of the functions $g_0, g_j \in \mathcal{F}(\Omega)$ Corollary 2.2 is an analogue of Lemma 2.1 in [8]

Proof. — We assume without loss of generality that all $g_j < 0$ and $g < 0$ in Ω . Since $d\mu$ vanishes on all pluripolar sets it is enough to prove that $\int_\Omega \phi g_j d\mu \rightarrow \int_\Omega \phi g d\mu$ as $j \rightarrow \infty$ for any $\phi \in C_0^\infty(\Omega)$. Given $\phi \in C_0^\infty(\Omega)$ we write

$$\begin{aligned} \int_\Omega \phi g_j d\mu - \int_\Omega \phi g d\mu &= \int_\Omega \phi (g_j - \max(g_j, -s)) d\mu \\ &+ \int_\Omega \phi (\max(g_j, -s) - \max(g, -s)) d\mu + \int_\Omega \phi (\max(g, -s) - g) d\mu, \end{aligned}$$

where the first and third term on the right-hand side are dominated by

$$\max |\phi| \int_{\text{supp } \phi \cap \{g_0 < -s\}} -g_0 d\mu,$$

which goes to zero as $s \rightarrow \infty$. On the other hand, by Theorem 6.3 in [7] there exist a bounded psh function ψ and a nonnegative integrable function f in Ω with respect to the measure $(dd^c\psi)^n$ such that $\chi_{\text{supp } \phi} d\mu = f (dd^c\psi)^n$, where $\chi_{\text{supp } \phi}$ is the characteristic function of $\text{supp } \phi$. So for any $\varepsilon > 0$ there exist $s, k > 0$ such that

$$\begin{aligned} & \left| \int_{\Omega} \phi g_j d\mu - \int_{\Omega} \phi g d\mu \right| \\ & \leq \left| \int_{\Omega} \phi (\max(g_j, -s) - \max(g, -s)) \min(f, k) (dd^c\psi)^n \right| + \varepsilon. \end{aligned}$$

Then, take a $h \in C(\Omega)$ such that $\int_{\text{supp } \phi} |\min(f, k) - h| (dd^c\psi)^n < \frac{\varepsilon}{s}$. Hence we have

$$\begin{aligned} & \left| \int_{\Omega} \phi g_j d\mu - \int_{\Omega} \phi g d\mu \right| \\ & \leq \left| \int_{\Omega} \phi (\max(g_j, -s) - \max(g, -s)) h (dd^c\psi)^n \right| + (2 \max |\phi| + 1) \varepsilon, \end{aligned}$$

where by Theorem 2.1 the last integral goes to zero as $j \rightarrow \infty$ and therefore we have completed the proof of Theorem 2.1. □

Now we discuss necessity of convergence in capacity in the convergence theorems. Our first result is the following one.

THEOREM 2.4. — *Let Ω be a bounded pseudoconvex domain and $u \in PSH(\Omega)$. Suppose that $\{u_j\}$ is a sequence of locally uniformly bounded functions in $PSH \cap L^\infty(\Omega)$ such that*

- i) $u_j \rightarrow u$ weakly in Ω ;
- ii) $\liminf_{z \rightarrow \partial\Omega} (u_j - u) \geq 0$ uniformly for all j ;
- iii) $(dd^c u_j)^n \rightarrow (dd^c u)^n$ weakly in Ω .

Then $u_j \rightarrow u$ in C_{n-1} on each $E \subset\subset \Omega$.

Recall that $\liminf_{z \rightarrow \partial\Omega} (u_j - u) \geq 0$ uniformly for all j means that for any constant $\varepsilon > 0$ there exists $E \subset\subset \Omega$ such that $u_j(z) - u(z) \geq -\varepsilon$ holds for any $z \in \Omega \setminus E$ and all j . To prove Theorem 2.4 we need two facts.

LEMMA 2.5. — [16] *If $u, v \in PSH(\Omega)$ satisfy $\liminf_{z \rightarrow \partial\Omega} (u(z) - v(z)) \geq 0$, then the inequality*

$$\int_{u < v} (v - u) dd^c w_1 \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n \leq \int_{u < v} (r - w_1) dd^c(u - v) \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n$$

holds for any bounded functions w_1, w_2, \dots, w_n in $PSH(\Omega)$ and for any constants $r \geq \sup_{\Omega} w_1$.

Following the proof of Proposition 4.2 in [4] we have

LEMMA 2.6. — *If $u, v, w_1, \dots, w_{n-1} \in PSH \cap L_{loc}^{\infty}(\Omega)$, then $dd^c \max(u, v) \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1} = dd^c u \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1}$ as measures on the set $\{u > v\}$.*

Remark 2.7. — For later reference we point out that the result of Lemma 2.6 is still true in the case of arbitrary functions $u, v, w_1, \dots, w_{n-1}$ in $\mathcal{F}(\phi, \Omega)$, since the proof in [4] works even for this class of functions.

Proof of Theorem 2.4. — Given $E \subset\subset \Omega$. Let $a_1 = \sup_{\Omega} |z|^2$. For any $\delta > 0$ and $w \in PSH(\Omega)$ with $0 < w < 1$ by the definition of C_{n-1} we get

$$\int_{E \cap \{|u_j - u| > \delta\}} (dd^c w)^{n-1} \wedge dd^c |z|^2 \leq \frac{1}{a_1} C_{n-1}(E \cap \{u_j > u + \delta\}) + \int_{u_j < u - \delta} (dd^c w)^{n-1} \wedge dd^c |z|^2.$$

Since $u_j \rightarrow u$ in $L_{loc}^1(\Omega)$, by Hartog’s Lemma and the quasicontinuity of psh functions ([3]) we have that $C_{n-1}(E \cap \{u_j > u + \delta\}) \rightarrow 0$ as $j \rightarrow \infty$.

On the other hand, it turns out from Lemma 2.5 that

$$\begin{aligned}
 & \int_{u_j < u - \delta} (dd^c w)^{n-1} \wedge dd^c |z|^2 \leq \frac{1}{\delta} \int_{u_j < u - \delta} (u - u_j) (dd^c w)^{n-1} \wedge dd^c |z|^2 \\
 & \leq \frac{2}{\delta} \int_{u_j < u - \frac{\delta}{2}} (u - u_j - \frac{\delta}{2}) (dd^c w)^{n-1} \wedge dd^c |z|^2 \\
 & \leq \frac{2}{\delta} \int_{u_j < u - \frac{\delta}{2}} (1 - w) dd^c(u - u_j) \wedge (dd^c w)^{n-2} \wedge dd^c |z|^2 \\
 & \leq \frac{2}{\delta} \int_{u_j < u - \frac{\delta}{2}} dd^c(u + u_j) \wedge (dd^c w)^{n-2} \wedge dd^c |z|^2.
 \end{aligned}$$

Repeating this argument $n - 2$ times we obtain that the right-hand side in the last inequality does not exceed

$$\begin{aligned}
 & 2^{\frac{(n-1)n}{2}} \delta^{1-n} \int_{u_j < u - 2^{1-n} \delta} dd^c(u + u_j)^{n-1} \wedge dd^c |z|^2 \\
 & \leq 2^{\frac{(n-1)n}{2}} (n - 1)! \delta^{1-n} \int_{u_j < u - 2^{1-n} \delta} \sum_{k=0}^{n-1} (dd^c u)^k \wedge (dd^c u_j)^{n-1-k} \wedge dd^c |z|^2 \\
 & \leq 2^{\frac{(n-1)(n+2)}{2}} (n - 1)! \delta^{-n} \int_{u_j < u - 2^{1-n} \delta} (u - u_j) \sum_{k=0}^{n-1} (dd^c u)^k \\
 & \qquad \qquad \qquad \wedge (dd^c u_j)^{n-1-k} \wedge dd^c |z|^2.
 \end{aligned}$$

Since $\liminf_{z \rightarrow \partial\Omega} (u_j - u) \geq 0$ uniformly for all j there exists a strictly pseudoconvex domain Ω_1 with a defining function ρ in $PSH(\bar{\Omega}_1) \cap C(\bar{\Omega}_1)$ such that $\{u_j < u - 2^{1-n} \delta\} \subset \subset \Omega_1 \subset \subset \Omega$ for all j . Take a constant a_2 large enough such that $a_2 \rho(z) < |z|^2 - a_1$ on $\{u_j < u - 2^{1-n} \delta\}$ for all j . Then for any $\varepsilon > 0$ choose a subdomain Ω_2 such that all the sets $\{u_j < u - 2^{1-n} \delta\} \subset \subset \Omega_2 \subset \subset \Omega_1$ and $-\varepsilon < \max(|z|^2 - a_1, a_2 \rho(z)) < 0$ in $\Omega_1 \setminus \Omega_2$. Let $a_3 = \inf_j \{u_j(z); z \in \Omega_1\}$. Take a function $\phi \in C_0^\infty(\Omega_1)$ with $\phi = 1$ on Ω_2 and take a constant $a_4 > 0$ such that $a_4 \rho < u - a_3$ and $a_4 \rho < u_j - a_3$ in $\text{supp } \phi$ for all j . Since $u_j \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$ and u_j are locally uniformly bounded in Ω , by Hartog’s lemma we get that for all j

large enough the last integral equals

$$\begin{aligned}
 & \int_{u_j < u - 2^{1-n}\delta} (\max(u - a_3, a_4\rho) - \max(u_j - a_3, a_4\rho)) \sum_{k=0}^{n-1} (dd^c u)^k \\
 & \qquad \qquad \qquad \wedge (dd^c u_j)^{n-1-k} \wedge dd^c \max(|z|^2 - a_1, a_2\rho) \\
 &= \int_{\Omega_1} (\max(u - a_3, a_4\rho) - \max(u_j - a_3, a_4\rho)) \sum_{k=0}^{n-1} (dd^c u)^k \\
 & \qquad \qquad \qquad \wedge (dd^c u_j)^{n-1-k} \wedge dd^c \max(|z|^2 - a_1, a_2\rho) + O(\varepsilon) \\
 &= \int_{\Omega_1} \max(|z|^2 - a_1, a_2\rho) \sum_{k=0}^{n-1} (dd^c u)^k \wedge (dd^c u_j)^{n-1-k} \\
 & \qquad \qquad \qquad \wedge dd^c (\max(u - a_3, a_4\rho) - \max(u_j - a_3, a_4\rho)) + O(\varepsilon) \\
 &= \int_{\Omega_1} \phi \max(|z|^2 - a_1, a_2\rho) \sum_{k=0}^{n-1} (dd^c u)^k \wedge (dd^c u_j)^{n-1-k} \\
 & \qquad \qquad \qquad \wedge dd^c (\max(u - a_3, a_4\rho) - \max(u_j - a_3, a_4\rho)) + O(\varepsilon) \\
 &= \int_{\Omega_1} \phi \max(|z|^2 - a_1, a_2\rho) ((dd^c u)^n - (dd^c u_j)^n) + O(\varepsilon) \longrightarrow 0
 \end{aligned}$$

as $j \rightarrow \infty$ and then $\varepsilon \rightarrow 0$,

where the equality before last follows from the inequality $-\varepsilon < \max(|z|^2 - a_1, a_2\rho(z)) < 0$ in $\Omega_1 \setminus \Omega_2$, and the last one follows from Lemma 2.6. Hence we have proved that $u_j \rightarrow u$ in C_{n-1} on each $E \subset\subset \Omega$ and the proof of Theorem 2.4 is complete. □

Under the assumptions of Theorem 2.4 one cannot expect the stronger convergence result that $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$. For instance, by [5] there exist uniformly bounded psh functions u_j in a $\Omega \subset\subset \mathbb{C}$ such that $u_j \rightarrow u$ weakly and $u_j dd^c u_j \not\rightarrow u dd^c u$ in Ω . Changing values of functions near the boundary $\partial\Omega$ we can assume that all the u_j coincide outside a compact subset in Ω . Hence the sequence $\{u_j\}$ satisfies the assumptions in Theorem 2.4, but $u_j \not\rightarrow u$ in C_1 on all $E \subset\subset \Omega$ since $u_j dd^c u_j \not\rightarrow u dd^c u$ and Theorem A. Therefore, we need some conditions stronger than weak convergence of Monge-Ampère measures to ensure convergence in C_n of the functions. Now our next result is

THEOREM 2.8. — *Let $u \in PSH(\Omega)$. Suppose that $\{u_j\}$ is a sequence of locally uniformly bounded functions in $PSH \cap L^\infty(\Omega)$ such that*

- i) $u_j \rightarrow u$ weakly in Ω ;
- ii) $\liminf_{z \rightarrow \partial\Omega} (u_j - u) \geq 0$ uniformly for all j ;
- iii) there exists a positive measure $d\mu$ in Ω such that $g(dd^c u_j)^n$ converges weakly to $g d\mu$ in Ω uniformly for all $g \in PSH(\Omega)$ with $0 \leq g \leq 1$.

Then $(dd^c u)^n = d\mu$ and $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$. Hence, if furthermore $\liminf_{z \rightarrow \partial\Omega} (u - u_j) \geq 0$ uniformly for all j then $u_j \rightarrow u$ in C_n on Ω .

To prove Theorem 2.8 we need a stronger version of the comparison theorem.

LEMMA 2.9. — [17] Let $u, v \in PSH(\Omega) \cap L^\infty(\Omega)$ satisfy $\liminf_{z \rightarrow \partial\Omega} (u(z) - v(z)) \geq 0$. Then for any constant $r \geq 1$ and all $w_j \in PSH(\Omega)$ with $0 \leq w_j \leq 1, j = 1, 2, \dots, n$, we have

$$\begin{aligned} \frac{1}{(n!)^2} \int_{u < v} (v - u)^n dd^c w_1 \wedge \dots \wedge dd^c w_n + \int_{u < v} (r - w_1) (dd^c v)^n \\ \leq \int_{u < v} (r - w_1) (dd^c u)^n. \end{aligned}$$

If moreover $(dd^c v)^n \geq (dd^c u)^n$ in Ω then $v \leq u$ in Ω .

Proof of Theorem 2.8. — Let $E \subset\subset \Omega$. Given $\delta > 0$ and any $w \in PSH(\Omega)$ with $0 < w < 1$ we have that

$$\int_{E \cap \{|u_j - u| > \delta\}} (dd^c w)^n \leq C_n(E \cap \{u_j > u + \delta\}) + \int_{u_j < u - \delta} (dd^c w)^n.$$

It follows from Hartog’s Lemma and the quasicontinuity of psh functions ([3]) that $C_n(E \cap \{u_j > u + \delta\}) \rightarrow 0$ as $j \rightarrow \infty$. By ii) and Lemma 2.9 we get

$$\begin{aligned} \int_{u_j < u - \delta} (dd^c w)^n &\leq \frac{1}{\delta^n} \int_{u_j < u - \delta} (u - u_j)^n (dd^c w)^n \\ &\leq \frac{(n!)^2}{\delta^n} \int_{u_j < u - \delta} (dd^c u_j)^n \leq \frac{(n!)^2}{\delta^{n+1}} \int_{u_j < u - \delta} (u - u_j) (dd^c u_j)^n. \end{aligned}$$

Given $\varepsilon > 0$, take subdomains $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ such that $u_j - u \geq -\varepsilon$ in $\Omega \setminus \Omega_1$ and $\{u_j < u - \delta\} \subset\subset \Omega_1$ for all j . It follows again from the quasicontinuity of psh functions and Hartog’s Lemma that there exist $j_0 > 0$ and a set $E \subset \Omega_2$ with $C_n(E) < \varepsilon$ such that $\varepsilon + u(z) - u_j(z) \geq 0$ in $\Omega_2 \setminus E$ for all $j \geq j_0$. Choose $0 \leq \phi \in C_0^\infty(\Omega_2)$ with $\phi = 1$ in Ω_1 . Since all

the u_j and u are uniformly bounded in Ω_2 , then for $j \geq j_0$ the last integral does not exceed

$$\begin{aligned} \int_{\Omega_1 \setminus E} \phi(\varepsilon + u - u_j)(dd^c u_j)^n + O(\varepsilon) &\leq \int_{\Omega_2} \phi(u - u_j)(dd^c u_j)^n + O(\varepsilon) \\ &= \int_{\Omega_2} \phi(u - u_j)((dd^c u_j)^n - d\mu) + \int_{\Omega_2} \phi(u - u_j) d\mu + O(\varepsilon), \end{aligned}$$

where by iii) and Corollary 2.2 the last two integrals tend to zero as $j \rightarrow \infty$. Hence we have proved that $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$. Then by Theorem A we get $(dd^c u)^n = d\mu$ and the proof of Theorem 2.8 is complete. \square

We write $\limsup_{z \rightarrow \partial\Omega} |f(z) - g(z)| = 0$ if for any $\varepsilon > 0$ there exists $E \subset\subset \Omega$ such that $|f - g| < \varepsilon$ in $\Omega \setminus E$. As a consequence of Theorem 2.8 we have the following stability theorem of solutions of Monge-Ampère equations.

COROLLARY 2.10. — *Suppose that $\{u_j\}$ is a sequence of locally uniformly bounded functions in $PSH \cap L^\infty(\Omega)$ such that*

- i) $\limsup_{z \rightarrow \partial\Omega} |u_j - u_i| = 0$ uniformly for all j and i ;
- ii) *there exists a positive measure $d\mu$ in Ω such that $g(dd^c u_j)^n$ converges weakly to $g d\mu$ in Ω uniformly for all $g \in PSH(\Omega)$ with $0 \leq g \leq 1$.*

Then there exists $u \in PSH \cap L^\infty_{loc}(\Omega)$ such that $(dd^c u)^n = d\mu$ and $u_j \rightarrow u$ in C_n on Ω .

Proof. — It is enough to prove that there exists $u \in PSH \cap L^\infty_{loc}(\Omega)$ such that from any subsequence of the given sequence $\{u_j\}$ one can extract sub-subsequence $\{u_{j_k}\}$ converging to u in C_n on Ω . Choose $\{u_{j_k}\}$ from any subsequence such that it converges weakly to a psh function u in Ω . It then follows from Theorem 2.8 that $u_{j_k} \rightarrow u$ in C_n on Ω and by Theorem A we have $(dd^c u)^n = d\mu$. By Lemma 2.9 such a function u is unique. This completes the proof. \square

In fact, following the proof of Theorem 2.8 one can prove another type of converse theorem.

THEOREM 2.11. — *Suppose that a sequence $\{u_j\}$ of locally uniformly bounded functions in $PSH \cap L^\infty(\Omega)$ satisfies*

- i) u_j converges weakly to a psh function u in Ω ;
- ii) $\liminf_{z \rightarrow \partial\Omega} (u_j - u) \geq 0$ uniformly for all j ;

- iii) $(dd^c u_j)^n$ converges weakly to a positive measure $d\mu$ in Ω ;
- iv) there exists a positive measure $d\nu$ vanishing on all pluripolar sets in Ω such that $(dd^c u_j)^n \leq d\nu$ in Ω for all j .

Then $(dd^c u)^n = d\mu$ and $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$.

Proof. — Given $E \subset\subset \Omega$ and $\delta > 0$, similar to the proof of Theorem 2.8 we get that for any $\varepsilon > 0$ there exists $\Omega_1 \subset\subset \Omega$ such that the inequality $C_n(E \cap \{|u_j - u| > \delta\}) \leq (n!)^2 \delta^{-n-1} \int_{\Omega_1} |u - u_j| d\nu + \varepsilon$ holds for all j large enough. By Corollary 2.2 the last integral tends to zero as $j \rightarrow \infty$. Hence $u_j \rightarrow u$ in C_n on E and the proof is complete. \square

As a direct consequence of Theorem 2.11 we have

COROLLARY 2.12. — *Suppose that $\{u_j\}$ is a sequence of locally uniformly bounded functions in $PSH \cap L^\infty(\Omega)$ such that*

- i) $\limsup_{z \rightarrow \partial\Omega} |u_j - u_i| = 0$ uniformly for all j and i ;
- ii) $(dd^c u_j)^n$ converges weakly to a positive measure $d\mu$ in Ω ;
- iii) there exists a positive measure $d\nu$ vanishing on all pluripolar sets in Ω such that $(dd^c u_j)^n \leq d\nu$ in Ω for all j .

Then there exists $u \in PSH \cap L^\infty_{\text{loc}}(\Omega)$ such that $(dd^c u)^n = d\mu$ and $u_j \rightarrow u$ in C_n on Ω .

Remark 2.13. — We do not assume in Corollary 2.12 that all the functions u_j have the same continuous boundary data. In fact, under the assumptions of the stability theorem in [10] there exist functions h_1, h_2 in $PSH \cap L^\infty(\Omega)$ such that $(dd^c h_1)^n = d\mu, (dd^c h_2)^n = 0$ and $\lim_{\zeta \rightarrow z} h_1(\zeta) = \lim_{\zeta \rightarrow z} h_2(\zeta) = \phi(z)$ for $z \in \partial\Omega$. It then turns out from the comparison theorem that $h_2 \geq u_j \geq h_1$ in Ω and hence the functions u_j fulfill the hypotheses of Corollary 2.12, since $h_2 - h_1 = 0$ on the compact set $\partial\Omega$. Therefore, Stability Theorem follows from Corollary 2.12. Moreover, since Corollary 2.12 is valid for all type of bounded domains, it also implies the stability theorem for hyperconvex domains, see [10].

Sometimes it is difficult to find such a measure $d\nu$ which dominates all the $(dd^c u_j)^n$. For this reason, we feel useful to present a slightly different version of Theorem 2.11.

THEOREM 2.14. — *Suppose that a sequence $\{u_j\}$ of locally uniformly bounded functions in $PSH \cap L^\infty(\Omega)$ satisfies*

- i) u_j converges weakly to a psh function u in Ω ;
- ii) $\liminf_{z \rightarrow \partial\Omega} (u_j - u) \geq 0$ uniformly for all j ;

iii) $(dd^c u_j)^n$ converges weakly to a positive measure $d\mu$ in Ω .

If there exist locally uniformly bounded psh functions v_j in Ω such that $(dd^c u_j)^n \leq (dd^c v_j)^n$ for all j in Ω and v_j converges to some psh function v in C_n on each $E \subset\subset \Omega$, then $(dd^c u)^n = d\mu$ and $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$. Thus, if furthermore $\liminf_{z \rightarrow \partial\Omega} (u - u_j) \geq 0$ uniformly for all j then $u_j \rightarrow u$ in C_n on Ω .

We omit the proof of Theorem 2.14 since it is completely similar to the proof of Theorem 2.8. As an application we give a proof of the following well known result due to Kolodziej.

COROLLARY 2.15. — [12] *Let Ω be a strictly pseudoconvex domain. Suppose that $v \in PSH \cap L^\infty(\Omega)$ satisfies $\lim_{\varsigma \rightarrow z} v(\varsigma) = 0$ for $z \in \partial\Omega$. Then for any positive measure $d\mu \leq (dd^c v)^n$ there exists $u \in PSH \cap L^\infty(\Omega)$ such that $(dd^c u)^n = d\mu$ and $\lim_{\varsigma \rightarrow z} u(\varsigma) = 0$ for $z \in \partial\Omega$.*

Proof. — It is no loss of generality to assume that $\int_\Omega (dd^c v)^n < \infty$. Take a decreasing sequence $\{v_k\}$ of smooth psh functions in Ω such that $v_k \searrow v$ in Ω and $v_k = 0$ on $\partial\Omega$. By the Lebesgue-Radon-Nikodym theorem we have $d\mu = f (dd^c v)^n$, where $0 \leq f \leq 1$ and f is integrable with respect to the measure $(dd^c v)^n$. Take $f_j \in C_0^\infty(\Omega)$ with $0 \leq f_j \leq 1$ such that $\int_\Omega |f - f_j| (dd^c v)^n < \frac{1}{j}$, and then take a subsequence $\{v_{k_j}\}$ of the $\{v_k\}$ such that $\left| \int_\Omega f_j ((dd^c v)^n - (dd^c v_{k_j})^n) \right| < \frac{1}{j}$ for all j (such a subsequence exists since $(dd^c v_k)^n$ converges weakly to $(dd^c v)^n$ in Ω). By [2] there exist $u_j \in PSH \cap C(\bar{\Omega})$ such that $(dd^c u_j)^n = f_j (dd^c v_{k_j})^n$ and $u_j = 0$ on $\partial\Omega$. It follows from the comparison theorem that $v \leq v_{k_j} \leq u_j \leq 0$ in Ω . Since one can extract a subsequence of the $\{u_j\}$ such that it converges weakly to some psh function u , by Theorem 2.14 we get that $(dd^c u)^n = d\mu$ and $u = 0$ on $\partial\Omega$. The proof of Corollary 2.15 is complete. □

3. Convergence in the Class \mathcal{F}^a

In this section we first prove an approximation theorem for the Monge-Ampère operator on $\mathcal{F}^a(\Omega)$. Then we give a converse theorem which generalize Theorem 2.4 and Theorem 2.8 to functions in $\mathcal{F}^a(\phi, \Omega)$.

Throughout this section we assume that Ω is a hyperconvex domain in \mathbb{C}^n , that is, Ω is a bounded domain and there exists a negative psh function ρ in Ω such that $\{z \in \Omega; \rho(z) < -c\} \subset\subset \Omega$ for any $c > 0$. Recall that $\mathcal{E}_0(\Omega)$ is the set of bounded psh functions u in Ω with $\lim_{\varsigma \rightarrow z} u(\varsigma) = 0$

for any $z \in \partial\Omega$ and $\int_{\Omega}(dd^c u)^n < \infty$. Denote by $\mathcal{F}(\Omega)$ the set of psh functions u in Ω such that there exists a decreasing sequence $\{u_j\}$ in $\mathcal{E}_0(\Omega)$ satisfying $u_j \searrow u$ in Ω and $\sup_j \int_{\Omega}(dd^c u_j)^n < \infty$. We shall use the subclass $\mathcal{F}^a(\Omega)$ of functions from $\mathcal{F}(\Omega)$ whose Monge-Ampère measures have zero mass on all pluripolar subsets of Ω . The Monge-Ampère measure $(dd^c u)^n$ is a well defined finite positive measure in Ω for any $u \in \mathcal{F}(\Omega)$, see [9]. Recall that a sequence $\{\mu_j\}$ of positive measures is said to be uniformly absolutely continuous with respect to C_n in a set E if for any constant $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for all Borel subsets $E_1 \subset E$ with $C_n(E_1) < \delta$ the inequality $\mu_j(E_1) < \varepsilon$ holds for all j . We begin with several lemmas.

LEMMA 3.1. — For any $w_0, w_1, w_2, \dots, w_{n-1} \in \mathcal{F}(\Omega)$ and any $g \in \mathcal{E}_0(\Omega)$ the measures

$$(-w_0) dd^c \max(w_1, -j) \wedge dd^c w_2 \wedge \dots \wedge dd^c w_{n-1} \wedge dd^c g$$

are absolutely continuous with respect to C_n in Ω uniformly for all $j = 1, 2, \dots$

Proof. — Write $T = dd^c w_2 \wedge \dots \wedge dd^c w_{n-1} \wedge dd^c g$. Since w_0 is upper semicontinuous and $dd^c \max(w_1, -j) \wedge T$ converges weakly to $dd^c w_1 \wedge T$ in Ω as $j \rightarrow \infty$, we have

$$\liminf_{j \rightarrow \infty} \int_{\Omega} (-w_0) dd^c \max(w_1, -j) \wedge T \geq \int_{\Omega} (-w_0) dd^c w_1 \wedge T.$$

On the other hand, an integration by parts yields

$$\int_{\Omega} (-w_0) dd^c \max(w_1, -j) \wedge T \leq \int_{\Omega} (-w_0) dd^c w_1 \wedge T.$$

Hence we get

$$\lim_{j \rightarrow \infty} \int_{\Omega} (-w_0) dd^c \max(w_1, -j) \wedge T = \int_{\Omega} (-w_0) dd^c w_1 \wedge T.$$

Therefore, for any $a > 0$ by the remark of Lemma 2.6 we have

$$\begin{aligned} & \lim_{j \rightarrow \infty} \int_{w_1 < -a} (-w_0) dd^c \max(w_1, -j) \wedge T \\ &= \lim_{j \rightarrow \infty} \int_{\Omega} (-w_0) dd^c \max(w_1, -j) \wedge T - \lim_{j \rightarrow \infty} \int_{w_1 \geq -a} (-w_0) dd^c \max(w_1, -j) \wedge T \\ &= \int_{\Omega} (-w_0) dd^c w_1 \wedge T - \int_{w_1 \geq -a} (-w_0) dd^c w_1 \wedge T = \int_{w_1 < -a} (-w_0) dd^c w_1 \wedge T. \end{aligned}$$

So for any $\varepsilon > 0$ there exist $a_0 > 0$ and $j_0 > a_0$ such that for all $j \geq j_0$ and any $E \subset \Omega$ we have

$$\int_E (-w_0) dd^c \max(w_1, -j) \wedge T \leq \int_{E \cap \{w_1 \geq -a\}} (-w_0) dd^c \max(w_1, -j) \wedge T + \int_{w_1 < -a} (-w_0) dd^c \max(w_1, -j) \wedge T \leq \int_E (-w_0) dd^c w_1 \wedge T + \varepsilon.$$

Using the inequality

$$\int_{w_k < -s} dd^c w_1 \wedge T \leq s^{-1} \int_{\Omega} (-w_k) dd^c w_1 \wedge T \leq s^{-1} \int_{\Omega} (-g) dd^c w_1 \wedge \dots \wedge dd^c w_{n-1} \wedge dd^c w_k,$$

we have that the measure $dd^c w_1 \wedge T$ is absolutely continuous with respect to C_n in Ω , and together with

$$\int_{\Omega} (-w_0) dd^c w_1 \wedge T = \int_{\Omega} (-g) dd^c w_0 \wedge dd^c w_1 \wedge \dots \wedge dd^c w_{n-1} < \infty$$

we get that $\int_E (-w_0) dd^c w_1 \wedge T$ is small when $C_n(E)$ is small. Since for each j the measure $(-w_0) dd^c \max(w_1, -j) \wedge T$ is absolutely continuous with respect to C_n , we have proved that $(-w_0) dd^c \max(w_1, -j) \wedge T$ are absolutely continuous with respect to C_n in Ω uniformly for all j . The proof of Lemma 3.1 is complete. \square

Now we can prove a generalization of Lemma 2.5 for functions in $\mathcal{F}(\Omega)$.

LEMMA 3.2. — *If $u, v, w_1, w_2, \dots, w_{n-1} \in \mathcal{F}(\Omega)$ and $g \in \mathcal{E}_0(\Omega)$, then we have*

$$\int_{u < v} (v - u) dd^c w_1 \wedge \dots \wedge dd^c w_{n-1} \wedge dd^c g \leq \int_{u < v} (-w_1) dd^c (u - v) \wedge dd^c w_2 \wedge \dots \wedge dd^c w_{n-1} \wedge dd^c g.$$

Proof. — Write $u_j = \max(u, -j)$, $v_k = \max(v, -k)$ and $w_1^l = \max(w_1, -l)$ and $T = dd^c w_2 \wedge \dots \wedge dd^c w_{n-1} \wedge dd^c g$. By a similar proof of Lemma 2.5 we have

$$\int_{u_j < v_k} (v_k - u_j) dd^c w_1^l \wedge T + \int_{u_j < v_k} (-w_1^l) dd^c v_k \wedge T \leq \int_{u_j < v_k} (-w_1^l) dd^c u_j \wedge T.$$

Let $j \rightarrow \infty$ and by Fatou lemma we get

$$\int_{u < v} (v_k - u) dd^c w_1^l \wedge T + \int_{u < v} (-w_1^l) dd^c v_k \wedge T \leq \liminf_{j \rightarrow \infty} \int_{u \leq v_k} (-w_1) dd^c u_j \wedge T.$$

By Lemma 3.1 and the quasicontinuity of psh functions, we can assume without loss of generality that $\{u < v\}$ is open and $\{u \leq v_k\}$ is closed. Hence, letting $l \rightarrow \infty, k \rightarrow \infty$ and by the weak convergences of these currents we obtain the inequality

$$\int_{u < v} (v - u) dd^c w_1 \wedge T \leq \int_{u \leq v} (-w_1) dd^c (u - v) \wedge T.$$

Applying the last inequality for $v + \varepsilon g$ instead of v and then letting the constants $\varepsilon \searrow 0$ we get the required inequality. The proof is complete. \square

LEMMA 3.3. — *The following assertions hold.*

- a) *If $u_0 \in \mathcal{F}^a(\Omega)$, then the measures $(-u) (dd^c u)^{n-1} \wedge (dd^c g)$ are absolutely continuous with respect to C_n in Ω uniformly for all $u \in PSH(\Omega)$ with $0 \geq u \geq u_0$ and for all $g \in \mathcal{E}_0(\Omega)$ with $0 \geq g \geq -1$.*
- b) *If $u_0 \in \mathcal{F}(\Omega)$, then for each fixed $g \in \mathcal{E}_0(\Omega)$ the measures $(-u) (dd^c u)^{n-1} \wedge (dd^c g)$ are absolutely continuous with respect to C_n in Ω uniformly for all $u \in PSH(\Omega)$ with $0 \geq u \geq u_0$.*

Proof. — To prove a), from $u_0 \in \mathcal{F}^a(\Omega)$ it turns out that $u \in \mathcal{F}(\Omega)$. The result is trivial when the u is bounded. Otherwise, for any $E \subset \Omega$ and $a > 0$ we have

$$\begin{aligned} \int_E (-u) (dd^c u)^{n-1} \wedge dd^c g &\leq \int_{u < -a} (-u) (dd^c u)^{n-1} \wedge dd^c g \\ &\quad + \int_{E \cap \{u > -a-1\}} (-u) (dd^c u)^{n-1} \wedge dd^c g \end{aligned}$$

By Remark 2.7 we get that the last integral equals

$$\begin{aligned} \int_{E \cap \{u > -a-1\}} (-\max(u, -a-1)) (dd^c \max(u, -a-1))^{n-1} \wedge dd^c g \\ \leq (a+1)^{-n} C_n(E). \end{aligned}$$

From Lemma 3.2 it turns out that

$$\begin{aligned} \int_{u < -a} (-u) (dd^c u)^{n-1} \wedge dd^c g &\leq \int_{u_0 < -a} (-u_0) (dd^c u)^{n-1} \wedge dd^c g \\ &\leq \int_{2u_0 < \max(2u_0, -a)} (-2u_0 + \max(2u_0, -a)) (dd^c u)^{n-1} \wedge dd^c g \\ &\leq \int_{2u_0 < -a} -2u (dd^c u)^{n-2} \wedge dd^c u_0 \wedge dd^c g. \end{aligned}$$

Continuing in this manner $n - 1$ more times we obtain that the last integral is dominated by

$$2^n \int_{2^n u_0 < -a} (-g) (dd^c u_0)^n \leq 2^n \int_{2^n u_0 < -a} (dd^c u_0)^n \rightarrow 0 \quad \text{as } a \rightarrow \infty,$$

since $u_0 \in \mathcal{F}^a(\Omega)$. Hence we have proved a).

To prove b), using the same method as the proof of a) we get that for any $E \subset \Omega$ and $a > 0$ the inequality

$$\begin{aligned} \int_E (-u) (dd^c u)^{n-1} \wedge dd^c g &\leq 2^{n-1} \int_{2^{n-1} u_0 < -a} (-u_0) (dd^c u_0)^{n-1} \wedge dd^c g \\ &\quad + (a + 1)^{-n} C_n(E) \end{aligned}$$

holds. From $u_0 \in \mathcal{F}(\Omega)$ it turns out that the measure $(dd^c u_0)^{n-1} \wedge dd^c g$ is absolutely continuous with respect to C_n in Ω and hence the last integral tends to zero as $a \rightarrow \infty$, which has proved (b). The proof of Lemma 3.3 is complete. \square

THEOREM 3.4. — *Let B be a family of locally uniformly bounded psh functions in Ω . Suppose that $u_0 \in \mathcal{F}^a(\Omega)$ and that the functions $u_j \in PSH(\Omega)$ satisfy $0 \geq u_j \geq u_0$. If $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$ then $g (dd^c u_j)^n$ converges weakly to $g (dd^c u)^n$ in Ω uniformly for all $g \in B$. Furthermore, if $g_j \in B$ converges weakly to $g \in B$, then $g_j (dd^c u_j)^n$ converges weakly to $g (dd^c u)^n$ in Ω .*

Proof. — It is no restriction to assume that $B \subset \mathcal{E}_0(\Omega)$. For any $a > 0$ we write $(dd^c u_j)^n - (dd^c u)^n = ((dd^c u_j)^n - (dd^c \max(u_j, -a))^n) + ((dd^c \max(u_j, -a))^n - (dd^c \max(u, -a))^n) + ((dd^c \max(u, -a))^n - (dd^c u)^n) := A_j^1(a) + A_j^2(a) + A_j^3(a)$. From Theorem 2.1 it turns out that for each fixed $a > 0$ the currents $g A_j^2(a)$ converges weakly to zero in Ω uniformly for all $g \in B$. On the other hand, for $\phi \in C_0^\infty(\Omega)$ and any $g \in B$ we write $\phi g = \phi (g - \min_\Omega g) + \phi \min_\Omega g$ and then, following the proof of Theorem 2.1, the current

$dd^c(\phi g)$ can be written as a sum of finite terms of the form $\pm f_1 dd^c f_2$ where f_1, f_2 are locally uniformly bounded psh functions depending only on ϕ and g . Hence there exists a family B_ϕ of locally uniformly bounded psh functions in Ω such that the current $dd^c(\phi g)$ is dominated by $dd^c g_\phi$ for some g_ϕ in B_ϕ . Therefore, using an integration by parts we have

$$\begin{aligned} \left| \int_{\Omega} \phi g A_j^1(a) \right| &= \left| \int_{\Omega} (u_j - \max(u_j, -a)) dd^c(\phi g) \wedge \sum_{k=0}^{n-1} (dd^c u_j)^k \right. \\ &\quad \left. \wedge (dd^c \max(u_j, -a))^{n-1-k} \right| \\ &= \left| \int_{u_j < -a} (u_j + a) dd^c(\phi g) \wedge (dd^c u_j)^{n-1} \right| \leq \int_{u_0 < -a} (-u_j) dd^c g_\phi \wedge (dd^c u_j)^{n-1} \end{aligned}$$

which by a) of Lemma 3.3 tends to zero as $a \rightarrow \infty$ uniformly for all $g \in B$. Similarly, $g A_j^3(a)$ converges weakly to zero uniformly for all $g \in B$. Hence we have proved that $g (dd^c u_j)^n$ converges weakly to $g (dd^c u)^n$ in Ω uniformly for all $g \in B$. The second assertion follows from the first one, and the proof of Theorem 3.4 is complete.

Following the proof of Theorem 3.4 and using b) instead of a) in Lemma 3.3, we obtain the following theorem which is a slightly stronger version of Theorem B due to Cegrell [8]. □

THEOREM 3.5. — *Suppose that $u_0 \in \mathcal{F}(\Omega)$ and that the functions $u_j \in PSH(\Omega)$ satisfy $0 \geq u_j \geq u_0$. If $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$ then for each fixed $g \in PSH \cap L^\infty(\Omega)$ we have that $g (dd^c u_j)^n \rightarrow g (dd^c u)^n$ weakly in Ω .*

We shall show that the assumption of convergence in capacity of Theorem 3.4 is necessary in some case. Let $\phi \in C(\partial\Omega)$ and h be a maximal psh function in Ω such that $\lim_{\varsigma \rightarrow z} h(\varsigma) = \phi(z)$ for all $z \in \partial\Omega$. Denote by $\mathcal{F}(\phi, \Omega)$ the class of those $u \in PSH(\Omega)$ for which $h \geq u \geq h + v$ for some $v \in \mathcal{F}(\Omega)$. We use the subclass $\mathcal{F}^a(\phi, \Omega)$ of functions from $\mathcal{F}(\phi, \Omega)$ whose Monge-Ampère measures put no mass on all pluripolar subsets of Ω , see [1][10]. Clearly, $\mathcal{F}(\Omega) = \mathcal{F}(0, \Omega)$ and $\mathcal{F}^a(\Omega) = \mathcal{F}^a(0, \Omega)$. We need the following fact.

LEMMA 3.6. — *Suppose that $u_0 \in \mathcal{F}^a(\phi, \Omega)$ satisfies $\int_{\Omega} (dd^c u_0)^n < \infty$ and $\lim_{\varsigma \rightarrow z} u_0(\varsigma) = \phi(z)$ for all $z \in \partial\Omega$. If $u \in \mathcal{F}(\phi, \Omega)$ with $u \geq u_0$ then $u \in \mathcal{F}^a(\phi, \Omega)$ and $\int_{\Omega} (dd^c u)^n < \infty$. Moreover, the measures $(dd^c u)^n$ are absolutely continuous with respect to C_n in Ω uniformly for all $u \in \mathcal{F}(\phi, \Omega)$ with $u \geq u_0$.*

Proof. — It is no loss of generality to assume that $\phi < 0$ on $\partial\Omega$. By the definition of $u \in \mathcal{F}(\phi, \Omega)$ there exists $\bar{u} \in \mathcal{F}(\Omega)$ such that $h \geq u \geq \bar{u} + h$ in Ω . Take a sequence $\{\bar{u}_j\}$ in $\mathcal{E}_0(\Omega)$ such that $\bar{u}_j \searrow \bar{u}$ and define $u_j = \max(u, \bar{u}_j + h)$. Then $u_j \searrow u$ in Ω and each $u_j \geq u_0$ with equality on $\partial\Omega$. we claim that $\int_{\Omega} (dd^c u_j)^n \leq \int_{\Omega} (dd^c u_0)^n < \infty$ for each j . If the claim is true, then for any $k \in \mathcal{E}_0(\Omega)$ we have

$$\begin{aligned} \int_{\Omega} k (dd^c u_0)^n - \int_{\Omega} k (dd^c u_j)^n &= \int_{\Omega} k ((dd^c u_0)^n - (dd^c u_j)^n) \\ &= \int_{\Omega} (u_0 - u_j) dd^c k \wedge \sum_{l=0}^{n-1} (dd^c u_0)^l \wedge (dd^c u_j)^{n-l} \leq 0. \end{aligned}$$

Using Theorem 2.1 in [9] we get that $\int_{\Omega} -k (dd^c u_j)^n \leq \int_{\Omega} -k (dd^c u_0)^n$ for any negative $k \in PSH \cap L^\infty(\Omega)$ and all j . Let $j \rightarrow \infty$ and, since $(dd^c u_j)^n \rightarrow (dd^c u)^n$ weakly in Ω and k is upper semicontinuous, we get that $\int_{\Omega} -k (dd^c u)^n \leq \int_{\Omega} -k (dd^c u_0)^n$ for any negative $k \in PSH \cap L^\infty(\Omega)$ and hence $\int_{\Omega} (dd^c u)^n \leq \int_{\Omega} (dd^c u_0)^n < \infty$. Particularly, for $k = \max(u/t, -1)$ with $t \geq 1$ we obtain

$$\begin{aligned} \int_{u < -t} (dd^c u)^n &\leq \int_{\Omega} -\max(u/t, -1) (dd^c u)^n \\ &\leq \int_{\Omega} -\max(u_0/t, -1) (dd^c u_0)^n \rightarrow 0 \end{aligned}$$

as $t \rightarrow \infty$, since $(dd^c u_0)^n$ put no mass on $\{u_0 = -\infty\}$. This implies that $u \in \mathcal{F}^a(\phi, \Omega)$ and the measures $(dd^c u)^n$ are absolutely continuous with respect to C_n in Ω uniformly for all $u \in \mathcal{F}(\phi, \Omega)$ with $u \geq u_0$.

It remains to prove the claim that $\int_{\Omega} (dd^c u_j)^n \leq \int_{\Omega} (dd^c u_0)^n$ for each j . Given $b > 0$, since $\lim_{\varsigma \rightarrow z} (u_0(\varsigma) - u_j(\varsigma)) = 0$ for $z \in \partial\Omega$ there exists a closed subset $F_b \subset \Omega$ such that $A_{a,b} := \{\max(u_0, -a) + 1/b \leq u_j\} \subset F_b$ for all $a > \sup_{\Omega} |u_j|$. So by the comparison theorem we have

$$\begin{aligned} \int_{A_{a_1,b}} (dd^c u_j)^n &\leq \int_{A_{a_2,b}} (dd^c u_j)^n \leq \int_{A_{a_2,b}} (dd^c \max(u_0, -a_2))^n \\ &\leq \int_{F_b} (dd^c \max(u_0, -a_2))^n \end{aligned}$$

for all $a_2 > a_1 > \sup_{\Omega} |u_j|$. Letting $a_2 \rightarrow \infty$ we get that $\int_{A_{a_1,b}} (dd^c u_j)^n \leq \int_{F_b} (dd^c u_0)^n \leq \int_{\Omega} (dd^c u_0)^n$. Then, letting $a_1 \rightarrow \infty$ and $b \rightarrow \infty$ we have

that $\int_{u_0 < u_j} (dd^c u_j)^n \leq \int_{\Omega} (dd^c u_0)^n$. Take $g_0 \in \mathcal{E}_0(\Omega)$ with $g_0 \neq 0$. Using $u_j + \varepsilon g_0$ instead of u_j in the last proof and by $u_j \in L^\infty(\Omega)$ we obtain that $\int_{\Omega} (dd^c u_j)^n \leq \int_{\Omega} (dd^c u_0)^n$, which concludes the proof of Lemma 3.6. \square

THEOREM 3.7. — *Suppose that $u_0 \in \mathcal{F}^a(\phi, \Omega)$ satisfies $\int_{\Omega} (dd^c u_0)^n < \infty$ and $\lim_{\zeta \rightarrow z} u_0(\zeta) = \phi(z)$ for all $z \in \partial\Omega$, and suppose that $u, u_j \in \mathcal{F}(\phi, \Omega)$ such that $u_j \geq u_0$ in Ω and $u_j \rightarrow u$ weakly in Ω . Then the following assertions hold.*

- a) *If $g(dd^c u_j)^n$ converges weakly to $g(dd^c u)^n$ in Ω uniformly for all $g \in PSH(\Omega)$ with $0 \leq g \leq 1$, then $u_j \rightarrow u$ in C_n on Ω .*
- b) *If $(dd^c u_j)^n \rightarrow (dd^c u)^n$ weakly in Ω , then $u_j \rightarrow u$ in C_{n-1} on Ω .*

Proof. — By Lemma 3.6 we have that $u, u_j \in \mathcal{F}^a(\phi, \Omega)$ and $\int_{\Omega} (dd^c u)^n + \int_{\Omega} (dd^c u_j)^n < \infty$. To prove a), write $f_s = 1 + \max(u_0/s, -1)$ for $s = 1, 2, \dots$. Then $f_s \in PSH(\Omega)$ and $0 \leq f_s \leq \chi_{\{u_0 > -s\}} \leq 1$. Thus, using the equality $2gf_s = (g + f_s)^2 - g^2 - f_s^2$ we get that $gf_s(dd^c u_j)^n \rightarrow gf_s(dd^c u)^n$ as $j \rightarrow \infty$ uniformly for all $g \in PSH(\Omega)$ with $0 \leq g \leq 1$. Since $f_s(dd^c u_j)^n \leq \chi_{\{u_0 > -s\}}(dd^c u_j)^n = \chi_{\{u_0 > -s\}}(dd^c \max(u_j, -s))^n \leq (dd^c \max(u_j, -s))^n$, by Theorem 8.1 in [7] there exist $v_j^s \in PSH \cap L^\infty(\Omega)$ such that $(dd^c v_j^s)^n = f_s(dd^c u_j)^n$ and $\lim_{\zeta \rightarrow z} v_j^s(\zeta) = \phi(z)$ for $z \in \partial\Omega$ and $s > \max_{\Omega} |h|$. From the comparison theorem it follows that $\max(u_0, -s) \leq \max(u_j, -s) \leq v_j^s \leq h = \max(h, -s)$ for $s > \max_{\Omega} |h|$. Hence by Corollary 2.10 there exist $v^s \in PSH \cap L^\infty(\Omega)$ such that $(dd^c v^s)^n = f_s(dd^c u)^n$ and $v_j^s \rightarrow v^s$ in C_n on Ω as $j \rightarrow \infty$. Since $(dd^c v^s)^n \nearrow (dd^c u)^n$ as $s \nearrow \infty$ then $v^s \searrow v$ for some $v \in \mathcal{F}^a(\phi, \Omega)$ with $u_0 \leq v \leq h$ in Ω . It then follows that $(dd^c v^s)^n \rightarrow (dd^c v)^n$ and hence $(dd^c v)^n = (dd^c u)^n$, which by Lemma 2 in [18] implies that $v = u$ in Ω . On the other hand, by Lemma 5.14 in [9] there exist $h_j^s \in \mathcal{F}^a(\Omega)$ such that $(dd^c h_j^s)^n = -\max(u_0/s, -1)(dd^c u_j)^n$. Therefore, $(dd^c v_j^s)^n \leq (dd^c u_j)^n = (dd^c v_j^s)^n + (dd^c h_j^s)^n \leq (dd^c(v_j^s + h_j^s))^n$ which by the comparison theorem gives that $v_j^s + h_j^s \leq u_j \leq v_j^s$. Choose a sequence $\{\phi_k\}$ of functions in $\mathcal{E}_0(\Omega)$ such that $\phi_k \searrow h_j^s$ in Ω and $\int_{\Omega} (dd^c \phi_k)^n \rightarrow \int_{\Omega} (dd^c h_j^s)^n$. Given $\varepsilon > 0$ and $w \in PSH(\Omega)$ with $0 \leq w \leq 1$, by the comparison theorem we have

$$\begin{aligned} \varepsilon^n \int_{\phi_k < -\varepsilon} (dd^c w)^n &\leq \int_{\phi_k < \varepsilon(w-1)} (dd^c \varepsilon(w-1))^n \leq \int_{\phi_k < \varepsilon(w-1)} (dd^c \phi_k)^n \\ &\leq \int_{\Omega} (dd^c \phi_k)^n. \end{aligned}$$

Letting $k \rightarrow \infty$ and taking supremum over all such w , we get

$$\begin{aligned} \varepsilon^n C_n(\{h_j^s < -\varepsilon\}) &\leq \int_{\Omega} -\max(u_0/s, -1) (dd^c u_j)^n \\ &\leq \int_{\Omega} -\max(u_0/s, -1) (dd^c u_0)^n, \end{aligned}$$

where the last inequality follows from the proof of Lemma 3.6. This implies that on each $E \subset\subset \Omega$ the functions h_j^s uniformly tend to zero in C_n as $s \rightarrow \infty$. Finally, we have that $|u_j - u| \leq |u_j - v_j^s| + |v_j^s - v^s| + |v^s - u| \leq |h_j^s| + |v_j^s - v^s| + |v^s - u|$ where on each $E \subset\subset \Omega$ the third and first terms in the last sum tend to zero in C_n as $s \rightarrow \infty$ uniformly for all j , and for each fixed s the second one tends to zero in C_n as $j \rightarrow \infty$. Thus, we have obtained that $u_j \rightarrow u$ in C_n on each $E \subset\subset \Omega$ and hence we have proved a). We omit the proof of b) since it is similar to the proof of a). The proof of Theorem 3.7 is complete. \square

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