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ON THE ZETA FUNCTIONS OF PREHOMOGENEOUS VECTOR SPACES FOR A PAIR OF SIMPLE ALGEBRAS

by Takashi TANIGUCHI

ABSTRACT. — In this paper we consider the prehomogeneous vector space for a pair of simple algebras which are inner forms of the D_4 type and the E_6 type. We mainly study the non-split cases. The main purpose of this paper is to determine the principal parts of the global zeta functions associated with these spaces when the simple algebras are non-split. We also give a description of the sets of rational orbits of these spaces, which clarifies the expected density theorems arising from the properties of these zeta functions.

RÉSUMÉ. — Dans cet article, nous considérons l'espace vectoriel préhomogène associé à une paire d'algèbres simples qui sont des formes intérieures de types D_4 et E_6. Nous traitons principalement les cas non-déployées. Le but principal de cet article est de déterminer les parties principales de la fonction zêta globale de ces espaces quand les algèbres simples sont non-déployés. Nous donnons aussi une description des ensembles des orbites rationnelles de ces espaces, qui clarifie les théorèmes de densité provenant des propriétés de ces fonctions zêta.

1. Introduction

This is the first part of a series of works on the zeta functions of inner forms of D_4 type and E_6 type prehomogeneous vector spaces. A density theorem for the D_4 type is proved in [15], and a similar theorem will be treated in a forthcoming paper for the E_6 type.

Let k be a field and $\mathcal{D}$ a simple algebra of dimension 4 or 9 over k. We denote by $\mathcal{D}^{\mathrm{op}}$ the opposite algebra of $\mathcal{D}$. In this paper, we consider the prehomogeneous vector space $(G, V) = (G, \rho, V)$ where

$$(1.1) \qquad G = \mathcal{D}^\times \times (\mathcal{D}^{\mathrm{op}})^\times \times \mathrm{GL}(2), \qquad V = \mathcal{D} \otimes k^2,$$

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and

$$\rho(g)(a \otimes v) = (g_{11} a g_{12}) \otimes (g_2 v) \quad \text{for} \quad g = (g_{11}, g_{12}, g_2) \in G, a \in \mathcal{D}, v \in k^2.$$

We say that a prehomogeneous vector space is of type D_4 or of type E_6 if the dimension of $\mathcal{D}$ is 4 or 9, respectively. This representation is an inner form of

$$G' = \mathrm{GL}(n) \times \mathrm{GL}(n) \times \mathrm{GL}(2), \qquad V' = k^n \otimes k^n \otimes k^2$$

for $n = 2$ and $n = 3$ if the dimension of $\mathcal{D}$ is 4 and 9, respectively. If $\mathcal{D}$ is split then (G, V) is equivalent to (G', V') over k. In this paper we give a certain description of k-rational orbits and determine the structure of the stabilizers for semi-stable points. Also we determine the principal parts of the global zeta functions for the non-split cases of (G, V) over an algebraic number field k.

Before stating our result, we briefly recall the definition of prehomogeneous vector spaces and their applications to number theory. For simplicity, we give here a definition of a certain restricted class instead of the general case.

Definition 1.1. — *An irreducible representation of a connected reductive group (G, V) over k is called a* regular prehomogeneous vector space *if*

(1) *over the algebraic closure of k, there exists a Zariski open G-orbit in V and*
(2) *there exists a non-constant polynomial $P \in k[V]$ and a rational character χ of G such that $P(gx) = \chi(g)P(x)$ for all $g \in G$ and $x \in V$.*

Irreducible prehomogeneous vector spaces over an arbitrary characteristic 0 algebraically closed field were classified by Sato and Kimura in [11].

We next recall the theory of the global zeta function. Sato and Shintani [12] defined global zeta functions for prehomogeneous vector spaces if (G, V) is defined over a number field. Information on the principal part at the rightmost pole of the global zeta function for a prehomogeneous vector space together with an appropriate local theory yields interesting density theorems. For example, using Shintani's result [13] for the space of binary cubic forms $(\mathrm{GL}(2), \mathrm{Sym}^3 k^2)$, Datskovsky and Wright [2, 3] gave a zeta function theoretic proof of the Davenport and Heilbronn [4] density theorem

$$\sum_{\substack{[F:\mathbb{Q}]=3 \\ |\Delta_F| \leqslant x}} 1 \sim \frac{x}{\zeta(3)} \qquad (x \to \infty),$$

where F runs through all the cubic fields in $\overline{\mathbb{Q}}$ with the absolute value of its discriminant $|\Delta_F|$ not bigger than x, and generalized it to cubic extensions of an arbitrary number field. Also recent work on the space of pairs of binary Hermitian forms [7] by Kable and Yukie, combined with Yukie's global theory [20], gave some new density theorems. For the statement of the density theorem, see the introduction of [7]. Note that this case is another k-form of the D_4 type. These k-forms are listed in H. Saito's classification [9].

We return to our prehomogeneous vector space (1.1). The following theorem is the main result of this paper.

Theorem 1.2. — *Let $\mathcal{D}$ be a non-split simple algebra of dimension $m = 4$ or 9. Then the global zeta function $Z(\Phi, s, \omega)$, associated with the prehomogeneous vector space (1.1), can be continued meromorphically to the region $\Re(s) > 2m - 2$ with an only possible simple pole at $s = 2m$ with the residue $\delta(\omega)\tau(G_1)\varrho^{-1}\int_{V_{\mathbb{A}}}\Phi(x)dx$.*

The constants $\delta(\omega)$, $\tau(G_1)$, ϱ and the measure dx on $V_{\mathbb{A}}$ are defined in Section 4. All the other poles are likewise described by means of certain distributions in Theorem 4.24, but the above theorem is enough to get density theorems. The expected density theorems from our cases will be discussed after Remark 3.10 using a result in Section 3. They require not only the standard tauberian theorem, but also an appropriate local theory and what is called the "filtering process". This was carried out in [15] in the case of D_4 type and a density theorem for "square of class number times regulator" of quadratic extensions was proved. We quote the result in Theorem 3.11. The density theorem in the case of E_6 type should be for "class number times regulator" of cubic extensions; it will be studied in a forthcoming paper. For a general transition process from the tauberian theorem for global zeta function of prehomogeneous vector spaces to density theorems, see [19].

One advantage of non-split cases is that the global theory becomes much easier. The analysis of the global zeta function becomes much more complicated as the split rank of the group grows, and we have not yet succeeded in establishing the global theory for the split D_4 and E_6 cases. Especially in the split E_6 case when the rank of the group is 5, the complexity of computing the principal part of the zeta function seems formidable.

In the recent work of Kable-Wright [6], they discovered that the idele characters of the zeta functions of prehomogeneous vector spaces segregate field extensions via their Steinitz class. This idea was also used by the author [14]. Taking this and other possible applications into account, we choose to carry out the global theory with not only principal characters but also general ones.

For the rest of this section, we will describe the contents of this paper and the notations used in this paper. More specific notations will be introduced in each section. In Section 2, we will define the space of a pair of simple algebras and summarize its basic properties. Before starting the global theory in Section 4, we will give a certain description of rational orbits in Section 3. The split cases are treated in [18, §3], and this is a slight generalization of those cases. We prove that the set of rational orbits has one-to-one correspondence to a certain set of quadratic extensions and cubic extensions for the D_4 case and the E_6 case, respectively. Also we determine the structure of the stabilizers for semi-stable points. The expected density theorems from our cases will be discussed after Remark 3.10 and the result for the D_4 case [15] is stated in Theorem 3.11.

In Section 4, we study the global theory for the non-split cases. In § 4.1, we introduce notations used in this section and review some basic facts on adelic analysis. In § 4.2, we define the global zeta function. Also we will give an estimate of an incomplete theta series. Although H. Saito [10] proved the convergence of all global zeta functions associated with prehomogeneous vector spaces, we need the estimate in order to use Shintani's lemma. In § 4.3, we divide the global zeta function into the "entire part" and the "principal part" by using the Poisson summation formula. We study the "principal part" in later subsections.

In § 4.4, we introduce a stratification of unstable points. To separate the contribution from unstable strata, we use Shintani's lemma. In § 4.5, we review Shintani's lemma and apply it to our cases. Since $\mathcal{D}^\times$ is of rank 0, the smoothed Eisenstein series for our case is essentially the same as that of GL(2). In § 4.6, we review some analytic properties of the zeta function associated with a (single) simple algebra, because this zeta function appears in the induction process. In § 4.7, we compute contributions from unstable points. By putting together the results we have obtained in § 4.4–§ 4.7, we determine the principal part of the global zeta function in § 4.8.

The standard symbols $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ will denote respectively the sets of the rational, real and complex numbers and the rational integers. If R is any ring then $R^\times$ is the set of invertible elements of R and if V is a variety defined over R then V_R denotes its R-points. In Sections 2 and 3, k denotes arbitrary field. In Section 4, k denotes an algebraic number field.

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2. The space of a pair of simple algebras

In this section, we define the representation of the space of a pair of simple algebras, and discuss its basic properties.

Let k be an arbitrary field and $\mathcal{D}$ a simple algebra over k of dimension $m = n^2, n \geqslant 1$. Let $\mathcal{T}$ and $\mathcal{N}$ be the reduced trace and reduced norm, respectively. We denote by $\mathcal{D}^{\mathrm{op}}$ the opposite algebra of $\mathcal{D}$. We introduce a group G_1 and its representation space W as follows. Let

$$G_1 = \mathcal{D}^\times \times (\mathcal{D}^{\mathrm{op}})^\times.$$

That is, G_1 is equal to $\mathcal{D}^\times \times \mathcal{D}^\times$ set theoretically and the multiplication law is given by $(g_{11}, g_{12})(h_{11}, h_{12}) = (g_{11}h_{11}, h_{12}g_{12})$. We regard G_1 as an algebraic group over k. The simple algebra $\mathcal{D}$ can be considered as a vector space over k. When we regard $\mathcal{D}$ as a vector space over k, we denote this space as W. We define the action of G_1 on W as follows:

$$(g_1, w) \longmapsto g_{11} w g_{12}, \qquad g_1 = (g_{11}, g_{12}) \in G_1, w \in W.$$

This defines a representation W of G_1. Clearly, (G_1, W) is a prehomogeneous vector space. We discuss the properties of the zeta function associated with this space in §4.6, which will be used in the analysis of the zeta function associated with the space of a pair of simple algebras.

Let $G_2 = \mathrm{GL}(2)$ and k^2 the standard representation of G_2. The group $G = G_1 \times G_2$ acts naturally on $V = W \otimes k^2$. This is a k-form of $(\mathrm{GL}(n) \times \mathrm{GL}(n) \times \mathrm{GL}(2), k^n \otimes k^n \otimes k^2)$, and it is proved in [11] that this is a prehomogeneous vector space if and only if $n = 2$ or $n = 3$. Since we are interested in prehomogeneous vector space, *we consider the case $n = 2, 3$ for the rest of this paper*. That is, $\mathcal{D}$ is a simple algebra of dimension 4 or 9. We call this representation D_4 type and E_6 type for $n = 2$ and $n = 3$, respectively following [18].

We describe the action more explicitly. Throughout of this paper, we express elements of $V \cong W \oplus W$ as $x = (x_1, x_2)$. We identify $x = (x_1, x_2) \in$

V with $x(v) = v_1x_1 + v_2x_2$ which is an element of simple algebra with entries in linear forms in two variables $v = (v_1, v_2)$. Then the action of $g = (g_{11}, g_{12}, g_2) \in G$ on $x \in V$ is defined by

$$(gx)(v) = g_{11}x(vg_2)g_{12}.$$

We put $F_x(v) = \mathcal{N}(x(v))$. This is a binary quadratic form (resp. cubic form) in variables $v = (v_1, v_2)$ if $n = 2$ (resp. $n = 3$), and the discriminant $P(x)$ $(x \in V)$ is a polynomial in V. The polynomial $P(x)$ is characterized by

$$P(x) = \prod_{i<j}(\alpha_i\beta_j - \alpha_j\beta_i)^2 \quad \text{for} \quad F_x(v) = \prod_{1\leqslant i\leqslant n}(\alpha_i v_1 - \beta_i v_2), \quad x \in V_{\bar{k}}.$$

Let χ_i $(i = 1, 2)$ be the character of G_i defined by

$$\chi_1(g_1) = \mathcal{N}(g_{11})\mathcal{N}(g_{12}), \quad \chi_2(g_2) = \det g_2,$$

respectively. We define $\chi(g) = \chi_1(g_1)^2\chi_2(g_2)^2$ for $n = 2$ and $\chi(g) = \chi_1(g_1)^4\chi_2(g_2)^6$ for $n = 3$. Then one can easily see that

$$P(gx) = \chi(g)P(x)$$

and hence $P(x)$ is a relative invariant polynomial with respect to the character χ. Let $S = \{x \in V \mid P(x) = 0\}$ and $V^{\mathrm{ss}} = \{x \in V \mid P(x) \neq 0\}$ and call them the set of unstable points and semi-stable points, respectively. That is, $x \in V$ is semi-stable if and only if $F_x(v)$ does not have a multiple root in $\mathbb{P}^1 = \{(v_1 : v_2)\}$.

3. Rational orbit decomposition

3.1. Rational orbit decomposition

In this section, we will interpret the rational orbit space $G_k\backslash V_k^{\mathrm{ss}}$ and determine the structure of the stabilizers for semi-stable points. The split cases are treated in [18, §3], and here is a slight generalization of that treatment. For the expected density theorems and the result from the D_4 case, see after Remark 3.10. For $x \in V_k^{\mathrm{ss}}$, let G_x be the stabilizer of x and G_x° its identity component.

Recall that we put n as the degree of $\mathcal{D}$, which is either 2 or 3. Let $\mathfrak{A}_2^{\mathrm{sep}}$ (resp. $\mathfrak{A}_3^{\mathrm{sep}}$) be set of isomorphism classes of separable commutative k-algebras of dimension 2 (resp. 3). For example, $\mathfrak{A}_2^{\mathrm{sep}}$ can be regarded as the disjoint union of $\{k \times k\}$ and the set of separable quadratic extensions of k.

DEFINITION 3.1. — *For $x \in V_k^{\mathrm{ss}}$, we define*

$$Z_x = \operatorname{Proj} k[v_1, v_2]/(F_x(v)),$$
$$\tilde{k}(x) = \Gamma(Z_x, \mathcal{O}_{Z_x}).$$

Also we define $k(x)$ to be the splitting field of $F_x(v)$.

Note that $\tilde{k}(x)$ may not be a field. Since V_k^{ss} is the set of x such that F_x does not have a multiple root, Z_x is a reduced scheme over k and $\tilde{k}(x)$ is an element of $\mathfrak{A}_2^{\mathrm{sep}}$ (resp. $\mathfrak{A}_3^{\mathrm{sep}}$) for $n = 2$ (resp. $n = 3$). Since

$$F_{gx}(v) = \chi_1(g_1) F_x(v g_2),$$

the isomorphism classes of $Z_x, \tilde{k}(x)$ and $k(x)$ depend only on the G_k-orbit of x.

We let

$$\tilde{\alpha}_V : G_k \backslash V_k^{\mathrm{ss}} \longrightarrow \mathfrak{A}_2^{\mathrm{sep}} \quad (\text{resp.}\, \mathfrak{A}_3^{\mathrm{sep}}) \quad x \longmapsto \tilde{k}(x)$$

for the D_4 case (resp. the E_6 case). We first determine the image of $\tilde{\alpha}_V$.

DEFINITION 3.2.

(1) *For $n = 2$, we denote by $\mathfrak{A}_2^{\mathrm{sep}}(\mathcal{D})$ the subset of $\mathfrak{A}_2^{\mathrm{sep}}$ consisting of algebras which have an embedding into $\mathcal{D}_k$.*
(2) *For $n = 3$, we denote by $\mathfrak{A}_3^{\mathrm{sep}}(\mathcal{D})$ the subset of $\mathfrak{A}_3^{\mathrm{sep}}$ consisting of algebras which have an embedding into $\mathcal{D}_k$.*

LEMMA 3.3.

(1) *Let (G, V) be of D_4 type. Then the image of the map $\tilde{\alpha}_V$ is $\mathfrak{A}_2^{\mathrm{sep}}(\mathcal{D})$.*
(2) *Let (G, V) be of E_6 type. Then the image of the map $\tilde{\alpha}_V$ is $\mathfrak{A}_3^{\mathrm{sep}}(\mathcal{D})$.*
(3) *Moreover, any orbit $G_k x \subset V_k^{\mathrm{ss}}$ contains an element of the form $y = (1, y_2)$.*

Proof. — Here we consider the E_6 case. The D_4 case can be treated similarly. First note that for $x = (1, w) \in V_k^{\mathrm{ss}}$, $F_x(v_1, 1) = \mathcal{N}(v_1 + w)$ is the characteristic polynomial of $-w \in \mathcal{D}_k$ that does not have a multiple root, and hence the algebra $\tilde{k}(x)$ is isomorphic to the subalgebra $k[w] \subset \mathcal{D}_k$ generated by w over k in $\mathcal{D}_k$.

Let $L \in \mathfrak{A}_3^{\mathrm{sep}}(\mathcal{D})$. We regard L as a subalgebra of $\mathcal{D}_k$ and take an element $u \in \mathcal{D}_k$ so that $L = k[u]$. Let $x = (1, -u) \in V_k$. Then since $\dim_k L = 3 = \deg F_x(v_1, 1)$, the characteristic polynomial $F_x(v_1, 1)$ of u is also the minimum polynomial of u. Hence $F_x(v_1, 1)$ does not have a multiple root since u is separable. This shows that $x = (1, -u) \in V_k^{\mathrm{ss}}$ and now by the remark above we have $\tilde{\alpha}_V(G_k x) = L$. This proves that the image of $\tilde{\alpha}_V$ contains $\mathfrak{A}_3^{\mathrm{sep}}(\mathcal{D})$.

Since (3) implies the opposite inclusion, we consider (3). If $\mathcal{D}_k$ is non-split, this is obvious because $\mathcal{D}_k$ is a division algebra. We consider the split cases. Since the argument is similar, we consider the E_6 case here. In this case, $\mathcal{D} = \mathrm{M}(3)$ be the algebra of 3×3 matrices. For $a \in \mathcal{D}$, let $\mathrm{rank}(a)$ denote the rank of the matrix a.

Let $x = (x_1, x_2) \in V_k^{\mathrm{ss}}$. If either the rank of x_1 or x_2 is equal to 3, the element is invertible and hence, there exists a $g \in G_k$ such that $gx = (1, *)$. Also if both the rank of x_1 and x_2 are less than or equal to 1, we have $F_x(v) = \det(x_1v_1 + x_2v_2) = 0$ which contradicts to $x \in V_k^{\mathrm{ss}}$. Hence, by interchanging x_1 and x_2 if necessary, we assume that $\mathrm{rank}(x_1) = 2$. Then there exists a $g_1 \in G_{1k}$ such that $x' = g_1x = (e, y)$, where

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}.$$

If $y_{33} = 0$, then it is easy to see that $F_{x'}(v)$ has a multiple root, and so $y_{33} \neq 0$. Hence, again we can take an element $g_1' \in G_{1k}$ such that

$$g_1'(e, y) = (e, z), \quad z = \begin{pmatrix} z_{11} & z_{12} & 0 \\ z_{21} & z_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now it is easy to see that there exist $\alpha, \beta \in k$ such that $\mathrm{rank}(\alpha e + \beta z) = 3$, hence we have (3). □

We later show that the map $\tilde{\alpha}_V$ is in fact injective. Next we consider the structure of the stabilizers for semi-stable points. Note that for $x \in V_k^{\mathrm{ss}}$,

$$\dim G_x^\circ = \dim G_x = \dim G - \dim V = 4.$$

Lemma 3.4. — *Let $x \in V_k^{\mathrm{ss}}$.*

(1) *Let (G, V) be of D_4 type. Then $G_x^\circ \cong (\mathrm{GL}(1)_{\tilde{k}(x)})^2$ as a group over k.*

(2) *Let (G, V) be of E_6 type. Then $G_x^\circ \cong \mathrm{GL}(1)_{\tilde{k}(x)} \times \mathrm{GL}(1)_k$ as a group over k.*

Proof. — By Lemma 3.3 (3), any G_k-orbit in V_k^{ss} contains an element of the form $x = (1, w)$ with $w \in \mathcal{D}_k \setminus k$. Hence it is enough to show the lemma for these elements. We identify $\tilde{k}(x)$ with $k[w] \subset \mathcal{D}_k$.

In order to prove an isomorphism between two algebraic groups G_1 and G_2 over k, it is enough to construct isomorphisms between the sets G_{1R} and G_{2R} of R-rational points of G_1 and G_2 for all commutative k-algebras R which satisfy the usual functorial property. For this, the reader should see [8, p.17].

We first consider (1). For the D_4 case, $\tilde{k}(x) = k[w]$ is a separable k-algebra of dimension 2. Let R be any commutative k-algebra. We put $\widetilde{R}(x) = \tilde{k}(x) \otimes R$. Note that $\widetilde{R}(x) = R[w]$ is a subalgebra of $\mathcal{D}_R = \mathcal{D} \otimes R$ and is commutative. Since $\{1, w\}$ is a k-basis of $\tilde{k}(x)$, this is also an R-basis of $\widetilde{R}(x)$. Let $s, t \in \widetilde{R}(x)^\times$. Then $\{st, stw\}$ is also an R-basis of $\widetilde{R}(x)$, and so there exists a unique element $g = g_{st} \in \mathrm{GL}(2)_R$ such that $g\, {}^t(st, stw) = {}^t(1, w)$. Hence $(s, t) \mapsto (s, t, g_{st})$ gives an injective homomorphism from $(\widetilde{R}(x)^\times)^2$ to G_{xR}.

This shows that there exists an injective homomorphism

$$(\mathrm{GL}(1)_{\tilde{k}(x)})^2 \longrightarrow G_x.$$

Since $(\mathrm{GL}(1)_{\tilde{k}(x)})^2$ is a connected algebraic group of dimension 4, we have $(\mathrm{GL}(1)_{\tilde{k}(x)})^2 \cong G_x^\circ$.

Next we consider (2). Again we let R be any algebra and put $\widetilde{R}(x) = \tilde{k}(x) \otimes R$. Then we have a injective homomorphism from $\widetilde{R}(x)^\times \times R^\times$ to G_{xR} by sending (s, t) to $(s, s^{-1}t^{-1}, t)$. This shows that there exists an injective homomorphism

$$\mathrm{GL}(1)_{\tilde{k}(x)} \times \mathrm{GL}(1)_k \longrightarrow G_x.$$

Since $\mathrm{GL}(1)_{\tilde{k}(x)} \times \mathrm{GL}(1)_k$ is a connected algebraic group of dimension 4, we have the isomorphism $\mathrm{GL}(1)_{\tilde{k}(x)} \times \mathrm{GL}(1)_k \cong G_x^\circ$. □

Finally, we show the injectivity of $\tilde{\alpha}_V$.

Lemma 3.5. — *In both cases, the map $\tilde{\alpha}_V$ is injective.*

Proof. — Since the split case is already proven in [18], we only consider the non-split cases here. Let $x, y \in V_k^{\mathrm{ss}}$ satisfy $\tilde{k}(x) \cong \tilde{k}(y)$. By Lemma 3.3 (3), we may assume $x = (1, u_1), y = (1, u_2)$. Then $k[u_1]$ and $k[u_2]$ are isomorphic subfields of $\mathcal{D}_k$. By the Skolem-Noether theorem [1, Chap. 8 §10], there exists an element $\theta \in \mathcal{D}_k^\times$ such that

$$k[u_1] \longrightarrow k[u_2], \quad p \longmapsto \theta p \theta^{-1}$$

gives an isomorphism from $k[u_1]$ to $k[u_2]$.

Let (G, V) be of D_4 type. Then $k[u_1]$ is a quadratic extension over k. Hence there exist $a, b \in k$ with $b \neq 0$ such that $u_2 = \theta(a + bu_1)\theta^{-1}$. Hence for

$$g = \left(\theta, \theta^{-1}, \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}\right) \in G_k,$$

we have $y = gx$.

Let (G, V) be of E_6 type. There exists $p \in k[u_1]$ so that $u_2 = \theta p \theta^{-1}$. We claim that there exist $a, b, c, d \in k$ with $ad - bc \neq 0$ such that $p =$

$(c+du_1)/(a+bu_1)$. In fact, if we consider the k-linear map

$$\psi\colon k^4 \longrightarrow k[u_1], \qquad (a,b,c,d) \longmapsto (a+bu_1)p-(c+du_1),$$

the kernel of ψ is non-trivial. Therefore there exists $(a,b,c,d)\in k^4\setminus\{0\}$ so that $(a+bu_1)p-(c+du_1)=0$, and since $p\notin k$, we have $ad-bc\neq 0$.

Hence for

$$g=\left(\theta(a+bu_1)^{-1},\theta^{-1},\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)\in G_k,$$

we have $y=gx$. □

We summarize the result in this subsection as follows.

Proposition 3.6. — *Let (G,V) be of D_4 type.*

(1) *The map*

$$G_k\backslash V_k^{\mathrm{ss}} \longrightarrow \mathfrak{A}_2^{\mathrm{sep}}(\mathcal{D}), \quad x\longmapsto \tilde{k}(x)$$

is bijective.

(2) *Let $x\in V_k^{\mathrm{ss}}$. As a group over k, $G_x^\circ\cong(\mathrm{GL}(1)_{\tilde{k}(x)})^2$.*

Proposition 3.7. — *Let (G,V) be of E_6 type.*

(1) *The map*

$$G_k\backslash V_k^{\mathrm{ss}} \longrightarrow \mathfrak{A}_3^{\mathrm{sep}}(\mathcal{D}), \quad x\longmapsto \tilde{k}(x)$$

is bijective.

(2) *Let $x\in V_k^{\mathrm{ss}}$. As a group over k, $G_x^\circ\cong\mathrm{GL}(1)_{\tilde{k}(x)}\times\mathrm{GL}(1)_k$.*

3.2. Application to global fields

If k is a global field, it is well known that the sets $\mathfrak{A}_2^{\mathrm{sep}}(\mathcal{D}),\mathfrak{A}_3^{\mathrm{sep}}(\mathcal{D})$ can be described by means of local conditions. Here, we review the argument. We assume that k is a global field in this subsection. Also if $\mathcal{D}$ is split, $\mathfrak{A}_2^{\mathrm{sep}}(\mathcal{D})=\mathfrak{A}_2^{\mathrm{sep}}$ and $\mathfrak{A}_3^{\mathrm{sep}}(\mathcal{D})=\mathfrak{A}_3^{\mathrm{sep}}$ for $n=2$ and $n=3$, respectively. Hence we assume $\mathcal{D}$ is non-split in this subsection. Recall that $m=n^2$ is the dimension of $\mathcal{D}$.

Let $\mathfrak{M}$ be the set of places of k. For $v\in\mathfrak{M}$, let k_v be the completion of k at v. We denote by $\mathrm{Inv}_v(\mathcal{D})$ the Hasse invariant of $\mathcal{D}\otimes k_v$ over k_v.

Definition 3.8. — *We define $\mathfrak{M}_{\mathcal{D}}$ to be the set of elements $v\in\mathfrak{M}$ which satisfy* $\mathrm{Inv}_v(\mathcal{D})\neq 0$.

It is well known that $\mathfrak{M}_{\mathcal{D}}$ is a finite set.

Proposition 3.9.

(1) *For $n = 2$, the set $\mathfrak{A}_2^{\mathrm{sep}}(\mathcal{D})$ consists of elements $L \in \mathfrak{A}_2^{\mathrm{sep}}$ such that $L \otimes k_v$ is a quadratic extension of k_v for all $v \in \mathfrak{M}_{\mathcal{D}}$.*

(2) *For $n = 3$, the set $\mathfrak{A}_3^{\mathrm{sep}}(\mathcal{D})$ consists of elements $L \in \mathfrak{A}_3^{\mathrm{sep}}$ such that $L \otimes k_v$ is a cubic extension of k_v for all $v \in \mathfrak{M}_{\mathcal{D}}$.*

Proof. — We will prove the proposition in the case $n = 3$. The case $n = 2$ can be treated similarly.

Let L be an arbitrary separable cubic extension of k. We denote by $\mathfrak{M}_L$ the set of places of L. The field L is an element of $\mathfrak{A}_3^{\mathrm{sep}}(\mathcal{D})$ if and only if $\mathcal{D}$ is split over L. By the Hasse principle, this condition is equivalent to that $\mathcal{D} \otimes_k L_w \cong \mathrm{M}(3,3)_{L_w}$ for all $w \in \mathfrak{M}_L$. Since $\mathcal{D} \otimes_k k_v \cong \mathrm{M}(3,3)_{k_v}$ for all $v \notin \mathfrak{M}_{\mathcal{D}}$, we only need to consider w which divides an element $v \in \mathfrak{M}_{\mathcal{D}}$. For this v, $\mathcal{D}_v = \mathcal{D} \otimes k_v$ is a division algebra. Hence for a separable extension F/k_v with $[F : k_v] \leqslant 3$, $\mathcal{D}_v \otimes_{k_v} F \cong \mathrm{M}(3,3)_F$ if and only if $[F : k_v] = 3$. Therefore $\mathcal{D}_v \otimes_{k_v} L_w \cong \mathrm{M}(3,3)_{L_w}$ if and only if $[L_w : k_v] = 3$. □

Remark 3.10. — Let $\widetilde{T} = \ker(G \to \mathrm{GL}(V))$. Then it is easy to that

$$\widetilde{T} = \{(t_{11}, t_{12}, t_2) \mid t_{11}, t_{12}, t_2 \in \mathrm{GL}(1)_k, t_{11}t_{12}t_2 = 1\} \cong \mathrm{GL}(1)_k \times \mathrm{GL}(1)_k,$$

and hence

$$G_x^\circ/\widetilde{T} \cong \begin{cases} (\mathrm{GL}(1)_{\tilde{k}(x)}/\,\mathrm{GL}(1)_k)^2 & \text{the } D_4 \text{ case,} \\ \mathrm{GL}(1)_{\tilde{k}(x)}/\,\mathrm{GL}(1)_k & \text{the } E_6 \text{ case.} \end{cases}$$

For the non-split cases, $\tilde{k}(x)$ is a quadratic or cubic field over k for any $x \in V_k^{\mathrm{ss}}$, and hence $G_x^\circ/\widetilde{T}$ does not contain a split torus. This shows that the spaces (G, V) are of complete type for the non-split cases.

We conclude this subsection with a brief discussion of the density theorems which we can derive from the theory of the zeta function for our cases.

Let k be a number field. Roughly speaking, the global zeta function is a counting function for the unnormalized Tamagawa numbers of $G_x^\circ/\widetilde{T}$ of points in $x \in G_k \backslash V_k^{\mathrm{ss}}$. Let $F = \tilde{k}(x)$. We denote by h_F and R_F the class number and the regulator of F, respectively. If we consider the canonical measure on the adelization of $G_x^\circ/\widetilde{T}$, by Remark 3.10 the unnormalized Tamagawa number of this group is $(\mathrm{Res}_{s=1}\,\zeta_F(s))^2$ (resp. $\mathrm{Res}_{s=1}\,\zeta_F(s)$) for the D_4 case (resp. the E_6 case) where $\zeta_F(s)$ is the Dedekind zeta function. This leads us to believe that the theory of the zeta function will eventually yield the average density of $h_F^2 R_F^2$ for $F \in \mathfrak{A}_2^{\mathrm{sep}}(\mathcal{D})$ from the D_4 case, and the average density of $h_F R_F$ for $F \in \mathfrak{A}_3^{\mathrm{sep}}(\mathcal{D})$ from the E_6 case.

In fact, the necessary local theory for the D_4 case is carried out in [15], and combined with the global theory of this paper, the density theorem for that case is proved. To make our discussion above more comprehensible, we quote the density theorem proved in [15]. The corresponding theory for the E_6 case will be treated in a forthcoming paper.

Let k be a number field. Let $\mathfrak{M}$, $\mathfrak{M}_\infty$ and $\mathfrak{M}_{\mathrm{f}}$ denote respectively the set of all places of k, all infinite places and all finite places. For $v \in \mathfrak{M}$ let k_v denote the completion of k at v and if $v \in \mathfrak{M}_{\mathrm{f}}$ then let q_v denote the order of the residue field of k_v. We let Δ_k, r_1, r_2, and e_k be respectively the absolute discriminant, the number of real places, the number of complex places, and the number of roots of unity contained in k. We denote by $\zeta_k(s)$ the Dedekind zeta function of k.

Let $S \supset \mathfrak{M}_\infty$ be a finite set of places. We fix an S-tuple $L_S = (L_v)_{v\in S}$ where each L_v is a separable quadratic algebra of k_v, *i.e.*, either $k_v \times k_v$ or a quadratic extension of k_v. We put

$$\mathcal{Q}(L_S, X) = \left\{ F \,\middle|\, \begin{array}{ll} [F:k]=2, & \mathcal{N}(\Delta_{F/k}) \leqslant X, \\ F\otimes k_v \cong L_v, & \forall v \in S \end{array} \right\},$$

where we denote by $\Delta_{F/k}$ the relative discriminant of F/k and by $\mathcal{N}(\Delta_{F/k})$ its absolute norm. Then the following is proved in [15].

Theorem 3.11. — *Let $L_S = (L_v)_{v\in S}$ be an S-tuple such that L_v is a field for at least two places of S. Then the limit*

$$\lim_{X\to\infty} \frac{1}{X^2} \sum_{F\in\mathcal{Q}(L_S,X)} h_F^2 R_F^2$$

exists, and its value is equal to

$$\frac{(\mathrm{Res}_{s=1}\,\zeta_k(s))^3 \Delta_k^2 e_k^2 \zeta_k(2)^2}{2^{r_1+r_2+1} 2^{2r_1(L_S)} (2\pi)^{2r_2(L_S)}} \prod_{v\in S\cap\mathfrak{M}_{\mathrm{f}}} \mathfrak{e}_v(L_v) \prod_{v\in\mathfrak{M}_{\mathrm{f}}} (1-3q_v^{-3}+2q_v^{-4}+q_v^{-5}-q_v^{-6}).$$

Here we denote by $r_1(L_S)$ and $r_2(L_S)$ respectively the number of real and complex places of $F \in \mathcal{Q}(L_S, X)$ (these numbers do not depend on the choice of F) and also for $v \in \mathfrak{M}_{\mathrm{f}}$ we put

$$\mathfrak{e}_v(L_v) = \begin{cases} 2^{-1}(1+q_v^{-1})(1-q_v^{-2}) & L_v \cong k_v \times k_v, \\ 2^{-1}(1-q_v^{-1})^3 & L_v \text{ is quadratic unramified}, \\ 2^{-1}\mathcal{N}(\Delta_{L_v/k_v})^{-1}(1-q_v^{-1})(1-q_v^{-2})^2 & L_v \text{ is quadratic ramified}. \end{cases}$$

The condition on L_S that two of L_v are fields corresponds to Proposition 3.9 (1). Let L_{v_1} and L_{v_2} are fields. We prove Theorem 3.11 by choosing $\mathcal{D}$ such that $\mathfrak{M}_{\mathcal{D}} = \{v_1, v_2\}$. Combined with the result of Kable-Yukie [7] we also obtain the limit of correlation coefficients of $h_F R_F$ for certain families of quadratic extensions of k. For more details including this result, see [15, Introduction].

4. The global zeta function

In this section, we study analytic properties of the global zeta function for non-split cases. The main result is Theorem 4.24 which describes the principal parts of the global zeta function.

4.1. Preliminaries

In this subsection, we collect basic notations that we use in this section. Also, we review some basic facts concerning adelic analysis that we need later. Throughout this section, k is a number field. Let $\mathcal{D}$ be a non-split simple algebra over k of dimension 4 or 9. Then $\mathcal{D}$ is a division algebra. Since the argument is similar for the two cases, we treat them simultaneously. Recall that $m = n^2$ is the dimension of $\mathcal{D}$.

Suppose that G is a locally compact group and Γ a discrete subgroup of G contained in the maximal unimodular subgroup of G. For any left invariant measure dg on G, we choose a left invariant measure dg (we use the same notation, but the meaning will be clear from the context) on $X = G/\Gamma$ so that

$$\int_G f(g)\,dg = \int_X \sum_{\gamma\in\Gamma} f(g\gamma)\,dg.$$

Let r_1, r_2, h_k, R_k and Δ_k be the number of real places, the number of complex places, the class number, the regulator and the discriminant of k, respectively. Let e_k be the number of roots of unity contained in k. We set

$$\mathfrak{C}_k = 2^{r_1}(2\pi)^{r_2} h_k R_k e_k^{-1}.$$

We refer to [16] as the basic reference for fundamental properties of adeles. The ring of adeles and the group of ideles are denoted by $\mathbb{A}$ and $\mathbb{A}^\times$, respectively. The adelic absolute value $|\ |$ on $\mathbb{A}^\times$ is normalized so that, for $t \in \mathbb{A}^\times$, $|t|$ is the module of multiplication by t with respect to any Haar measure dx on $\mathbb{A}$, *i.e.*, $|t| = d(tx)/dx$. Let $\mathbb{A}^0 = \{t \in \mathbb{A}^\times \mid |t| = 1\}$. We fix a non-trivial additive character $\langle\ \rangle$ of $\mathbb{A}/k$. The set of positive real numbers is denoted $\mathbb{R}_+$. Suppose $[k : \mathbb{Q}] = n$. For $\lambda \in \mathbb{R}_+$, $\underline{\lambda} \in \mathbb{A}^\times$ is the idele whose component at any infinite place is $\lambda^{1/n}$ and whose component at any finite place is 1. Then we have $|\underline{\lambda}| = \lambda$.

We choose a Haar measure dx on $\mathbb{A}$ so that $\int_{\mathbb{A}/k} dx = 1$. We define a Haar measure $d^\times t^0$ on $\mathbb{A}^0$ so that $\int_{\mathbb{A}^0/k^\times} d^\times t^0 = 1$. Using this measure, we choose a Haar measure $d^\times t$ on $\mathbb{A}^\times$ so that

$$\int_{\mathbb{A}^\times} f(t)\, d^\times t = \int_0^\infty \int_{\mathbb{A}^0} f(\underline{\lambda} t^0)\, d^\times t^0 d^\times \lambda,$$

where $d^\times \lambda = \lambda^{-1} d\lambda$ and $d\lambda$ is the usual Lebesgue measure.

Let $\zeta_k(s)$ be the Dedekind zeta function of k. We define

$$Z_k(s) = |\Delta_k|^{s/2} \left(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right)^{r_1} \left((2\pi)^{1-s}\Gamma(s)\right)^{r_2} \zeta_k(s)\,.$$

This definition differs from that in [16], p.129 by the factor of $|\Delta_k|^{s/2}$ and from that in [19] by the factor of $(2\pi)^{r_2}$. It is adopted here as the most convenient for our purposes. It is well known that $\operatorname{Res}_{s=1} Z_k(s) = \mathfrak{C}_k$. We define

$$\phi(s) = \frac{Z_k(s)}{Z_k(s+1)}, \quad \text{and} \quad \varrho = \operatorname{Res}_{s=1} \phi(s) = \frac{\mathfrak{C}_k}{Z_k(2)},$$

which will play an important role in this section. For a complex variable s, we denote by $\Re(s)$ the real part of s.

For a vector space V over k, $V_{\mathbb{A}}$ denotes its adelization. Let $\mathscr{S}(V_{\mathbb{A}})$ be the spaces of Schwartz-Bruhat functions on $V_{\mathbb{A}}$. We define the Haar measure dx on $V_{\mathbb{A}}$ so that $\int_{V_{\mathbb{A}}/V_k} dx = 1$.

We denote elements of $G_2 = \mathrm{GL}(2)$ as follows:

$$a(t_1, t_2) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix},\ n(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},\ \nu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We recall the following well known facts concerning adelic analysis. The proof may be found in [19, Chapter 1].

Lemma 4.1.

(1) *Let $C \subset \mathrm{GL}(V)_{\mathbb{A}}$ be a compact set, and $\Phi \in \mathscr{S}(V_{\mathbb{A}})$. Then there exists $\Psi \in \mathscr{S}(V_{\mathbb{A}})$ such that*

$$|\Phi(gx)| \leqslant \Psi(x)$$

for all $g \in C, x \in V_{\mathbb{A}}$.

(2) *Let Φ be a Schwartz-Bruhat function on $\mathbb{A}^n$. Then there exist Schwartz-Bruhat functions $\Phi_1, \ldots, \Phi_n \geqslant 0$ on $\mathbb{A}$ such that*

$$|\Phi(x_1, \ldots, x_n)| \leqslant \Phi_1(x_1) \cdots \Phi_n(x_n).$$

(3) *Suppose* $\Phi \in \mathscr{S}(\mathbb{A})$. *Then for any* $N \geqslant 1$,

$$\sum_{x \in k} \Phi(tx) \ll \max\{1, |t|^{-1}\}, \qquad \sum_{x \in k^\times} \Phi(tx) \ll |t|^{-N}.$$

4.2. The global zeta function

In this subsection, we define the global zeta function. Also we give an estimate of an incomplete theta series in order to use Shintani's lemma.

Recall that $G_2 = \mathrm{GL}(2)$. Let $T_2 \subset G_2$ be the set of diagonal matrices and $N_2 \subset G_2$ be the set of lower-triangular matrices whose diagonal entries are 1. Then $B_2 = T_2 N_2$ is a Borel subgroup of G_2.

Let

$$\begin{aligned}
G^0_{1\mathbb{A}} &= \{g_1 = (g_{11}, g_{12}) \in G_{1\mathbb{A}} \mid |\mathcal{N}(g_{11})| = |\mathcal{N}(g_{12})| = 1\},\\
G^0_{2\mathbb{A}} &= \{g_2 \in G_{2\mathbb{A}} \mid |\det g_2| = 1\},\\
G^0_{\mathbb{A}} &= G^0_{1\mathbb{A}} \times G^0_{2\mathbb{A}}, \quad \overline{G}_{\mathbb{A}} = \mathbb{R}_+ \times G^0_{\mathbb{A}},\\
\widehat{T}^0_{2\mathbb{A}} &= \{a(t_{21}, t_{22}) \mid t_{21}, t_{22} \in \mathbb{A}^0\},\\
T^0_{2\mathbb{A}} &= \{a(\underline{\mu}^{-1}, \underline{\mu}) t_2 \mid \mu \in \mathbb{R}_+, t_2 \in \widehat{T}^0_{2\mathbb{A}}\},\\
B^0_{2\mathbb{A}} &= T^0_{2\mathbb{A}} N_{2\mathbb{A}}, \quad P^0_{\mathbb{A}} = G^0_{1\mathbb{A}} \times B^0_{2\mathbb{A}}.
\end{aligned}$$

By assuming that $\lambda \in \mathbb{R}_+$ acts on $V_{\mathbb{A}}$ by multiplication by $\underline{\lambda}$, we may assume that $\overline{G}_{\mathbb{A}}$ acts on $V_{\mathbb{A}}$. Throughout this section, we write elements $\bar{g} \in \overline{G}_{\mathbb{A}}$, $g^0 \in G^0_{\mathbb{A}}$ as

$$\bar{g} = (\lambda, g_1, g_2), g^0 = (g_1, g_2)$$

where $\lambda \in \mathbb{R}_+$, $g_1 \in G^0_{1\mathbb{A}}$, and $g_2 \in G^0_{2\mathbb{A}}$. We identify an element $g^0 \in G^0_{\mathbb{A}}$ with $(1, g^0) \in \overline{G}_{\mathbb{A}}$ and $g_1 \in G^0_{1\mathbb{A}}$, $g_2 \in G^0_{2\mathbb{A}}$ with $(1, g_1, 1), (1, 1, g_2)$. We may also write $\bar{g}$ as $\bar{g} = \underline{\lambda} g^0$.

Let K_2 be the standard maximal compact subgroup of $G^0_{2\mathbb{A}}$, *i.e.*,

$$K_2 = \prod_{v \in \mathfrak{M}_\mathbb{R}} \mathrm{O}(2, \mathbb{R}) \times \prod_{v \in \mathfrak{M}_\mathbb{C}} \mathrm{U}(2, \mathbb{C}) \times \prod_{v \in \mathfrak{M}_\mathrm{f}} \mathrm{GL}(2, \mathcal{O}_v).$$

Let $d\kappa_2$ be the Haar measure on K_2 such that the total volume of K_2 is 1.

Let

$$t_2 = a_2(\underline{\mu}^{-1} t_{21}, \underline{\mu} t_{22}), b_2 = t_2 n_2(u)$$

where $\mu \in \mathbb{R}_+, t_{21}, t_{22} \in \mathbb{A}^0, u \in \mathbb{A}$. Throughout this section, we assume that

$$g_2 = \kappa_2 b_2 = \kappa_2 a_2(\underline{\mu}^{-1} t_{21}, \underline{\mu} t_{22}) n_2(u)$$

is the Iwasawa decomposition of $g_2 \in G^0_{2\mathbb{A}}$.

The measure du on $\mathbb{A}$ induces an invariant measure on $N_{\mathbb{A}}$. We put

$$d^\times t_2 = d^\times \mu \, d^\times t_{21} \, d^\times t_{22}, \quad db_2 = \mu^2 \, d^\times t_2 \, du_2.$$

We use $dg_2 = d\kappa_2 db_2$ as the Haar measure on $G^0_{2\mathbb{A}}$. It is well known that the volume of $G^0_{2\mathbb{A}}/G_{2k}$ with respect to the measure dg_2 is ϱ^{-1}.

We fix an arbitrary Haar measure dg_1 on $G^0_{1\mathbb{A}}$. Since the rank of the group G_1 is 0, $G^0_{1\mathbb{A}}/G_{1k}$ is compact. We put

$$\tau(G_1) = \int_{G^0_{1\mathbb{A}}/G_{1k}} dg_1.$$

We choose $dg^0 = dg_1 dg_2, d\bar{g} = d^\times \lambda dg^0$ as Haar measures on $G^0_{\mathbb{A}}, \overline{G}_{\mathbb{A}}$, respectively.

For $\eta > 0$, we define

$$T^0_{2\eta+} = \{a(\underline{\mu}^{-1}, \underline{\mu}) \mid \mu \in \mathbb{R}_+, \mu \leqslant \eta\}.$$

Let $\mathfrak{C}_2 \subset \widehat{T}^0_{2\mathbb{A}} N_{2\mathbb{A}}$ be a compact subset. We define $\mathfrak{S}^0_2 = K_2 T^0_{2\eta+} \mathfrak{C}_2$. It is well known that for a suitable choice of η and $\mathfrak{C}_2$, $\mathfrak{S}^0_2$ surjects to $G^0_{2\mathbb{A}}/G_{2k}$. Also there exists another compact set $\widehat{\mathfrak{C}}_2 \in G^0_{2\mathbb{A}}$ such that $\mathfrak{S}^0_2 \subset \widehat{\mathfrak{C}}_2 T^0_{2\eta+}$. We fix a compact subset $\widehat{\mathfrak{C}}_1 \subset G^0_{1\mathbb{A}}$ which surjects to $G^0_{1\mathbb{A}}/G_{1k}$. Let $\widehat{\mathfrak{C}} = \widehat{\mathfrak{C}}_1 \times \widehat{\mathfrak{C}}_2$.

Definition 4.2. — *Let $r \in \mathbb{R}$. We define $C(G^0_{\mathbb{A}}/G_k, r)$ to be the set of continuous functions $f(g^0)$ on $G^0_{\mathbb{A}}/G_k$ satisfying*

$$\sup_{g^0 \in \widehat{\mathfrak{C}} T^0_{2\eta+}} f(g^0)\mu^{-r} < \infty.$$

A function f on $G^0_{\mathbb{A}}/G_k$ is said to be slowly increasing *if $f \in C(G^0_{\mathbb{A}}/G_k, r)$ for some $r \in \mathbb{R}$.*

Note that $C(G^0_{\mathbb{A}}/G_k, r_1) \subset C(G^0_{\mathbb{A}}/G_k, r_2)$ if $r_1 > r_2$ and $C(G^0_{\mathbb{A}}/G_k, r) \subset L^1(G^0_{\mathbb{A}}/G_k, dg^0)$ if $r > -2$.

Lemma 4.3. — *For any $N \geqslant 1$,*

$$\sum_{x \in V^{\mathrm{ss}}_k} \Phi(\bar{g}x) \ll \begin{cases} \lambda^{-2N-2}\mu^{-1} & \text{if} \quad \lambda \geqslant 1, \\ \lambda^{-2-m}\mu^{-1} & \text{if} \quad \lambda \leqslant 1, \end{cases}$$

for $\bar{g} \in \mathbb{R}_+ \times \mathfrak{S}^0$.

Proof. — By (1) of Lemma 4.1, we may assume $\bar{g} = \underline{\lambda}(1, a_2(\underline{\mu}^{-1}, \underline{\mu}))$, $\mu \ll 1$. For $x = (x_1, x_2) \in V^{\mathrm{ss}}_k$, we have $x_1 \neq 0$ and $x_2 \neq 0$. Note that the weight of $a(t_1, t_2) \in T_2$ with respect to each k-coordinate of x_1 and

x_2 is t_1 and t_2, respectively. Hence, by (2) and (3) of Lemma 4.1, for any $N_1, N_2 \geqslant 1$,

$$\begin{aligned}\sum_{x\in V_k^{\mathrm{ss}}} \Phi(\bar{g}x) &\ll (\lambda^{-1}\mu)^{N_1}(\lambda^{-1}\mu^{-1})^{N_2} \max(1,\lambda^{-1}\mu)^{m-1}\max(1,\lambda^{-1}\mu^{-1})^{m-1} \\ &\leqslant \lambda^{-N_1-N_2}\mu^{N_1-N_2}\max(1,\lambda^{2-2m})\max(1,\mu^{m-1})\max(1,\mu^{1-m}) \\ &\ll \lambda^{-N_1-N_2}\max(1,\lambda^{2-2m})\mu^{N_1-N_2+1-m}.\end{aligned}$$

Note that $\max\{1,ab\} \leqslant \max\{1,a\}\cdot\max\{1,b\}$ for $a,b\geqslant 0$. For $\lambda \geqslant 1$, take $N_1 = N+m-2, N_2 = N$. For $\lambda \leqslant 1$, take $N_1 = m-1, N_2 = 1$. Then we have the lemma. □

We will introduce notations for characters.

Definition 4.4.

(1) *Let Ω_1 and Ω_2 be the groups of characters on $G^0_{1\mathbb{A}}/G_{1k}$ and $\mathbb{A}^0/k^\times$, respectively. We put $\Omega = \Omega_1 \times \Omega_2$ and express elements of Ω as $\omega = (\omega_1, \omega_2)$. We put $\omega(g^0) = \omega_1(g_1)\omega_2(\det g_2)$.*
(2) *For $\omega_1 \in \Omega_1$, we define $\omega_1^\iota \in \Omega_1$ by $\omega_1^\iota((g_{11}, g_{12})) = \omega_1((g_{12}^{-1}, g_{11}^{-1}))$. We put $\omega_2^\iota = \omega_2^{-1}$ and $\omega^\iota = (\omega_1^\iota, \omega_2^\iota)$.*
(3) *We put $\delta(\omega_i) = 1$ if ω_i is trivial and $\delta(\omega_i) = 0$ otherwise. Further we let $\delta(\omega) = \delta(\omega_1)\delta(\omega_2)$.*

Now we define the global zeta function.

Definition 4.5. — *For $\Phi \in \mathscr{S}(V_\mathbb{A})$, $s \in \mathbb{C}$ and $\omega \in \Omega$, we define*

$$Z(\Phi, s, \omega) = \int_{\overline{G}_\mathbb{A}/G_k} \lambda^s \omega(g^0) \sum_{x\in V_k^{\mathrm{ss}}} \Phi(\bar{g}x)\, d\bar{g},$$

$$Z_+(\Phi, s, \omega) = \int_{\substack{\overline{G}_\mathbb{A}/G_k \\ \lambda\geqslant 1}} \lambda^s \omega(g^0) \sum_{x\in V_k^{\mathrm{ss}}} \Phi(\bar{g}x)\, d\bar{g}.$$

The integral $Z(\Phi, s, \omega)$ is called the *global zeta function*. By Lemma 4.3, the integral $Z(\Phi, s, \omega)$ converges absolutely and locally uniformly on a certain right half-plane and the integral $Z_+(\Phi, s, \omega)$ is an entire function. Since the global zeta function has an absolute convergence domain for V_k^{ss}, by Theorem (0.3.7) in [19] (which is due to Shintani), $Z(\Phi, s, \omega)$ can be continued meromorphically to the entire plane and satisfies a functional equation

$$Z(\Phi, s, \omega) = Z(\widehat{\Phi}, 2m - s, \omega^\iota).$$

The purpose of this section is to determine the pole structure and to describe the residues by means of certain distributions.

Remark 4.6. — The above definition of the zeta function looks slightly different from the original definition in [12], but the two definitions are essentially the same. We briefly compare these functions. For simplicity we assume ω is trivial. Let $\widetilde{G} = G/\widetilde{T}$. Recall that we put $\widetilde{T} = \ker(G \to \mathrm{GL}(V))$. Let $d\tilde{g}$ be an invariant measure on $\widetilde{G}_{\mathbb{A}}$. The original definition of the global zeta function is as follows:

$$Z^*(\Phi, s) = \int_{\widetilde{G}_{\mathbb{A}}/\widetilde{G}_k} |\chi(\tilde{g})|^s \sum_{x \in V_k^{\mathrm{ss}}} \Phi(\tilde{g}x)\, d\tilde{g}.$$

Since $\widetilde{T} \cong \mathrm{GL}(1) \times \mathrm{GL}(1)$ is a split torus, the first Galois cohomology set $H^1(k', \widetilde{T})$ is trivial for any field k' containing k. This implies that the set of k'-rational point of $\widetilde{G}$ coincides with $G_{k'}/\widetilde{T}_{k'}$. Therefore $\widetilde{G}_{\mathbb{A}} = G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$ and $\widetilde{G}_{\mathbb{A}}/\widetilde{G}_k = G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}} G_k$. Let $\widetilde{T}^0_{\mathbb{A}} = G^0_{\mathbb{A}} \cap \widetilde{T}_{\mathbb{A}}$. Then we have

$$(\mathbb{R}_+ \times G^0_{\mathbb{A}})/\widetilde{T}^0_{\mathbb{A}} \cong G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$$

via the map which sends the class of $(\lambda, g_{11}, g_{12}, g_2)$ to class of $(g_{11}, g_{12}, \underline{\lambda} g_2)$. Moreover, this map is compatible with their actions on $V_{\mathbb{A}}$. If we identify $\overline{G}_{\mathbb{A}}/\widetilde{T}^0_{\mathbb{A}}$ with $\widetilde{G}_{\mathbb{A}}$ via the isomorphism, then we have $|\chi(\bar{g})| = \lambda^4$ for the D_4 case and $|\chi(\bar{g})| = \lambda^{12}$ for the E_6 case. Also the volume of $\widetilde{T}^0_{\mathbb{A}}/\widetilde{T}_k \cong (\mathbb{A}^0/k^\times)^2$ is finite. Hence it ω is trivial, $Z(\Phi, 4s, \omega)$ is a constant multiple of $Z^*(\Phi, s)$ for the D_4 case and $Z(\Phi, 12s, \omega)$ is a constant multiple of $Z^*(\Phi, s)$ for the E_6 case. (The constant depends on the choice of the measure.) Our choice of $Z(\Phi, s, \omega)$ is for the conventions of our global theory.

Let

$$\mathrm{M}_{\omega_2}\Phi(x) = \int_{K_2} \omega_2(\kappa_2)\Phi(\kappa_2 x)\, d\kappa_2.$$

Then $Z(\Phi, s) = Z(\mathrm{M}_{\omega_2}\Phi, s)$ and $\mathrm{M}_{\omega_2}(\mathrm{M}_{\omega_2}\Phi) = \mathrm{M}_{\omega_2}\Phi$. Therefore, we may assume the following for the rest of this section.

ASSUMPTION 4.7. — *The Schwartz-Bruhat function* Φ *satisfies* $\mathrm{M}_{\omega_2}\Phi = \Phi$.

4.3. The principal part

For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we define

$$[x, y] = \mathcal{T}(x_1 y_2) + \mathcal{T}(x_2 y_1).$$

This is a non-degenerate bilinear form on V. For $\bar{g} = (\lambda, g_{11}, g_{12}, g_2) \in \mathbb{R}_+ \times G^0_{\mathbb{A}}$, we define

$$\bar{g}^\iota = (\lambda^{-1}, g_{12}^{-1}, g_{11}^{-1}, \nu^t g_2^{-1} \nu).$$

This is an involution and the above bilinear form satisfies

$$[\bar{g}x, \bar{g}^{\iota}y] = [x, y].$$

Recall that $\langle\ \rangle$ is a non-trivial additive character of $\mathbb{A}/k$. For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$, we define its Fourier transform by

$$\widehat{\Phi}(x) = \int_{V_{\mathbb{A}}} \Phi(y)\langle[x, y]\rangle\, dy.$$

It is easy to see that the Fourier transform of $\Phi(\bar{g}\cdot)$ is $\lambda^{-2m}\widehat{\Phi}(\bar{g}^{\iota}\cdot)$.

For $\lambda \in \mathbb{R}_{+}$, we define $\Phi_{\lambda}(x) = \Phi(\underline{\lambda}x)$.

Definition 4.8. — *For $\Phi \in \mathscr{S}(V_{\mathbb{A}}), s \in \mathbb{C}$ and $g^0 \in G^0_{\mathbb{A}}$, we define*

$$J(\Phi, g^0) = \sum_{x\in S_k} \widehat{\Phi}((g^0)^{\iota}x) - \sum_{x\in S_k} \Phi(g^0 x),$$

$$I^0(\Phi, \omega) = \int_{G^0_{\mathbb{A}}/G_k} \omega(g^0) J(\Phi, g^0)\, dg^0,$$

$$I(\Phi, s, \omega) = \int_0^1 \lambda^s I^0(\Phi_{\lambda}, \omega)\, d^{\times}\lambda.$$

Then by the Poisson summation formula, we have the following.

Proposition 4.9. — *We have*

$$Z(\Phi, s, \omega) = Z_{+}(\Phi, s, \omega) + Z_{+}(\widehat{\Phi}, 2m - s, \omega^{\iota}) + I(\Phi, s, \omega).$$

We study the last term for the rest of this section.

4.4. Stratification

In this subsection, we consider a stratification of V_k. Let

$$Y_1 = \{x \in V \mid x_1 = 0\}, \quad Y_1^{\mathrm{ss}} = \{x \in Y_1 \mid x_2 \neq 0\}.$$

We define $S_1 = GY_1^{\mathrm{ss}}$. Let $P = G_1 \times B_2$.

Lemma 4.10. — *We have*

(1) $V_k \setminus \{0\} = V_k^{\mathrm{ss}} \amalg S_{1k}$,
(2) $S_{1k} = G_k \times_{P_k} Y_{1k}^{\mathrm{ss}}$.

Proof. — We consider (1). Let $x \in V_k \setminus \{0\}$ and $x \notin V_k^{\mathrm{ss}}$. Since either $x_1 \neq 0$ or $x_2 \neq 0$, there exists an element $g \in G_k$ such that the first coordinate of gx is 1. Replacing x by gx, we may assume that x is of the form $x = (1, -x_2)$, where $x_2 \in W_k$. Then $F_x(v)$ is the characteristic polynomial of x_2 and the condition $P(x) = 0$ is equivalent to that the characteristic polynomial of x_2 has a multiple root.

Let $L = k[x_2]$ be the subalgebra of $\mathcal{D}_k$ generated by x_2 over k. Since $\mathcal{D}_k$ has no zero divisor, $k[x_2]$ is a (commutative) integral domain which is finite over the field k. So it is a field. Then since the degree of extension $[L : k]$ divides $\dim_k \mathcal{D}_k = n^2$, it is either 1 or n. Note that we are assuming $n = 2$ or 3. Assume $[L : k] = n$. Then $F_x(v)$ is a minimum polynomial of x_2 over k because the degree of $F_x(v)$ is n. Since any field extension of an algebraic number field is separable, we conclude that $F_x(v)$ does not have a multiple root. This is an contradiction and hence $[L : k] = 1$, which implies $x_2 \in k$. Therefore, there exists an element $g_2 \in G_{2k}$ such that $g_2 x \in Y_{1k}^{\mathrm{ss}}$. This proves (1).

It is easy to see that $P_{1k} Y_{1k}^{\mathrm{ss}} = Y_{1k}^{\mathrm{ss}}$ and that if $x \in Y_{1k}^{\mathrm{ss}}, g \in G_k$ and $gx \in Y_{1k}^{\mathrm{ss}}$ then $g \in P_k$. This proves (2). □

4.5. The smoothed Eisenstein series

To compute $I^0(\Phi, \omega)$, it seems natural to divide the index set S_k of the summation into its G_k-orbits and perform integration separately. However, we can not put this into practice because the corresponding integrals diverge. This is the main difficulty when one calculates the global zeta functions of the prehomogeneous vector spaces. To surmount this problem Shintani [13] introduced the smoothed Eisenstein series of GL(2). He used this series to determine the principal parts of the global zeta functions for the space of binary cubic forms and the space of binary quadratic forms. Later A. Yukie [19] generalized the theory of Eisenstein series to the products of GL(n)'s, and applied it to determine the principal parts of the global zeta functions in some cases. In this subsection, we essentially repeat the argument of Shintani and Yukie in our settings.

We express the Iwasawa decomposition of $g_2 \in G_{2\mathbb{A}}^0$ as

$$g_2 = \kappa_2(g_2) a_2(t_{21}(g_2), t_{22}(g_2)) n_2(u(g_2)).$$

Let $s \in \mathbb{C}$. The Eisenstein series of $G_{2\mathbb{A}}^0$ for B_2 is defined as

$$E(g_2, s) = \sum_{\gamma \in G_{2k}/B_{2k}} |t_{21}(g_2 \gamma)|^{s+1}$$

It is well known that the summation defining $E(g_2, s)$ converges absolutely and locally uniformly in $\Re(s) > 1$ and can be continued meromorphically to the whole complex plane. For analytic properties of $E(g_2, s)$, see [17], [19].

Let $\psi(s)$ be an entire function of s such that

$$\sup_{c_1<\Re(s)<c_2}(1+|s|^N)|\psi(s)|<\infty$$

for all $c_1<c_2, N>0$. Moreover, we assume $\psi(1)\neq 0$.

Definition 4.11. — *For a complex variable w, we define*

$$\mathscr{E}(g^0,w,\psi)=\frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=r_1}E(g_2,s)\frac{\psi(s)}{w-s}\,ds.$$

for some $r_1>1$.

Note that the above definition does not depend on the choice of r_1. The function $\mathscr{E}(g^0,w,\psi)$ is called the smoothed Eisenstein series. When there is no confusion, we drop ψ and use the notation $\mathscr{E}(g^0,w)$ instead.

The following proposition is known as Shintani's lemma.

Proposition 4.12.

(1) *The function $\mathscr{E}(g^0,w)$ is holomorphic for $\Re(w)>0$ except for a simple pole at $w=1$ with the residue $\varrho\psi(1)$.*

(2) *Let $f\in C(G^0_{\mathbb{A}}/G_k,r)$ for some $r>-2$. Then the integral*

$$\int_{G^0_{\mathbb{A}}/G_k}f(g^0)\mathscr{E}(g^0,w)\,dg^0$$

is a holomorphic function of w in the region $\Re(w)>1-\epsilon$ for a constant $\epsilon>0$ except possibly for a simple pole at $w=1$ with residue

$$\varrho\psi(1)\int_{G^0_{\mathbb{A}}/G_k}f(g^0)\,dg^0.$$

(3) *For a slowly increasing function $f(g^0)$ on $G^0_{\mathbb{A}}/G_k$, the integral*

$$\int_{G^0_{\mathbb{A}}/G_k}f(g^0)\mathscr{E}(g^0,w)\,dg^0$$

is a holomorphic function of w in a certain right half-plane.

(4) *We have*

$$\int_{G^0_{\mathbb{A}}/G_k}\omega(g^0)\mathscr{E}(g^0,w)\,dg^0=\delta(\omega)\tau(G_1)\frac{\psi(1)}{w-1}.$$

The above proposition was first proved for GL(2) by Shintani [13, pp. 172, 173, 177]. The adelic proof is given in [17, pp. 527, 528]. In our case, we included the first factor $G^0_{1\mathbb{A}}$ in the statement instead of just considering GL(2), but exactly the same proof works because $G^0_{1\mathbb{A}}/G_{1k}$ is compact. For the convenience of the reader, we indicate the following lemma on the Eisenstein series for $G_2 = \mathrm{GL}(2)$. For the proof, see [13], [17], or [19]. Proposition 4.12 can be proved by the standard argument from this lemma.

Lemma 4.13.

(1) *Let $E_{N_2}(g_2, s)$ be the constant term of $E(g_2, s)$ with respect to N_2, i.e.,*

$$E_{N_2}(g_2, s) = \int_{N_{2\mathbb{A}}/N_{2k}} E(g_2 n_2(u), s)\, du.$$

Then

$$E_{N_2}(g_2, s) = \mu^{-s-1} + \mu^{s-1}\phi(s).$$

(2) *Let $\widetilde{E}(g_2, s) = E(g_2, s) - E_{N_2}(g_2, s)$ be the non-constant term. Then $\widetilde{E}(g_2, s)$ is holomorphic for $\Re(s) > 0$. Moreover, for any s in this region and $l > 1$,*

$$|\widetilde{E}(g_2, s)| \ll \mu^{2l-1}.$$

(3) *We have*

$$\int_{G^0_{2\mathbb{A}}/G_{2k}} \omega_2(g_2)\mathscr{E}(g_2, w)\, dg_2 = \delta(\omega_2)\frac{\psi(1)}{w-1}.$$

Let $\mathscr{E}_{N_2}(g^0, w)$ be the constant term of $\mathscr{E}(g^0, w)$ with respect to N_2 *i.e.,*

$$\mathscr{E}_{N_2}(g^0, w) = \int_{N_{2\mathbb{A}}/N_{2k}} \mathscr{E}(g^0 n(u), w)\, du.$$

By Lemma 4.13 (1), we have

$$\mathscr{E}_{N_2}(g^0, w) = \frac{1}{2\pi\sqrt{-1}} \int_{\Re(s)=r_1} (\mu^{-s-1} + \mu^{s-1}\phi(s))\frac{\psi(s)}{w-s}\, ds.$$

Definition 4.14. — *Let $f(w), g(w)$ be holomorphic functions of $w \in \mathbb{C}$ in some right half-plane. We use the notation $f(w) \sim g(w)$ if $f(w) - g(w)$ can be continued meromorphically to $\{w \mid \Re(w) > 1 - \epsilon\}$ for some $\epsilon > 0$ and is holomorphic at $w = 1$.*

We define

$$I^0(\Phi, \omega, w) = \int_{G^0_{\mathbb{A}}/G_k} \omega(g^0)J(\Phi, g^0)\mathscr{E}(g^0, w)\, dg^0.$$

By Lemma 4.3, $J(\Phi, g) \in C(G^0_{\mathbb{A}}/G_k, -1)$. Hence, by Proposition 4.12 (2), we have the following.

Proposition 4.15. — *We have*

$$I^0(\Phi, \omega, w) \sim \frac{\varrho\psi(1)}{w-1} I^0(\Phi, \omega).$$

DEFINITION 4.16. — *For a complex variable w, we define*

$$\Xi_1(\Phi,\omega,w)=\int_{G^0_{\mathbb{A}}/G_k}\omega(g^0)\sum_{x\in S_{1k}}\Phi(g^0x)\mathscr{E}(g^0,w)\,dg^0,$$

$$\Xi_{\#}(\Phi,\omega,w)=\Phi(0)\int_{G^0_{\mathbb{A}}/G_k}\omega(g^0)\mathscr{E}(g^0,w)\,dg^0.$$

Since $\sum_{x\in S_{1k}}\Phi(g^0x)$ is a slowly increasing function, by Proposition 4.12 (3), the integral $\Xi_1(\Phi,w)$ converges absolutely for sufficiently large $\Re(w)$. It is proved in [19] that $E(g_2,s)=E({}^tg_2^{-1},s)$ for $g_2\in G^0_{2\mathbb{A}}$. Hence, $\mathscr{E}(g^0,w)=\mathscr{E}((g^0)^\iota,w)$ for $g^0\in G^0_{\mathbb{A}}$. Therefore, by Lemma 4.10, we have the following.

PROPOSITION 4.17. — *We have*

$$I^0(\Phi,\omega,w)=\Xi_1(\widehat{\Phi},\omega^\iota,w)+\Xi_{\#}(\widehat{\Phi},\omega^\iota,w)-\Xi_1(\Phi,\omega,w)-\Xi_{\#}(\Phi,\omega,w).$$

For $\Xi_{\#}(\Phi,\omega,w)$, Proposition 4.12 immediately leads to the following.

PROPOSITION 4.18. — *We have*

$$\Xi_{\#}(\Phi,\omega,w)=\delta(\omega)\Phi(0)\tau(G_1)\frac{\psi(1)}{w-1}.$$

We study $\Xi_1(\Phi,w)$ in § 4.7.

4.6. The zeta function associated with the space of division algebra

Since the prehomogeneous vector space (G_1,W) of (single) division algebra appears in the induction process, we have to know the principal part of the zeta function for this case. This function is essentially the same as that of Godement-Jacquet [5]. In this subsection we describe the principal part of the zeta function in this case.

We put $P_1(x)=\mathcal{N}(x)$ for $x\in W$ and $W^{\mathrm{ss}}=\{x\in W\mid P_1(x)\neq 0\}$. Note that $W^{\mathrm{ss}}_k=\{x\in W_k\mid x\neq 0\}$. By assuming that $\lambda\in\mathbb{R}_+$ acts on $W_{\mathbb{A}}$ by multiplication by $\underline{\lambda}$, we may assume that $\mathbb{R}_+\times G^0_{1\mathbb{A}}$ acts on $W_{\mathbb{A}}$.

DEFINITION 4.19. — *For $\Psi\in\mathscr{S}(W_{\mathbb{A}})$, $s\in\mathbb{C}$ and $\omega_1\in\Omega_1$, set*

$$Z_W(\Psi,s,\omega_1)=\int_{\mathbb{R}_+\times G^0_{1\mathbb{A}}/G_{1k}}\lambda^s\omega_1(g_1)\sum_{x\in W^{\mathrm{ss}}_k}\Psi(\underline{\lambda}g_1x)\,d^\times\lambda dg_1,$$

$$Z_{W+}(\Psi,s,\omega_1)=\int_{\substack{\mathbb{R}_+\times G^0_{1\mathbb{A}}/G_{1k}\\ \lambda\geqslant 1}}\lambda^s\omega_1(g_1)\sum_{x\in W^{\mathrm{ss}}_k}\Psi(\underline{\lambda}g_1x)\,d^\times\lambda dg_1.$$

The following lemma is a direct consequence of Lemma 4.1.

Lemma 4.20. — *The integral defining $Z_W(\Psi, s, \omega_1)$ converges absolutely and locally uniformly in the region $\Re(s) > m$, and the integral defining $Z_{W+}(\Psi, s, \omega_1)$ is an entire function.*

For $x, y \in W$, we put

$$[x, y]_W = \mathcal{T}(xy).$$

This defines a non-degenerate bilinear form on W. We note that this bilinear form satisfies $[g_1 x, g_1^{\iota} y]_W = [x, y]_W$ where $(g_{11}, g_{12})^{\iota} = (g_{12}^{-1}, g_{11}^{-1})$.

We define the Fourier transform on $\mathscr{S}(W_{\mathbb{A}})$ by

$$\Psi^*(x) = \int_{W_{\mathbb{A}}} \Psi(y)\langle [x, y]_W\rangle \, dy.$$

Then by the Poisson summation formula, we have

$$\sum_{x \in W_k^{\mathrm{ss}}} \Psi(\underline{\lambda} g_1 x) = \lambda^{-m} \sum_{x \in W_k^{\mathrm{ss}}} \Psi^*(\underline{\lambda}^{-1} (g_1)^{\iota} x) + \lambda^{-m}\Psi^*(0) - \Psi(0).$$

Applying the above equation, we obtain the following principal part formula for this zeta function.

Proposition 4.21. — *We have*

$$Z_W(\Psi, s, \omega_1) = Z_{W+}(\Psi, s, \omega_1) + Z_{W+}(\Psi^*, m - s, \omega_1^{\iota}) + \delta(\omega_1)\tau(G_1)\left(\frac{\Psi^*(0)}{s - m} - \frac{\Psi(0)}{s}\right),$$

where $Z_{W+}(\Psi, s, \omega_1)$ and $Z_{W+}(\Psi^*, m - s, \omega_1^{\iota})$ are entire functions.

4.7. Contribution from unstable strata

In this subsection, we express the residue of $\Xi_1(\Phi, w, \omega)$ in terms of Z_W defined in the previous subsection. We identify Y_1 (see § 4.4) with the space W of single division algebras in § 4.6.

Definition 4.22. — *For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$, we define a Schwartz-Bruhat function $\mathcal{R}_W\Phi$ on $W_{\mathbb{A}}$ by restricting Φ to $Y_{1\mathbb{A}}$.*

Proposition 4.23. — *By changing ψ if necessary, we have*

$$\Xi_1(\Phi, \omega, w) \sim \frac{\varrho\psi(1)}{w - 1}\delta(\omega_2) Z_W(\mathcal{R}_W\Phi, 2, \omega_1).$$

Proof. — We have

$$\begin{aligned}\Xi_1(\Phi,\omega,w) &= \int_{G^0_{\mathbb{A}}/G_k} \omega(g^0) \sum_{x\in S_{1k}} \Phi(g^0x)\mathscr{E}(g^0,w)\,dg^0 \\ &= \int_{G^0_{\mathbb{A}}/P_k} \omega(g^0) \sum_{x\in Y^{\mathrm{ss}}_{1k}} \Phi(g^0x)\mathscr{E}(g^0,w)\,dg^0 \\ &= \int_{P^0_{\mathbb{A}}/P_k} \omega(p^0) \sum_{x\in Y^{\mathrm{ss}}_{1k}} \Phi(p^0x)\mathscr{E}(p^0,w)\,dp^0 \\ &= \int_{G^0_{1\mathbb{A}}/G_{1k}\times B^0_{2\mathbb{A}}/B_{2k}} \omega_1(g_1)\omega_2(b_2) \sum_{x\in Y^{\mathrm{ss}}_{1k}} \Phi((g_1,b_2)x) \\ &\qquad \times \mathscr{E}(b_2,w)\,dg_1db_2 \\ &= \int_{G^0_{1\mathbb{A}}/G_{1k}\times T^0_{2\mathbb{A}}/T_{2k}} \omega_1(g_1)\omega_2(t_2)\mu^2 \sum_{x\in Y^{\mathrm{ss}}_{1k}} \Phi((g_1,t_2)x) \\ &\qquad \times \mathscr{E}_{N_2}(t_2,w)\,dg_1d^\times t_2.\end{aligned}$$

The last step is because N_2 acts on Y_1 trivially. By changing g_{11} to $g_{11}t_{22}^{-1}$, we have

$$\begin{aligned}\Xi_1(\Phi,\omega,w) = \delta(\omega_2)\int_{\mathbb{R}_+\times G^0_{1\mathbb{A}}/G_{1k}} \mu^2\omega_1(g_1) \sum_{x\in W^{\mathrm{ss}}_k} \mathcal{R}_W\Phi((\mu,g_1)x) \\ \times \mathscr{E}_{N_2}(a(\underline{\mu}^{-1},\underline{\mu}),w)\,d^\times\mu dg_1.\end{aligned}$$

By the definition of $Z_W(\Psi,s)$, we have

$$\begin{aligned}&\int_{\mathbb{R}_+\times G^0_{1\mathbb{A}}/G_{1k}} \mu^{\mp s+1}\omega_1(g_1) \sum_{x\in W^{\mathrm{ss}}_k} \mathcal{R}_W\Phi((\mu,g_1)x)\,d^\times\mu dg_1 \\ &= Z_W(\mathcal{R}_W\Phi,\mp s+1,\omega_1)\end{aligned}$$

for $\Re(s) < -m+1, \Re(s) > m-1$, respectively. Since

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=r_2<-m+1} Z_W(\mathcal{R}_W\Phi,-s+1,\omega_1)\frac{\psi(s)}{w-s}\,ds \sim 0,$$

we have

$$\Xi_1(\Phi,\omega,w) \sim \delta(\omega_2)\frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=r_3>m-1} Z_W(\mathcal{R}_W\Phi,s+1,\omega_1)\phi(s)\frac{\psi(s)}{w-s}\,ds.$$

By Proposition 4.21, $Z_W(\mathcal{R}_W\Phi,s+1,\omega_1)$ has a possible simple pole at $s=m-1$ and is holomorphic for $\Re(s)>0$ except for that point. If we consider $(s-m+1)\psi(s)$ instead of $\psi(s)$, this function still satisfies the property we have assumed. Namely,

$$\sup_{c_1<\Re(s)<c_2}(1+|s|^N)|(s-m+1)\psi(s)|<\infty$$

for all $c_1 < c_2, N > 0$ and $(s-m+1)\psi(s)|_{s=1} \neq 0$. Therefore, by changing $\psi(s)$ to $(s-m+1)\psi(s)$, we may assume that $Z_W(\mathcal{R}_W\Phi, s+1, \omega_1)\psi(s)/(w-s)$ is holomorphic for $\Re(s) > 0$. (This is the passing principle (3.6.1) of [19].) Hence,

$$\begin{aligned}\Xi_1(\Phi,\omega_1,w) &\sim \delta(\omega_2)\frac{1}{2\pi\sqrt{-1}}\int_{\Re(s)=1/2} Z_W(\mathcal{R}_W\Phi, s+1,\omega_1)\phi(s)\frac{\psi(s)}{w-s}\,ds\\ &\quad+\frac{\varrho\psi(1)}{w-1}\delta(\omega_2)Z_W(\mathcal{R}_W\Phi,2,\omega_1)\\ &\sim \frac{\varrho\psi(1)}{w-1}\delta(\omega_2)Z_W(\mathcal{R}_W\Phi,2,\omega_1).\end{aligned}$$

This proves the proposition. □

4.8. The principal part formula

THEOREM 4.24. — *Suppose that $\Phi = \mathrm{M}_{\omega_2}\Phi$. Then*

$$\begin{aligned}Z(\Phi,s,\omega) &= Z_+(\Phi,s,\omega)+Z_+(\widehat{\Phi},2m-s,\omega^\iota)\\ &\quad+\delta(\omega)\tau(G_1)\varrho^{-1}\left(\frac{\widehat{\Phi}(0)}{s-2m}-\frac{\Phi(0)}{s}\right)\\ &\quad+\delta(\omega_2)\left(\frac{Z_W(\mathcal{R}_W\widehat{\Phi},2,\omega_1^\iota)}{s-(2m-2)}-\frac{Z_W(\mathcal{R}_W\Phi,2,\omega_1)}{s-2}\right),\end{aligned}$$

where the first two terms in the right hand side are entire functions.

Proof. — By Propositions 4.17, 4.18 and 4.23,

$$\begin{aligned}I^0(\Phi,\omega,w) \sim \frac{\varrho\psi(1)}{w-1}\Big(&\delta(\omega)\tau(G_1)\varrho^{-1}(\widehat{\Phi}(0)-\Phi(0))\\ &+\delta(\omega_2)(Z_W(\mathcal{R}_W\widehat{\Phi},2,\omega_1^\iota)-Z_W(\mathcal{R}_W\Phi,2,\omega_1))\Big)\end{aligned}$$

for a suitable choice of $\psi(s)$. Hence, together with Proposition 4.15, we obtain

$$\begin{aligned}I^0(\Phi,\omega) = \delta(\omega)\tau(G_1)\varrho^{-1}(\widehat{\Phi}(0)-\Phi(0))+\delta(\omega_2)(&Z_W(\mathcal{R}_W\widehat{\Phi},2,\omega_1^\iota)\\ &-Z_W(\mathcal{R}_W\Phi,2,\omega_1)).\end{aligned}$$

Recall that $I(\Phi, s, \omega) = \int_0^1 \lambda^s I^0(\Phi, \omega) d^\times \lambda$ where $\Phi_\lambda(x) = \Phi(\underline{\lambda} x)$. It is easy to see that

$$\Phi_\lambda(0) = \Phi(0), \quad \widehat{\Phi_\lambda}(0) = \lambda^{-2m} \widehat{\Phi}(0).$$

Since

$$Z_W(\mathcal{R}_W \Phi_\lambda, s, \omega_1) = \lambda^{-s} Z_W(\mathcal{R}_W \Phi, s, \omega_1),$$
$$Z_W(\mathcal{R}_W \widehat{\Phi_\lambda}, s, \omega_1^\iota) = \lambda^{2m-s} Z_W(\mathcal{R}_W \widehat{\Phi}, s, \omega_1^\iota),$$

we get

$$Z_W(\mathcal{R}_W \Phi_\lambda, 2, \omega_1) = \lambda^{-2} Z_W(\mathcal{R}_W \Phi, 2, \omega_1),$$
$$Z_W(\mathcal{R}_W \widehat{\Phi_\lambda}, 2, \omega_1^\iota) = \lambda^{2m-2} Z_W(\mathcal{R}_W \widehat{\Phi}, 2, \omega_1^\iota).$$

Then the theorem follows by integrating $\lambda^s I^0(\Phi_\lambda, \omega)$ over $\lambda \in (0, 1]$. □

Theorem 1.2 in the introduction immediately follows from the above theorem.

BIBLIOGRAPHY

[1] N. Bourbaki, *Algèbre. Éléments de mathématique*, Hermann, 1958.

[2] B. Datskovsky & D. J. Wright, "The adelic zeta function associated with the space of binary cubic forms II: Local theory.", *J. Reine Angew. Math.* **367** (1986), p. 27-75.

[3] B. Datskovsky & D. J. Wright., "Density of discriminants of cubic extensions", *J. Reine Angew. Math.* **386** (1988), p. 116-138.

[4] H. Davenport & H. Heilbronn., "On the density of discriminants of cubic fields. II", *Proc. Royal Soc.* **A322** (1971), p. 405-420.

[5] R. Godement & H. Jacquet, "Zeta Functions of Simple Algebras", in *Lecture Notes in Mathematics* (Berlin, Heidelberg, New York) (Springer-Verlag, ed.), vol. 260, 1972.

[6] A. C. Kable & D. J. Wright, "Uniform distribution of the Steinitz invariants of quadratic and cubic extensions", *Compos. Math.* **142** (2006), p. 84-100.

[7] A. C. Kable & A. Yukie, "The mean value of the product of class numbers of paired quadratic fields, I", *Tohoku Math. J.* **54** (2002), p. 513-565.

[8] D. Mumford, "Lectures on curves on an algebraic surface", in *Annales of Mathematical Studies* (Princeton, New Jersey) (P. U. Press, ed.), vol. 59, 1966.

[9] H. Saito, "On a classification of prehomogeneous vector spaces over local and global fields", *Journal of Algebra* **187** (1997), p. 510-536.

[10] ———, "Convergence of the zeta functions of prehomogeneous vector spaces", *Nagoya. Math. J.* **170** (2003), p. 1-31.

[11] M. Sato & T. Kimura, "A classification of irreducible prehomogeneous vector spaces and their relative invariants", *Nagoya Math. J.* **65** (1977), p. 1-155.

[12] M. Sato & T. Shintani, "On zeta functions associated with prehomogeneous vector spaces.", *Ann. of Math.* **100** (1974), p. 131-170.

[13] T. Shintani, "On Dirichlet series whose coefficients are class-numbers of integral binary cubic forms", *J. Math. Soc. Japan* **24** (1972), p. 132-188.

[14] T. Taniguchi, “Distributions of discriminants of cubic algebras”, Preprint 2006, math.NT/0606109.

[15] ———, “A mean value theorem for the square of class number times regulator of quadratic extensions”, Preprint 2006, math.NT/0410531.

[16] A. Weil, *Basic number theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[17] D. J. Wright, “The adelic zeta function associated to the space of binary cubic forms part I: Global theory”, *Math. Ann.* **270** (1985), p. 503-534.

[18] D. J. Wright & A. Yukie, “Prehomogeneous vector spaces and field extensions”, *Invent. Math.* **110** (1992), p. 283-314.

[19] A. Yukie, “Shintani Zeta Functions”, in *Lecture Note Series*, vol. 183, London Math. Soc., cambridge university press ed., 1993.

[20] ———, “On the Shintani zeta function for the space of pairs of binary Hermitian forms”, *J. Number Theory* **92** (2002), p. 205-256.

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