PETER LOEB
BERTRAM WALSH

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THE EQUIVALENCE OF HARNACK’S PRINCIPLE
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IN THE AXIOMATIC SYSTEM OF BRELOT
by PETER A. LOEB (1) AND BERTRAM WALSH (2)

During the last ten years, Marcel Brelot [2] and others have investigated elliptic differential equations in an abstract setting, a setting in which the Harnack principle is assumed to be valid. When necessary, the Harnack principle has been replaced by another axiom which establishes a form of the Harnack inequality. In 1964, Gabriel Mokobodzki showed that the two axioms are equivalent when the underlying space has a countable base for its topology (see [1], pp. 16-18). We shall show that this restriction is unnecessary. First we recall some basic definitions.

Let W be a locally compact Hausdorff space which is connected and locally connected but not compact. Let \( \mathcal{H} \) be a class of real-valued continuous functions with open domains in W such that for each open set \( \Omega \subseteq W \) the set \( \mathcal{H}_\Omega \), (consisting of all functions in \( \mathcal{H} \)) with domains equal to \( \Omega \), is a real vector space. An open subset \( \Omega \) of W is said to be regular for \( \mathcal{H} \) or regular iff its closure in W is compact and for every continuous real-valued function \( f \) defined on \( \partial \Omega \) there is a unique continuous function \( h \) defined on \( \overline{\Omega} \) such that

\[
h|_{\partial \Omega} = f, \quad h|_\Omega \in \mathcal{H}, \quad \text{and} \quad h \geq 0 \quad \text{if} \quad f \geq 0.
\]

Moreover, the class \( \mathcal{H} \) is called a harmonic class on W if it satisfies the following three axioms which are due to Brelot [2]:

**Axiom I.** — A function \( g \) with an open domain \( \Omega \subseteq W \) is an element of \( \mathcal{H} \) if for every point \( x \in \Omega \) there is a function \( h \in \mathcal{H} \) and an open set \( \omega \) with \( x \in \omega \subseteq \Omega \) such that \( g|_\omega = h|_\omega \).

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AXIOM II. — There is a base for the topology of $W$ such that each set in the base is a regular region (non empty connected open set).

AXIOM III. — If $\mathcal{F}$ is a subset of $\mathcal{H}_\Omega$, where $\Omega$ is a region in $W$, and $\mathcal{F}$ is directed by increasing order on $\Omega$, then the upper envelope of $\mathcal{F}$ is either identically $+\infty$ or is a function in $\mathcal{H}_\Omega$.

It follows immediately from Axiom I that if $h$ is in $\mathcal{H}_\Omega$, then the restriction of $h$ to any nonempty open subset of its domain is again in $\mathcal{H}$. Given Axioms I and II, Constantinescu and Cornea ([3], p. 344 and p. 378) have shown that the following axioms are equivalent to Axiom III:

AXIOM III$_1$. — If $\Omega$ is a region in $W$ and $\{h_n\}$ is an increasing sequence of functions in $\mathcal{H}_\Omega$, then either $\lim_{n \to \infty} h_n$ is identically $+\infty$ or $\lim_{n \to \infty} h_n$ is in $\mathcal{H}_\Omega$.

AXIOM III$_2$. — If $\Omega$ is a region in $W$, $K$ a compact subset of $\Omega$, and $x_0$ a point in $K$, then there is a constant $M \geq 1$ such that every nonnegative function $h \in \mathcal{H}_\Omega$ satisfies the inequality

$$h(x) \leq M \cdot h(x_0)$$

at every point $x \in K$.

Given Axioms I and II, we shall show that the following axiom is equivalent to Axiom III.

AXIOM III$_3$. — If $\Omega$ is a region in $W$ then every nonnegative function in $\mathcal{H}_\Omega$ is either identically 0 or has no zeros in $\Omega$. Furthermore, for any point $x_0 \in \Omega$ the set

$$\Phi_{x_0} = \{h \in \mathcal{H}_\Omega : h \geq 0 \quad \text{and} \quad h(x_0) = 1\}$$

is equicontinuous at $x_0$.

Axiom III$_1$ is, of course, just the Harnack principle, and Axiom III$_2$ gives a « weak » Harnack inequality for $\mathcal{H}_\Omega$. On the other hand, a consequence of Axiom III$_3$ is the fact that for any region $\Omega$ and any compact subset $K \subset \Omega$ there is a constant $M \geq 1$ such that for every nonnegative $h \in \mathcal{H}_\Omega$ and every pair of points $x_1$ and $x_2$ in $K$ the relation

$$\frac{1}{M} \cdot h(x_1) \leq h(x_2) \leq M \cdot h(x_1)$$

(1)
holds. Moreover, for any point $x$ in $\Omega$ and any constant $M > 1$ there is a compact neighborhood $K$ of $x$ in which (1) holds. Thus Axiom III$_3$ establishes a strong Harnack inequality for $\mathcal{S}_\Omega$. Mokobodzki has established the equivalence of III$_3$ and III for the case in which the topology of $W$ has a countable base; it is this restriction which we shall now remove.

That Axioms III and III$_3$ are equivalent follows from the

Theorem. — Let $\mathcal{S}$ be a harmonic class and $\Omega$ be a region in $W$. Let $x_0$ be a point in $\Omega$, and set $\Phi = \{ h \in \mathcal{S}_\Omega : h \geq 0$ and $h(x_0) = 1 \}$. Then $\Phi$ is equicontinuous at $x_0$.

Proof. — Let $\omega$ be a regular region and $K$ a compact neighborhood of $x_0$ such that $x_0 \in K \subset \omega \subset \overline{\omega} \subset \Omega$. Each continuous function $f$ on $\partial \omega$ has a unique extension $H(f) \in \mathcal{S}_\omega$, and for each $x \in \omega$ the mapping $f \rightarrow H(f)(x)$ from $C(\partial \omega)$ into the reals is a nonnegative Radon measure on $\partial \omega$, which we denote by $\rho_x$. Axiom III$_3$ (which follows from Axiom III) gives for each pair of points $x_1$ and $x_2$ in $\omega$ a constant $M$ (depending on those points) for which $H(f)(x_1) \leq M \cdot H(f)(x_2)$, i.e.

$$\rho_{x_1} \leq M \cdot \rho_{x_2}$$

in the usual ordering of measures on $\partial \omega$. Hence all the measures $\{ \rho_x \}_{x \in \omega}$ are absolutely continuous with respect to one another, and the Radon-Nikodym density of any one with respect to any other is essentially bounded (« essentially » being unambiguous because all the measures have the same null sets). Following an idea of Mokobodzki’s, we now consider for each $x \in \omega$ the Radon-Nikodym density of $\rho_x$ with respect to $\rho_{x_0}$, which we denote by $g_x$; each $g_x$ is essentially bounded, and $d\rho_x = g_x \cdot d\rho_{x_0}$.

Let $A = \{ h|\omega : h \in \Phi \}$. Axiom III$_3$ states that the functions in $A$ are uniformly bounded on $\partial \omega$, and of course they are continuous there. Thus, if $S$ is any countably infinite subset of $A$, there is a function $f \in L^\infty(\rho_{x_0})$ which is an accumulation point of $S$ with respect to the weak* topology of $L^\infty(\rho_{x_0})$ (i.e. the topology determined by $L^1(\rho_{x_0})$; see [4], p. 424). Since $L^\infty(\rho_{x_0}) \subset L^1(\rho_{x_0})$, $f$ is also an accumulation point of $S$ with respect to the weak topology of $L^1(\rho_{x_0})$ (i.e. the topology determined by $L^\infty(\rho_{x_0})$). Thus by the Eberlein-Šmulian theorem.
([4], p. 430), any sequence in A has a subsequence which converges weakly to an element of \( L^1(\varphi_{x_0}) \). Since each

\[ g_x \in L^\infty(\varphi_{x_0}) = L^1(\varphi_{x_0})^* , \]

it follows that any sequence \( \{h_n\} \) in \( \Phi \) has a subsequence (which we may also denote by \( \{h_n\} \)) for which

\[ h_n(x) = \int_{\delta_{x_0}} h_n(y) g_x(y) d\varphi_{x_0}(y) \]

converges for each \( x \in \omega \); the pointwise limit function \( h \) on \( \omega \) belongs to \( \mathcal{H}_{\omega_0} \) since it is the extension in \( \mathcal{H}_{\omega_0} \) of the weak limit (in \( L^1(\varphi_{x_0}) \)) of the sequence \( \{h_n|_{\delta\omega}\} \). By a result of R.-M. Hervé ([5], p. 432)

\[
\hat{h} = \sup_n \left( \inf_{k > n} h_n \right)
\]

where \( \hat{f}(x) = \sup_{\delta} \left( \inf_{y \in \delta} f(y) \right) \) as \( \delta \) ranges over the neighborhood system of \( x \). Thus \( \hat{h} \) is the limit of the increasing sequence of lower-semicontinuous functions \( \inf_{k > n} h_n \), and that limit is attained uniformly on the compact set \( K \). It follows that \( h_n \to \hat{h} \) uniformly on \( K \), and thus \( \Phi|K \) is relatively sequentially compact, hence relatively compact, in the uniform norm topology of \( C(K) \). So \( \Phi|K \) is equicontinuous (Arzelà; see [4], p. 266), whence \( \Phi \) is equicontinuous at the interior points of \( K \), and in particular at \( x_0 \).

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Peter A. Loeb and B. Walsh,
Department of Mathematics,
University of California,
Los Angeles, Calif. (U.S.A.).