

ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 16, n° 2 (1966), p. 159-166

[<http://www.numdam.org/item?id=AIF_1966__16_2_159_0>](http://www.numdam.org/item?id=AIF_1966__16_2_159_0)

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ON RESTRICTED MEASURABILITY

by A. K. MOOKHOPADHYAYA

1. Introduction and Definitions.

The purpose of the present paper is to study some properties of the restricted measurability [5] and to show that a Radon measure similar to that of [4] can be constructed with the help of the notion of the restricted measurability. Before we go into details, we write out, for the sake of completeness, a few definitions and notations some of which are borrowed from the above papers and the standard texts such as Halmos [1] and Kelley [2].

1.1. DEFINITION. — μ is a measure (Carathéodory) on X if μ is a function on the family of all subsets of X to $0 \leq t \leq \infty$ such that

- (i) $\mu(0) = 0$
- (ii) $0 \leq \mu A \leq \sum_{n=1}^{\infty} \mu B_n$, whenever $A \subset \bigcup_{n=1}^{\infty} B_n \subset X$.

1.2. DEFINITION. — $A \subset X$ is μ -measurable if for every $T \subset X$

$$\mu T = \mu(T \cap A) + \mu(T \sim A)$$

where μ is a measure on X .

1.3. DEFINITION. — A partition is a finite or infinite disjoint sequence $\{E_i\}$ of sets such that $\bigcup_i E_i = X$.

1.4. DEFINITION. — A partition $\{E_i\}$ is called a μ -partition if

$$\mu A = \sum_{i=1}^{\infty} \mu(A \cap E_i)$$

for every A in X and where μ is a measure on X .

1.5. DEFINITION. — If μ is a measure on X , then a partition $\{E_i\}$ is called a μ -partition F if for every E of F

$$\mu(TE) = \sum_{i=1}^{\infty} \mu(TE \cap E_i) \quad \text{whenever} \quad T \subset X.$$

1.6. DEFINITION. — If $\{E_i\}$ and $\{F_j\}$ are partitions, then $\{E_i\}$ is called a subpartition of $\{F_j\}$ if each E_i is contained in some F_j .

1.7. DEFINITION. — A set E is a μ -set if the partition $\{E, E'\}$ is a μ -partition.

1.8. DEFINITION. — A set D is a μ -set F if the partition $\{D, D'\}$ is a μ -partition F .

1.9. DEFINITION. — A is μ -measurable F if μ is a measure and, for each member E of F

$$\mu(TE) = \mu(TE \cap A) + \mu(TE \sim A)$$

whenever $T \subset X$.

1.10. DEFINITION. — F is μ -convenient if μ is a measure, F is hereditary, and corresponding to each T of finite μ measure there exists such a sequence C that $\mu\left(T \sim \bigcup_{j=0}^{\infty} C_j\right) = 0$ and for each integer n , $C_n \subset C_{n+1} \in F$ and C_n is a μ -set F .

1.11. DEFINITION. — Sect (μ, B) is the function f on the subsets of X such that $f(\alpha) = \mu(\alpha B)$ for $\alpha \subset X$.

1.12. DEFINITION. — If ρ metrizes X , then

$$\text{dist}(A, B) = \inf \{\rho(x, y); x \in A, y \in B\}.$$

1.13. DEFINITION. — If X is a topological space, then μ is a Radon measure on X if μ is a measure and

- (i) open sets are μ -measurable
- (ii) if C is compact, then $\mu C < \infty$
- (iii) if α is open, then $\mu\alpha = \sup\{\mu C; C \text{ compact, } C \subset \alpha\}$
- (iv) if $A \subset X$, then $\mu A = \inf\{\mu\alpha, \alpha \text{ open, } A \subset \alpha\}$.

1.14. DEFINITION. — $(D, <)$ is a directed set if $D \neq \emptyset$, D is partially ordered by $<$ such that for any $i, j \in D$, there exists $k \in D$ with $i < k, j < k$.

Let X be a regular topological space; $\mathcal{B}$ be a base for the topology; $(D, <)$ be a directed set and for each $i \in D$, μ_i be a Radon measure on X .

For each $\alpha \in \mathcal{B}$, let

$$g(\alpha, E) = \frac{\sum_{i \in D} \mu_i(E \cap \alpha)}{\sum_{i \in D} \mu_i(\alpha)} \quad \text{where } E \text{ is a member of } \mathcal{F}.$$

Let $\varphi(A, E) = \inf_{\alpha \in \mathcal{B}} \sum_{\alpha \in H} g(\alpha, E)$; H countable, $H \subset \mathcal{B}$, $A \subset \bigcup_{\alpha \in H} \alpha$
and $\varphi^*(A, E) = \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi(C, E)$ where $A \subset X$.

Then φ is a measure on X generated by g and $\mathcal{B}$ [3].

2. Theorems and Corollaries.

2.1. THEOREM. — Product of two μ -partitions F is a μ -partition F .

Proof. — Let $\{E_i\}$ and $\{F_i\}$ be two μ -partitions F , then for every E of F

$$\mu(TE) = \sum_{i=1}^{\infty} \mu(TE \cap E_i) \quad \text{and} \quad \mu(TE) = \sum_{i=1}^{\infty} \mu(TE \cap F_i)$$

whenever $T \subset X$.

Since

$$\begin{aligned} \sum_{i,j} \mu(TE \cap E_i \cap F_j) &= \sum_j \left\{ \sum_i \mu(TE \cap F_j \cap E_i) \right\} \\ &= \sum_j \mu(TE \cap F_j) \\ &= \mu(TE), \end{aligned}$$

the proof is complete.

2.2. THEOREM. — *If a subpartition $\{F_i\}$ of a partition $\{E_i\}$ is a μ -partition F , then $\{E_i\}$ is a μ -partition F .*

Proof. — For $T \subset X$ and any member E of F , we have

$$\sum_i \mu(TE \cap E_i) = \sum_i \mu[TE \cap \{\bigcup_j E_{ji}\}]$$

where $\bigcup_j E_{ji} = E_i$ and E_{ji} is a member of $\{F_i\}$

$$\leq \sum_i \sum_j \mu(TE \cap E_{ji}) = \mu(TE),$$

since $\{F_i\}$ is a μ -partition F . The reverse inequality is, however, clear. This proves the theorem.

2.3. THEOREM. — *A partition $\{E_i\}$ is a μ -partition F if each E_i is a μ -set F .*

Proof. — Suppose that each E_i is a μ -set F . Then for E in F and $T \subset X$, we have

$$\begin{aligned} \mu(TE) &= \mu(TE \cap E_1) + \mu(TE \cap E'_1) \\ &= \mu(TE \cap E_1) + \mu(TE \cap \{E_2 \cup E_3 \cup \dots\}). \end{aligned}$$

And

$$\begin{aligned} \mu(TE \cap \{E_2 \cup E_3 \cup \dots\}) &= \mu(TE \cap \{E_2 \cup E_3 \cup \dots\} \cap E_2) \\ &\quad + \mu(TE \cap \{E_2 \cup E_3 \cup \dots\} \cap E'_2) \\ &= \mu(TE \cap E_2) + \mu(TE \cap \{E_3 \cup E_4 \cup \dots\}). \end{aligned}$$

So,

$$\mu(TE) = \mu(TE \cap E_1) + \mu(TE \cap E_2) + \mu(TE \cap \{E_3 \cup E_4 \cup \dots\}).$$

Proceeding in this way, we ultimately obtain

$$\mu(TE) = \mu(TE \cap E_1) + \mu(TE \cap E_2) + \dots = \sum_i \mu(TE \cap E_i).$$

This proves that $\{E_i\}$ is a μ -partition F .

Conversely, suppose that $\{E_i\}$ is a μ -partition F . Then for every E of F and $T \subset X$, we have $\mu(TE) = \sum_i \mu(TE \cap E_i)$.

Replacing T by $T \cap \{E_2 \cup E_3 \cup \dots\}$, we obtain

$$\begin{aligned} \mu(TE \cap \{E_2 \cup E_3 \cup \dots\}) &= \sum_i \mu(TE \cap \{E_2 \cup E_3 \cup \dots\} \cap E_i) \\ &= \mu(TE \cap E_2) + \mu(TE \cap E_3) + \dots \end{aligned}$$

So,

$$\begin{aligned}\mu(TE) &= \mu(TE \cap E_1) + \mu(TE \cap \{E_2 \cup E_3 \cup \dots\}) \\ &= \mu(TE \cap E_1) + \mu(TE \cap E'_1).\end{aligned}$$

This shows that E_1 is a μ -set F . Similarly, it can be shown that each E_i , $i = 2, 3, \dots$ is a μ -set F .

COROLLARY. — *If F is μ -convenient, then any μ -partition F is a μ -partition.*

Proof. — Let the partition $\{E_i\}$ be a μ -partition F , then each E_i is a μ -set F . Since F is μ -convenient, by Theorem 3.4 [5], a μ -set F is a μ -set. So, each E_i is a μ -set and consequently the partition $\{E_i\}$ is a μ -partition [p. 48 [1]].

In the following two theorems, we shall suppose that ρ metrizes X .

2.4. THEOREM. — *If F is hereditary and $\mu(A \cup B) = \mu A + \mu B$ whenever A and B are such members of F that $d(A, B) > 0$, then each open set is a μ -set F .*

Proof. — This theorem is due to Trevor J. McMinn [5].

2.5. THEOREM. — *If F is μ -convenient and every open set is a μ -set F , then μ is a metric outer measure.*

Proof. — It follows from Theorem 3.4 [5] that each open set is a μ -set. Let A and B be two sets with $d(A, B) > 0$.

Let α be an open set such that $A \subset \alpha$ and $\alpha \cap B = \emptyset$. Then

$$\begin{aligned}\mu(A \cup B) &= \mu(\{A \cup B\} \cap \alpha) + \mu(\{A \cup B\} \sim \alpha) \\ &= \mu A + \mu B.\end{aligned}$$

In the following theorems, we shall suppose that X is a regular topological space and $\mathcal{B}$ be a base for the topology.

2.6. THEOREM. — *If A and B are disjoint, closed compact sets, then*

$$\varphi(A \cup B, E) = \varphi(A, E) + \varphi(B, E) \quad \text{for each } E \text{ of } F.$$

Proof. — Let α and β be two open sets such that $A \subset \alpha$, $B \subset \beta$ and $\alpha \cap \beta = \emptyset$. This is possible, since X is regular. If $\varepsilon > 0$ is arbitrary, there exists a sequence $\{\nu_n\}$ of open

sets such that

$$A \cup B \subset \bigcup_n v_n \quad \text{and} \quad \sum_n g(v_n, E) \leq \varphi(A \cup B, E) + \varepsilon.$$

Let $v'_n = v_n \cap \alpha$ and $v''_n = v_n \cap \beta$, then v'_n, v''_n are open and

$$A \subset \bigcup_n v'_n, \quad B \subset \bigcup_n v''_n.$$

So,

$$\begin{aligned} \varphi(A, E) + \varphi(B, E) &\leq \sum_n \{g(v'_n, E) + g(v''_n, E)\} \\ &= \sum_n \{g(v_n \cap \alpha, E) + g(v_n \cap \beta, E)\} \\ &= \sum_n \left\{ \frac{\mathcal{L}t}{i \in D} \text{Sect}(\mu_i, E)(v_n \cap \alpha) \right. \\ &\quad \left. + \frac{\mathcal{L}t}{i \in D} \text{Sect}(\mu_i, E)(v_n \cap \beta) \right\} \\ &\leq \sum_n \left\{ \frac{\mathcal{L}t}{i \in D} \text{Sect}(\mu_i, E)v_n \right\} \\ &= \sum_n g(v_n, E) \\ &\leq \varphi(A \cup B, E) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\varphi(A, E) + \varphi(B, E) \leq \varphi(A \cup B, E).$$

The reverse inequality is clear, because φ is a Carathéodory measure. This proves the theorem.

2.7. THEOREM. — *For each E of F , φ^* is a Radon measure on X .*

Proof. — (i) If α is any open set, by definition

$$\varphi^*(\alpha, E) = \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi(C, E) \leq \varphi(\alpha, E).$$

So, for any $A \subset X$, we have

$$\begin{aligned} \varphi^*(A, E) &= \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi(C, E) = \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \varphi^*(\alpha, E) \\ &\leq \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \varphi(\alpha, E) = \varphi(A, E). \end{aligned}$$

If C is compact and α is open, $C \subset \alpha$, we have

$$\varphi(C, E) \leq \varphi^*(\alpha, E), \quad \text{so} \quad \varphi(C, E) \leq \varphi^*(C, E).$$

Therefore, if C is compact, $\varphi(C, E) = \varphi^*(C, E)$. But, it is clear that for any compact C , $\varphi(C, E) < \infty$ and hence $\varphi^*(C, E) < \infty$.

(ii) Let α be an open set, $T \subset X$ and $\varepsilon > 0$ arbitrary. Since for any $A \subset X$, $\varphi^*(A, E) = \inf_{\substack{V \text{ open} \\ A \subset V}} \varphi(V, E)$, there exists

open set T' , $T \subset T'$ and $\varphi^*(T', E) < \varphi^*(T, E) + \varepsilon$.

Also, $\varphi^*(\alpha, E) = \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi(C, E)$.

Therefore, since X is regular, there exists a closed compact set $C_1 \subset T' \cap \alpha$ such that $\varphi^*(T' \cap \alpha, E) \leq \varphi(C_1, E) + \varepsilon$. Similarly, there exists a closed compact set $C_2 \subset T' \sim C_1$ such that $\varphi^*(T' \sim C_1, E) \leq \varphi(C_2, E) + \varepsilon$.

So,

$$\begin{aligned} \varphi^*(T \cap \alpha, E) + \varphi^*(T \sim \alpha, E) &\leq \varphi^*(T' \cap \alpha, E) + \varphi^*(T' \sim C_1, E) \\ &\leq \varphi(C_1, E) + \varphi(C_2, E) + 2\varepsilon \\ &= \varphi^*(C_1 \cup C_2, E) + 2\varepsilon, \text{ by Theorem 2.6} \\ &\leq \varphi^*(T', E) + 2\varepsilon \\ &\leq \varphi^*(T, E) + 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows that α is φ^* -measurable.

The other properties are evident. This proves the theorem.

2.8. THEOREM. — If A and B are sets of which any one of them is open and $A \cap B = \emptyset$, then

$$\varphi^*(A \cup B, E) = \varphi^*(A, E) + \varphi^*(B, E) \quad \text{for each } E \text{ of } F.$$

Proof. — Let A be open and so it is φ^* -measurable. Hence

$$\begin{aligned} \varphi^*(A \cup B, E) &= \varphi^*\{(A \cup B) \cap A, E\} + \varphi^*\{(A \cup B) \sim A, E\} \\ &= \varphi^*(A, E) + \varphi^*(B, E). \end{aligned}$$

2.9. THEOREM. — If X is a metric space, then φ^* is a metric outer measure.

Proof. — This is clear.

In conclusion, I offer my best thanks to Dr. B. K. Lahiri of Calcutta University for his helpful guidance in the preparation of this paper.

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Manuscrit reçu le 24 septembre 1965.

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