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Francisco MARTÍN CABRERA & Andrew SWANN

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THE INTRINSIC TORSION OF ALMOST QUATERNION-HERMITIAN MANIFOLDS

by Francisco MARTÍN CABRERA & Andrew SWANN

ABSTRACT. — We study the intrinsic torsion of almost quaternion-Hermitian manifolds via the exterior algebra. In particular, we show how it is determined by particular three-forms formed from simple combinations of the exterior derivatives of the local Kähler forms. This gives a practical method to compute the intrinsic torsion and is applied in a number of examples. In addition we find simple characterisations of HKT and QKT geometries entirely in the exterior algebra and compute how the intrinsic torsion changes under a twist construction.

RÉSUMÉ. — Nous étudions la torsion intrinsèque des variétés presque hermitiennes quaternioniennes via l'algèbre extérieur. En particulier, nous montrons comment elle est déterminée par trois-formes particulières, formées à partir de simples combinaisons des différentielles extérieures des formes kählériennes locales. Ceci donne une méthode pratique pour calculer la torsion intrinsèque qui s'applique dans de nombreux exemples. En plus, nous trouvons des caractérisations simples des géométries HKT et QKT en utilisant l'algèbre extérieur et nous calculons la modification de la torsion intrinsèque pour une construction twistée.

1. Introduction

An almost quaternion-Hermitian manifold M is a Riemannian $4n$ -manifold which admits an $Sp(n)Sp(1)$ -structure, i.e., a reduction of its frame bundle to the subgroup $Sp(n)Sp(1)$ of $SO(4n)$. This is equivalent to the presence of a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and a rank-three subbundle \mathcal{G} of the endomorphism bundle $\text{End}TM$, locally generated by three almost complex structures I, J, K satisfying the identities of the imaginary unit quaternions. Almost quaternion-Hermitian manifolds are of special interest because $Sp(n)Sp(1)$ is included in Berger's list [2] of possible holonomy groups of locally irreducible Riemannian manifolds that are

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not locally symmetric. Also in the field of theoretical physics, the study of supersymmetric sigma models and their couplings to supergravity is very related with the study of complex and quaternionic structures defined on Riemannian manifolds [10, 16].

Since $Sp(n)Sp(1)$ is a closed and connected subgroup of $SO(4n)$, there exists a unique metric $Sp(n)Sp(1)$ -connection $\nabla^{\text{aqH}} = \nabla^{\text{LC}} + \xi$, where ∇^{LC} is the Levi-Civita connection and ξ is a tensor, called the *intrinsic $Sp(n)Sp(1)$ -torsion*, in $T^*M \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp$. Here $(\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp$ denotes the orthogonal complement in $\mathfrak{so}(4n)$ of the Lie algebra $\mathfrak{sp}(n) + \mathfrak{sp}(1)$.

Under the action of $Sp(n)Sp(1)$, the space $T^*M \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp$ of possible intrinsic torsion tensors ξ decomposes into irreducible $Sp(n)Sp(1)$ -modules, giving rise to a natural classification of almost quaternion-Hermitian manifolds. In [27] it was shown that, in general dimensions, ξ has six components and $2^6 = 64$ classes of such manifolds potentially arise. An almost quaternion-Hermitian manifold is said to be quaternion-Kähler, if the intrinsic torsion ξ vanishes. In this case, the reduced holonomy group is a subgroup of $Sp(n)Sp(1)$ and the manifold is Einstein. On the other hand, if the three almost complex structures are globally defined, then M is said to be endowed with an almost hyperHermitian structure (an $Sp(n)$ -structure). When the three Kähler forms $\omega_I, \omega_J, \omega_K$ of the $Sp(n)$ -structure are covariant constant, the manifold is called hyperKähler. HyperKähler manifolds have reduced holonomy group contained in $Sp(n)$ and their Ricci curvature vanishes.

By identifying the intrinsic $Sp(n)Sp(1)$ -torsion ξ with the Levi-Civita covariant derivative of a certain four-form Ω , defined below in equation (2.1), one obtains an analogue of the method of Gray & Hervella [13] for finding conditions for classes of almost quaternion-Hermitian manifolds. Detailed conditions describing classes in this way were given in [20].

In the present paper, we will take another approach. In fact, we will show how the intrinsic torsion ξ can be determined by means of the exterior derivatives $d\omega_I, d\omega_J$ and $d\omega_K$ of the local Kähler forms corresponding to the almost complex structures I, J, K . In the process, there will arise additional, detailed information about the components of ξ which will be very useful in working on examples of almost quaternion-Hermitian manifolds. For all of this, we give expressions for the covariant derivatives $\nabla^{\text{LC}}\omega_A$ in terms of $d\omega_I, d\omega_J$ and $d\omega_K$, see Proposition 4.3. Such expressions contribute to a better understanding of Hitchin's result [14] saying that if ω_I, ω_J and ω_K are closed, then they are covariant constant. Indeed in Proposition 4.3, we show how the Nijenhuis tensor N_I in general is determined

by the difference $Jd\omega_J - Kd\omega_K$. Let us briefly explain one application of this result, cf. §6.

It is known that the geometry of the target space of $(4, 0)$ supersymmetric even-dimensional sigma models without Wess-Zumino term (torsion) is a hyperKähler manifold. In presence of torsion, the geometry of the target space is a hyperKähler manifold with torsion, usually called an HKT-manifold [17].

Grancharov & Poon [11] showed that an almost hyperHermitian manifold $(M, I, J, K, \langle \cdot, \cdot \rangle)$ is HKT if and only if:

- (i) the three almost complex structures I, J and K are integrable, and
- (ii) $Id\omega_I = Jd\omega_J = Kd\omega_K$.

A direct consequence of our expression for N_A is that condition (ii) is sufficient to characterise HKT geometry, and in particular (ii) implies the integrability condition (i). Similarly, we also show how Grancharov & Poon's holomorphic characterisation for HKT-manifolds may be simplified, see §6.

It is known that an almost quaternion-Hermitian manifold the three covariant derivatives $\nabla^{\text{LC}}\omega_I, \nabla^{\text{LC}}\omega_J$ and $\nabla^{\text{LC}}\omega_K$ are not independent, but rather any two determine the third [8, 20], see equation (4.5). The corresponding statement for the exterior derivatives $d\omega_I, d\omega_J, d\omega_K$ is not true. However, we find that there are still relations expressed by symmetries of the three-forms

$$\beta_I = Jd\omega_J + Kd\omega_K, \quad \text{etc.}$$

see (4.12). These symmetries are equivalent to requiring β_A to be of type $\{2, 1\} = (2, 1) + (1, 2)$ with respect to the almost complex structure A . Algebraically the β_I, β_J and β_K are independent three-forms of these types and we find that the space of possible triples of covariant derivatives $(\nabla\omega_I, \nabla\omega_J, \nabla\omega_K)$ is isomorphic to the space of possible triples of three-forms $(\beta_I, \beta_J, \beta_K)$.

The relevance of the three-forms $\beta_I, \beta_J, \beta_K$ is clearly seen in Proposition 5.3, where we demonstrate how they may be used to compute the components of the intrinsic $Sp(n)Sp(1)$ -torsion ξ . This gives a practical way to compute ξ via the exterior algebra and will be used in the study of concrete examples in §10.

In §7, we focus attention on quaternion-Kähler manifolds with torsion, also known as QKT-manifolds. Motivation for studying these structures can be also found in the field of theory of supersymmetric sigma models, see [15]. Our results lead to a new characterisation (7.5) of QKT-manifolds that is simpler than that provided by Ivanov [18, Theorem 2.2]. We also

obtain new expressions for the torsion three-form and torsion one-form and study of the integrability properties of the almost complex structures.

In §9, we consider the intrinsic torsion of quaternion-Hermitian manifolds obtained by the twist interpretation of T-duality given in [28]. Using the exterior algebra is particularly advantageous here. We see that in many cases the QKT condition is preserved.

Finally, in §10, we give an number of examples of types of almost quaternion-Hermitian manifolds. We particularly mention one of the quaternionic structures considered on the manifold $S^3 \times T^9$ which is a non-QKT-manifold admitting an $Sp(n)Sp(1)$ -connection with skew-symmetric torsion $(0, 3)$ -tensor. In the various examples, we have also determined the types of almost Hermitian structure. Because it is an advantage to handle Lie brackets instead of directly using ∇^{LC} , we determine such types by means of the exterior derivative $d\omega_I$ and the Nijenhuis tensor N_I . Therefore, in §8, we include Table 8.1 showing conditions in terms of $d\omega_I$ and N_I to characterise the Gray-Hervella classes.

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2. Definitions and notation

Let G be a subgroup of the linear group $GL(m, \mathbb{R})$. A manifold M is said to be equipped with a G -structure, if there is a principal G -subbundle P of the principal frame bundle. In such a case, there always exist connections, called G -connections, defined on the subbundle P . Moreover, if $(M^m, \langle \cdot, \cdot \rangle)$ is an orientable m -dimensional Riemannian manifold and G a closed and connected subgroup of $SO(m)$, then there exists a unique metric G -connection ∇^G such that $\xi = \nabla^G - \nabla^{\text{LC}}$ takes its values in \mathfrak{g}^\perp , where \mathfrak{g}^\perp denotes the orthogonal complement in $\mathfrak{so}(m)$ of the Lie algebra \mathfrak{g} of G [25, 4]. The tensor ξ is said to be the *intrinsic G -torsion* and ∇^G is called the *minimal G -connection*.

A $4n$ -dimensional manifold M is said to be *almost quaternion-Hermitian*, if M is equipped with an $Sp(n)Sp(1)$ -structure. This is equivalent to the presence of a Riemannian metric $\langle \cdot, \cdot \rangle$ and a rank-three subbundle \mathcal{G} of the endomorphism bundle $\text{End}TM$, such that locally \mathcal{G} has an *adapted basis* I, J, K satisfying $I^2 = J^2 = -1$ and $K = IJ = -JI$, and $\langle AX, AY \rangle =$

$\langle X, Y \rangle$, for all $X, Y \in T_x M$ and $A = I, J, K$. An almost quaternion-Hermitian manifold with a global adapted basis is called an *almost hyper-Hermitian* manifold. In such a case the structure group reduces to $Sp(n)$. We note that if I, J, K is an adapted basis then so are J, K, I and K, I, J ; thus formulæ derived for an arbitrary adapted I, J, K will also apply to cyclic permutations of these almost complex structures.

There are three local Kähler-forms $\omega_A(X, Y) = \langle X, AY \rangle$, $A = I, J, K$. From these one may define a global, non-degenerate four-form Ω , the *fundamental form*, via the local formula

$$(2.1) \quad \Omega = \sum_{A=I,J,K} \omega_A \wedge \omega_A.$$

We will write

$$\Lambda_I : \Lambda^p T^* M \rightarrow \Lambda^{p-2} T^* M$$

for the adjoint of $\cdot \mapsto \cdot \wedge \omega_I$ with respect to the metrics

$$\langle a, b \rangle = \frac{1}{p!} a(e_{i_1}, \dots, e_{i_p}) b(e_{i_1}, \dots, e_{i_p}).$$

In particular, for a three-form β we have $\Lambda_I \beta = \langle \lrcorner \beta, \omega_I \rangle$, and for a one-form ν , we have

$$\Lambda_I(\nu \wedge \omega_A) = -\frac{1}{2} \sum_{i=1}^{4n} (\nu(e_i) \omega_A(Ie_i, \cdot) + \omega_A(e_i, Ie_i) \nu + \nu(Ie_i) \omega_A(\cdot, e_i)).$$

In the next section, we will explicitly describe the intrinsic torsion of almost quaternion-Hermitian manifolds. For such a purpose, we need some basic tools related with almost quaternion-Hermitian manifolds in a context of representation theory. We will follow the *E-H*-formalism used in [23, 27] and we refer to [3] for general information on representation theory. Thus, E is the fundamental representation of $Sp(n)$ on $\mathbb{C}^{2n} \cong \mathbb{H}^n$ via left multiplication by quaternionic matrices, considered in $GL(2n, \mathbb{C})$, and H is the representation of $Sp(1)$ on $\mathbb{C}^2 \cong \mathbb{H}$ given by $q \cdot \zeta = \zeta \bar{q}$, for $q \in Sp(1)$ and $\zeta \in \mathbb{H}$. An $Sp(n) Sp(1)$ -structure on a manifold M gives rise to local bundles E and H associated to these representation and identifies $TM \otimes_{\mathbb{R}} \mathbb{C} \cong E \otimes_{\mathbb{C}} H$.

On E , there is an $Sp(n)$ -invariant complex symplectic form ω_E and a Hermitian inner product given by $\langle x, y \rangle_{\mathbb{C}} = \omega_E(x, \tilde{y})$, where $y \mapsto \tilde{y} = jy$ is a quaternionic structure map on $E = \mathbb{C}^{2n}$ considered as left complex vector space. The mapping $x \mapsto x^\omega = \omega_E(\cdot, x)$ gives us an identification of E with its dual E^* . If $\{u_1, \dots, u_n, \tilde{u}_1, \dots, \tilde{u}_n\}$ is a complex orthonormal basis for E , then $\omega_E = u_i^\omega \wedge \tilde{u}_i^\omega = u_i^\omega \tilde{u}_i^\omega - \tilde{u}_i^\omega u_i^\omega$, where we have used the summation

convention and omitted tensor product signs. These conventions will be used throughout the paper.

The $Sp(1)$ -module H will be also considered as a left complex vector space. Regarding H as a 4-dimensional real space with the Euclidean metric $\langle \cdot, \cdot \rangle$ such that $\{1, i, j, k\}$ is an orthonormal basis. The complex symplectic form is given by $\omega_H = 1^b \wedge j^b + k^b \wedge i^b + i(1^b \wedge k^b + i^b \wedge j^b)$, where h^b is given by $q \mapsto \langle h, q \rangle$. We also have the identification, $h \mapsto h^\omega = \omega_H(\cdot, h)$, of H with its dual H^* as complex space. On H , we have a quaternionic structure map given by $q = z_1 + z_2j \mapsto \tilde{q} = jq = -\bar{z}_2 + \bar{z}_1j$, where $z_1, z_2 \in \mathbb{C}$ and \bar{z}_1, \bar{z}_2 are their conjugates. If $h \in H$ is such that $\langle h, h \rangle = 1$, then $\{h, \tilde{h}\}$ is a basis of the complex vector space H and $\omega_H = h^\omega \wedge \tilde{h}^\omega$.

The irreducible representations of $Sp(1)$ are the symmetric powers $S^k H \cong \mathbb{C}^{k+1}$. An irreducible representation of $Sp(n)$ is determined by its dominant weight $(\lambda_1, \dots, \lambda_n)$, where λ_i are integers with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. This representation will be denoted by $V^{(\lambda_1, \dots, \lambda_r)}$, where r is the largest integer such that $\lambda_r > 0$. We will only need to use some of these representations and use more familiar notation for these: $S^k E = V^{(k)}$, the k th symmetric power of E ; $\Lambda_0^r E = V^{(1, \dots, 1)}$, where there are r ones in exponent and $\Lambda_0^r E$ is the $Sp(n)$ -invariant complement to $\omega_E \Lambda^{r-2} E$ in $\Lambda^r E$; also $K = V^{(21)}$, which arises in the decomposition $E \otimes \Lambda_0^2 E \cong \Lambda_0^3 E + K + E$, where $+$ denotes direct sum.

Most of the time in this paper, if V is a complex G -module equipped with a real structure, V will also denote the real G -module which is $(+1)$ -eigenspace of the structure map. The context should tell us which space we are referring to. However, when a risk of confusion arise, we will denote the second mentioned space by $[V]$. Likewise, the following conventions will be used in this paper. If ψ is a $(0, s)$ -tensor, for $A = I, J, K$, we write

$$\begin{aligned}
 A_{(i)}\psi(X_1, \dots, X_i, \dots, X_s) &= -\psi(X_1, \dots, AX_i, \dots, X_s), \\
 A_{(ij\dots k)} &= A_{(i)}A_{(j)} \dots A_{(k)}, \quad \text{and} \\
 A\psi(X_1, \dots, X_s) &= (-1)^s \psi(AX_1, \dots, AX_s).
 \end{aligned}$$

3. The intrinsic torsion via differential forms

The intrinsic $Sp(n) Sp(1)$ -torsion ξ , $n > 1$, is in $T^*M \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp \cong EH \otimes \Lambda_0^2 ES^2H \subset T^*M \otimes \Lambda^2 T^*M$. The space $EH \otimes \Lambda_0^2 ES^2H$ consists of tensors ζ such that

- (i) $(1 + I_{(23)} + J_{(23)} + K_{(23)})\zeta = 0$;
- (ii) $\Lambda_A(\zeta_X) = 0$, for $A = I, J, K, X \in TM$,

where I, J, K is an adapted basis of \mathcal{G} . We recall that $\Lambda^2 T^*M = S^2E + S^2H + \Lambda_0^2 ES^2H$, where $S^2E \cong \mathfrak{sp}(n)$ and $S^2H \cong \mathfrak{sp}(1)$ are the Lie algebras of $Sp(n)$ and $Sp(1)$, respectively.

A connection $\tilde{\nabla}$ is an $Sp(n) Sp(1)$ -connection if $\tilde{\nabla}\Omega = 0$. This is the same as saying that $\tilde{\nabla}$ is metric, $\tilde{\nabla}g = 0$, and *quaternionic*, meaning that for any local adapted basis I, J, K of \mathcal{G} we have

$$(3.1) \quad (\tilde{\nabla}_X I)Y = \gamma_K(X)JY - \gamma_J(X)KY, \quad \text{etc.},$$

where γ_I, γ_J and γ_K are locally defined one-forms. Here and throughout the rest of this paper, ‘etc.’ means the equations obtained by cyclically permuting I, J, K .

PROPOSITION 3.1. — *The minimal $Sp(n) Sp(1)$ -connection is given by*

$$\nabla^{\text{aqH}} = \nabla^{\text{LC}} + \xi,$$

where ∇^{LC} is the Levi-Civita connection and the intrinsic $Sp(n) Sp(1)$ -torsion ξ is given by

$$\xi_X Y = -\frac{1}{4} \sum_{A=I,J,K} A(\nabla_X^{\text{LC}} A)Y + \frac{1}{2} \sum_{A=I,J,K} \lambda_A(X)AY,$$

for all vectors X, Y . Here the one-forms λ_I, λ_J and λ_K are defined by

$$(3.2) \quad \lambda_I(X) = \frac{1}{2n} \langle \nabla_X^{\text{LC}} \omega_J, \omega_K \rangle, \quad \text{etc.}$$

Note that if $n = 1$, then $\nabla^{\text{aqH}} = \nabla^{\text{LC}}$ and $\xi = 0$.

Proof. — It is not hard to check $\nabla^{\text{aqH}}g = 0$, so ∇^{aqH} is metric. Now, computing $(\xi_X I)Y = \xi_X IY - I\xi_X Y$, it is straightforward to obtain

$$(3.3) \quad (\nabla_X^{\text{LC}} I)Y = \lambda_K(X)JY - \lambda_J(X)KY - \xi_X IY + I\xi_X Y, \quad \text{etc.}$$

Hence $(\nabla_X^{\text{aqH}} I)Y = (\nabla_X^{\text{LC}} I)Y + (\xi_X I)Y = \lambda_K(X)JY - \lambda_J(X)KY$. Therefore, ∇^{aqH} is an $Sp(n) Sp(1)$ -connection.

Furthermore, the tensor ξ satisfies

$$\sum_{A=I,J,K} A\xi_X AY = \xi_X Y \quad \text{and} \quad \sum_{i=1}^{4n} \langle \xi_X e_i, Ae_i \rangle = 0,$$

for $A = I, J, K$. Since these conditions imply $\xi \in T^*M \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp = T^*M \otimes \Lambda_0^2 ES^2H$, then $\nabla^{\text{aqH}} = \nabla^{\text{LC}} + \xi$ is the minimal $Sp(n) Sp(1)$ -connection. □

The next result describes the decomposition of the space of possible intrinsic torsion tensors $T^*M \otimes \Lambda_0^2 ES^2H$ into irreducible $Sp(n) Sp(1)$ -modules.

THEOREM 3.2 (Swann [27]). — *The intrinsic torsion ξ of an almost quaternion-Hermitian manifold M of dimension at least 8, has the property*

$$(3.4) \quad \begin{aligned} \xi \in T^*M \otimes \Lambda_0^2 ES^2H &= \Lambda_0^3 ES^3H + KS^3H + ES^3H \\ &+ \Lambda_0^3 EH + KH + EH. \end{aligned}$$

If the dimension of M is at least 12, all the modules of the sum are non-zero. For an eight-dimensional manifold M , we have $\Lambda_0^3 E = \{0\}$. Therefore, for $\dim M \geq 12$, we have $2^6 = 64$ classes of almost quaternion-Hermitian manifolds, whereas there are $2^4 = 16$ classes when $\dim M = 8$. The map $\xi \mapsto \nabla^{LC}\Omega = -\xi\Omega$ is an isomorphism, and in [20] this was exploited to give explicit conditions characterising these classes in terms of conditions on $\nabla^{LC}\Omega$. However, from such conditions, it is not hard to derive descriptions for the corresponding $Sp(n)Sp(1)$ -components of ξ as we will now demonstrate.

Firstly, the space of three-forms $\Lambda^3 T^*M$ decomposes under the action of $Sp(n)Sp(1)$ as

$$\Lambda^3 T^*M = \Lambda_0^3 ES^3H + ES^3H + KH + EH.$$

Consider the operator

$$(3.5) \quad \mathcal{L} = \mathcal{L}_I + \mathcal{L}_J + \mathcal{L}_K$$

on $\Lambda^3 T^*M$, where

$$\mathcal{L}_A = A_{(12)} + A_{(13)} + A_{(23)}.$$

The operator \mathcal{L} has eigenvalues $+3$ and -3 with corresponding eigenspaces $(K + E)H$ and $(\Lambda_0^3 E + E)S^3H$. For $\psi \in \Lambda^3 T^*M$, we have $\psi = \psi_H + \psi_{S^3H}$ with

$$(3.6) \quad \psi_H = \frac{1}{6}(3\psi + \mathcal{L}\psi),$$

$$(3.7) \quad \psi_{S^3H} = \frac{1}{6}(3\psi - \mathcal{L}\psi).$$

The component ψ_H is characterised by $\mathcal{L}_A\psi_H = \psi_H$, for $A = I, J, K$. On the other hand ψ_{S^3H} satisfies $\sum_{A=I,J,K} A_{(12)}\psi_{S^3H} = -\psi_{S^3H}$. Writing $\psi_H = \psi^{(KH)} + \psi^{(EH)} \in KH + EH$ and $\psi_{S^3H} = \psi^{(33)} + \psi^{(E3)} \in \Lambda_0^3 ES^3H + ES^3H$, one computes

$$\begin{aligned} \psi^{(EH)} &= -\frac{1}{2n+1} \sum_{A=I,J,K} A\theta^\psi \wedge \omega_A, \\ \psi^{(E3)} &= -\frac{1}{2(n-1)} \sum_{A=I,J,K} A(\theta_A^\psi - \theta^\psi) \wedge \omega_A, \end{aligned}$$

where $\theta_A^\psi(X) = A\Lambda_A\psi$ and $\theta^\psi = \frac{1}{3} \sum_{A=I,J,K} \theta_A^\psi$.

Let us now describe the $Sp(n) Sp(1)$ -components of the intrinsic torsion ξ , and include characterisations via three-forms. We will write ξ_{33} , ξ_{K3} , ξ_{E3} , ξ_{3H} , ξ_{KH} and ξ_{EH} for the components of ξ corresponding to the modules in the sum (3.4). We have the following descriptions:

(i) ξ_{33} is a tensor characterised by the conditions:

- (a) $\sum_{A=I,J,K} (\xi_{33})_A A = -\sum_{A=I,J,K} A (\xi_{33})_A = -\xi_{33}$,
- (b) $\langle \cdot, (\xi_{33}) \cdot \rangle$ is a skew-symmetric three-form.

Or equivalently, ξ_{33} is given by $\langle Y, (\xi_{33})_X Z \rangle = \psi^{(3)}(X, Y, Z)$, where $\psi^{(3)}$ lies in the module $\Lambda^3_0 ES^3H \subset \Lambda^3 T^*M$.

(ii) ξ_{K3} is a tensor characterised by the conditions:

- (a) $\sum_{A=I,J,K} (\xi_{K3})_A A = -\sum_{A=I,J,K} A (\xi_{K3})_A = -\xi_{K3}$,
- (b) $\mathfrak{G}_{XYZ} \langle Y, (\xi_{K3})_X Z \rangle = 0$.

Or equivalently, ξ_{K3} is expressed by

$$\langle Y, (\xi_{K3})_X Z \rangle = \sum_{A=I,J,K} A_{(23)} \psi_A^{(K)},$$

where $\psi_A^{(K)}$, $A = I, J, K$, are local three-forms in the module KH such that $\sum_{A=I,J,K} \psi_A^{(K)} = 0$.

(iii) ξ_{E3} is given by

$$\langle Y, (\xi_{E3})_X Z \rangle = \frac{1}{n} \sum_{A=I,J,K} (nA(\theta_A^\xi - \theta^\xi) \wedge \omega_A - (n-1)A(\theta_A^\xi - \theta^\xi) \otimes \omega_A)(X, Y, Z),$$

where θ^ξ is the one-form defined by

$$(3.8) \quad \frac{6}{n}(2n+1)(n-1)\theta^\xi(X) = -\langle \xi_{e_i} e_i, X \rangle = -\sum_{A=I,J,K} \langle A\xi_{e_i} A e_i, X \rangle,$$

and $\theta_I^\xi, \theta_J^\xi, \theta_K^\xi$ are the local one-forms given by

$$\frac{2}{n}(2n+1)(n-1)\theta_A^\xi(X) = -\langle A\xi_{e_i} A e_i, X \rangle.$$

Note that $3\theta^\xi = \theta_I^\xi + \theta_J^\xi + \theta_K^\xi$.

(iv) ξ_{3H} is a tensor characterised by the conditions:

- (a) $(\xi_{3H})_A A - A(\xi_{3H})_A - A\xi_{3H}A = \xi_{3H}$, for $A = I, J, K$,
- (b) $\mathfrak{G}_{X,Y,Z} \langle Y, (\xi_{3H})_X Z \rangle = 0$.

Or equivalently, ξ_{3H} is expressed by

$$\langle Y, (\xi_{3H})_X Z \rangle = \sum_{A=I,J,K} A_{(23)} \psi_A^{(3)},$$

where $\psi_A^{(3)}$, $A = I, J, K$ are local three-forms such that

- (p) $\psi_A^{(3)}$ is in $\Lambda_0^3 ES^3H$,
- (q) $\psi_A^{(3)}$ is of type $\{2, 1\}$ with respect to the almost complex structure A , i.e., $\mathcal{L}_A \psi_A^{(3)} = \psi_A^{(3)}$, $A = I, J, K$, and
- (r) $\sum_{A=I,J,K} \psi_A^{(3)} = 0$.

One may check that one of these three-forms is sufficient to determine the others. Indeed

$$\psi_J^{(3)} = -\frac{1}{2}(3 + \mathcal{L}_J)\psi_I^{(3)} \quad \text{and} \quad \psi_K^{(3)} = -\frac{1}{2}(3 + \mathcal{L}_K)\psi_I^{(3)}.$$

(v) ξ_{KH} is a tensor characterised by the conditions:

- (a) $(\xi_{KH})_A A - A(\xi_{KH})_A - A\xi_{KH}A = \xi_{KH}$, for $A = I, J, K$;
- (b) there exists a skew-symmetric three-form $\psi^{(K)}$ such that

$$\langle Y, (\xi_{KH})_X Z \rangle = (3\psi^{(K)} - \sum_{A=I,J,K} A_{(23)} \psi^{(K)})(X, Y, Z);$$

- (c) $\sum_{i=1}^{4n} (\xi_{KH})_{e_i} e_i = 0$.

Note that conditions (v.a) and (v.c) can be replaced by saying that $\psi^{(K)}$ is in KH , i.e., for each $A = I, J, K$, the form $\psi^{(K)}$ is of type $\{2, 1\}_A$ and satisfies $\Lambda_A \psi^{(K)} = 0$.

(vi) ξ_{EH} is given by

$$\begin{aligned} \langle Y, (\xi_{EH})_X Z \rangle &= 3e_i \otimes e_i \wedge \theta^\xi(X, Y, Z) \\ &\quad - \sum_{A=I,J,K} (e_i \otimes Ae_i \wedge A\theta^\xi + \frac{2}{n}A\theta^\xi \otimes \omega_A)(X, Y, Z), \end{aligned}$$

where θ^ξ is the global one-form defined by (3.8).

(vii) The part $\xi_{S^3H} = \xi_{33} + \xi_{K3} + \xi_{E3}$ of ξ in $(\Lambda_0^2 E + K + E)S^3H$ is characterised by the condition

$$\sum_{A=I,J,K} (\xi_{S^3H})_A A = - \sum_{A=I,J,K} A(\xi_{S^3H})_A = -\xi_{S^3H}.$$

(viii) The part $\xi_H = \xi_{3H} + \xi_{KH} + \xi_{EH}$ of ξ in $(\Lambda_0^2 E + K + E)H$ is characterised by the condition

$$(\xi_H)_A A - A(\xi_H)_A - A(\xi_H)A = \xi_H,$$

for $A = I, J, K$.

4. Use of exterior derivatives

Here we will find out how the intrinsic torsion ξ can be determined by means of the exterior derivatives of the Kähler forms $d\omega_I, d\omega_J$ and $d\omega_K$.

In [21] it was shown that the covariant derivatives $\nabla\omega_I, \nabla\omega_J$ and $\nabla\omega_K$ are given by

$$(4.1) \quad \nabla^{\text{LC}}\omega_I = \lambda_K \otimes \omega_J - \lambda_J \otimes \omega_K + J_{(2)}\alpha_K - K_{(2)}\alpha_J, \quad \text{etc.}$$

where λ_A are given by equation (3.2) and $\alpha_I, \alpha_J, \alpha_K \in T^*M \otimes \Lambda_0^2 E \subset T^*M \otimes S^2 T^*M$ are defined by

$$(4.2) \quad \begin{aligned} \alpha_I &:= -\lambda_I \otimes g + \frac{1}{2}(J_{(2)} - J_{(3)})\nabla^{\text{LC}}\omega_K \\ &= -\lambda_I \otimes g + \frac{1}{2}(K_{(3)} - K_{(2)})\nabla^{\text{LC}}\omega_J, \quad \text{etc.} \end{aligned}$$

We may rewrite the intrinsic torsion ξ from Proposition 3.1 using equation (4.1) giving

$$(4.3) \quad \xi_X Y = -\frac{1}{2}((\alpha_I)_X IY) + (\alpha_J)_X JY + (\alpha_K)_X KY,$$

where (α_A) is given by $\langle Y, (\alpha_A)_X Z \rangle = \alpha_A(X; Y, Z)$. Thus the intrinsic torsion ξ only depends on the α_A 's; the λ_A 's have no influence.

Note that the dimension of the space of possible triples $(\alpha_I, \alpha_J, \alpha_K)$ coincides with the dimension $\dim T^*M \otimes \Lambda_0^2 ES^2H = 12n(2n+1)(n-1) = 3 \dim T^*M \otimes \Lambda_0^2 E$ of the space of possible intrinsic torsion tensors.

As $EH \otimes \Lambda_0^2 E = \Lambda_0^3 EH + KH + EH$, one may decompose α_I into three $Sp(n) Sp(1)$ -components

$$\alpha_I = \alpha_I^{(3)} + \alpha_I^{(K)} + \alpha_I^{(E)} \in \Lambda_0^3 EH + KH + EH.$$

If $\dim M = 8$, the module $\Lambda_0^3 E$ is trivial and the corresponding component $\alpha_I^{(3)}$ is not present. The component $\alpha_I^{(E)}$ is determined from a one-form η_I which is defined by

$$(4.4) \quad \eta_I(X) = \alpha_I(e_i, e_i, X).$$

Furthermore, the components of the α_A 's can be used to characterise classes of almost quaternion-Hermitian manifolds.

PROPOSITION 4.1 (Cabrera & Swann [21]). — *Let M be an almost quaternion-Hermitian manifold with intrinsic torsion ξ . If I, J, K is an adapted basis of \mathfrak{G} , then for $V = 3, K, E$,*

- (i) *each component ξ_{VH} is linearly determined by $I_{(1)}\alpha_I^{(V)} + J_{(1)}\alpha_J^{(V)} + K_{(1)}\alpha_K^{(V)}$,*
- (ii) *each component ξ_{V3} is linearly determined by $I_{(1)}\alpha_I^{(V)} - J_{(1)}\alpha_J^{(V)}$ and $J_{(1)}\alpha_J^{(V)} - K_{(1)}\alpha_K^{(V)}$.*

Observing that $A_{(1)}\alpha_A^{(E)} = A\alpha_A^{(E)}$ is linearly determined by the one-form $A\eta_A$, we have the following result.

COROLLARY 4.2. — *Under the same conditions as Proposition 4.1, we have:*

- (i) ξ_{EH} is linearly determined by $I\eta_I + J\eta_J + K\eta_K$,
- (ii) ξ_{E3} is linearly determined by $I\eta_I - J\eta_J$ and $J\eta_J - K\eta_K$.

We now proceed to express the covariant derivatives $\nabla_X^{LC}\omega_I$, $\nabla_X^{LC}\omega_J$ and $\nabla_X^{LC}\omega_K$ in terms of $d\omega_I$, $d\omega_J$ and $d\omega_K$. We use a relation between these covariant derivatives found in [8, 20], which may be symmetrically expressed by

$$(4.5) \quad (\nabla_X^{LC}\omega_I)(JY, KZ) + (\nabla_X^{LC}\omega_J)(KY, IZ) + (\nabla_X^{LC}\omega_K)(IY, JZ) = 0$$

and the following identity given by Gray [12]

$$(4.6) \quad 2\nabla^{LC}\omega_I = d\omega_I - I_{(23)}d\omega_I - I_{(3)}N_I,$$

where $N_I(X, Y, Z) = \langle X, N_I(Y, Z) \rangle$ and the $(1, 2)$ -tensor N_I is the Nijenhuis tensor for I , i.e., $N_I(X, Y) = [X, Y] + I[IX, Y] + I[X, IY] - [IX, IY]$.

Under the action of $U(2n)_I$ the space of three-forms decomposes in to irreducible modules as

$$\Lambda^3 T^*M = \Lambda_I^{\{3,0\}} T^*M + \Lambda_{0,I}^{\{2,1\}} T^*M + \Lambda_I^{\{1,0\}} T^*M \wedge \omega_I = \mathcal{W}_{1+3+4,I},$$

where \mathcal{W}_{iI} are isomorphic to the Gray-Hervella modules described in [13] and the subscript I indicates the almost complex structure considered. Note that a three-form ψ lies in $\mathcal{W}_{3+4,I} = \Lambda_I^{\{2,1\}} T^*M \subset \Lambda^3 T^*M$ if and only if

$$(I_{(12)} + I_{(13)} + I_{(23)})\psi = \psi, \quad \text{i.e., } \mathcal{L}_I\psi = \psi.$$

PROPOSITION 4.3. — *For an adapted basis I, J, K the exterior derivatives $d\omega_I$, etc., determine*

- (i) the covariant derivative $\nabla^{LC}\omega_I$ by

$$(4.7) \quad 2\nabla^{LC}\omega_I = (1 - I_{(23)})d\omega_I + (I_{(2)} + I_{(3)})J_{(1)}d\omega_J - (1 - I_{(23)})J_{(1)}d\omega_K$$

$$(4.8) \quad = (1 - I_{(23)})d\omega_I + (I_{(2)} + I_{(3)})K_{(1)}d\omega_K + (1 - I_{(23)})K_{(1)}d\omega_J,$$

- (ii) the Nijenhuis $(0, 3)$ -tensor N_I by

$$(4.9) \quad 2N_I = (I_{(12)} + I_{(13)} + I_{(23)} - 1)J_{(23)}(Jd\omega_J - Kd\omega_K) = (1 - I_{(12)})(K_{(23)} - J_{(23)})(Jd\omega_J - Kd\omega_K),$$

(iii) the one-form $I\lambda_I$ of equation (3.2) by

$$(4.10) \quad 2nI\lambda_I = I\Lambda_K d\omega_J + J\Lambda_K d\omega_I + J\Lambda_I d\omega_K$$

$$(4.11) \quad = I\Lambda_K d\omega_J + \Lambda_I d\omega_I - \Lambda_K d\omega_K,$$

and moreover

(iv) $Jd\omega_J + Kd\omega_K \in \mathcal{W}_{3+4,I}$, i.e.,

$$(4.12) \quad (I_{(12)} + I_{(13)} + I_{(23)})(Jd\omega_J + Kd\omega_K) = Jd\omega_J + Kd\omega_K.$$

The corresponding expressions with respect to J and K are obtained by cyclic permutations of I, J, K .

Proof. — Equation (4.7) is derived from its right-hand side, taking into account that $d\omega_A(X, Y, Z) = \mathfrak{S}_{XYZ}(\nabla_X^{LC}\omega_A)(Y, Z)$, $A = I, J, K$, and making repeated use of equation (4.5). The proof for equation (4.8) is similar. Now, (ii), (iii) and (iv) are immediate consequences of (4.7), (4.8) and Gray’s identity (4.6). \square

The expressions for the intrinsic $Sp(n) Sp(1)$ -torsion ξ given in next result are consequences of the last proposition and equation (3.3).

PROPOSITION 4.4. — *The intrinsic $Sp(n) Sp(1)$ -torsion ξ is determined by the exterior derivatives $d\omega_I$, etc., by*

$$\begin{aligned} \xi_X Y &= \frac{1}{4n} \mathfrak{S}_{IJK} (\Lambda_K d\omega_J - I\Lambda_I d\omega_I + I\Lambda_K d\omega_K)(X) IY \\ &+ \frac{1}{8} \mathfrak{S}_{IJK} \left((I_{(2)} + I_{(3)} + (J_{(12)} + J_{(13)} + K_{(23)} - 1)I_{(1)}) d\omega_I \right) (X, Y, e_i) e_i. \end{aligned}$$

Proof. — One computes first

$$\begin{aligned} \xi_X Y &= \frac{1}{4n} \mathfrak{S}_{IJK} \{ \langle X \lrcorner d\omega_J, \omega_K \rangle + \langle IX \lrcorner d\omega_I, \omega_I \rangle - \langle IX \lrcorner d\omega_K, \omega_K \rangle \} IY \\ &- \frac{1}{8} \mathfrak{S}_{IJK} \{ d\omega_I(X, Y, Ie_i) + d\omega_I(X, IY, e_i) - d\omega_J(JX, Y, e_i) \\ &\quad + d\omega_J(JX, IY, Ie_i) + d\omega_K(JX, Y, Ie_i) e_i + d\omega_K(JX, IY, e_i) \} e_i; \end{aligned}$$

and then takes advantage of the second cyclic sum to rearrange terms. \square

5. A minimal description

Motivated by Proposition 4.3(iv), let us introduce the three-forms

$$\beta_I = Jd\omega_J + Kd\omega_K, \quad \text{etc.}$$

These determine the exterior derivatives $d\omega_I, d\omega_J, d\omega_K$ as follows

$$2d\omega_I = I(\beta_I - \beta_J - \beta_K), \quad \text{etc..}$$

and $\beta_A \in \mathcal{W}_{3+4,A} = \Lambda_A^{\{2,1\}} T^*M$, so the dimension of the space of possible exterior derivatives $d\omega_I, d\omega_J, d\omega_K$ is at most $3 \dim(\mathcal{W}_{3+4}) = 12n^2(2n - 1)$ [13]. On the other hand, equation (4.1) implies that the dimension of the space of covariant derivatives $\nabla\omega_I, \nabla\omega_J, \nabla\omega_K$ is determined by the possible triples of λ 's and α 's. This dimension is $12n + 12n(2n + 1)(n - 1) = 12n^2(n - 1)$, which coincides with the above one computed for the β 's. Therefore, algebraically, the three-forms $\beta_I, \beta_J, \beta_K$ are independent.

We will now show how, components of, the β 's determine the intrinsic torsion. Consider the action of the group $Sp(n) U(1)_I$, which is the intersection of $U(2n)_I$ with $Sp(n) Sp(1)$, on the module $\mathcal{W}_{3+4,I} = \Lambda_I^{\{2,1\}} T^*M \subset \Lambda^3 T^*M$. It was shown in [21] that $\mathcal{W}_3 \otimes \mathbb{C} = (\Lambda_0^3 E + K + E)(L_I + \overline{L_I})$ and $\mathcal{W}_4 \otimes \mathbb{C} = E(L_I + \overline{L_I})$, where we write L_I for the standard representation of $U(1)_I$ on \mathbb{C} . Since $\Lambda_0^3 E, K, E$ are representations of quaternionic type and L_I is of complex type, the tensor products $\Lambda_0^3 EL, \text{etc.}$, in the above decompositions are all of quaternionic type. The underlying real modules $[V]_{\mathbb{R}}$ obtained by regarding the modules V as real vector spaces give real representations of $Sp(n) U(1)$ and

$$\mathcal{W}_{3I} = [\Lambda_0^3 EL_I]_{\mathbb{R}} + [KL_I]_{\mathbb{R}} + [EL_I]_{\mathbb{R}3}, \quad \mathcal{W}_{4I} = [EL_I]_{\mathbb{R}4}.$$

Using these decompositions, the tensor β_I splits into four components

$$(5.1) \quad \beta_I = \beta_I^{(3)} + \beta_I^{(K)} + \beta_{3I}^{(E)} + \beta_{4I},$$

with one-form parts

$$(5.2) \quad \beta_{3I}^{(E)} = -\frac{1}{2} J \nu_3^I \wedge \omega_J - \frac{1}{2} K \nu_3^I \wedge \omega_K + \frac{1}{2n-1} I \nu_3^I \wedge \omega_I,$$

$$(5.3) \quad \beta_{4I} = -\frac{1}{2n-1} I \nu_4^I \wedge \omega_I,$$

where ν_3^I and ν_4^I are one-forms, which we will now specify. We have

$$(5.4) \quad \nu_4^I = I \Lambda_I \beta_I.$$

A computation gives the following formula determining ν_3^I from β_I :

$$(5.5) \quad J \Lambda_J \beta_I = K \Lambda_K \beta_I = \frac{1}{2n-1} (\nu_4^I + (2n + 1)(n - 1) \nu_3^I).$$

Here the first equality in (5.5) is equivalent to

$$(5.6) \quad I \Lambda_K d\omega_J + I \Lambda_J d\omega_K = -\Lambda_J d\omega_J + \Lambda_K d\omega_K$$

which is an immediate consequence of equations (4.10) and (4.11) of Proposition 4.3 and the fact that $\beta_A \in \mathcal{W}_{3+4,A}$. We may now find the other components of β_I via (3.6) and (3.7):

$$(5.7) \quad \beta_I^{(3)} = \frac{1}{6} (2 - \mathcal{L}_J - \mathcal{L}_K) \beta_I^{(3+K)}, \quad \beta_I^{(K)} = \frac{1}{6} (4 + \mathcal{L}_J + \mathcal{L}_K) \beta_I^{(3+K)},$$

where $\beta_I^{(3+K)} = \beta_I - \beta_{3I}^{(E)} - \beta_{4I}$.

Remark 5.1. — The expressions for the one-form parts are a little simpler in dimension four, i.e., $n = 1$. Recall that the Lee form of ω_I is $Id^*\omega_I = -\Lambda_I d\omega_I$, where d^* is the co-derivative. For $n = 1$, we have $\mathcal{W}_3 = \{0\}$, so $\beta_I \in \mathcal{W}_{4I}$, $I\Lambda_I\beta_I = J\Lambda_J\beta_I = K\Lambda_K\beta_I = \nu_4^I$, and

$$K\Lambda_J d\omega_I = -J\Lambda_K d\omega_I = -\Lambda_I d\omega_I = Id^*\omega_I, \quad \text{etc.}$$

Remark 5.2. — In order to apply Proposition 4.1 to classify almost quaternion-Hermitian manifolds, the tensors $\alpha_A^{(3)}$ and $\alpha_A^{(K)}$ can be computed from the triples $\beta_I^{(3)}, \beta_J^{(3)}, \beta_K^{(3)}$ and $\beta_I^{(K)}, \beta_J^{(K)}, \beta_K^{(K)}$ respectively. In fact, we would begin with equations (4.2) which define α_A and then use Proposition 4.3.

To analyse the ES^3H and EH components of the intrinsic torsion ξ , we wish to apply Corollary 4.2 which requires knowledge of the one-forms η_I . Let us see how these are determined by the β_I 's. Equation (4.1) gives

$$\begin{aligned} \beta_I &= -J\lambda_I \wedge \omega_K + J\lambda_K \wedge \omega_I + J \text{Alt}(K_{(2)}\alpha_I) - J \text{Alt}(I_{(2)}\alpha_K) \\ &\quad - K\lambda_J \wedge \omega_I + K\lambda_I \wedge \omega_J + K \text{Alt}(I_{(2)}\alpha_J) - K \text{Alt}(J_{(2)}\alpha_I), \end{aligned}$$

where $\text{Alt}(\phi)(X, Y, Z) = \mathfrak{S}_{XYZ} \phi(X, Y, Z)$, for $\phi \in T^*M \otimes \Lambda^2 T^*M$. This combined with (4.11) leads to

$$\begin{aligned} 4nI\eta_I &= 2(n-1)J\Lambda_J\beta_I + I\Lambda_I((n-1)\beta_I + \beta_J + \beta_K) \\ &\quad - nJ\Lambda_J\beta_J - nK\Lambda_K\beta_K, \\ 4nI\lambda_I &= 2J\Lambda_J\beta_I + I\Lambda_I(\beta_I - \beta_J - \beta_K), \quad \text{etc.} \end{aligned}$$

Note that the right-hand sides of these equations are linear combinations of ν_3^A and ν_4^A , $A = I, J, K$, so

$$\begin{aligned} I\eta_I &= \frac{(2n+1)(n-1)}{4n(2n-1)} ((2(n-1)\nu_3^I + \nu_3^J + \nu_3^K) + (\nu_4^I - \nu_4^J - \nu_4^K)), \\ I\lambda_I &= \frac{(2n+1)(n-1)}{4n(2n-1)} (2\nu_3^I - \nu_3^J - \nu_3^K) + \frac{1}{4n(2n-1)} ((2n+1)\nu_4^I - \nu_4^J - \nu_4^K), \end{aligned}$$

etc.

The next proposition shows clearly the rôles played by the three-forms β_I, β_J and β_K in determining the components of the intrinsic torsion ξ . This provides a practical way to compute ξ using the tools of the exterior algebra.

PROPOSITION 5.3. — *For an almost quaternion-Hermitian $4n$ -manifold, $n > 1$, we have:*

(i) *The three-form $\psi^{(3)}$, which determines ξ_{33} , is given by*

$$(5.8) \quad \psi^{(3)} = \frac{1}{12}(\beta_I^{(3)} + \beta_J^{(3)} + \beta_K^{(3)}).$$

(ii) The local three-forms $\psi_I^{(3)}, \psi_J^{(3)}, \psi_K^{(3)}$ each of which determine ξ_{3H} , are given by

$$(5.9) \quad \psi_A^{(3)} = -\frac{1}{8}\beta_A^{(3)} + \frac{1}{48}(3 + \mathcal{L}_A) \sum_{B=I,J,K} \beta_B^{(3)}.$$

(iii) The three-form $\psi^{(K)}$, which determines ξ_{KH} , is given by

$$(5.10) \quad \psi^{(K)} = -\frac{1}{48}(\beta_I^{(K)} + \beta_J^{(K)} + \beta_K^{(K)}).$$

(iv) The local three-forms $\psi_I^{(K)}, \psi_J^{(K)}, \psi_K^{(K)}$ which determine ξ_{K3} , are given by

$$(5.11) \quad \psi_A^{(K)} = -\frac{1}{2}\beta_A^{(K)} + \frac{1}{6} \sum_{B=I,J,K} \beta_B^{(K)}.$$

(v) The one-form θ^ξ , which determines ξ_{EH} , is given by

$$(5.12) \quad \theta^\xi = \frac{n}{24(2n-1)} \sum_{A=I,J,K} (\nu_3^A - 2A\lambda_A),$$

(vi) The local three-forms $\theta_I^\xi, \theta_J^\xi, \theta_K^\xi$ whose differences $\theta_A^\xi - \theta^\xi$ determine ξ_{E3} , are given by

$$(5.13) \quad \theta_A^\xi = \frac{n}{4(n+1)} \left((\nu_3^A - 2A\lambda_A) - \frac{n-1}{2(2n-1)} \sum_{B=I,J,K} (\nu_3^B - 2B\lambda_B) \right).$$

Proof. — For the covariant derivative of the local Kähler forms ω_I , we have

$$(5.14) \quad \begin{aligned} \nabla_X^{\text{LC}} \omega_I(Y, Z) &= \lambda_K(X)\omega_J(Y, Z) - \lambda_J(X)\omega_K(Y, Z) \\ &\quad - \langle Y, \xi_X IZ \rangle - \langle IY, \xi_X Z \rangle, \end{aligned}$$

from which one derives

$$Id\omega_I = -I\lambda_K \wedge \omega_J + I\lambda_J \wedge \omega_K - \mathfrak{S}_{XYZ} (\langle Y, \xi_{IX} IZ \rangle + \langle IY, \xi_{IX} Z \rangle)$$

and

$$(5.15) \quad \begin{aligned} \beta_I &= K\lambda_I \wedge \omega_J - J\lambda_I \wedge \omega_K - I\lambda_I^+ \wedge \omega_I \\ &\quad - \mathfrak{S}_{XYZ} (\langle Y, \xi_{JX} JZ \rangle + \langle JY, \xi_{JX} Z \rangle + \langle Y, \xi_{KX} KZ \rangle + \langle KY, \xi_{KX} Z \rangle), \end{aligned}$$

where $\lambda_I^+ = J\lambda_J + K\lambda_K$.

For parts (i) and (ii), equation (5.15) gives

$$\begin{aligned} -\beta_I^{(3)}(X, Y, Z) &= \mathfrak{S}_{XYZ} \langle Y, (\xi_{33} + \xi_{3H})_{JX} JZ \rangle + \langle JY, (\xi_{33} + \xi_{3H})_{JX} Z \rangle \\ &\quad + \langle Y, (\xi_{33} + \xi_{3H})_{KX} KZ \rangle + \langle KY, (\xi_{33} + \xi_{3H})_{KX} Z \rangle. \end{aligned}$$

We now get

$$\beta_A^{(3)} = 6\psi^{(3)} + 2\mathcal{L}_A\psi^{(3)} - 8\psi_A^{(3)}$$

which leads to equations (5.8) and (5.9) as required.

For parts (iii) and (iv), we use (5.15) to get

$$\beta_A^{(K)} = -16\psi^{(K)} - 2\psi_A^{(K)},$$

which gives equations (5.10) and (5.11).

Finally, for parts (v) and (vi), use (5.15) to find

$$\begin{aligned} n\beta_I^{(E)} &= -I(n\lambda_I^+ + 6(n-1)\theta^\xi - 2(3n+1)\theta_I^\xi) \wedge \omega_I \\ &\quad - J(nI\lambda_I + 6(n-1)\theta^\xi + 2(n+1)\theta_I^\xi) \wedge \omega_J \\ &\quad - K(nI\lambda_I + 6(n-1)\theta^\xi + 2(n+1)\theta_I^\xi) \wedge \omega_K, \quad \text{etc.} \end{aligned}$$

Using equations (5.2), (5.3) and (5.5), this gives

$$(5.16) \quad \nu_3^A = 2A\lambda_A + \frac{4}{n}(3(n-1)\theta^\xi + (n+1)\theta_A^\xi),$$

$$(5.17) \quad \nu_4^A = (2n-1)\lambda_A^+ + 2A\lambda_A + \frac{6(n-1)(2n+1)}{n}(\theta^\xi - \theta_A^\xi),$$

for $A = I, J, K$. As $3\theta^\xi = \theta_I^\xi + \theta_J^\xi + \theta_K^\xi$, equations (5.12) and (5.13) follow. □

We may now quickly record what happens under conformal changes of metric.

PROPOSITION 5.4. — *On a almost quaternion-Hermitian $4n$ -manifold, if we consider a conformal change of metric $\langle \cdot, \cdot \rangle^o = e^{2\sigma} \langle \cdot, \cdot \rangle$, with $\sigma \in C^\infty(M)$, then*

$$\begin{aligned} \omega_A^o &= e^{2\sigma}\omega_A, & d\omega_A^o &= e^{2\sigma}(2d\sigma \wedge \omega_A + d\omega_A), \\ Ad^*\omega_A^o &= Ad^*\omega_A - 2(2n-1)d\sigma, & A\lambda_A^o &= A\lambda_A - \frac{1}{n}d\sigma, \\ \beta_I^o &= e^{2\sigma}(\beta_I + 2Jd\sigma \wedge \omega_J + 2Kd\sigma \wedge \omega_K), & \text{etc.}, \\ \nu_3^{A^o} &= \nu_3^A - 4d\sigma, & \nu_4^{A^o} &= \nu_4^A - 4d\sigma, & \theta^{\xi^o} &= \theta^\xi - \frac{1}{4}d\sigma, & \theta_A^{\xi^o} &= \theta_A^\xi - \frac{1}{4}d\sigma. \end{aligned}$$

In particular, the only component of the intrinsic torsion that changes is ξ_{EH}^o .

Proof. — The identities follow from the definitions of each tensor involved. For the intrinsic torsion, use these identities, Proposition 5.3 and the descriptions of the components of ξ given at the end of §3. □

6. HyperKähler manifolds with torsion

In this section we will see some consequences of Proposition 4.3 in HKT-geometry. This geometry arises on the target space of a $N = 2$ supersymmetric $(4, 0)$ σ -models with Wess-Zumino term.

DEFINITION 6.1 (Howe & Papadopoulos [17]). — *An almost hyperHermitian manifold $(M, I, J, K, g = \langle \cdot, \cdot \rangle)$ is an HKT-manifold (hyperKähler with torsion), if the following conditions are satisfied:*

- (i) *the almost complex structures I, J, K are integrable;*
- (ii) *M admits a linear connection $\nabla^{\text{HKT}} = \nabla^{\text{LC}} + \frac{1}{2}T$, such that*
 - (a) *$\nabla^{\text{HKT}}I = \nabla^{\text{HKT}}J = \nabla^{\text{HKT}}K = 0$, and*
 - (b) *$\nabla^{\text{HKT}}g = 0$;*
- (iii) *the $(0, 3)$ -tensor field, also denoted by T , defined by $T(X, Y, Z) = \langle X, T(Y, Z) \rangle$ is a skew-symmetric three-form.*

A result of Grancharov & Poon [11] says that an almost hyperHermitian manifold M is HKT if and only if $(I, J, K, \langle \cdot, \cdot \rangle)$ is hyperHermitian (i.e., $N_I = N_J = N_K = 0$) and $Id\omega_I = Jd\omega_J = Kd\omega_K$. We now give the following improvement of this result, showing that the integrability assumption is redundant.

PROPOSITION 6.2. — *Let (M, I, J, K, g) be an almost hyperHermitian manifold. Then the following conditions are equivalent:*

- (i) *M is an HKT-manifold;*
- (ii) *$Id\omega_I = Jd\omega_J = Kd\omega_K$;*
- (iii) *$\beta_I = \beta_J = \beta_K$.*

Proof. — If M is a HKT-manifold, we have a connection $\nabla^{\text{HKT}} = \nabla^{\text{LC}} + \frac{1}{2}T$ satisfying the conditions given in Definition 6.1. The integrability condition gives $N_I = 0 = N_J = N_K$ and $\nabla^{\text{LC}}\omega_A \in \mathcal{W}_{3+4,A}$. Now, using equation (4.6), we obtain $T = Id\omega_I = Jd\omega_J = Kd\omega_K = \frac{1}{2}\beta_I \in \mathcal{W}_{3+4}$.

Conversely, suppose $Id\omega_I = Jd\omega_J = Kd\omega_K$. Proposition 4.3(ii) gives $N_I = N_J = N_K = 0$. The connection $\nabla^{\text{HKT}} = \nabla^{\text{LC}} + \frac{1}{2}T$, where $\langle X, T(Y, Z) \rangle = Id\omega_I(X, Y, Z)$ now satisfies the HKT conditions. □

Grantcharov & Poon [11] give a second characterisation of HKT manifolds in terms of the complex geometry of I . Let us define as usual the operators ∂_A and $\bar{\partial}_A$ acting on a p -form ψ by

$$\partial_A\psi = \frac{1}{2}(d + (-1)^p iAdA)\psi, \quad \bar{\partial}_A\psi = \frac{1}{2}(d - (-1)^p iAdA)\psi.$$

Assuming integrability of I, J and K , Grantcharov & Poon show that M is HKT if and only if the $(2, 0)$ -form $\omega_J + i\omega_K$ is ∂_I -closed. Once again we may weaken the integrability requirements.

PROPOSITION 6.3. — *Let (M, I, J, K, g) be an almost hyperHermitian manifold. Then the following conditions are equivalent:*

- (i) M is an HKT-manifold;
- (ii) $Jd\omega_J = Kd\omega_K$ and $N_J = 0$;
- (iii) $\partial_I(\omega_J + i\omega_K) = 0$ and $N_J = 0$;
- (iv) $\bar{\partial}_I(\omega_J - i\omega_K) = 0$ and $N_J = 0$.

Proof. — It is easy to see that the three conditions $Jd\omega_J = Kd\omega_K$, $\partial_I(\omega_J + i\omega_K) = 0$ and $\bar{\partial}_I(\omega_J - i\omega_K) = 0$, are equivalent. Moreover, by Proposition 4.3(ii), the condition $Jd\omega_J = Kd\omega_K$ implies $N_I = 0$. Now, the integrability of I and J implies that K is integrable (see [22] or the newer proof [21]). Hence, any of the last three conditions gives that the manifold is hyperHermitian and $\partial_I(\omega_J + i\omega_K) = 0$ and we obtain HKT from Grantcharov & Poon.

Alternatively, we may prove the result just using tools contained in the present paper. Suppose $N_J = 0$ and $Jd\omega_J = Kd\omega_K$. Then $Jd\omega_J$ and hence $Kd\omega_K$ lie in $\mathcal{W}_{3+4,J}$. However, Proposition 4.3(iv) gives that $Kd\omega_K + Id\omega_I \in \mathcal{W}_{3+4,J}$, so we have $Id\omega_I \in \mathcal{W}_{3+4,J}$ too. Now let us use Proposition 4.3(ii) for the integrability of J . We have $0 = K_{(23)}(-J_{(12)} - J_{(13)} + J_{(23)} - 1)(Kd\omega_K - Id\omega_I)$. But $J_{(12)} + J_{(23)} + J_{(13)} = 1$ on $\mathcal{W}_{3+4,J}$ and $Jd\omega_J = Kd\omega_K$, so

$$J_{(23)}(Jd\omega_J - Id\omega_I) = Jd\omega_J - Id\omega_I.$$

Skew-symmetrising both sides of the identity, we find that $Jd\omega_J - Id\omega_I = 3(Jd\omega_J - Id\omega_I)$. So, $Id\omega_I = Jd\omega_J = Kd\omega_K$. □

Next we describe the very special situation for four-dimensional HKT-manifolds.

PROPOSITION 6.4. — *If M is an almost hyperHermitian 4-manifold, then the following conditions are equivalent:*

- (i) M is an HKT-manifold;
- (ii) the three Lee one-forms are equal, i.e., $Id^*\omega_I = Jd^*\omega_J = Kd^*\omega_K$;
- (iii) the almost complex structures I and J are integrable;
- (iv) the almost Hermitian structures corresponding to I and J are locally conformally Kähler, so M is locally conformally hyperKähler.

Proof. — For dimension 4, the Gray-Hervella modules \mathcal{W}_1 and \mathcal{W}_3 are zero, we have $I\theta \wedge \omega_I = J\theta \wedge \omega_J = K\theta \wedge \omega_K$, for all one-forms θ , and any three-form may be written in this way. If M^4 is an HKT-manifold, we see that the almost Hermitian structures are of type \mathcal{W}_4 and that

$$T = Ad\omega_A = -At \wedge \omega_A,$$

where $t = A\Lambda_A T = -\Lambda_A d\omega_A = Ad^*\omega_A$. On the other hand, if the three Lee forms are equal to a one-form t , then

$$Id\omega_I = -It \wedge \omega_I = -Jt \wedge \omega_J = Jd\omega_J.$$

Hence $Id\omega_I = Jd\omega_J = Kd\omega_K$ and M is HKT.

For conditions (iii) or (iv), the three almost complex structures are integrable, so the almost Hermitian structures have a common Lee form, by [21]. □

In §4 it was shown that for any almost quaternion-Hermitian manifold, the exterior derivatives of the three local Kähler forms of an adapted basis I, J, K satisfy the identities (5.6). When the manifold is HKT, additional identities are also satisfied.

LEMMA 6.5. — *For a $4n$ -dimensional HKT-manifold, the exterior derivatives $d\omega_I, d\omega_J$ and $d\omega_K$ satisfy*

$$(6.1) \quad t = -\Lambda_I d\omega_I = K\Lambda_J d\omega_I = -J\Lambda_K d\omega_I, \quad \text{etc.},$$

where $t = Id^*\omega_I = Jd^*\omega_J = Kd^*\omega_K$. Furthermore, $I\lambda_I = J\lambda_J = K\lambda_K = \frac{1}{2n}t$, $\theta^\xi = \theta_I^\xi = \theta_J^\xi = \theta_K^\xi$ and the one-forms corresponding to the E -parts of β_A are such that

$$(6.2) \quad \nu_4^I = \nu_4^J = \nu_4^K = 2t,$$

$$(6.3) \quad \nu_3^I = \nu_3^J = \nu_3^K = 32\theta^\xi = \frac{4}{2n+1}t,$$

where the second line holds for $n > 1$.

Proof. — Since $2T = \beta_I = \beta_J = \beta_K$, we have

$$I\Lambda_I\beta_A = J\Lambda_J\beta_A = K\Lambda_K\beta_A,$$

from which we obtain (6.1) and (6.2), via (5.4). Now, using equation (5.5), we have $(2n + 1)(n - 1)\nu_3^A = 4(n - 1)Ad^*\omega_A$ and hence (6.3). □

7. Quaternion-Kähler manifolds with torsion

A genuinely quaternionic analogue of HKT geometry also arises in the physics literature via the theory of super-symmetric sigma models. In this section we give a definition in terms of intrinsic torsion, relate this definition to the existence of connections with skew-symmetric torsion, provide different characterisations of the geometry and describe the relationship with HKT geometry. Important mathematical work in this direction was

previously done by Ivanov [18]. Here we concentrate on the intrinsic geometry, fit the geometry into our general formalism and improve and clarify a number of his results.

DEFINITION 7.1. — *An almost quaternion-Hermitian manifold of dimension $4n \geq 8$ is QKT (quaternion-Kähler with torsion) if its intrinsic torsion lies in $(K + E)H$.*

As in other cases, we may write this condition on the intrinsic torsion in terms of three-forms.

LEMMA 7.2. — *The intrinsic torsion ξ lies in $(K + E)H$ precisely when it is given by a three-form $\psi \in (K + E)H \subset \Lambda^3 T^*M$ via*

$$(7.1) \quad \langle Y, \xi_X Z \rangle = \left(3\psi + \sum_{A=I,J,K} (-A_{(23)}\psi + \frac{2}{n}A\theta^\psi \otimes \omega_A) \right) (X, Y, Z),$$

where $\theta^\psi = I\Lambda_I\psi = J\Lambda_J\psi = K\Lambda_K\psi$. Moreover, for a given intrinsic torsion $\xi \in (K + E)H$ we have that $\psi \in \Lambda^3 T^*M$ is unique and given by

$$48(n - 1)\psi = (n - 1)d^*\Omega + \frac{1}{2} \sum_{A=I,J,K} A\theta^{d^*\Omega} \wedge \omega_A,$$

where Ω is the fundamental four-form (2.1) and d^* is the co-derivative.

When applying this result it is often useful to recall the formula [20]

$$d^*\Omega = 2 \sum_{A=I,J,K} (d^*\omega_A \wedge \omega_A - Ad\omega_A).$$

Let us now demonstrate how the QKT condition relates to connections with skew-symmetric torsion and so the original definition of Howe, Opfermann & Papadopoulos [15]. Recall that $(K + E)H \subset \Lambda^3 T^*M$ is the $(+3)$ -eigenspace of the operator \mathcal{L} given in (3.5).

THEOREM 7.3. — *An almost quaternion-Hermitian manifold M is QKT if and only if there exists a metric connection $\nabla^{\text{QKT}} = \nabla^{\text{LC}} + \frac{1}{2}T$ that is quaternionic and whose $(0, 3)$ -torsion $T(X, Y, Z) = \langle X, T(Y, Z) \rangle$ is a three-form satisfying in $\mathcal{L}T = -3T$. When M is QKT, ∇^{QKT} is the unique $Sp(n)Sp(1)$ -connection on M with skew-symmetric torsion.*

Concretely, we claim that the intrinsic torsion ξ of the QKT structure is given by (7.1) with $\psi = -\frac{1}{8}T$ and that so

$$T = -\frac{1}{6}d^*\Omega - \frac{1}{12(n-1)} \sum_{A=I,J,K} A\theta^{d^*\Omega} \wedge \omega_A.$$

Note that in §10 we will provide examples of almost quaternion-Hermitian manifolds that are not QKT but none-the-less admit $Sp(n)Sp(1)$ -connections with skew-symmetric torsion.

Proof. — If $\xi \in (K + E)H$, then ξ is given by equation (7.1) for some ψ in $(K + E)H$. Putting $T = -8\psi$ we find that $\nabla^{\text{QKT}} = \nabla^{\text{LC}} + \frac{1}{2}T$ is metric and, via (3.3), quaternionic.

Conversely, if M has such a connection ∇^{QKT} , then

$$(7.2) \quad \nabla^{\text{LC}}\omega_I = \gamma_K \otimes \omega_J - \gamma_J \otimes \omega_K - \frac{1}{2}(I_{(2)} + I_{(3)})T,$$

where $\gamma_A, A = I, J, K$, are the one-forms given by (3.1) for $\tilde{\nabla} = \nabla^{\text{QKT}}$. Using equation (5.14), we find

$$(7.3) \quad \begin{aligned} \langle Y, \xi_X Z \rangle &= -\frac{1}{8} \left(3T - \sum_{A=I,J,K} A_{(23)}T \right) (X, Y, Z) \\ &+ \frac{1}{2} \sum_{A=I,J,K} (\lambda_A - \gamma_A) \otimes \omega_A (X, Y, Z). \end{aligned}$$

As $\xi \in T^*M \otimes \Lambda_0^2 ES^2H$, we have $\langle Ae_i, \xi_X e_i \rangle = 0$ and find

$$(7.4) \quad \lambda_A - \gamma_A = -\frac{1}{2n}At,$$

with $t = I\Lambda_I T = J\Lambda_J T = K\Lambda_K T$. Using (7.3) and (7.4), we obtain equation (7.1) with $\psi = -\frac{1}{8}T \in (K + E)H$. □

Remark 7.4. — The situation for 4-dimensional almost quaternion-Hermitian manifolds is very special. Here the Levi-Civita connection is always quaternionic, i.e.,

$$\nabla^{\text{LC}}I = \lambda_K \otimes J - \lambda_J \otimes K, \quad \text{etc.}$$

Also in this dimension, we have $\Lambda^3 T^*M \cong T^*M$ and any three-form T may be written as $T = -At \wedge \omega_A$ for some one-form t valid for $A = I, J$ and K . In this way, given any $t \in \Omega^1(M)$, we may construct $\tilde{\nabla} = \nabla^{\text{LC}} + \frac{1}{2}T$ and find

$$\tilde{\nabla}I = \gamma_K \otimes J - \gamma_J \otimes K, \quad \text{etc.,}$$

where $\gamma_A = \lambda_A + \frac{1}{2}At$. Hence $\tilde{\nabla}$ is a connection with skew-symmetric torsion preserving the almost quaternion-Hermitian structure. However, in this case, $\tilde{\nabla}$ is not unique.

Forgetting the metric of an almost quaternion-Hermitian structure we are left with an almost quaternionic structure. This is an *integrable* quaternionic structure if there is a torsion-free quaternionic connection ∇^q , i.e., $\tilde{\nabla} = \nabla^q$ satisfies (3.1); this is a weaker condition than integrability of I, J

and K . In the presence of a compatible metric, integrability of the quaternionic structure is equivalent to the vanishing of ξ_{S^3H} , the $(\Lambda_0^3 E + K + H)S^3H$ -part of the intrinsic torsion, cf. [24]. In dimension four, this condition is just self-duality of the conformal structure. In dimension 8, the module $\Lambda_0^3 E$ is zero, so $\xi \in (K + E)(S^3H + H)$ and integrability implies that $\xi \in (K + E)H$. We thus have:

PROPOSITION 7.5. — *Any compatible metric on an eight-dimensional quaternionic manifold is QKT.*

This applies for example to any metric compatible with Joyce’s invariant hypercomplex structure on $SU(3)$ [19].

The one-form t in the proof of Theorem 7.3 has independent importance.

DEFINITION 7.6. — *For a QKT manifold with torsion three-form T the torsion one-form t is defined by*

$$t = I\Lambda_I T,$$

for any compatible almost complex structure I .

There are many alternative expressions for t :

LEMMA 7.7. — *For a $4n$ -dimensional QKT-manifold, $n > 1$, the torsion one-form t satisfies*

$$-\frac{3(n-1)}{4n}t = (\xi_{e_i} e_i)^{\flat} = -\frac{3}{2}A\eta_A = \frac{1}{16n} * (*d\Omega \wedge \Omega) = -\frac{3}{8n}A\Lambda_A d^* \Omega,$$

where η_A is given by (4.4) and $X^{\flat} = \langle X, \cdot \rangle$.

Remark 7.8. — The one-form

$$*(*d\Omega \wedge \Omega) = -2 \sum_{A=I,J,K} A\Lambda_A d^* \Omega$$

was considered in [20] in relation with the EH -component of ξ in general. One finds that $*(*d\Omega \wedge \Omega) = 16n (\xi_{e_i} e_i)^{\flat}$. There it was noted that

$$4n(J\eta_J + K\eta_K) = 2I\Lambda_I d^* \Omega = - * (*d\Omega \wedge \omega_I \wedge \omega_I).$$

Proof. — Using (4.3), one finds $(\xi_{e_i} e_i)^{\flat} = -\frac{1}{2} \sum_{A=I,J,K} A\eta_A$. When ξ is in $(K + E)H$, we have $I\eta_I = J\eta_J = K\eta_K$. Therefore, $(\xi_{e_i} e_i)^{\flat} = -\frac{3}{2}I\eta_I$. On the other hand, using $\Psi = -\frac{1}{8}T$ in equation (7.1), we obtain $(\xi_{e_i} e_i)^{\flat} = -\frac{3(n-1)}{4n}t$.

The remaining equalities follow from Remark 7.8 □

Further relations between the torsion three-form T and the four-form Ω follow from equation (7.2). In particular, we have

$$\nabla^{\text{LC}}\Omega = - \sum_{A=I,J,K} (A_{(2)} + A_{(3)})T \wedge \omega_A, \quad d\Omega = -2 \sum_{A=I,J,K} AT \wedge \omega_A.$$

Let us now give a characterisation of QKT manifolds.

THEOREM 7.9. — *An almost quaternion-Hermitian manifold M is QKT if and only if for each local adapted basis I, J, K there are local one-forms $\gamma_I, \gamma_J, \gamma_K$ such that*

$$(7.5) \quad \begin{aligned} \beta_J - \beta_I &= Id\omega_I - Jd\omega_J \\ &= I(K\gamma_K) \wedge \omega_I - J(K\gamma_K) \wedge \omega_J + K\gamma_K^- \wedge \omega_K, \quad \text{etc.}, \end{aligned}$$

where $\gamma_K^- = I\gamma_I - J\gamma_J$;

In this case, the skew-symmetric torsion three-form T is given by

$$(7.6) \quad 2T = 2(Id\omega_I + J(K\gamma_K) \wedge \omega_J + K(J\gamma_J) \wedge \omega_K)$$

$$(7.7) \quad = \beta_I + J(I\gamma_I) \wedge \omega_J + K(I\gamma_I) \wedge \omega_K + I\gamma_I^+ \wedge \omega_I, \quad \text{etc.}$$

where $\gamma_I^+ = J\gamma_J + K\gamma_K$ and

$$(7.8) \quad 2(n-1)I\gamma_I = (\Lambda_J + I\Lambda_K)d\omega_J = (\Lambda_K - I\Lambda_J)d\omega_K, \quad \text{etc..}$$

Remark 7.10. — Equation (7.7) also implies that, for dimensions strictly greater than four, the QKT-connection ∇^{QKT} is unique. This fact was already proved by Ivanov, who also characterised QKT-manifolds by the differences $(Id\omega_I)_{\mathcal{W}_{3+4,I}} - (Jd\omega_J)_{\mathcal{W}_{3+4,J}}$ [18, Theorem 2.2]. Our Theorem 7.9 can be considered as an improved version, based on the three-forms β_A , which are automatically in $\mathcal{W}_{3+4,A}$.

Remark 7.11. — Let us write $W_{3I}^{(E)}(\eta_I)$ and $W_{4I}(\eta_I)$ for the right-hand sides of equations (5.2) and (5.3), respectively. Equation (7.7) can then be written

$$2T = \beta_A - W_{3A}^{(E)}(2A\gamma_A) - W_{4A}(2A\gamma_A + (2n-1)\gamma_A^+).$$

Consequently, we see that QKT-manifolds have

$$\begin{aligned} \beta_A^{(3)} &= 0, \quad \beta_I^{(K)} = \beta_J^{(K)} = \beta_K^{(K)}, \\ \nu_3^A &= 2A\gamma_A + \frac{4}{2n+1}t \quad \text{and} \quad \nu_4^A = 2A\gamma_A + (2n-1)\gamma_A^+ + 2t, \end{aligned}$$

using equation (7.4) and Lemma 7.7. This should be compared with the results of Lemma 6.5.

Proof. — Suppose M is QKT. From $\nabla^{\text{QKT}} = \nabla^{\text{LC}} + \frac{1}{2}T$ and equation (7.2), we have

$$(7.9) \quad d\omega_I = -IT + \gamma_K \wedge \omega_J - \gamma_J \wedge \omega_K.$$

Multiplying by I gives (7.6) and equation (7.5) follows.

Conversely, if equation (7.5) is satisfied for some one-forms γ_A , then we may consider the three-form T given by (7.6) and use (7.5) to obtain the alternative expression (7.7), see that it is unchanged by a cyclic permutation of I, J, K , and hence globally defined. By Proposition 4.3(iv), we find that $T \in \mathcal{W}_{3+4,A}$, for $A = I, J, K$. Hence $T \in (K + E)H$ and M is a QKT-manifold for the connection $\nabla^{\text{QKT}} = \nabla^{\text{LC}} + \frac{1}{2}T$.

Finally, using equation (7.9), we get

$$K\Lambda_J d\omega_I = t + J\gamma_J + (2n - 1)K\gamma_K \quad \text{and} \quad -\Lambda_I d\omega_I = t + J\gamma_J + K\gamma_K.$$

Using the identity (5.6), now provides the claimed expressions for $I\gamma_I$. \square

The contraction identity (5.6) used above is valid for all almost quaternion-Hermitian manifolds. When the structure is QKT, there are additional identities of this type.

PROPOSITION 7.12. — *For a local adapted basis I, J, K of a QKT-manifold, the following identities are satisfied:*

$$(7.10) \quad Id^*\omega_I = t + J\gamma_J + K\gamma_K = J\lambda_J + K\lambda_K + 2I\eta_I, \quad \text{etc.},$$

$$(7.11) \quad J\gamma_J - K\gamma_K = J\lambda_J - K\lambda_K = -(Jd^*\omega_J - Kd^*\omega_K), \quad \text{etc.},$$

$$(7.12) \quad J\Lambda_K d\omega_I + K\Lambda_I d\omega_I = -2(n - 1)(\Lambda_J d\omega_J - \Lambda_K d\omega_K), \quad \text{etc.},$$

$$(7.13) \quad J\Lambda_I d\omega_K + K\Lambda_I d\omega_J = (2n - 1)(\Lambda_J d\omega_J - \Lambda_K d\omega_K), \quad \text{etc.}$$

Proof. — As M is a QKT-manifold, we have from equation (7.2)

$$d^*\omega_I = -(\nabla_{e_i}\omega_I)(e_i, \cdot) = -It - I(J\gamma_J + K\gamma_K)$$

giving the first equality of equation (7.10). On the other hand, it was shown in [21] that $Id^*\omega_I = J\lambda_J + K\lambda_K + J\eta_J + K\eta_K$. But, for manifolds of type $KH + EH$, we have $I\eta_I = J\eta_J = K\eta_K$ by Corollary 4.2, and we obtain the second equality of equation (7.10).

Equation (7.11) is an immediate consequence of equation (7.10).

Finally, equation (7.12) and equation (7.13) can be deduced from equations (7.8), (5.6) and (7.11), using $Id^*\omega_I = -\Lambda_I d\omega_I$. \square

Remark 7.13. — It is not hard to prove that an almost quaternion-Hermitian $4n$ -manifold, $n > 1$, is of type $\Lambda_0^3 ES^3H + KS^3H + \Lambda_0^3 EH + KH + EH$ if and only if equation (7.12) is satisfied for any local adapted basis I, J, K . In other words, (7.12) characterises the vanishing of the ES^3H -component of the intrinsic torsion.

Let us now turn to the question of integrability of compatible almost complex structures I, J and K . Taking the following identity

$$N_I(X, Y) = -(\nabla_X^{LC} I)IY - (\nabla_X^{LC} I)Y + (\nabla_Y^{LC} I)IX + (\nabla_Y^{LC} I)X,$$

into account, we obtain that the corresponding $(0, 3)$ -tensor $N_I = \langle \cdot, N_I(\cdot, \cdot) \rangle$ is given by

$$(7.14) \quad N_I = J\gamma_I^- \wedge \omega_J - K\gamma_I^- \wedge \omega_K - J\gamma_I^- \otimes \omega_J + K\gamma_I^- \otimes \omega_K, \quad \text{etc.}$$

Note that equation (7.4) gives $\gamma_A^- = \lambda_A^-$, where $\lambda_I^- = J\lambda_J - K\lambda_K$.

Fixing the almost complex structure I and under the action of the subgroup $Sp(n)U(1)$ of $U(2n)_I$, it was noted in [21] that, for $n > 1$, $\mathcal{W}_1 \otimes \mathbb{C} = (\Lambda_0^3 E + E)(L^3 + \bar{L}^3)$ and $\mathcal{W}_2 \otimes \mathbb{C} = (K + E)(L^3 + \bar{L}^3)$. As in § 5, we get

$$\mathcal{W}_1 = [\Lambda_0^3 EL^3]_{\mathbb{R}} + [EL^3]_{\mathbb{R}1}, \quad \mathcal{W}_2 = [KL^3]_{\mathbb{R}} + [EL^3]_{\mathbb{R}2}.$$

It is well known that, in general, the tensor N_I belongs to \mathcal{W}_{1+2} . In our case, equation (7.14) gives us that $N_I \in [EL^3]_{\mathbb{R}1} + [EL^3]_{\mathbb{R}2}$, i.e., the components of N_I in $[\Lambda_0^3 EL^3]_{\mathbb{R}}$ and $[KL^3]_{\mathbb{R}}$ vanish.

Using equations (7.14) and (7.11), we thus have:

COROLLARY 7.14. — *Let M be a QKT-manifold of dimension $4n > 4$. Then, for any local adapted basis I, J, K , the following conditions are equivalent:*

- (i) *the almost complex structure I is integrable;*
- (ii) *the Lee forms $Jd^*\omega_J$ and $Kd^*\omega_K$ are equal;*
- (iii) *the one-forms $J\lambda_J$ and $K\lambda_K$ are equal;*
- (iv) *the one-forms $J\gamma_J$ and $K\gamma_K$ are equal;*
- (v) *the equation $\Lambda_K d\omega_J = -\Lambda_J d\omega_K$ is satisfied;*
- (vi) *the equation $J\Lambda_K d\omega_I = -K\Lambda_J d\omega_I$ is satisfied;*
- (vii) *the equation $J\Lambda_I d\omega_K = -K\Lambda_I d\omega_J$ is satisfied.*

COROLLARY 7.15. — *Let M be a $4n$ -dimensional, ($n > 1$), almost quaternion-Hermitian with a global adapted basis I, J, K , i.e., M is equipped with an almost hyperHermitian structure. Then the following conditions are equivalent*

- (i) *M is an HKT-manifold;*

- (ii) M is a QKT-manifold such that $I\lambda_I = J\lambda_J = K\lambda_K = \frac{1}{2n}t$;
- (iii) M is a QKT-manifold such that $Id^*\omega_I = Jd^*\omega_J = Kd^*\omega_K$ and

$$I\Lambda_K d\omega_J = Id^*\omega_I, \quad \text{etc.}$$

Proof. — This is an immediate consequence of Theorem 7.9, Proposition 7.12 and Corollary 7.14. □

Finally, let us look at the two special types of QKT-manifolds with intrinsic torsion in one summand of $(K + E)H$.

LEMMA 7.16. — *An almost quaternion-Hermitian manifold M of dimension $4n > 4$ is of type $\Lambda_0^3 ES^3H + KS^3H + \Lambda_0^3 EH + KH$ if and only if, for any local adapted basis I, J, K , we have*

$$(7.15) \quad -I\Lambda_K d\omega_J = (n - 1)Id^*\omega_I - nJd^*\omega_J - (n - 1)Kd^*\omega_K, \quad \text{etc.}$$

Proof. — We have $\theta^\xi = \theta_I^\xi = \theta_J^\xi = \theta_K^\xi = 0$. Equations (5.16) and (5.17) then give

$$\nu_3^A = 2A\lambda_A, \quad \nu_4^A = (2n - 1)\lambda_A^+ + 2A\lambda_A.$$

From these equalities together with equations (5.3), (5.5) and (4.11), we deduce equation (7.15). □

THEOREM 7.17. — *Let M be an almost quaternion-Hermitian $4n$ -manifold, $n > 1$.*

- (i) M is of type KH if and only if M is a QKT-manifold and, for any local adapted basis I, J, K , equation (7.15) holds.
- (ii) M is of type EH if and only if there exists a global one-form t defined on M such that, for any local adapted basis I, J, K , we have

$$d\omega_I = -\frac{1}{2n+1}t \wedge \omega_I - K \left(K\lambda_K + \frac{1}{2n(2n+1)}t \right) \wedge \omega_J + J \left(J\lambda_J + \frac{1}{2n(2n+1)}t \right) \wedge \omega_K, \quad \text{etc.}$$

In this case, M is called a locally conformal quaternionic Kähler manifold and the one-form t is given by

$$2(n - 1)t = 2nId^*\omega_I - K\Lambda_J d\omega_I + J\Lambda_K d\omega_I, \quad \text{etc.}$$

Proof. — The first part follows directly from Lemma 7.16.

For M of type EH , then $T = -\frac{1}{2n+1} \sum_{A=I,J,K} At \wedge \omega_A$. Equation (7.16) then follows from equations (7.9) and (7.10) and Lemma 7.7. \square

8. Almost Hermitian structures

In this section, we will consider almost Hermitian manifolds M of dimension $2n$ and the classification of Gray & Hervella [13]. These manifolds are equipped with an almost complex structure I compatible with a Riemannian metric $\langle \cdot, \cdot \rangle$. Therefore, their orthogonal frame bundles can be reduced to the unitary group $U(n)$.

By identifying the intrinsic $U(n)$ -torsion ξ^{aH} with $\nabla^{\text{LC}}\omega_I$ via $\xi^{\text{aH}} \mapsto -\xi^{\text{aH}}\omega_I = \nabla^{\text{LC}}\omega_I$, Gray & Hervella [13] gave conditions characterising classes of almost Hermitian manifolds by means of the covariant derivative $\nabla^{\text{LC}}\omega_I$. The space of intrinsic $U(n)$ -torsions is then isomorphic to the space $\mathcal{W} \cong T^*M \otimes \Lambda^{\{2,0\}}$ of covariant derivatives of the Kähler form ω_I . Under the action of $U(n)$, $n > 2$, \mathcal{W} decomposes into four irreducible modules,

$$\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4 \cong \Lambda^{\{3,0\}} + [U^{3,0}]_{\mathbb{R}} + \Lambda_0^{\{2,1\}} + \Lambda^{\{1,0\}}.$$

Therefore, for $n > 2$, there are $2^4 = 16$ classes of almost Hermitian manifolds. For $n = 2$, $\mathcal{W}_1 = \mathcal{W}_3 = \{0\}$ and there are only 4 classes.

On the other hand, because $d\omega_I \in \Lambda^3 T^*M = \mathcal{W}_{1+3+4}$, only partial information about ξ^{aH} can be recovered from the exterior derivative $d\omega_I$. The remaining component can be found in the Nijenhuis $(0, 3)$ -tensor $N_I \in \mathcal{W}_{1+2} \subset T^*M \otimes \Lambda^2 T^*M$, which is often more convenient to work with than ∇^{LC} . Table 8.1 lists conditions characterising the classes of almost Hermitian manifolds in terms of tensors $d\omega_I$ and N_I . The symbol \mathcal{N}_I denotes the three-form obtained by skew-symmetrisation of N_I , i.e., $\mathcal{N}_I(X, Y, Z) = \mathfrak{S}_{XYZ} N_I(X, Y, Z)$. The conditions are found by studying the $U(n)$ -maps

$$\begin{aligned} \xi^{\text{aH}} \mapsto - \mathfrak{S}_{XYZ} (\xi_X^{\text{aH}} \omega_I)(Y, Z) &= d\omega_I(X, Y, Z), \\ \xi^{\text{aH}} \mapsto -(\xi_Z^{\text{aH}} \omega_I)(IX, Y) - (\xi_I^{\text{aH}} \omega_I)(X, Y) \\ &\quad - (\xi_Y^{\text{aH}} \omega_I)(IZ, X) - (\xi_{IY}^{\text{aH}} \omega_I)(Z, X) \end{aligned}$$

and recalling the following well-known identity [12]

$$\begin{aligned} N_I(X, Y, Z) &= (\nabla_Z^{\text{LC}} \omega_I)(IX, Y) + (\nabla_{IZ}^{\text{LC}} \omega_I)(X, Y) \\ &\quad + (\nabla_Y^{\text{LC}} \omega_I)(IZ, X) + (\nabla_{IY}^{\text{LC}} \omega_I)(Z, X). \end{aligned}$$

\mathcal{K}	$N_I = 0$ and $d\omega_I = 0$
$\mathcal{W}_1 = \mathcal{NK}$	$d\omega_I = -\frac{3}{4}IN_I$
$\mathcal{W}_2 = \mathcal{AK}$	$d\omega_I = 0$
\mathcal{W}_3	$N_I = 0$ and $d^*\omega_I = 0$
$\mathcal{W}_4 = \mathcal{LCK}$	$N_I = 0$ and $d\omega_I = -\frac{1}{n-1}Id^*\omega_I \wedge \omega_I$
\mathcal{W}_{1+2}	$d\omega_I = -\frac{1}{4}IN_I$
\mathcal{W}_{1+3}	$\mathcal{N}_I = 3N_I$ and $d^*\omega_I = 0$
\mathcal{W}_{1+4}	$d\omega_I = -\frac{3}{4}IN_I - \frac{1}{n-1}Id^*\omega_I \wedge \omega_I$
\mathcal{W}_{2+3}	$\mathcal{N}_I = 0$ and $d^*\omega_I = 0$
\mathcal{W}_{2+4}	$d\omega_I = -\frac{1}{n-1}Id^*\omega_I \wedge \omega_I$
\mathcal{W}_{3+4}	$N_I = 0$
\mathcal{W}_{1+2+3}	$d^*\omega_I = 0$
\mathcal{W}_{1+2+4}	$d\omega_I = -\frac{1}{4}IN_I - \frac{1}{n-1}Id^*\omega_I \wedge \omega_I$
\mathcal{W}_{1+3+4}	$\mathcal{N}_I = 3N_I$
\mathcal{W}_{2+3+4}	$\mathcal{N}_I = 0$
\mathcal{W}	no relation

Table 8.1. The Gray-Hervella classes of almost Hermitian structures characterised by the Nijenhuis tensor and the exterior derivative of the Kähler form.

9. Twisting

In this section we consider the effects of “twisting” an almost hyperHermitian in the sense of [28] where this construction was shown to be an interpretation of T-duality.

Let $(M, I, J, K, g = \langle \cdot, \cdot \rangle)$ be an almost hyperHermitian manifold. Suppose that X is a tri-holomorphic isometry, so

$$L_X g = 0, \quad L_X I = 0 = L_X J = L_X K.$$

Let F_θ be a closed 2-form with $L_X F_\theta = 0$ and choose a nowhere vanishing function $a \in C^\infty(M)$ so that $X^\theta = X \lrcorner F_\theta = -da$. If $P \rightarrow M$ is a principal \mathbb{R} -bundle with connection θ whose curvature is F_θ , then X may be lifted to a transverse vector field \tilde{X} on P that preserves θ : the vertical component is given by aY where Y generates the principal action. Locally the twist W of M by (X, F_θ, a) is the quotient $W = P/\langle \tilde{X} \rangle$ with the geometry induced from the horizontal space $\mathcal{H} = \ker \theta$. Each X -invariant $(0, p)$ -tensor κ on M

is \mathcal{H} -related to a unique $(0, p)$ -tensor κ^W on W defined by the condition that the pull-backs to P agree on \mathcal{H} . Exterior differentiation on W then corresponds to the twisted derivative $d^W = d - F_\theta \wedge \frac{1}{a} X \lrcorner$ on invariant forms on M .

If we twist an almost hyperHermitian structure the three-form $\beta_I = Jd\omega_J + Kd\omega_K$ transforms to

$$\beta_I^W = \beta_I - \frac{1}{a} X^\flat \wedge (J + K)F_\theta.$$

Thus to compute the change of the intrinsic torsion we should decompose β_I^W as in (5.1). We start by decomposing $F_\theta \in \Lambda^2 T^*M = S^2H + \Lambda_0^2 E S^2H + S^2E$:

$$F_\theta = (\mu_I \omega_I + \mu_J \omega_J + \mu_K \omega_K) + (I_{(1)} \kappa_I + J_{(1)} \kappa_J + K_{(1)} \kappa_K) + \alpha_\theta,$$

with each $\kappa_A \in \Lambda_0^2 E \subset S^2 T^*M$ and $\alpha_\theta \in S^2 E$. We have

$$\begin{aligned} \alpha_\theta &= \frac{1}{4}(1 + I + J + K)F_\theta, & \mu_A &= \frac{1}{n} \Lambda_A F_\theta, \\ A_{(1)} \kappa_A &= -\frac{1}{4} A(1 + I + J + K)A F_\theta - \mu_A \omega_A. \end{aligned}$$

We find that

$$\frac{1}{2}(J + K)F_\theta = -\mu_I \omega_I - I_{(1)} \kappa_I + \alpha_\theta \in \Lambda_I^{1,1}.$$

PROPOSITION 9.1. — *Suppose W is obtained from an almost hyperHermitian manifold M by a twist of the \mathbb{R} -action generated by a symmetry X and using a curvature form F_θ . Then W carries an almost hyperHermitian structure and the intrinsic almost quaternionic torsion is determined by the one-forms*

$$\begin{aligned} \nu_4^I &= \nu_4^I + \frac{2}{a} \{ \mu_I (2n - 1) I X^\flat - X \lrcorner \alpha_\theta - I X \lrcorner \kappa_I \}, \\ \nu_3^I &= \nu_3^I + \frac{4}{a(2n+1)(n-1)} (n I X \lrcorner \kappa_I - (n - 1) X \lrcorner \alpha_\theta). \end{aligned}$$

and three-form components

$$\begin{aligned} \beta_I^{(3)W} &= \beta_I^{(3)} + \frac{2}{3a} (2X^\flat \wedge I_{(1)} \kappa_I + JX^\flat \wedge K_{(1)} \kappa_I - KX^\flat \wedge J_{(1)} \kappa_I) \\ &\quad + \frac{2}{3a(n-1)} (2(X \lrcorner \kappa_I) \wedge \omega_I - (KX \lrcorner \kappa_I) \wedge \omega_J + (JX \lrcorner \kappa_I) \wedge \omega_K), \\ \beta_I^{(K)W} &= \beta_I^{(K)} - \frac{2}{a} X^\flat \wedge \alpha_\theta \\ &\quad + \frac{2}{3a} (X^\flat \wedge I_{(1)} \kappa_I - JX^\flat \wedge K_{(1)} \kappa_I + KX^\flat \wedge J_{(1)} \kappa_I) \\ &\quad - \frac{2}{3a(2n+1)} ((X \lrcorner \kappa_I) \wedge \omega_I + (KX \lrcorner \kappa_I) \wedge \omega_J + (JX \lrcorner \kappa_I) \wedge \omega_K) \\ &\quad - \frac{2}{a(2n+1)} ((IX \lrcorner \alpha_\theta) \wedge \omega_I + (JX \lrcorner \alpha_\theta) \wedge \omega_J + (KX \lrcorner \alpha_\theta) \wedge \omega_K). \end{aligned}$$

Proof. — One first computes the following contraction formulæ

$$\begin{aligned}
 I\Lambda_I(X^b \wedge \alpha_\theta) &= X \lrcorner \alpha_\theta, & I\Lambda_I(\gamma \wedge \omega_I) &= (2n - 1)I\gamma, & I\Lambda_I(\gamma \wedge \omega_J) &= J\gamma, \\
 I\Lambda_I(X^b \wedge I\kappa_A) &= -IX \lrcorner \kappa_A, & I\Lambda_I(X^b \wedge J\kappa_A) &= JX \lrcorner \kappa_A.
 \end{aligned}$$

These lead directly to the claimed expressions for ν_4^{IW} and ν_3^{IW} and then give

$$\begin{aligned}
 \beta_{4I}^W &= \beta_{4I} + \frac{2}{a(2n-1)}(\mu_I(2n-1)X^b + IX \lrcorner \alpha_\theta - X \lrcorner \kappa_I) \wedge \omega_I, \\
 \beta_{3I}^{(E)W} &= \beta_{3I}^{(E)} - \frac{4}{a(2n+1)(n-1)(2n-1)}(-nX \lrcorner \kappa_I - (n-1)IX \lrcorner \alpha_\theta) \wedge \omega_I \\
 &\quad - \frac{2}{a(2n+1)(n-1)}(-nKX \lrcorner \kappa_I - (n-1)JX \lrcorner \alpha_\theta) \wedge \omega_J \\
 &\quad - \frac{2}{a(2n+1)(n-1)}(-nJX \lrcorner \kappa_I - (n-1)KX \lrcorner \alpha_\theta) \wedge \omega_K.
 \end{aligned}$$

The remaining components of β_I^W are then found using (5.7) with

$$\begin{aligned}
 \mathcal{L}_I(\gamma \wedge \alpha_\theta) &= \gamma \wedge \alpha_\theta, & \mathcal{L}_I(\gamma \wedge \omega_J) &= 2I\gamma \wedge \omega_K - \gamma \wedge \omega_J, \\
 \mathcal{L}_I(\gamma \wedge \omega_I) &= \gamma \wedge \omega_I, & \mathcal{L}_I(\gamma \wedge J_{(1)}\kappa_A) &= 2I\gamma \wedge K_{(1)}\kappa_A - \gamma \wedge J_{(1)}\kappa_A. \quad \square
 \end{aligned}$$

COROLLARY 9.2. — *Twisting by $F_\theta \in S^2E + S^2H$ leaves ξ_{33} , ξ_{3H} and ξ_{K3} unchanged. Furthermore,*

- (i) *if $3X^\theta = -2(n+2) \sum_A \mu_A AX^b$, then ξ_{EH} is not affected;*
- (ii) *if $F_\theta \in S^2E$, then ξ_{E3} is unaltered.*

Proof. — The assumption on F_θ is equivalent to the vanishing of κ_I , κ_J and κ_K . We thus have $\beta_I^{(3)W} = \beta_I^{(3)}$ and that $\beta_I^{(K)W} - \beta_I^{(K)}$ is independent of I , from which the invariance of the components in $(\Lambda_0^3E+K)S^3E + \Lambda_0^3EH$ follows.

The change in $I\eta_I$ is $\frac{(n-1)}{2na}((2n+1)(\mu_I IX^b - \mu_J JX^b - \mu_K KX^b) - X \lrcorner \alpha_\theta)$. If $3X \lrcorner \alpha_\theta + (2n+1) \sum_A \mu_A AX^b = 0$, then there is no contribution to $\sum_A A\eta_A$ and hence no change in ξ_{EH} . On the other hand, if $F_\theta \in S^2E$ then each $\mu_A = 0$, and there is no contribution to ES^3H . □

Remark 9.3. — Case (ii) shows in particular that the QKT condition is preserved by twisting with $F_\theta \in S^2E$.

10. Examples

In this section we use the techniques developed in the previous sections to compute the intrinsic torsion in a number of particular examples. In the first instance we consider examples which are almost hyperHermitian with each Hermitian structure of the same Gray-Hervella type. From [21] we know

that certain combinations can not occur. Table 10.1 gives an overview of which types may be obtained. The $4n$ -torus $T^{4n} = \mathbb{H}^n / \mathbb{Z}^{4n}$ is hyperKähler and so has intrinsic torsion 0. The Hopf-like manifold $S^{4n-1} \times S^1 = \mathbb{H}^n \setminus \{0\} / (q \mapsto 2q)$, is locally, but not globally conformal, to the flat hyperKähler metric, so each almost Hermitian structure is of class \mathcal{W}_4 and the almost quaternionic type is EH . The other examples are described below.

I, J, K	$\{0\}$	\mathcal{W}_1	\mathcal{W}_2	\mathcal{W}_{1+2}
$\{0\}$	T^{4n}	impossible	impossible	impossible
\mathcal{W}_3	H_Q	$S^3 \times T^9$	$T^3 \times M(k)^3$	$T^3 \times (\Gamma \setminus H)^3$
\mathcal{W}_4	$S^{4n-1} \times S^1$	impossible	impossible	unknown
\mathcal{W}_{3+4}	$S^3 \times T^{4m+1}$	$S^3 \times T^9$	$T^3 \times (\Gamma \setminus H)^3, T^3 \times M(k)^3$	$T^3 \times (\Gamma \setminus H)^3$

Table 10.1. Examples with common almost Hermitian structures

10.1. The manifold $S^3 \times T^9$

The sphere S^3 is isomorphic to the Lie group $Sp(1)$. In the Lie algebra $\mathfrak{sp}(1)$ there is a basis x, y, z such that $[x, y] = 2z, [z, x] = 2y$ and $[y, z] = 2x$. From x, y, z one can determine the left invariant one-forms a, b, c which constitute a basis for one-forms and their exterior derivatives are given by $da = -2b \wedge c, db = -2c \wedge a$ and $dc = -2a \wedge b$.

In the product manifold $M = S^3 \times T^9$, we write a_1, b_1, c_1 for the one-forms corresponding to the factor S^3 and $a_i, b_i, c_i, i = 2, 3, 4$, for a basis of invariant one forms on $T^9 = (S^1)^9$. On M we consider an almost hyperHermitian structure I, J, K with compatible metric $\langle \cdot, \cdot \rangle = \sum_{i=1}^4 (a_i^2 + b_i^2 + c_i^2)$ and whose Kähler forms are given by

$$(10.1) \quad \omega_I = a_2a_1 + a_4a_3 + b_2b_1 + b_4b_3 + c_2c_1 + c_4c_3, \quad \text{etc.}(234),$$

where $a_2a_1 = a_2 \wedge a_1$ and “etc.(234)” denotes the corresponding equations obtained by simultaneously cyclically permuting (I, J, K) and $(2, 3, 4)$. Their respective exterior derivatives and the three-forms β_A are given by

$$d\omega_I = 2 \mathfrak{S}_{abc} a_2b_1c_1, \quad \beta_I = -2 \mathfrak{S}_{abc} (a_1b_3c_3 + a_1b_4c_4), \quad \text{etc.}(234).$$

Since $\Lambda_B d\omega_A = 0, A, B = I, J, K$, we have $\lambda_A = 0$ and $\beta_A^{(E)} = 0$, for $A = I, J, K$. Then, using equation (5.7), we obtain

$$\beta_I^{(K)} = -\frac{2}{3} \mathfrak{S}_{abc} (a_1b_1c_1 + a_1b_2c_2 + a_1b_3c_3 + a_1b_4c_4), \quad \text{etc.}$$

$$\beta_I^{(3)} = -\frac{2}{3} \mathfrak{S}_{abc} (-a_1b_1c_1 - a_1b_2c_2 + 2a_1b_3c_3 + 2a_1b_4c_4), \quad \text{etc.}(234).$$

Using Proposition 5.3 we get $\psi_A^{(3)} = 0, \psi_A^{(K)} = 0, \theta^\xi = 0 = \theta_A^\xi$, for $A = I, J, K, \psi^{(3)} \neq 0$ and $\psi^{(K)} \neq 0$. Therefore, we have

$$\xi \in \Lambda_0^3 ES^3H + KH.$$

Note also that $\xi_{33} \neq 0$ and $\xi_{KH} \neq 0$.

Furthermore, if we consider the connection $\tilde{\nabla} = \nabla^{LC} + \frac{1}{2}T$, where T is given by

$$\langle Y, T_X Z \rangle = \frac{1}{6} \sum_{A=I, J, K} (\beta_A^{(K)} - \beta_A^{(3)})(X, Y, Z),$$

we will obtain that this connection is metric and $\tilde{\nabla}I = \tilde{\nabla}J = \tilde{\nabla}K = 0$. Thus, we have got an example of a quaternion-Hermitian manifold, which is not QKT, admitting an $Sp(n)Sp(1)$ -connection with skew-symmetric $(0, 3)$ -torsion.

We compute the Nijenhuis $(0, 3)$ -tensors for I, J and K via Proposition 4.3:

$$N_I = 2a_1b_1c_1 - 2 \underset{abc}{\mathfrak{S}} a_1b_2c_2, \quad \text{etc. (234)}.$$

Since N_I, N_J, N_K are skew-symmetric, then the almost Hermitian structures are of a type that lies in \mathcal{W}_{1+3+4} . However, $Ad^*\omega_A = -\Lambda_A d\omega_A = 0$ implies that such structures are of type \mathcal{W}_{1+3} . The facts $4d\omega_A \neq 3AN_A$ and $N_A \neq 0$ respectively imply that the structures are not of the types \mathcal{W}_1 and \mathcal{W}_3 .

Finally, if we make a conformal change of metric $\langle \cdot, \cdot \rangle^o = e^{2\sigma} \langle \cdot, \cdot \rangle$ as in Proposition 5.4, we obtain a new quaternionic structure with

$$\xi^o \in \Lambda_0^3 ES^3H + KH + EH.$$

The new almost Hermitian structures are of type \mathcal{W}_{1+3+4} .

10.2. The manifold $S^3 \times T^{4m+1}$

In the product manifold $M = S^3 \times T^{4m+1}, m \geq 1$, we write $n = m+1, a_2, a_3, a_4$ to denote the one-forms corresponding to the factor S^3 and $a_1, a_i, i = 5, \dots, 4n$, for a basis of invariant one-forms on T^{4m+1} . On M we consider an almost hyperHermitian structure I, J, K with compatible metric $\langle \cdot, \cdot \rangle = \sum_{i=1}^{4n} a_i \otimes a_i$ and whose Kähler forms are given by the expressions

$$(10.2) \quad \omega_I = \sum_{i=0}^{n-1} (a_{4i+2}a_{4i+1} + a_{4i+4}a_{4i+3}), \quad \text{etc. (234)}.$$

Their respective exterior derivatives and the three-forms β_A are given by

$$d\omega_I = -2a_1a_3a_4, \quad \text{etc.}(234), \quad \beta_I = -4a_2a_3a_4, \quad \text{etc.}$$

Hence, by Proposition 6.3, M is an HKT-manifold. Although this example is already known, we wish to give a few more details. We have

$$\begin{aligned} Id^*\omega_I &= -2a_1, & I\lambda_I &= -\frac{1}{n} a_1, & \nu_3^I &= -\frac{8}{2n+1} a_1, \\ \nu_4^I &= -4a_1, & \theta^\xi &= \theta_I^\xi = -\frac{1}{4(2n+1)} a_1, & \text{etc.}, \end{aligned}$$

so

$$\xi \in (K + E)H$$

with $\xi_{KH} \neq 0$ and $\xi_{EH} \neq 0$. Also, the three almost Hermitian structures are of type $\mathcal{W}_{3+4} \setminus (\mathcal{W}_3 \cup \mathcal{W}_4)$.

Making a local conformal change of metric $\langle \cdot, \cdot \rangle^o = e^{2\sigma} \langle \cdot, \cdot \rangle$, with σ satisfying $d\sigma = 4\theta^\xi = -\frac{1}{(2n+1)} a_1$, we will obtain a new almost hyperHermitian structure $\{I, J, K; \langle \cdot, \cdot \rangle^o\}$ with

$$\xi^o \in KH.$$

However, by Proposition 5.4, we will have

$$\begin{aligned} Id^*\omega_I^o &= -\frac{4}{(2n+1)} a_1, & I\lambda_I^o &= -\frac{2}{2n+1} a_1, & \nu_3^{I^o} &= -\frac{4}{2n+1} a_1, \\ \nu_4^{I^o} &= -\frac{8n}{2n+1} a_1, & \theta^{\xi^o} &= \theta_I^{\xi^o} = 0, & \text{etc.}, \end{aligned}$$

so the identities given in Lemma 6.5 are not satisfied and structure is not HKT. It has $A\lambda_A^o = \frac{1}{2n} t^o \neq 0$, but the three almost Hermitian structures are still of type $\mathcal{W}_{3+4} \setminus (\mathcal{W}_3 \cup \mathcal{W}_4)$.

Finally, for a local conformal change of metric by σ satisfying $d\sigma = -\frac{1}{2n-1} a_1$, we obtain an almost hyperHermitian structure with three almost Hermitian structures of type \mathcal{W}_3 . However, such a structure is not HKT. The almost quaternion-Hermitian structure has

$$\xi^o \in (K + E)H$$

but not in any submodule. In fact, we will have

$$\begin{aligned} Id^*\omega_I^o &= 0, & I\lambda_I^o &= -\frac{2(n-1)}{2n-1} a_1, & \nu_3^{I^o} &= -\frac{4(2n-3)}{4n^2-1} a_1, \\ \nu_4^{I^o} &= -\frac{8(n-1)}{2n-1} a_1, & \theta^{\xi^o} &= \theta_I^{\xi^o} = \frac{1}{2(4n^2-1)} a_1, & \text{etc.} \end{aligned}$$

10.3. The quaternionic Heisenberg group

Now we consider the quaternionic Heisenberg group H_Q described by Cordero et al. [6]:

$$H_Q = \left\{ \begin{pmatrix} 1 & q_1 & q_3 \\ 0 & 1 & q_2 \\ 0 & 0 & 1 \end{pmatrix} : q_1, q_2, q_3 \in \mathbb{H} \right\},$$

which is a connected nilpotent Lie group. A basis for the left-invariant one-forms on H_Q is given by $a_i, b_i, c_i, i = 1, \dots, 4$, where

$$\begin{aligned} dq_1 &= a_1 + ia_2 + ja_3 + ka_4, & dq_2 &= b_1 + ib_2 + jb_3 + kb_4, \\ dq_3 - q_1dq_2 &= c_1 + ic_2 + jc_3 + kc_4. \end{aligned}$$

With Γ_Q be the subgroup of matrices of H_Q with $q_i \in \mathbb{Z}\{1, i, j, k\}$, we see that these forms descend to the compact manifold $M_Q = \Gamma_Q \backslash H_Q$. Consider the almost hyperHermitian structure on M_Q with metric $\langle \cdot, \cdot \rangle = \sum_{i=1}^4 (a_i^2 + b_i^2 + c_i^2)$, and Kähler forms given by (10.1).

PROPOSITION 10.1 (Cordero et al. [6]). — *The three almost Hermitian structures I, J, K defined on M_Q are of type \mathcal{W}_3 .*

Structures of type \mathcal{W}_3 are sometimes called *balanced* Hermitian.

The three-forms β_A are given by

$$\begin{aligned} \frac{1}{2}\beta_I &= a_1b_1c_1 + a_1b_2c_2 + a_1b_3c_3 + a_1b_4c_4 \\ &\quad - a_2b_1c_2 + a_2b_2c_1 - a_2b_3c_4 + a_2b_4c_3, \quad \text{etc.} \end{aligned} \tag{234}$$

Using equation (5.5), we obtain that $\nu_3^A = \nu_4^A = 0$, so $\xi_{EH} = \xi_{E3} = 0$ by Proposition 5.3.

On the other hand, using equation (5.7), we obtain $\beta_I^{(K)} = \beta_J^{(K)} = \beta_K^{(K)} \neq 0$ and $\beta_I^{(3)} + \beta_J^{(3)} + \beta_K^{(3)} = 0$, with each $\beta_A^{(3)} \neq 0$. Therefore, Proposition 5.3 gives $\xi_{33} = \xi_{K3} = 0, \xi_{3H} \neq 0$ and $\xi_{KH} \neq 0$. In summary,

$$\xi \in \Lambda_0^3 EH + KH.$$

10.4. The manifold $T^3 \times (\Gamma \backslash H)^3$

Let H be the real Heisenberg group of dimension 3:

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

A basis of left-invariant one-forms is given by $\{dx, dy, dz - xdy\}$. Let Γ be the discrete subgroup of matrices of H whose entries x, y, z are integers. The quotient space $\Gamma \backslash H$ is called the Heisenberg compact nilmanifold. The one-forms $\{dx, dy, dz - xdy\}$ descend to one-forms p, q, r on $\Gamma \backslash H$ with $dr = -p \wedge q$.

10.4.1. A first structure

Let M be the manifold $T^3 \times (\Gamma \backslash H)^3$. On M we consider a basis of invariant one-forms a_1, b_1, c_1 for T^3 and one-forms $a_2, a_3, a_4, b_2, b_3, b_4, c_2, c_3$ and c_4 corresponding to the factors $\Gamma \backslash H$ such that $da_2 = -a_3a_4, db_3 = -b_4b_2$ and $dc_4 = -c_2c_3$.

On M we consider an almost hyperHermitian structure I, J, K with compatible metric $\langle \cdot, \cdot \rangle = \sum_{i=1}^4 (a_i^2 + b_i^2 + c_i^2)$ and whose Kähler forms are given by the expressions (10.1).

Their respective exterior derivatives and three-forms β_A are given by

$$d\omega_I = -a_1a_3a_4, \quad \beta_I = -b_2b_3b_4 - c_2c_3c_4, \quad \text{etc.}(abc; 234).$$

Using equations (5.3), (5.5) and (4.11), we obtain

$$\nu_3^I = -\frac{2}{7}(b_1 + c_1), \quad \nu_4^I = -b_1 - c_1, \quad I\lambda_I = \frac{1}{6}(a_1 - b_1 - c_1), \quad \text{etc.}(abc).$$

Therefore, by Proposition 5.3, we get

$$56\theta_I^\xi = -3a_1 + b_1 + c_1, \quad -168\theta^\xi = a_1 + b_1 + c_1, \quad \text{etc.}(abc).$$

Thus, the ξ_{E3} and ξ_{EH} parts of the intrinsic torsion are not zero.

On the other hand, by equations (5.2) and (5.3), we have

$$7\beta_I^{(E)} = \sum_{A=I, J, K} A(b_1 + c_1)\omega_A, \quad \text{etc.}(abc).$$

Hence, by equation (5.7) on $\beta_A^{(3+K)} = \beta_A - \beta_A^{(E)}$, we have

$$\beta_I^{(3)} = 0, \quad \beta_I^{(K)} = -b_2b_3b_4 - c_2c_3c_4 - \frac{1}{7} \sum_{A=I, J, K} A(b_1 + c_1)\omega_A, \quad \text{etc.}(abc).$$

Proposition 5.3 then gives $\xi_{33} = \xi_{3H} = 0, \xi_{K3} \neq 0$ and $\xi_{KH} \neq 0$ and we conclude

$$\xi \in KS^3H + ES^3H + KH + EH.$$

Let us now analyse the almost Hermitian structures. We compute the Nijenhuis tensors using Proposition 4.3:

$$\begin{aligned} N_I &= b_3 \otimes b_1b_3 - b_3 \otimes b_2b_4 - b_4 \otimes b_1b_4 - b_4 \otimes b_2b_3 \\ &\quad - c_3 \otimes c_1c_3 + c_3 \otimes c_2c_4 + c_4 \otimes c_1c_4 + c_4 \otimes c_2c_3, \quad \text{etc.}(abc; 234). \end{aligned}$$

We see that the alternation of the Nijenhuis tensors N_A are zero. Therefore, the almost Hermitian structures are of type \mathcal{W}_{2+3+4} .

Moreover, $N_A \neq 0$ so the structure is not of type \mathcal{W}_{3+4} . Since $Id^*\omega_I = -a_1, \text{ etc.}(abc)$, the structures are not of type \mathcal{W}_{2+3} . It can be also checked that $5d\omega_A \neq -Ad^*\omega_A\omega_A$, so the structures also not of type \mathcal{W}_{2+4} .

Finally, making a conformal change of metric $\langle \cdot, \cdot \rangle = e^{2\sigma} \langle \cdot, \cdot \rangle^o$ with σ is a local function satisfying $d\sigma = 4\theta^\xi = -\frac{1}{42}(a_1 + b_1 + c_1)$, we obtain locally a new almost quaternion-Hermitian structure $\{I, J, K; \langle \cdot, \cdot \rangle^o\}$ with

$$\xi^o \in KS^3H + ES^3H + KH.$$

The three new almost Hermitian structures are still in \mathcal{W}_{2+3+4} .

10.4.2. A second structure

This time, on $M = T^3 \times (\Gamma \backslash H)^3$, consider a_1, b_1, c_1 a basis of invariant one-forms on T^3 , as before, and let $a_2, a_3, a_4, b_2, b_3, b_4, c_2, c_3, c_4$ denote linearly independent one-forms on the factors $\Gamma \backslash H$ now with $da_2 = -b_2c_2, db_3 = -c_3a_3$ and $dc_4 = -a_4b_4$.

Take the almost hyperHermitian structure I, J, K with compatible the metric $\langle \cdot, \cdot \rangle = \sum_{i=1}^4 (a_i^2 + b_i^2 + c_i^2)$ and with Kähler forms are respectively given by (10.1).

Their respective exterior derivatives and three-forms β_A are given by

$$d\omega_I = -a_1b_2c_2 + a_3b_4c_3 - a_4b_4c_3, \quad \text{etc.}(abc; 234),$$

$$\beta_I = -a_1b_1c_4 - a_1b_3c_1 - a_2b_2c_4 - a_2b_3c_2 - a_2b_3c_3 - a_2b_4c_4, \quad \text{etc.}(abc; 234).$$

Note that $\Lambda_B d\omega_A = 0$ and $\Lambda_B \beta_A = 0$, for $A, B = I, J, K$, so $A\lambda_A = 0$ and $\beta_A^{(E)} = 0$. This implies that $\xi_{E3} = \xi_{EH} = 0$. Furthermore, using equation (5.7) one computes the components $\beta_A^{(3)}$ and $\beta_A^{(K)}$ and obtains that $\sum_{A=I,J,K} \beta_A^{(K)} \neq 0$ and $3\beta_A \neq \sum_{B=I,J,K} \beta_B^{(K)}$. By Proposition 5.3, we find that $\xi_{K3} \neq 0$ and $\xi_{KH} \neq 0$. Also we compute $\sum_{A=I,J,K} \beta_A^{(3)} \neq 0$ and $6\beta_A^{(3)} \neq 3 \sum_{B=I,J,K} \beta_B^{(3)} + \mathcal{L}_A \left(\sum_{B=I,J,K} \beta_B^{(3)} \right)$, so $\xi_{33} \neq 0$ and $\xi_{3H} \neq 0$. In summary, we get

$$\xi \in \Lambda_0^3 ES^3H + KS^3H + KH + EH.$$

Now we turn to analysis of the almost Hermitian structures. Since $Ad^*\omega_A = -\Lambda_A d\omega_A = 0$, the almost Hermitian structures are of type \mathcal{W}_{1+2+3} . Moreover, using the expressions for N_A in Proposition 4.3, we have

$$\begin{aligned} N_I &= -a_1 \otimes b_1c_2 - a_1 \otimes b_2c_1 + a_2 \otimes b_2c_2 - a_2 \otimes b_1c_1 \\ &\quad + b_3 \otimes c_3a_3 - b_3 \otimes c_4a_4 - b_4 \otimes c_3a_4 - b_4 \otimes c_4a_3 \\ &\quad - c_3 \otimes a_3b_3 - c_3 \otimes a_4b_3 - c_4 \otimes a_3b_3 + c_4 \otimes a_4b_4, \quad \text{etc.}(abc; 234). \end{aligned}$$

We find that N_A is not skew-symmetric, the alternation \mathcal{N}_A of N_A is non-zero and that $4d\omega_A \neq -AN_A$. Therefore, the almost Hermitian structures are not of type \mathcal{W}_{i+j} , for $i, j = 1, 2, 3$.

Making a global conformal change we obtain almost Hermitian structures of the general type $\mathcal{W}_{1+2+3+4}$ whilst preserving the almost quaternion Hermitian type.

10.5. The manifold $T^3 \times M(k)^3$

Let us consider the manifolds $M(k)$ described in [1] as S_1/D and studied geometrically in [5, 7, 9]. For a fixed $k \in \mathbb{R} \setminus \{0\}$, let $G(k)$ be the three-dimensional connected and solvable (non-nilpotent) Lie group consisting of matrices:

$$G(k) = \left\{ \begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

A basis of right invariant one-forms on $G(k)$ is $\{dx - kx dz, dy + ky dz, dz\}$.

The Lie group $G(k)$ possesses a discrete subgroup $\Gamma(k)$ such that the manifold $M(k) = G(k)/\Gamma(k)$ is compact. One example of $\Gamma(k)$ is generated by choosing $r > 0$ so that $e^{kr} + e^{-kr} \in \mathbb{Z}$ and then taking the subgroup of $G(k)$ generated by $(x, y \in \mathbb{Z}, z = 0)$ and $(x = 0 = y, z = r)$. The given basis of one-forms on $G(k)$ descends to one-forms a, b, c on $M(k)$ satisfying $da = -kac$ and $db = kbc$.

10.5.1. A first structure

Let M be the manifold $M = T^3 \times M(k)^3$. Consider a basis of invariant one-forms a_1, b_1, c_1 on T^3 and one-forms $a_2, a_3, a_4, b_2, b_3, b_4, c_2, c_3, c_4$ corresponding on the factors $M(k)$ with

$$da_3 = -ka_3a_2, \quad da_4 = ka_4a_2, \quad \text{etc.}(abc; 234).$$

Now we consider on M an almost hyperHermitian structure I, J, K with compatible metric $\langle \cdot, \cdot \rangle = \sum_{i=1}^4 (a_i^2 + b_i^2 + c_i^2)$ and Kähler forms given by (10.1).

The respective exterior derivatives and three-forms β_A are given by

$$d\omega_I = kb_1b_2b_3 - kc_1c_2c_4, \quad \beta_I = -kb_1b_2b_4 - kc_1c_2c_3, \quad \text{etc.}(abc; 234).$$

Note that $\sum_A \beta_A = 0$. Therefore, $\sum_A \beta_A^{(3)} = 0$ and $\sum_A \beta_A^{(K)} = 0$. This implies that $\xi_{33} = 0$ and $\xi_{K3} = 0$. To compute the E -parts of β_A and ξ , we first find

$$\begin{aligned} Id^* \omega_I &= kb_3 - kc_4, & I\Lambda_K d\omega_J &= -ka_2 + kc_4, \\ I\Lambda_J d\omega_K &= -ka_2 + kb_3, & \text{etc.}(abc; 234). \end{aligned}$$

Using equation (4.11), we now obtain $I\lambda_I = \frac{k}{3}(-b_3 + c_4)$, etc.(abc;234). On the other hand, using $\nu_4^A = A\Lambda_A\beta_A$ and equation (5.5), we get

$$\nu_4^I = k(-b_3 + c_4), \quad \nu_3^I = \frac{2k}{7}(-b_3 + c_4), \quad \text{etc.}(abc; 234).$$

Finally, from the obtained expressions for $A\lambda_A$, ν_3^A and ν_4^I , using Proposition 5.3, we find

$$\theta^\xi = 0, \quad \theta_I^\xi = \frac{5k}{28}(-b_3 + c_4), \quad \text{etc.}(abc; 234).$$

Hence $\xi_{EH} = 0$ and $\xi_{E3} \neq 0$. Furthermore, since

$$\beta_I^{(E)} = \frac{k}{7} \sum_{A=I,J,K} A(b_3 - c_4) \wedge \omega_A, \quad \text{etc.}(abc; 234),$$

we have $\mathcal{L}_A(\beta_B - \beta_B^{(E)}) = \beta_B - \beta_B^{(E)}$, for $A, B = I, J, K$. Therefore, $\beta_A^{(K)} = \beta_A - \beta_A^{(E)} \neq 0$ and $\beta_I^{(3)} = \beta_J^{(3)} = \beta_K^{(3)} = 0$. These last claims imply $\xi_{K3} \neq 0$ and $\xi_{3H} = 0$. In conclusion,

$$\xi \in KS^3H + ES^3H.$$

Now we analyse the almost Hermitian structures. Using the expression for N_A in Proposition 4.3, one can obtain Nijenhuis $(0, 3)$ -tensor for I, J and K . These tensors N_A are not zero and but their alternations \mathcal{N}_A are zero. Therefore, the almost Hermitian structures are of type \mathcal{W}_{2+3+4} .

Moreover, $N_A \neq 0$, this implies that the structures are not of type \mathcal{W}_{3+4} . Additionally, we have seen above that the Lee one-forms $Ad^*\omega_A$ are non-zero, so the structures are not of type \mathcal{W}_{2+3} . Finally, one can easily check that $d\omega_A \neq -\frac{1}{5}Ad^*\omega_A\omega_A$, so the almost Hermitian structures are not of type \mathcal{W}_{2+4} .

10.5.2. A second structure

We again consider $M = T^3 \times M(k)^3$. We take a_1, b_1, c_1 to be a basis of invariant one-forms on T^3 and now $a_2, a_3, a_4, b_2, b_3, b_4, c_2, c_3, c_4$ is basis of one-forms on $M(k)^3$ with

$$db_2 = -kb_2a_2, \quad dc_2 = kc_2a_2, \quad \text{etc.}(abc; 234).$$

On M we consider an almost hyperHermitian structure I, J, K with compatible metric $\langle \cdot, \cdot \rangle = \sum_{i=1}^4 (a_i^2 + b_i^2 + c_i^2)$ and Kähler forms given by (10.1). The respective exterior derivatives and the three-forms β_A are given by

$$\begin{aligned} \frac{1}{k} d\omega_I &= a_3 a_4 b_3 - a_3 a_4 c_4 - b_1 b_2 a_2 + b_3 b_4 c_4 + c_1 c_2 a_2 - c_3 c_4 b_3, \\ \frac{1}{k} \beta_I &= -a_1 a_3 b_1 + a_1 a_4 c_1 - a_2 a_3 b_2 + a_2 a_4 c_2 - b_1 b_4 c_1 - b_2 b_3 a_3 \\ &\quad - b_2 b_4 a_4 - b_2 b_4 c_2 + c_1 c_3 b_1 + c_2 c_3 a_3 + c_2 c_3 b_2 + c_2 c_4 a_4, \end{aligned}$$

etc.(abc;234). Since $\Lambda_B d\omega_A = 0$, for $A, B = I, J, K$, we have $A\lambda_A = \nu_3^A = \nu_4^A = \theta^\xi = \theta_A^\xi = 0$, so $\xi_{E3} = 0$ and $\xi_{EH} = 0$. Now, using (5.7), one can compute the components $\beta_A^{(3)}$ and $\beta_I^{(K)}$. One checks that $\sum_{A=I,J,K} \beta_A^{(3)} = 0$, $\beta_A^{(3)} \neq 0$, the $\beta_A^{(K)}$, $A = I, J, K$, are distinct and $\sum_{A=I,J,K} \beta_A^{(K)} \neq 0$. Therefore, $\xi_{33} = 0$, $\xi_{EH} \neq 0$, $\xi_{K3} \neq 0$ and $\xi_{KH} \neq 0$. In summary,

$$\xi \in KS^3H + \Lambda_0^3EH + KH.$$

For the almost Hermitian structures, one computes N_A via Proposition 4.3 and find that they are non-zero but their alternations \mathcal{N}_A vanish, so the structures lie in \mathcal{W}_{2+3+4} . Since $\Lambda_B d\omega_A = 0$, for $A, B = I, J, K$, we have that the respective \mathcal{W}_4 -parts are zero. Thus, the almost Hermitian structures are of type \mathcal{W}_{2+3} . Because $N_A \neq 0$ and $d\omega_A \neq 0$, the structures are not of the simple types \mathcal{W}_2 or \mathcal{W}_3 .

10.6. Twisting tori

Up to this point we have obtained all the claimed examples of Table 10.1 and determined the corresponding types of the almost quaternion-Hermitian structures. Let us now use the twist construction of §9 to give some other examples of types of ξ .

Let us first demonstrate that the condition in Corollary 9.2(i) can be satisfied. Let $M = T^{4n}$, $n \geq 2$, with the standard flat hyperKähler structure. Take $F_\theta = \omega_I + \alpha_\theta$ and X an arbitrary isometry. Then $\mu_I = 1$, $\mu_J = 0 = \mu_K$ and $X^\theta = -IX^b + X \lrcorner \alpha_\theta$. The condition of Corollary 9.2(i) is now equivalent to $X \lrcorner \alpha_\theta = -\frac{(2n+1)}{3} IX^b$. This is satisfied if we take $\alpha_\theta = -\frac{(2n+1)}{3\|X\|^2} (X^b \wedge IX^b - JX^b \wedge KX^b)$. Locally, we may now twist to obtain a structure with

$$\xi^W \in KH + ES^3H.$$

Since $X^b \wedge \alpha_\theta \neq 0$, α is supported on $\mathbb{H}X$ and $n \geq 2$, we have $\xi_{KH}^W \neq 0$. On the other hand the values of the μ_A ensure that $\xi_{E3}^W \neq 0$.

As second example, consider T^{4n} , $n \geq 3$ with invariant basis a_1, \dots, a_{4n} and two-forms (10.2). Take $F_\theta = a_2a_1 + a_4a_3 - a_6a_5 - a_8a_7$. We have $F_\theta \in \Lambda_I^{1,1}$ orthogonal to ω_I and of type $\{2, 0\}$ for J and K , so $F_\theta = I_{(1)}\kappa_I$. Twisting via any X such that $X \lrcorner F_\theta = 0$ and each $AX \lrcorner F_\theta = 0$ yields an almost quaternion-Hermitian structure with

$$\xi^W \in (\Lambda_0^3 E + K)(S^3 H + H)$$

but not in any proper submodule. Making a conformal change we may also obtain structures with intrinsic torsion in $(\Lambda_0^3 E + K)S^3 H + (\Lambda_0^3 E + K + E)H$.

10.7. Twisting Salamon’s example

Recall that Salamon [26] gave an example of a non-quaternionic Kähler 8-manifold that has $d\Omega = 0$. The manifold is a compact nilmanifold $\Gamma \backslash G$, where G has a basis of left-invariant one-forms $a_1, \dots, a_4, b_1, \dots, b_4$ satisfying

$$\begin{aligned} da_2 &= -\sqrt{3}a_1a_4 - 3a_4b_1, & db_2 &= a_1a_4 + \sqrt{3}a_4b_1, \\ db_4 &= -a_1a_2 - \sqrt{3}a_2b_1 - \sqrt{3}a_1b_2 + 3b_1b_2, \end{aligned}$$

with the other basis elements closed. The Lie algebra \mathfrak{g} is a direct sum $\mathbb{R}^3 + \mathfrak{h}$, with \mathfrak{h} two-step nilpotent. Salamon’s structure is then given by (10.1) (with $c_i = 0$). In terms of intrinsic torsion this has

$$\xi \in KS^3H.$$

The two-form $F_\theta = a_1a_3 + a_2a_4$ is a closed element of S^2E and defines an integral cohomology class on M for an appropriate choice of Γ . We may thus twist using, for example, the central vector field X dual to b_3 to obtain an almost quaternion Hermitian 8-manifold W with

$$\xi^W \in K(S^3H + H).$$

Conformally scaling these two examples we may obtain structures with

$$\xi^o \in KS^3H + EH \quad \text{and} \quad \xi^{W^o} \in KS^3H + (K + E)H.$$

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Francisco MARTÍN CABRERA
University of La Laguna
Department of Fundamental Mathematics
38200 La Laguna
Tenerife (Spain)
fmartin@ull.es

Andrew SWANN
University of Southern Denmark
Department of Mathematics and Computer Science
Campusvej 55
5230 Odense M (Denmark)
swann@imada.sdu.dk