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Soeren FOURNAIS & Bernard HELFFER

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ACCURATE EIGENVALUE ASYMPTOTICS FOR THE MAGNETIC NEUMANN LAPLACIAN

by Soeren FOURNAIS & Bernard HELFFER (*)

ABSTRACT. — Motivated by the theory of superconductivity and more precisely by the problem of the onset of superconductivity in dimension two, many papers devoted to the analysis in a semi-classical regime of the lowest eigenvalue of the Schrödinger operator with magnetic field have appeared recently. Here we would like to mention the works by Bernoff-Sternberg, Lu-Pan, Del Pino-Felmer-Sternberg and Helffer-Morame and also Bauman-Phillips-Tang for the case of a disc. In the present paper we settle one important part of this question completely by proving an asymptotic expansion to all orders for low-lying eigenvalues for generic domains. The word 'generic' means in this context that the curvature of the boundary of the domain has a unique non-degenerate maximum.

RÉSUMÉ. — Motivés par la théorie de la supraconductivité et plus précisément par le problème de l'apparition de la supraconductivité à la surface, de nombreux articles ont été consacrés récemment à l'analyse semi-classique de la plus petite valeur propre de l'opérateur de Schrödinger avec champ magnétique (Bernoff-Sternberg, Lu-Pan, Del Pino-Felmer-Sternberg, Helffer-Morame et aussi Bauman-Phillips-Tang pour le cas du disque). Dans cet article, nous proposons des asymptotiques complètes pour les premières valeurs propres dans le cas d'un domaine de $\mathbb{R}^2$ dont la courbure du bord n'a qu'un unique maximum non-dégénéré.

1. Introduction

The object of study in this paper is a magnetic Schrödinger operator with Neumann boundary conditions in a smooth, bounded domain Ω. We are interested in finding an accurate description of the eigenvalues near the

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bottom of the spectrum. In particular, we will improve estimates given in [15] in the case of constant magnetic field.

Apart from its intrinsic mathematical interest, this question is important for applications to superconductivity. Precise knowledge of the lowest eigenvalues of this magnetic Schrödinger operator is crucial for a detailed description of the nucleation of superconductivity (on the boundary) for superconductors of Type II and for accurate estimates of the critical field H_{C_3}. These applications will be the subject of further work and will be published elsewhere. We refer the reader to the works of Bernoff-Sternberg [3], Lu-Pan [23, 22, 24, 25], and Helffer-Pan [17] for further discussion of this subject and to [33] and [29] for the physical motivation.

Let us fix the notations. The domain $\Omega \subset \mathbb{R}^2$ is supposed to be smooth, bounded and simply connected. Points (x_1, x_2) in $\mathbb{R}^2$ are denoted by z or x. At each point z of the boundary, we denote by $\nu(z)$ the interior unit normal vector to the boundary of Ω. We define the magnetic Neumann operator $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{H}) \ni u \mapsto \mathcal{H}u = \mathcal{H}_{h,\Omega}u = (-ih\nabla_z - A(z))^2 u(z)\ . \tag{1.1}$$

Here $A(z)$ is a vector potential generating a constant magnetic field; $\operatorname{curl} A = 1$. We will make a specific choice of gauge in Definition 1.4 below. The domain $\mathcal{D}(\mathcal{H})$ of the operator $\mathcal{H}$ is defined by

$$\mathcal{D}(\mathcal{H}) = \left\{ u \in H^2(\Omega) \,\middle|\, \nu \cdot (-ih\nabla_z - A(z))u\big|_{\partial\Omega} = 0 \right\}\ .$$

The case of the half-plane, $\Omega = \mathbb{R} \times \mathbb{R}_+$, will be important for fixing notations. After a gauge transformation and a partial Fourier transformation we get, in this case and with $h = 1$, the family of models on the half-line:

$$H^{N,\xi} = D_x^2 + (x+\xi)^2\ , \tag{1.2}$$

on $L^2(\mathbb{R}_+)$ and with Neumann boundary conditions at $x = 0$. Important results about the operators $H^{N,\xi}$ will be recalled in Appendix A; here we only define the notation that will be used throughout the text. Let $\hat{\mu}^{(1)}(\xi)$ be the lowest eigenvalue of $H^{N,\xi}$. Then $\xi \mapsto \hat{\mu}^{(1)}(\xi)$ has a unique minimum Θ_0 attained at a point that we will denote by ξ_0. The corresponding unique positive, normalized eigenfunction of H^{N,ξ_0} will be denoted by u_0. We also introduce:

$$C_1 = \frac{u_0^2(0)}{3}\ . \tag{1.3}$$

The main result of the paper gives the asymptotic expansion of the lowest eigenvalues of $\mathcal{H}$. We define $\mu^{(n)}(h)$ to be the n-th eigenvalue of $\mathcal{H}$, in particular,

$$\mu^{(1)}(h) = \inf \operatorname{Spec} \mathcal{H}_{h,\Omega} \;,$$

and prove the following result.

THEOREM 1.1. — *Suppose that Ω is a smooth bounded domain, that its curvature $\partial\Omega \ni s \mapsto \kappa(s)$ at the boundary has a unique maximum,*

$$\kappa(s) < \kappa(s_0) =: k_{\max} \;, \text{ for all } s \neq s_0 \;, \tag{1.4}$$

and that the maximum is non-degenerate, i.e.

$$k_2 := -\kappa''(s_0) \neq 0 \;. \tag{1.5}$$

Then, for all $n \in \mathbb{N} \setminus \{0\}$, there exists a sequence $\{\zeta_j^{(n)}\}_{j=1}^{\infty} \subset \mathbb{R}$ (which can be calculated recursively to any order) such that $\mu^{(n)}(h)$ admits the following asymptotic expansion (for $h \searrow 0$):

$$\mu^{(n)}(h) \sim \Theta_0 h - k_{\max} C_1 h^{3/2} + C_1 \Theta_0^{1/4} \sqrt{\tfrac{3k_2}{2}}(2n-1)h^{7/4} + h^{15/8} \sum_{j=0}^{\infty} h^{j/8} \zeta_j^{(n)} \;. \tag{1.6}$$

Remarks 1.2.

- The semiclassical limit $h \searrow 0$ is clearly equivalent to a large magnetic field limit, since

$$\int_\Omega |(-i\nabla_z - BA(z))u(z)|^2 \, dz = B^2 \int_\Omega |(-i\tfrac{1}{B}\nabla_z - A(z))u(z)|^2 \, dz \;.$$

- Previous results on the bottom of the spectrum of $\mathcal{H}_{h,\Omega}$ were obtained in [15], where the two first terms in the expansion of $\mu^{(1)}(h)$ were given (see [15, Theorems 10.3 and 11.1]):

$$\mu^{(1)}(h) = \Theta_0 h - k_{\max} C_1 h^{3/2} + \mathcal{O}(h^{5/3}) \;. \tag{1.7}$$

- It is rather reasonable to believe that the proof of Theorem 1.1 can be adapted for getting a similar result under the weaker assumption that there exists $J \in \mathbb{N}$, such that

$$\begin{cases} \kappa^{(2j)}(s_0) = 0 \;, & \text{for } j = 1, 2, \dots, J-1 \;, \\ \kappa^{(2J)}(s_0) \neq 0 \;, & \end{cases} \tag{1.8}$$

  i.e. the maximum is non-degenerate of order $2J$. However we will not pursue this further.

  If the uniqueness condition in (1.4) is replaced by the assumption that there is a finite number of maxima (for which (1.5) is assumed to hold), we expect the existence of sequences of eigenvalues $z^{(n)}(h)$ corresponding to each maximum. This also follows from the techniques applied in the present paper with a little extra work.

- The assumption that Ω is bounded is included for convenience only. It will only be used once (in the proof of Theorem 4.4) in order to allow us to refer directly to a result by Baumann-Phillips-Tang [2]. An adaptation of the techniques present in this paper (and already in [15]) would permit the omission of this assumption.
- We only consider here the case of constant magnetic field, since this is the natural setting for the application that we have in mind (superconductivity). However, the non-constant field case is also interesting —both from a mathematical and a physical point of view— and has been considered in many of the works mentioned above, such as [15] and [23]. It follows easily from those papers that our main result, Theorem 1.1, holds without change in the case of a non-constant field $B(z) = \operatorname{curl} A(z)$ provided $B(z)$ is constant $= B$ on a neighborhood of the boundary $\partial\Omega$ and satisfies outside this neighborhood
$$\inf_{z\in\Omega} B(z) > \Theta_0 B\ .$$
However, we do not pursue this direction further here.
- Our proof of Theorem 1.1 also gives an approximation of the eigenfunction u_h by explicit quasi-modes modulo $\mathcal{O}(h^\infty)$ both in $L^2(\Omega)$- and in $H^2(\Omega)$-norm. This result is not needed in the present paper, but is probably useful for computing the asymptotics of quantities like $\int |u_h(x)|^4 dx$ (occuring in the analysis of the bifurcation in the problem in superconductivity).

For applications to bifurcations from the normal state in superconductivity it seems important to calculate the splitting between the ground and first excited states of $\mathcal{H}(h)$. Let us define

$$\Delta(h) = \mu^{(2)}(h) - \mu^{(1)}(h)\ . \tag{1.9}$$

Corollary 1.3. — *Under the hypothesis from Theorem 1.1, $\Delta(h)$ admits the following asymptotics:*

$$\Delta(h) \sim C_1 \Theta_0^{1/4}\sqrt{6k_2}\, h^{7/4} + h^{15/8}\sum_{j=0}^{\infty} h^{j/8}\xi_j\ . \tag{1.10}$$

where $\xi_j = \zeta_j^{(2)} - \zeta_j^{(1)}$.

The case where Ω is a disc has been analyzed in great detail in [2], using the radial symmetry to reduce the problem to ordinary differential equations. In this case the splitting $\Delta(h)$ turns out to become zero for a sequence of values of h tending to 0. This is a complication in the analysis of bifurcation. Thus, in some sense, the more 'generic' situation considered in this paper has a nicer property. We recall that for the disc it is reasonable to conjecture from [2] that:

$$0 = \liminf_{h\to 0} \frac{\Delta(h)}{h^2} < \limsup_{h\to 0} \frac{\Delta(h)}{h^2} < +\infty\ .$$

We recall also that in the case of a domain with a unique corner, with a sufficiently small angle, one has ([6], [7]):

$$\liminf_{h\to 0} \frac{\Delta(h)}{h} > 0\ .$$

In our case, (1.10) implies:

$$\lim_{h\to 0} \frac{\Delta(h)}{h^{\frac{7}{4}}} > 0\ .$$

Of course (see Bonnaillie [5] for a discussion inspired by Helffer-Sjöstrand [18, 19]), if there are multiple minima and symmetries, one expects an exponentially small gap between the two lowest eigenvalues. This has been confirmed recently by numerical computations by Bonnaillie-Dauge in the case of the square.

The plan of the paper is as follows. In Section 2 we prove a simple non-optimal upper bound to the ground state energy. This calculation motivates the more systematic treatment in Section 3, where we introduce a 'Grushin problem' in order to reduce the analysis to an effective model on the boundary. The effective model allows us to construct quasimodes whose energy corresponds to the lowest eigenvalues of $\mathcal{H}$ to any order in h. Thus we get the upper bound inherent in Theorem 1.1. In order to prove that the Grushin approach also gives a lower bound, we need to prove suitable localization results in phase space. That is carried through in sections 4 and 5. Finally, in Section 6 we finish the proof of Theorem 1.1. Appendix A recalls a number of results from the analysis of the half-plane model that are needed in the calculations. Appendix B contains definitions concerning the coordinate system near the boundary in which all the calculations will take place.

We end this introduction by fixing the gauge in which the actual calculations will be made.

Definition 1.4 (Gauge choice). — *We use the boundary coordinates* (s,t) *defined in Appendix B and chosen such that* $\kappa(0) = k_{\max}$. *Using Lemma B.1, we may make a global gauge change* ϕ *such that on* $(-\frac{|\partial\Omega|}{4}, \frac{|\partial\Omega|}{4}) \times (0, t_0)$, $\tilde{A}$ *has the form*

$$\tilde{A} = \begin{pmatrix} -t + \frac{t^2 k(s)}{2} \\ 0 \end{pmatrix} . \tag{1.11}$$

2. A simple upper bound to the ground state energy

This section contains a simple variational estimate of the ground state energy $\mu^{(1)}(h)$. The motivation for giving this result is a number of remarks and calculations appearing in the literature. It turns out that the 'obvious' choice of trial functions does not give as good energy estimates as one might expect. This motivates the more systematic approach in later sections.

Recall that we have defined the constants Θ_0 and C_1 in the introduction.

Theorem 2.1. — *Suppose* Ω *is a smooth bounded domain. Let*

$$k_{\max} = \sup_s \kappa(s) = \max_s \kappa(s) ,$$

be the maximal curvature of the boundary and let

$$k_2 = \inf_{s \in \kappa^{-1}(k_{\max})} (-\kappa''(s)) .$$

Then the ground state energy $\mu^{(1)}(h)$ *of the operator* $\mathcal{H}$ *(defined in* (1.1)*) satisfies*

$$\limsup_{h \to 0_+} h^{-7/4} \Big\{ \mu^{(1)}(h) - \Big(\Theta_0 h - k_{\max} C_1 h^{3/2} + \sqrt{\frac{k_2 C_1}{2}} h^{7/4} \Big) \Big\} \leqslant 0 .$$

Remark 2.2. — Theorem 2.1 does not give the correct coefficient to the $h^{7/4}$-term (compare with Theorem 1.1). The trial function used in the proof below is too simple since it only uses the ground state u_0 in the normal variable. Note that, when quoting [3] in [26, Remark 4.2], del Pino, Felmer and Sternberg forget to mention that one needs more terms in the [BeSt] expansion of the formal solution to capture the correct coefficient.

Proof. — The proof consists of an explicit calculation with a suitably chosen test function. (This is the same test function as mentioned in [26, Remark 4.2]).

Let us consider a point x_0 on the boundary $\partial\Omega$ such that the curvature of $\partial\Omega$ at x_0 is $k_{\max}$, the maximal curvature of the boundary. We choose our boundary coordinates (s,t) (see Appendix B) such that x_0 has coordinates $(0,0)$. Let $\chi \in C_0^\infty(\mathbb{R})$ be a standard cut-off function:

$$\chi(t) = 1 \text{ for } |t| \leqslant 1/2\ , \quad \text{and } \operatorname{supp}\chi \subset (-1,1)\ .$$

Consider now the test function,

$$\phi(s,t;h) = \phi_0(t,s;h)\,\chi(2s/|\partial\Omega|)\,\chi(t/t_0)\ , \tag{2.1}$$

where, for $\alpha > 0$ to be chosen below,

$$\phi_0(t,s;h) := (2\alpha)^{1/4} h^{-5/16} e^{-\alpha s^2/h^{1/4}} e^{i\xi_0 s/h^{1/2}} u_0(h^{-1/2}t)\ . \tag{2.2}$$

and t_0 is the constant from Appendix B defining the tubular neighborhood of the boundary on which one may use boundary coordinates. The function u_0 satisfies, $u_0'(0) = 0$, so in the gauge given by Definition 1.4, ϕ satisfies the magnetic Neumann boundary condition and therefore $\phi \in \mathcal{D}(\mathcal{H})$.

We will get an upper bound to the ground state energy of the Neumann problem by calculating the Rayleigh quotient $\langle \phi \mid \mathcal{H}\phi\rangle / \|\phi\|^2$ for a suitable ϕ in the domain of $\mathcal{H}$. Actually, one could also work with ϕ in the form domain of the corresponding quadratic form $q_{\mathcal{H}}$. From now on, we fix the gauge such that this property is satisfied.

Then

$$\begin{aligned}\langle \phi \mid \mathcal{H}\phi\rangle = \int_{-|\partial\Omega|/2}^{|\partial\Omega|/2} \int_0^\infty & \Big\{ |(hD_t - \tilde{A}_2)\phi|^2 \\ & + (1 - t\kappa(s))^{-2} |(hD_s - \tilde{A}_1)\phi|^2 \Big\} (1 - t\kappa(s))\, ds\, dt\ .\end{aligned}$$

Now, using the decay properties of u_0 and the exponential decay of the Gaussian, we first get:

$$\begin{aligned}\langle \phi \mid \mathcal{H}\phi\rangle = \int_{-|\partial\Omega|/2}^{|\partial\Omega|/2} \int_0^\infty & \Big\{ |(hD_t)\phi_0|^2 + (1 - t\kappa(s))^{-2} |(hD_s - \tilde{A}_1)\phi_0|^2 \Big\} \\ & \times (1 - t\kappa(s))\chi(2s/|\partial\Omega|)^2\, \chi(t/t_0)^2\, ds\, dt + \mathcal{O}(h^\infty)\ .\end{aligned}$$

Again using the properties of u_0 (see (A.15)) and of the Gaussian, we get

$$(2.3)\quad \begin{aligned}\langle \phi \mid \mathcal{H}\phi\rangle = h^{-5/8}\sqrt{2\alpha}\int_{-|\partial\Omega|/2}^{|\partial\Omega|/2}\int_0^{\infty} e^{-2\alpha s^2/h^{1/4}} h|u_0'(h^{-1/2}t)|^2 \\ \times (1-t\kappa(s))\,ds\,dt \\ + h^{-5/8}\sqrt{2\alpha}\int_{-|\partial\Omega|/2}^{|\partial\Omega|/2}\int_0^{\infty} e^{-2\alpha s^2/h^{1/4}}|u_0(h^{-1/2}t)|^2 \\ \times \left|h^{1/2}\xi_0 + i2\alpha s h^{3/4} + t(1-\frac{t}{2}\kappa(s))\right|^2 \\ \times \ (1-t\kappa(s))^{-1}\,\chi(2s/|\partial\Omega|)^2\,\chi(t/t_0)^2\,ds\,dt \\ + \mathcal{O}(h^\infty)\,.\end{aligned}$$

It is then clear that by interpreting $(1-t\kappa(s))^{-1}$ as $\sum_{n\geqslant 0} t^n\kappa(s)^n$ and computing term by term, the cut-off function in t does not affect the computation modulo $\mathcal{O}(h^\infty)$. So we get

$$(2.4)\quad \begin{aligned}\langle \phi \mid \mathcal{H}\phi\rangle \sim h^{-5/8}\sqrt{2\alpha}\int_{-|\partial\Omega|/2}^{|\partial\Omega|/2}\int_0^{\infty} e^{-2\alpha s^2/h^{1/4}} h|u_0'(h^{-1/2}t)|^2 \\ \times (1-t\kappa(s))\,ds\,dt \\ + h^{-5/8}\sqrt{2\alpha}\int_{-|\partial\Omega|/2}^{|\partial\Omega|/2}\int_0^{\infty} e^{-2\alpha s^2/h^{1/4}}\ |u_0(h^{-1/2}t)|^2 \\ \times \left|h^{1/2}\xi_0 + i2\alpha s h^{3/4} + t(1-\frac{t}{2}\kappa(s))\right|^2 \Big(\sum_n t^n\kappa(s)^n\Big) \\ \times \chi(2s/|\partial\Omega|)^2\ ds\,dt\,.\end{aligned}$$

The next step is to replace $\kappa(s)$ by its Taylor expansion $\kappa^{\mathrm{Tay}}(s)$ at 0, which leads to the equality (modulo $\mathcal{O}(h^\infty)$):

$$(2.5)\quad \begin{aligned}\langle \phi \mid \mathcal{H}\phi\rangle \sim h^{-5/8}\sqrt{2\alpha}\int_{-\infty}^{+\infty}\int_0^{\infty} e^{-2\alpha s^2/h^{1/4}} h|u_0'(h^{-1/2}t)|^2 \\ \times (1-t\kappa^{\mathrm{Tay}}(s))\,ds\,dt \\ + h^{-5/8}\sqrt{2\alpha}\int_{-\infty}^{+\infty}\int_0^{\infty} e^{-2\alpha s^2/h^{1/4}}|u_0(h^{-1/2}t)|^2 \\ \times \left|h^{1/2}\xi_0 + i2\alpha s h^{3/4} + t(1-\frac{t}{2}\kappa^{\mathrm{Tay}}(s))\right|^2 \\ \times \Big(\sum_n t^n\kappa^{\mathrm{Tay}}(s)^n\Big)\,ds\,dt\,.\end{aligned}$$

Here the cut-off functions have completely disappeared and the integration in the s variable is now over $(-\infty,+\infty)$.

We omit in what follows the reference to Taylor expansions written in superscript "Tay" for κ and we use for shortness $(1-t\kappa(s))^{-1}$ instead of $\sum_n t^n\kappa(s)^n$ in the next computations.

With the change of variables $\sigma=\sqrt{2\alpha}h^{-1/8}s$, $\tau=h^{-1/2}t$, we can continue the calculation as

$$\begin{aligned}
\langle\phi\mid\mathcal{H}\phi\rangle &\sim h\int_{-\infty}^{+\infty}\int_0^{\infty} e^{-\sigma^2}|u_0'(\tau)|^2\left(1-h^{1/2}\tau\kappa(\tfrac{h^{1/8}\sigma}{\sqrt{2\alpha}})\right)d\sigma\,d\tau\\
&\quad+h\int_{-\infty}^{+\infty}\int_0^{\infty} e^{-\sigma^2}|u_0(\tau)|^2\\
&\quad\times\left|\xi_0+\tau(1-\frac{h^{1/2}\tau}{2}\kappa(\tfrac{h^{1/8}\sigma}{\sqrt{2\alpha}}))+i\sqrt{2\alpha}h^{3/8}\sigma\right|^2\\
&\quad\times(1-h^{1/2}\tau\kappa(\tfrac{h^{1/8}\sigma}{\sqrt{2\alpha}}))^{-1}\,d\sigma d\tau\\
&=h\left\{T_1+T_2+T_3+T_4+T_5+\mathcal{O}(h^{\frac{7}{8}})\right\},
\end{aligned}$$

with (using that $\kappa'(0)=0$, since $\kappa(0)=k_{\max}$)

$$\begin{aligned}
T_1&=\int_{-\infty}^{+\infty}e^{-\sigma^2}\int_0^{\infty}|u_0'(\tau)|^2+(\xi_0+\tau)^2|u_0(\tau)|^2\,d\tau\,d\sigma\ ,\\
T_2&=-h^{1/2}\int_{-\infty}^{+\infty}\int_0^{\infty}e^{-\sigma^2}\tau|u_0'(\tau)|^2\Big(\kappa(0)+\frac{1}{2}\kappa''(0)\frac{h^{1/4}\sigma^2}{2\alpha}\Big)\,d\tau\,d\sigma\ ,\\
T_3&=2\alpha h^{3/4}\int_{-\infty}^{+\infty}\int_0^{\infty}e^{-\sigma^2}|u_0(\tau)|^2\sigma^2\,d\tau\,d\sigma\ ,\\
T_4&=h^{1/2}\int_{-\infty}^{+\infty}\int_0^{\infty}e^{-\sigma^2}|u_0(\tau)|^2(\xi_0+\tau)^2\tau\Big(\kappa(0)+\frac{1}{2}\kappa''(0)\frac{h^{1/4}\sigma^2}{2\alpha}\Big)\,d\tau\,d\sigma\ ,\\
T_5&=-h^{1/2}\int_{-\infty}^{+\infty}\int_0^{\infty}e^{-\sigma^2}|u_0(\tau)|^2(\xi_0+\tau)\tau^2\Big(\kappa(0)+\frac{1}{2}\kappa''(0)\frac{h^{1/4}\sigma^2}{2\alpha}\Big)\,d\tau\,d\sigma\ .
\end{aligned}$$

Therefore, up to $\mathcal{O}(h^{\frac{15}{8}})$, we get the equivalence

$$\langle\phi\mid\mathcal{H}\phi\rangle\sim h\big\{S_0+h^{1/2}S_{1/2}+h^{3/4}S_{3/4}\big\}\ ,$$

with

$$\begin{aligned}
S_0 &= \Theta_0 \int e^{-\sigma^2}\, d\sigma\ ,\\
S_{1/2} &= \kappa(0) \int e^{-\sigma^2}\, d\sigma \Big[- \int_0^\infty \tau |u_0'(\tau)|^2\, d\tau + \int_0^\infty \tau(\xi_0+\tau)^2 |u_0(\tau)|^2\, d\tau\\
&\qquad - \int_0^\infty \tau^2(\xi_0+\tau)|u_0(\tau)|^2\, d\tau \Big]\ ,\\
S_{3/4} &= 2\alpha \int \sigma^2 e^{-\sigma^2}\, d\sigma + \kappa''(0) \int \frac{\sigma^2}{4\alpha} e^{-\sigma^2}\, d\sigma \Big[- \int_0^\infty \tau |u_0'(\tau)|^2\, d\tau\\
&\qquad + \int_0^\infty \tau(\xi_0+\tau)^2 |u_0(\tau)|^2\, d\tau - \int_0^\infty \tau^2(\xi_0+\tau)|u_0(\tau)|^2\, d\tau \Big]\ .
\end{aligned}$$

From the known moments of u_0 (see Lemma A.1 below or Fournais-Helffer [11, (6.15), (6.16) and (6.17)]) we have

$$\begin{aligned}
&\int_0^\infty \tau |u_0(\tau)|^2\, d\tau = \sqrt{\Theta_0}, && \int_0^\infty \tau(\xi_0+\tau)^2 |u_0(\tau)|^2\, d\tau = \frac{1}{2}(C_1 + \Theta_0^{3/2}),\\
&\int_0^\infty \tau |u_0'(\tau)|^2\, d\tau = C_1 + \frac{\Theta_0^{3/2}}{2}, && \int_0^\infty \tau^2(\xi_0+\tau)|u_0(\tau)|^2\, d\tau = \frac{C_1}{2} + \Theta_0^{3/2}.
\end{aligned}$$

So with $I_0 = \int e^{-\sigma^2}\, d\sigma$, $I_2 = \int \sigma^2 e^{-\sigma^2}\, d\sigma$, we get

$$\begin{aligned}
S_0 &= \Theta_0 I_0\ ,\\
S_{1/2} &= \kappa(0) I_0 \Big[- (C_1 + \frac{\Theta_0^{3/2}}{2}) + \frac{1}{2}(C_1 + \Theta_0^{3/2}) - (\frac{C_1}{2} + \Theta_0^{3/2}) \Big]\\
&= -\kappa(0) I_0 (C_1 + \Theta_0^{3/2})\ ,\\
S_{3/4} &= I_2 \Big[2\alpha - \frac{\kappa''(0)}{4\alpha}(C_1 + \Theta_0^{3/2}) \Big]\ .
\end{aligned}$$

Therefore, we finally find

$$\begin{aligned}
\langle \phi \mid \mathcal{H}\phi \rangle &= h\Theta_0 I_0 - h^{3/2}\kappa(0) I_0 (C_1 + \Theta_0^{3/2})\\
&\qquad + h^{7/4} I_2 \big[2\alpha - \frac{\kappa''(0)}{4\alpha}(C_1 + \Theta_0^{3/2}) \big] + \mathcal{O}(h^{\frac{15}{8}})\ .
\end{aligned}$$

We now compute the asymptotics of $\|\phi\|_2^2$. Along the same lines as the previous computations and with the same conventions, we obtain

$$\begin{aligned}\|\phi\|_2^2 &\sim \int_{-\infty}^{+\infty}\int_0^{\infty} e^{-\sigma^2}|u_0(\tau)|^2(1-h^{1/2}\tau\kappa(\tfrac{h^{1/8}\sigma}{\sqrt{2\alpha}}))\,d\sigma d\tau + \mathcal{O}(h^\infty)\\ &= \int_{-\infty}^{+\infty} e^{-\sigma^2}\Big(1-h^{1/2}\kappa(\tfrac{h^{1/8}\sigma}{\sqrt{2\alpha}})\int_0^{\infty}\tau|u_0(\tau)|^2\,d\tau\Big)\,d\sigma + \mathcal{O}(h^\infty)\\ &= \int_{-\infty}^{+\infty} e^{-\sigma^2}\Big(1-h^{1/2}\kappa(\tfrac{h^{1/8}\sigma}{\sqrt{2\alpha}})\sqrt{\Theta_0}\Big)\,d\sigma + \mathcal{O}(h^\infty)\\ &= I_0 - h^{1/2}\sqrt{\Theta_0}\kappa(0)I_0 - h^{3/4}\sqrt{\Theta_0}\frac{\kappa''(0)}{4\alpha}I_2 + \mathcal{O}(h^{\frac{7}{8}})\,.\end{aligned}$$

So the Rayleigh quotient becomes

$$\frac{\langle\phi\mid\mathcal{H}\phi\rangle}{\|\phi\|_2^2} = \Theta_0 h - \kappa(0)C_1h^{3/2} + (2\alpha - \frac{\kappa''(0)C_1}{4\alpha})\frac{I_2}{I_0}h^{7/4} + \mathcal{O}(h^{\frac{15}{8}})\,.$$

Since the curvature κ has a maximum at $s=0$, we see that $\kappa''(0)\leqslant 0$. We recall that ϕ depends on α and that we can now optimize over α. We recover first the fact that the term in $\mathcal{O}(h^{\frac{3}{2}})$ is obtained without having to specify α. In the case when $k_2=-\kappa''(0)\neq 0$, which is our main interest, the optimal choice of α is

$$\alpha = \sqrt{\frac{k_2C_1}{8}}$$

and we get

$$\frac{\langle\phi\mid\mathcal{H}\phi\rangle}{\|\phi\|_2^2} = \Theta_0 h - \kappa(0)C_1h^{3/2} + \sqrt{\frac{k_2C_1}{2}}\frac{I_2}{I_0}h^{7/4} + \mathcal{O}(h^{\frac{15}{8}})\,.$$

In the case where $\kappa''(0)=0$, we can choose α as small as we wish and therefore get

$$\frac{\langle\phi\mid\mathcal{H}\phi\rangle}{\|\phi\|_2^2} = \Theta_0 h - \kappa(0)C_1h^{3/2} + o(h^{7/4})\,.$$

Using

$$I_0 = \int e^{-\sigma^2}\,d\sigma = \sqrt{\pi}\,, \qquad I_2 = \int \sigma^2 e^{-\sigma^2}\,d\sigma = \frac{\sqrt{\pi}}{2}\,,$$

we therefore get the result of the theorem. □

Remark 2.3. — In the case where $k_2=0$, one would expect that the error term $o(h^{7/4})$ could be replaced by (the stronger) $\mathcal{O}(h^s)$ for some $s\in(7/4,2]$ depending on the order to which the Taylor expansion of $\kappa(s)-\kappa(0)$ vanishes at 0. We will not pursue this further. See however also Remark 4.6.

3. Grushin type approach for upper bounds

3.1. Main statements

In this section we will prove the following accurate upper bound to the n-th eigenvalue of $\mathcal{H}$.

Theorem 3.1. — *Let Ω satisfy the assumptions of Theorem 1.1, and let $n \in \mathbb{N} \setminus \{0\}$. There exist a sequence $\{\zeta_j^{(n)}\}_{j=0}^{\infty} \subset \mathbb{R}$ and a sequence of functions $\{\phi_j^{(n)}\}_{j=0}^{\infty}$ in $\mathcal{D}(\mathcal{H})$ such that, for all $N > 0$, there exists $M > 0$ such that, if*

$$z_M^{(n)}(h) = \Theta_0 h - k_{\max} C_1 h^{3/2} + C_1 \sqrt{\frac{3}{2}} \Theta_0^{1/4} \sqrt{k_2} (2n-1) h^{7/4} + h^{15/8} \sum_{j=0}^{M} h^{j/8} \zeta_j^{(n)} , \tag{3.1}$$

and

$$\phi_M^{(n)}(x,h) = \sum_{j=0}^{M} h^{j/8} \phi_j^{(n)}(x) , \tag{3.2}$$

then (for $h \searrow 0$)

$$\|(\mathcal{H} - z_M^{(n)}) \phi_M^{(n)}\|_{L^2} = \mathcal{O}(h^N) \|\phi_M^{(n)}\|_{L^2} . \tag{3.3}$$

With the notations of the theorem, we define $z_\infty^{(n)}(h)$ as the asymptotic sum

$$z_\infty^{(n)}(h) := \Theta_0 h - k_{\max} C_1 h^{3/2} + C_1 \sqrt{\frac{3}{2}} \Theta_0^{1/4} \sqrt{k_2} (2n-1) h^{7/4} + h^{15/8} \sum_{j=0}^{\infty} h^{j/8} \zeta_j^{(n)} . \tag{3.4}$$

Consequently, $z_M^{(n)}(h)$ is the truncated sum of $z_\infty^{(n)}(h)$ at rank M.

Remark 3.2. — The lowest approximate eigenvalue $z^{(1)}(h)$ agrees with the calculation from Bernoff-Sternberg [3] (see also [32]) up to the order that they calculate (term of order $h^{7/4}$).

Since the operator $\mathcal{H}$ is self-adjoint, we can deduce the existence of eigenvalues near the points with asymptotics $z_\infty^{(n)}$.

Corollary 3.3. — *Let $n \in \mathbb{N} \setminus \{0\}$, $M \in \mathbb{N}$ and let $z_M^{(n)}(h)$ be as above. Then there exist $C > 0$ and $h_0 > 0$ such that*

$$\operatorname{dist}(z_M^{(n)}(h), \operatorname{Spec}(\mathcal{H})) \leqslant C h^{\frac{15+M}{8}} , \ \forall h \in (0, h_0] .$$

Proof. — This is clear by the Spectral Theorem. □

Remark 3.4. — In particular, the upper bound announced in Theorem 1.1 is a direct consequence of Theorem 3.1.

Proof of Theorem 3.1. — The proof is fairly long; so we will split it in different steps described in the next subsections. From now on we will assume that the maximum of κ, $k_{\max}$, is attained at $s = 0$.

3.2. Expanding operators in fractional powers of h

From [15, (B.8)] we get that the operator $\mathcal{H}$ in boundary coordinates becomes

$$\mathcal{H} = a^{-1}\Big[(hD_s - \tilde{A}_1)a^{-1}(hD_s - \tilde{A}_1) + (hD_t - \tilde{A}_2)a(hD_t - \tilde{A}_2)\Big] , \tag{3.5}$$

with

$$a(s,t) = 1 - t\kappa(s) . \tag{3.6}$$

Remark 3.5. — The representation of $\mathcal{H}$ given in (3.5) is only defined on functions with support in $[0, t_0) \times [-|\partial\Omega|/2, +|\partial\Omega|/2]$. We will only apply our operator on functions which are a product of cut-off functions with functions in the form of linear combination of terms like $h^\nu w(h^{-\frac{1}{4}}s, h^{-\frac{1}{2}}t)$, with w in $\mathcal{S}(\mathbb{R} \times \overline{\mathbb{R}^+})$. These functions are consequently $O(h^\infty)$ outside a fixed neighborhood of $(0,0)$. This is similar to the calculations in the previous section. We will do the computations formally in the sense that:

- Everything is determined modulo $\mathcal{O}(h^\infty)$;
- $a^{-1}(s,t)$ will be replaced by $\sum_{n\geqslant 0}(t\kappa(s))^n$;
- $\kappa(s)$ will be replaced by its Taylor's expansion.

For any n and N, we will find M and construct trial functions $\tilde{\phi}_M^{(n)}$ (expressed in boundary coordinates (s,t) and in the form (3.2)), localized near $(s,t) = (0,0)$ and satisfying

$$\|(\mathcal{H} - z_M^{(n)})\tilde{\phi}_M^{(n)}\|_{L^2} = \mathcal{O}(h^N)\|\tilde{\phi}_M^{(n)}\|_{L^2} , \quad (hD_t - \tilde{A}_2)\tilde{\phi}_M^{(n)}\big|_{(s,t)=(s,0)} = 0 . \tag{3.7}$$

By changing back to the original coordinates, this clearly implies (3.3) and that the involved functions satisfy the magnetic Neumann condition (and therefore lies in $\mathcal{D}(\mathcal{H})$). We will omit the tilda's in the following and thus denote by ϕ the trial function in boundary coordinates.

Using Definition 1.4, we work in the gauge where,

$$\tilde{A}_1 = -ta_2(s,t)\ , \qquad \tilde{A}_2 = 0\ ; \qquad a_2(s,t) = 1 - t\kappa(s)/2\ .$$

We make the scaling $\tau = h^{-1/2}t$, $\sigma = h^{-1/8}s$. Then $\mathcal{H}$ becomes

$$\tilde{P} = \tilde{a}^{-1}(h^{7/8}D_\sigma + h^{1/2}\tau\tilde{a}_2)\tilde{a}^{-1}(h^{7/8}D_\sigma + h^{1/2}\tau\tilde{a}_2) + h\tilde{a}^{-1}D_\tau\tilde{a}D_\tau\ , \tag{3.8}$$

with

$$\tilde{a}(\sigma,\tau) = 1 - h^{1/2}\tau\kappa(h^{1/8}\sigma)\ , \quad \tilde{a}_2(\sigma,\tau) = 1 - h^{1/2}\tau\kappa(h^{1/8}\sigma)/2\ . \tag{3.9}$$

Thus

$$h^{-1}\tilde{P} = \tilde{a}^{-1}(h^{3/8}D_\sigma + \tau\tilde{a}_2)\tilde{a}^{-1}(h^{3/8}D_\sigma + \tau\tilde{a}_2) + \tilde{a}^{-1}D_\tau\tilde{a}D_\tau\ .$$

We now define

$$P = e^{-i\sigma\xi_0/h^{3/8}}h^{-1}\tilde{P}e^{i\sigma\xi_0/h^{3/8}} - \Theta_0\ ,$$

and get, after removing the tilda's from the a's,

$$\begin{aligned} P = {} & a^{-1}\big((\tau+\xi_0) + h^{3/8}D_\sigma - \tau(1-a_2)\big) \\ & \times a^{-1}\big((\tau+\xi_0) + h^{3/8}D_\sigma - \tau(1-a_2)\big) + a^{-1}D_\tau a D_\tau - \Theta_0\ . \end{aligned} \tag{3.10}$$

We assume that κ is C^∞ and has a non-degenerate maximum at $s = 0$. Then, in the sense of asymptotic series in powers of $h^{\frac{1}{8}}$, we obtain

$$\begin{aligned} a(\sigma,\tau) &= 1 - h^{1/2}\tau\kappa(h^{1/8}\sigma) \\ &= 1 - h^{1/2}\tau\kappa(0) - \tau\sum_{j=2}^{\infty} h^{1/2+j/8}\frac{\sigma^j\kappa^{(j)}(0)}{j!}\ , \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} a_2(\sigma,\tau) &= 1 - h^{1/2}\tau\frac{\kappa(h^{1/8}\sigma)}{2} \\ &= 1 - h^{1/2}\tau\frac{\kappa(0)}{2} - \tau\sum_{j=2}^{\infty} h^{1/2+j/8}\sigma^j\frac{\kappa^{(j)}(0)}{2(j!)}\ . \end{aligned} \tag{3.12}$$

From the asymptotics of a and a_2, we get (remember the definition of k_2 from (1.5))

$$\begin{aligned} a(\sigma,\tau)^{-1} &= 1 + h^{1/2}\tau\kappa(0) - \tau h^{3/4}\frac{\sigma^2 k_2}{2} + \mathcal{O}(h^{7/8})\ , \\ a(\sigma,\tau)^{-2} &= 1 + 2h^{1/2}\tau\kappa(0) - \tau h^{3/4}\sigma^2 k_2 + \mathcal{O}(h^{7/8})\ , \\ -\tau(1 - a_2(\sigma,\tau)) &= -h^{1/2}\tau^2\frac{\kappa(0)}{2} + \tau^2 h^{3/4}\sigma^2\frac{k_2}{4} + \mathcal{O}(h^{7/8})\ . \end{aligned} \tag{3.13}$$

Thus, we can write

$$P = P_0 + h^{3/8} P_1 + h^{1/2} P_2 + h^{3/4} P_3 + h^{7/8} Q(h) \,, \tag{3.14}$$

where

$$P_0 = D_\tau^2 + (\tau + \xi_0)^2 - \Theta_0 \,, \tag{3.15}$$

$$P_1 = 2D_\sigma (\tau + \xi_0) \,, \tag{3.16}$$

$$\begin{aligned} P_2 &= -2\tau^2 \frac{\kappa(0)}{2} (\tau + \xi_0) + 2\tau\kappa(0)(\tau + \xi_0)^2 \\ &\qquad + \kappa(0)(\tau D_\tau^2 - D_\tau \tau D_\tau) \\ &= \kappa(0)\big(2\tau(\tau + \xi_0)^2 - \tau^2(\tau + \xi_0)\big) + i\kappa(0) D_\tau \,, \end{aligned} \tag{3.17}$$

$$\begin{aligned} P_3 &= D_\sigma^2 - \tau\sigma^2 k_2 (\tau + \xi_0)^2 + 2\tau^2\sigma^2 \frac{k_2}{4} (\tau + \xi_0) \\ &\qquad - \frac{k_2 \sigma^2}{2} (\tau D_\tau^2 - D_\tau \tau D_\tau) \\ &= D_\sigma^2 - \big(2\tau(\tau + \xi_0)^2 - \tau^2(\tau + \xi_0)\big) \frac{k_2\sigma^2}{2} - \frac{k_2\sigma^2}{2} i D_\tau \,, \end{aligned} \tag{3.18}$$

and where $Q(h)$ admits a complete expansion:

$$Q(h) \sim \sum_{j=0}^{\infty} h^{j/8} Q_j \,.$$

We define δP by

$$\delta P = P - P_0 \,, \tag{3.19}$$

We search for functions $\phi^{(n)}(h)$ having an asymptotic expansion in $h^{1/8}$ and such that

$$(P - \frac{z^{(n)}(h) + \Theta_0 h}{h}) \phi^{(n)}(h) \sim 0 \,, \qquad D_\tau \phi^{(n)}(h; \sigma, 0) = 0 \,. \tag{3.20}$$

The constructed functions will have sufficient decay properties to allow interpreting (3.20) in the L^2 sense and therefore, after multiplying by the cutoff appearing in (2.1), we get (3.7) (which implies (3.3)).

3.3. Reduction to the boundary

We will now explain the strategy initiated by Grushin [12] (and references therein) and Sjöstrand in the context of hypoellipticity [31]. Here we use this strategy for producing good trial functions and thereby results for the magnetic Neumann Laplacian.

Let us define the operators R_0^+, R_0^- and E_0 by:

$$\begin{aligned}(3.21)\qquad R_0^+ :\quad & \mathcal{S}(\mathbb{R}_\sigma) \rightarrow \mathcal{S}(\mathbb{R}_\sigma \times (\overline{\mathbb{R}_+})_\tau) \\ & \phi(\sigma) \mapsto \phi(\sigma) u_0(\tau) = \phi \otimes u_0 \ ,\end{aligned}$$

$$\begin{aligned}(3.22)\qquad R_0^- :\quad & \mathcal{S}(\mathbb{R}_\sigma \times (\overline{\mathbb{R}_+})_\tau) \rightarrow \mathcal{S}(\mathbb{R}_\sigma) \\ & f \mapsto \int_0^\infty f(\sigma,\tau) u_0(\tau)\, d\tau \ ,\end{aligned}$$

$$\begin{aligned}(3.23)\qquad E_0 :\quad & \mathcal{S}(\mathbb{R}_\sigma \times (\overline{\mathbb{R}_+})_\tau) \rightarrow \mathcal{S}(\mathbb{R}_\sigma \times (\overline{\mathbb{R}_+})_\tau) \\ & f \otimes \phi \mapsto \begin{cases} f \otimes (P_0^{-1}\phi) \ , & \text{if } \phi \perp u_0 \ , \\ 0 \ , & \text{if } \phi \parallel u_0 \ . \end{cases}\end{aligned}$$

Here we abused notation and considered P_0 as an operator on $L^2((\mathbb{R}_+)_\tau)$ in order to define E_0. That E_0 respects the Schwartz space $\mathcal{S}(\mathbb{R}_\sigma \times (\overline{\mathbb{R}_+})_\tau)$ follows from Lemma A.5.

Notice that R_0^+ is the Hilbertian adjoint of R_0^- (seen as an operator from $L^2(\mathbb{R}_\sigma \times \mathbb{R}^+_\tau; d\sigma d\tau)$ into $L^2(\mathbb{R}_\sigma)$). On the other hand $(P - z)$ is for $z \in \mathbb{R}$ formally selfadjoint for the original (h-dependent) L^2 scalar product inherited from the change of variable $z \mapsto (s,t) \mapsto (\sigma,\tau)$ (that is associated to the measure $(1 - h^{-\frac{1}{2}}\tau\kappa(h^{-\frac{1}{4}}\sigma) d\sigma d\tau))$ but not for the usual L^2 associated to the standard Lebesgue measure $d\sigma d\tau$.

With the above notations we define matrices of operators

$$(3.24)\qquad \mathcal{P}(z) = \begin{pmatrix} P - z & R_0^+ \\ R_0^- & 0 \end{pmatrix} , \qquad \mathcal{E}_0 = \begin{pmatrix} E_0 & R_0^+ \\ R_0^- & 0 \end{pmatrix} .$$

These operators act on $\mathcal{S}(\mathbb{R}_\sigma \times (\overline{\mathbb{R}_+})_\tau) \times \mathcal{S}(\mathbb{R}_\sigma)$. Actually we should better think of operators applied to formal expansions in suitable fractional powers of h with coefficients in these $\mathcal{S}$ spaces. These infinite formal expansions will then be truncated at a suitable rank for defining our quasimodes. So we prefer to write formal expansions to infinite order, having in mind that we could actually go back to truncated expansions if we want a given, arbitrarily small, remainder estimate.

We note first that:

$$(3.25)\qquad \begin{pmatrix} P_0 & R_0^+ \\ R_0^- & 0 \end{pmatrix} \circ \mathcal{E}_0 = I \ .$$

An easy calculation gives then:

$$\mathcal{P}(z)\mathcal{E}_0 = I + \mathcal{K} \ ,$$

where

$$\mathcal{K} = \begin{pmatrix} (\delta P - z)E_0 & (\delta P - z)R_0^+ \\ 0 & 0 \end{pmatrix} .$$

If we look for $z = z(h)$ satisfying

$$z(h) \sim \sum_{\ell \geqslant 3} \hat{z}_\ell h^{\frac{\ell}{8}} , \tag{3.26}$$

and having in mind the expansion (3.14), we observe that $(\delta P - z) = \mathcal{O}(h^{3/8})$, when acting on a fixed function in $\mathcal{S}(\mathbb{R}_\sigma \times (\overline{\mathbb{R}^+})_\tau)$ and can actually be expanded in powers of $h^{\frac{1}{8}}$, starting from $h^{\frac{3}{8}}(P_1 - \hat{z}_3)$. So, if we define

$$\mathcal{Q}_\infty \sim \sum_{j=0}^{+\infty} (-1)^j \mathcal{K}^j ,$$

then the operator is well defined (after reordering) as a formal expansion in powers of $h^{\frac{1}{8}}$ and

$$\mathcal{P}(z)\mathcal{E}_0\mathcal{Q}_\infty \sim I . \tag{3.27}$$

Now,

$$\mathcal{K}^j \sim \begin{pmatrix} [(\delta P - z)E_0]^j & [(\delta P - z)E_0]^{j-1}(\delta P - z)R_0^+ \\ 0 & 0 \end{pmatrix} ,$$

and therefore, if we write

$$\mathcal{E}(z) := \mathcal{E}_0\mathcal{Q}_\infty = \begin{pmatrix} E_\infty(z) & E_\infty^+(z) \\ E_\infty^-(z) & E_\infty^\pm(z) \end{pmatrix} ,$$

we get, in the sense of formal expansions in powers of $h^{\frac{1}{8}}$,

$$(P - z)E_\infty(z) + R_0^+ E_\infty^-(z) \sim 1 , \tag{3.28a}$$

$$(P - z)E_\infty^+(z) + R_0^+ E_\infty^\pm(z) \sim 0 , \tag{3.28b}$$

$$R_0^- E_\infty(z) \sim 0 , \tag{3.28c}$$

$$R_0^- E_\infty^+(z) \sim 1 . \tag{3.28d}$$

So, in particular, if $\sigma \mapsto \phi_\infty(\sigma; h)$ is a function[(1)] such that

$$E_\infty^\pm(z)\phi_\infty \sim 0 , \tag{3.29}$$

(1) More precisely $\phi_\infty(\,\cdot\,; h) \sim \sum_{j\in\mathbb{N}} h^{\frac{j}{8}} \phi_j(\,\cdot\,)$, so all the computations have to be expanded in powers of $h^{\frac{1}{8}}$.

then inserting ϕ_∞ in (3.28b) (i.e. inserting $\begin{pmatrix} 0 \\ \phi_\infty \end{pmatrix}$ in (3.27)), we find

$$(P-z)E^+_\infty(z)\phi_\infty \sim 0\,, \tag{3.30}$$

where everything is well defined modulo $\mathcal{O}(h^\infty)$.

3.4. Construction of trial functions

From the above, we see that $E^\pm_\infty(z)$ is the following asymptotic series,

$$E^\pm_\infty(z) = \sum_{j=1}^{\infty}(-1)^j R_0^-[(\delta P - z)E_0]^{j-1}(\delta P - z)R_0^+\,. \tag{3.31}$$

We look as before for

$$\phi_\infty(\sigma;h) \sim \sum_{j=0}^{\infty}\phi_j(\sigma)h^{j/8}\,,$$

$$z_\infty(h) \sim h^{3/8}z_1 + h^{1/2}z_2 + h^{3/4}z_3 + h^{7/8}\sum_{j=0}^{\infty}\zeta_j h^{j/8}\,,$$

such that

$$E^\pm_\infty(z_\infty(h))\phi_\infty(\sigma;h) \sim 0\,, \tag{3.32}$$

in the sense of asymptotic series in powers of $h^{\frac{1}{8}}$. Here the functions ϕ_j are supposed to be in $\mathcal{S}(\mathbb{R}_\sigma)$.

Lemma 3.6. — *For each $n \in \mathbb{N}\setminus\{0\}$, there exists a unique solution $(z^{(n)}(h), \phi^{(n)}(h))$ to equation* (3.32), *in the sense of asymptotic series, and such that*

$$z^{(n)}(h) \sim C_1\sqrt{\frac{3}{2}\sqrt{\Theta_0}k_2}\,(2n-1)h^{3/4} + h^{7/8}\sum_{j=0}^{\infty}\zeta_j^{(n)}h^{j/8}\,.$$

Conversely, for any pair $(z(h),\varphi(h))$ such that (3.32) *is satisfied, with $z(h) \sim Ch^{\frac{3}{4}} + h^{\frac{7}{8}}\sum_{j\geqslant 0}\zeta_j h^{\frac{j}{8}}$ and $\varphi(h) \sim \sum_{j\geqslant 0}h^{\frac{j}{8}}\varphi_j$, there exist n and $c(h) \sim \sum_j c_j h^{\frac{j}{8}}$ such that $z(h) = z^{(n)}(h)$ and $\varphi(h) = c(h)\phi^{(n)}(h)$.*

Proof of Lemma 3.6. — Let us write

$$E^\pm_\infty(z_\infty(h)) \sim h^{3/8}E_1 + h^{1/2}E_2 + h^{3/4}E_3 + h^{7/8}\sum_{j=0}^{\infty}h^{j/8}F_j\,. \tag{3.33}$$

The terms in this sum will be given in (3.34), (3.35), (3.38), and (3.45) below. Using the definitions (3.14) and (3.19), and the fact that $R_0^- E_0 = 0$ and $E_0 R_0^+ = 0$, we get modulo terms of order $\mathcal{O}(h^{\frac{7}{8}})$,

$$E_\infty^\pm(z_\infty(h)) \sim -R_0^-\Big(h^{3/8}(P_1 - z_1) + h^{1/2}(P_2 - z_2) + h^{3/4}(P_3 - z_3)\Big)R_0^+ \\ + h^{\frac{3}{4}} R_0^- P_1 E_0 P_1 R_0^+ + \mathcal{O}(h^{7/8}) \,.$$

Since also $R_0^-(\tau + \xi_0)R_0^+ = 0$, we find, using (3.16),

$$E_1 = -R_0^-(P_1 - z_1)R_0^+ = z_1 \,. \tag{3.34}$$

Furthermore, using again (3.17),

$$E_2 = -R_0^-(P_2 - z_2)R_0^+ = z_2 - \kappa(0)(I_{1,1} + I_{1,2}) \,,$$

with

$$I_{1,1} = \int_0^\infty [2\tau(\tau + \xi_0)^2 - \tau^2(\tau + \xi_0)]u_0^2(\tau)\, d\tau \,,$$

and

$$I_{1,2} = i\int_0^\infty u_0(\tau)D_\tau u_0(\tau)\, d\tau \,.$$

Using Proposition A.2, we get

$$I_{1,1} + I_{1,2} = -C_1 \,,$$

and therefore

$$E_2 = z_2 + \kappa(0)C_1 \,. \tag{3.35}$$

The term E_3 becomes, inserting P_1 and P_3 from (3.16) and (3.18),

$$\begin{aligned} E_3 &= -R_0^-(P_3 - z_3)R_0^+ + R_0^- P_1 E_0 P_1 R_0^+ \\ &= z_3 - D_\sigma^2 + \frac{k_2\sigma^2}{2}(I_{1,1} + I_{1,2}) + 4D_\sigma^2 I_2 \,, \end{aligned} \tag{3.36}$$

where we have introduced

$$I_2 = \int_0^\infty (\tau + \xi_0)u_0(\tau)P_0^{-1}(\tau + \xi_0)u_0(\tau)\, d\tau \,. \tag{3.37}$$

Using Proposition A.2 and Proposition A.3, we have

$$1 - 4I_2 = 3C_1\sqrt{\Theta_0} \,, \qquad I_{1,1} + I_{1,2} = -C_1 \,,$$

and we therefore get

$$E_3 = z_3 - 3C_1\sqrt{\Theta_0}D_\sigma^2 - C_1\frac{k_2\sigma^2}{2} \,. \tag{3.38}$$

Remember that $\kappa(s)$ has a non-degenerate maximum at $s = 0$, so $k_2 = -\kappa''(0) > 0$.

The first terms. In order to get the equation (3.32) to be satisfied, we choose

$$z_1 = 0 \,, \qquad z_2 = -\kappa(0)\, C_1 \,, \tag{3.39}$$

which implies

$$E_1 = 0 \,, \qquad E_2 = 0 \,. \tag{3.40}$$

With this choice, (3.32) becomes

$$0 \sim h^{3/4} E_3 \phi_0 + \mathcal{O}(h^{7/8}) \,.$$

So we determine z_3 and ϕ_0 by

$$E_3 \phi_0 = 0 \,. \tag{3.41}$$

Let us solve the equation $E_3\phi = 0$. It reads, with $k_2 = -\kappa''(0)$,

$$z_3 \phi = C_1 (3\sqrt{\Theta_0} D_\sigma^2 + \frac{k_2 \sigma^2}{2})\, \phi \,.$$

So, after the scaling $\tilde{s} = \sqrt[4]{k_2/6\sqrt{\Theta_0}}\; \sigma$, we find that z_3 should be an eigenvalue of the harmonic oscillator

$$C_1 \sqrt{\frac{3}{2}\sqrt{\Theta_0} k_2}\; \left(D_{\tilde{s}}^2 + \tilde{s}^2\right) \,.$$

Thus the possible values of z_3 are:

$$z_3^{(n)} = C_1 \sqrt{\frac{3}{2}\sqrt{\Theta_0} k_2}\; (2n-1) \,, \text{ where } n \in \mathbb{N} \setminus \{0\} \,. \tag{3.42}$$

In particular, using the inequality $3C_1\sqrt{\Theta_0} = 1 - 4I_2 < 1$ (see Proposition A.3), we get that $z_3^{(1)}$ is smaller than the value in Theorem 2.1.

Remark 3.7. — A second look at the calculations above (comparing with Section 2) shows why Theorem 2.1 does not give the correct ground state energy to order $h^{7/4}$. By using a trial state which has the simple form (2.1) and (2.2), we would not see the term $R_0^- P_1 E_0 P_1 R_0^+$ in the first line of (3.36) and therefore the last term, $4D_\sigma^2 I_2$, would be missing in the second line of (3.36). Thus the harmonic oscillator discussed above would become

$$D_{\tilde{s}}^2 + \frac{k_2 C_1}{2} \tilde{s}^2 \,,$$

instead. This harmonic oscillator has ground state energy $\sqrt{k_2 C_1/2}$ in agreement with the result of Theorem 2.1.

The iteration procedure. Let us define Π to be the orthogonal projection on $\{\phi_0\}^\perp$. Notice that ϕ_0 depends on the n chosen in (3.42) even though we do not explicitly recall this dependence in the notation. We will choose ϕ_j such that

$$\phi_j \perp \phi_0 \text{ for all } j > 0\ . \tag{3.43}$$

The term of order $h^{\frac{3}{4}+\frac{j}{8}}$ in (3.32) becomes

$$E_3\phi_j + \sum_{k=0}^{j-1} F_k\phi_{j-1-k} = 0\ . \tag{3.44}$$

Notice that

$$F_j = \zeta_j - R_0^- Q_j R_0^+ + \tilde{F}_j\ , \tag{3.45}$$

where $\tilde{F}_j$ only depends on z_1, z_2, z_3 and $\{\zeta_k\}_{k=0}^{j-1}$. By taking the scalar product with ϕ_0 in the equation (3.44), we therefore get, by using (3.41) and the property that E_3 is self adjoint,

$$\zeta_{j-1}\|\phi_0\|^2 = \langle \phi_0\ , R_0^- Q_{j-1} R_0^+ \phi_0\rangle - \langle \phi_0 \mid \tilde{F}_{j-1}\phi_0\rangle - \langle \phi_0 \mid \sum_{k=0}^{j-2} F_k\phi_{j-1-k}\rangle\ .$$

Since $\phi_0 \neq 0$, this equation determines $\zeta_{j-1} \in \mathbb{C}$ as a function of z_1, z_2, z_3, $\{\zeta_k\}_{k=0}^{j-2}$ and $\{\phi_k\}_{k=0}^{j-1}$. The property that $\zeta_{j-1} \in \mathbb{R}$ will be only proved later.

Upon projecting the equation (3.44) on $\{\phi_0\}^\perp$, and remembering the choice (3.43), we get

$$\Pi E_3 \Pi \phi_j = -\Pi\Big(\sum_{k=0}^{j-1} F_k\phi_{j-1-k}\Big)\ . \tag{3.46}$$

Since $\Pi E_3 \Pi$ is invertible on $\{\phi_0\}^\perp$, (3.46) together with (3.43) determines ϕ_j.

Uniqueness. Suppose that z_1, z_2 are not given by the choice in (3.39). For concreteness, let us suppose that $z_1 \neq 0$. Then the equation (3.32) implies that $\phi \sim 0$. Thus (3.39) is the only nontrivial choice.

Furthermore, in the construction above we imposed that $\phi_j \perp \phi_0$ for all $j > 0$. Suppose we do not impose that condition. Let $\overline{\phi}_j$ be the solution constructed above and let ϕ_j be the new solution. Then we can write each ϕ_j as

$$\phi_j = \phi_j' + c_j\phi_0\ , \qquad \text{with } \phi_j' \perp \phi_0, \quad c_j \in \mathbb{C}\ . \tag{3.47}$$

We now write

$$\phi(h) \sim \phi_0 + \sum_{j\geqslant 1} h^{j/8}\phi_j \sim c(h)\phi_0 + \sum_{j\geqslant 1} h^{j/8}\phi'_j \ ,$$

with

$$c(h) \sim 1 + \sum_{j\geqslant 1} h^{j/8} c_j \ ,$$

and

$$\phi'_j \perp \phi_0 \ .$$

By linearity, we therefore find that $\phi(h)/c(h)$ is the solution $\phi_0+\sum_{j\geqslant 1} h^{j/8}\overline{\phi}_j$ constructed above.

This finishes the proof of Lemma 3.6 □

Using Lemmas 3.6, A.4, and A.5, we can finish the proof of Theorem 3.1. Let $(z^{(n)}(h), \phi^{(n)}(h))$ be one of the formal solutions from Lemma 3.6. By stopping the formal sum at a finite number of terms we obtain partial sums $(z_M^{(n)}(h), \phi_M^{(n)}(h))$, solutions to

$$E_M^{\pm}(z_M^{(n)}(h))\phi_M^{(n)}(h) = h^M R_M(h) \ , \tag{3.48}$$

where $E_M^{\pm}$ is also defined by stopping the expansion of $E_\infty^{\pm}$. Since the ϕ_j's are Schwartz functions and all involved operators respect the space $\mathcal{S}$ (they are differential operators whose coefficients are smooth with polynomially bounded derivatives), the remainder $R_M(h)$ in (3.48) is bounded in $\mathcal{S}$. Using Lemma A.5, Lemma A.4 and the fact that all terms in E_M^+ preserve the Schwartz space (differential operators with polynomially bounded, smooth derivatives), we see that $E_M^+\phi_M^{(n)}(h)$ defines a finite sum, whose coefficients are in the space $\mathcal{S}(\mathbb{R}\times\overline{\mathbb{R}_+})$. Thus, the procedure described in Subsection 3.3 above (reduction to the boundary) gives a solution $\psi_M(h) = E_M^+\phi_M^{(n)}(h)$ to the equation

$$(P - z_M(h))\psi_M(h) = \mathcal{O}(h^M) \ .$$

Here the right hand side is in $\mathcal{S}$ and controlled in $\mathcal{O}(h^M)$ for any semi-norm on $\mathcal{S}$, thus in particular in the L^2 norm.

Moreover, $\mathcal{H}$ being selfadjoint, we can now prove that $\zeta_j \in \mathbb{R}$ and this finishes the proof of Theorem 3.1. □

4. Space localization

In this section we will prove that the ground state is well localized both in s and t. In the following Section 5 we will prove a similar (slightly weaker) localization result in the frequency variable ξ corresponding to s.

4.1. Agmon estimates in the normal direction

If ϕ is a function with compact support in Ω, i.e. $\phi \in C_0^\infty(\Omega)$, then

$$\begin{aligned} \int_\Omega |(-ih\nabla - A)\phi|^2\,dx &= \int_{\mathbb{R}^2} |(-ih\nabla - A)\phi|^2\,dx \\ &\geqslant h\|\phi\|^2_{L^2(\mathbb{R}^2)} = h\|\phi\|^2_{L^2(\Omega)}\,. \end{aligned} \tag{4.1}$$

Since $1 > \Theta_0$, this implies (compare $1\cdot h$ with $\Theta_0 \cdot h$) that functions with energy below our upper bounds from Theorem 3.1 cannot be localized in the interior of Ω (i.e. away from the boundary), as $h \to 0$. The powerful method of Agmon estimates can be applied to strengthen this property into an exponential localization of the eigenfunctions (corresponding to the bottom of the spectrum) in a neighborhood of the boundary. This is the content of the following proposition.

Theorem 4.1 (Normal Agmon estimates). — *Let $h_0 > 0, M \in (\Theta_0, 1)$. Then there exists $C, \alpha > 0$ and $h_1 \in (0, h_0]$ such that if $(u_h)_{h\in(0,h_0]}$ is a family of normalized eigenfunctions of $\mathcal{H}_{h,\Omega}$ with corresponding eigenvalue $\mu(h)$ satisfying $\mu(h) \leqslant Mh$, then, for all $h \in (0, h_1]$,*

$$\int_\Omega e^{2\alpha \operatorname{dist}(x,\partial\Omega)/h^{1/2}} \big(|u_h(x)|^2 + h^{-1}|(-ih\nabla - A)u_h(x)|^2\big)\,dx \leqslant C\,. \tag{4.2}$$

Proof. — The proof is similar to (but easier than) the proof of Theorem 4.9 below. We omit the details and refer to Helffer-Morame [15, Section 6.4, p. 621-623] or Helffer-Pan [17]. In [15] only the ground state is considered, but it is immediate to see that the analysis goes through for the eigenfunctions corresponding to higher eigenvalues. □

As a corollary, we get the weaker but useful estimate for u_h near the boundary.

Corollary 4.2 (Weak normal Agmon estimates). — *Let the assumptions be as in Theorem 4.1. For any integer k, there exist $C > 0$ and h_0, such that*

$$||\operatorname{dist}(x,\partial\Omega)^k u_h||_{L^2(\Omega)} \leqslant C\,h^{\frac{k}{2}}\,, \quad \forall h \in (0, h_0]\,. \tag{4.3}$$

Remark 4.3. — The L^2 statement in Theorem 4.1 can be converted to an L^∞ result using the Sobolev imbedding theorem. See [15, Theorem 6.3] for details.

4.2. First lower bound

In order to get good localization properties of the eigenfunctions in the variable parallel to the boundary (the s variable), we need to improve the lower bound on the ground state energy from (1.7). We will prove the following improvement of [15, Theorem 10.3].

THEOREM 4.4. — *Let Ω be a bounded region with smooth boundary satisfying the assumptions of Theorem 1.1. Then*

$$\mu^{(1)}(h) \geqslant \Theta_0 h - C_1 k_{\max} h^{3/2} + \mathcal{O}(h^{7/4}) . \tag{4.4}$$

Proof. — Since Ω is bounded, we have $k_{\max} > 0$. Using the results from [14, Section 10], we may localize to the region near boundary points with $\kappa(s) > 0$, so we may assume without loss of generality in the proof that $\kappa(s) > 0$ for all s.

The proof of Theorem 4.4 is similar to the proof of [14, Theorem 10.3]. We just need to do one of the estimates slightly more carefully. In particular, the proof goes by comparison with the case of a disc. For a disc, one can calculate the ground state energy with great precision by using the rotational symmetry. This was carried through in [2]. We state one of their results in the following form.

THEOREM 4.5. — *Let $\mu^{(1)}(h,b,D(0,R))$ be the ground state energy of the operator in* (1.1) *in the case where* $\operatorname{curl} A = b$ *(independent of $x \in \Omega$) and $\Omega = D(0,R)$; the disc of radius R. Then there exists $C > 0$ such that if*

$$bR^2/h \geqslant C ,$$

then

$$\mu^{(1)}(h,b,D(0,R)) \geqslant \Theta_0 bh - C_1 b^{1/2} h^{3/2}/R - Ch^2 R^{-2} . \tag{4.5}$$

Notice that in the case of a disc, the curvature κ is constant, $\kappa = R^{-1}$.

Let $\rho > 0$. We can find a sequence $\{s_{j,h}\}_{j=0}^{N(h)}$ in $\mathbb{R}/|\partial\Omega|$ and a partition of unity $\{\tilde{\chi}_{j,h}\}_{j=0}^{N(h)}$ on $\mathbb{R}/|\partial\Omega|$ such that $\operatorname{supp}\tilde{\chi}_{j,h} \cap \operatorname{supp}\tilde{\chi}_{k,h} = \emptyset$ if

$j \notin \{k-1, k, k+1\}$ (with the convention that $N(h)+1 = 0\,, 0-1 = N(h)$). Furthermore, we may impose the conditions

$$\operatorname{supp}\tilde{\chi}_{j,h} \subset s_{j,h} + [-h^\rho, h^\rho]\,,$$

$$\sum_j \tilde{\chi}_{j,h}^2 = 1\,, \quad \sum_j |\nabla\tilde{\chi}_{j,h}|^2 \leqslant Ch^{-2\rho}\,. \tag{4.6}$$

We will always choose the $s_{j,h}$ such that $|s_{j,h}| \leqslant |\partial\Omega|/2$.

Let χ_1, χ_2 be a standard partition of unity on $\mathbb{R}$:

$$\chi_1^2 + \chi_2^2 = 1\,, \quad \operatorname{supp}\chi_1 \subset (-2,2)\,, \quad \chi_1 = 1 \text{ on a nbhd of } [-1,1]\,. \tag{4.7}$$

Let us define

$$\chi_{j,h}(s,t) = \tilde{\chi}_{j,h}(s)\chi_1(t/h^\rho)\,.$$

We will also consider $\chi_{j,h}$ as a function on Ω (by passing to boundary coordinates) without changing the notation. For $\psi \in D(\mathcal{H})$, we can write

$$\psi = \sum_j \chi_{j,h}^2\psi + \theta_{2,h}^2\psi\,, \tag{4.8}$$

with

$$\theta_{j,h}(x) = \chi_j(t(x)/h^\rho)\,, \text{ for } j = 1,\, 2\,. \tag{4.9}$$

We get by the 'IMS'-formula:

$$\langle\psi\,|\,\mathcal{H}\psi\rangle = \big\langle\psi\,|\,\mathcal{H}\big(\sum_j \chi_{j,h}^2\psi + \theta_{2,h}^2\psi\big)\big\rangle \tag{4.10}$$

$$= \sum_j \langle\chi_{j,h}\psi\,|\,\mathcal{H}\chi_{j,h}\psi\rangle - h^2\int|\nabla\chi_{j,h}|^2|\psi|^2\,dx$$

$$+ \langle\theta_{2,h}\psi\,|\,\mathcal{H}\theta_{2,h}\psi\rangle - h^2\int|\nabla\theta_{2,h}|^2|\psi|^2\,dx\,. \tag{4.11}$$

In particular, for $\psi = u_h^{(1)}$ being a normalized ground state wave function, we get using the weak normal Agmon estimates (in the t variable) and the condition $\rho < 1/2$ (we will choose $\rho = 1/8$ in the end).

$$\mu^{(1)}(h) = \sum_j \langle\chi_{j,h}u_h^{(1)}\,|\,\mathcal{H}\chi_{j,h}u_h^{(1)}\rangle + \mathcal{O}(h^{2-2\rho})\,. \tag{4.12}$$

If $|s_{j,h}| > |\partial\Omega|/4$, we have $\kappa(s_{j,h}) > \delta > 0$. Therefore we get from [14, Proposition 10.5] that there exists a constant $C > 0$ such that for all j with $|s_{j,h}| > |\partial\Omega|/4$,

$$\langle\chi_{j,h}u_h^{(1)}\,|\,\mathcal{H}\chi_{j,h}u_h^{(1)}\rangle \geqslant \big(\Theta_0 h - C_1(k_{\max}+\delta)\,h^{3/2} - C'h^{5/3}\big)\|\chi_{j,h}u_h^{(1)}\|^2$$

$$\geqslant \big(\Theta_0 h - C_1 k_{\max}h^{3/2} - Ch^{7/4}\big)\|\chi_{j,h}u_h^{(1)}\|^2\,. \tag{4.13}$$

For the remaining terms in (4.12), we will write the inner products, $\langle\cdot\,|\,\cdot\rangle$, in boundary coordinates and compare with the similar term with fixed curvature. Remember the gauge choice fixed in Definition 1.4. Define

$$\tilde{A}_1(s,t) = -t(1-t\kappa(s)/2)\;, \qquad a(s,t) = 1-t\kappa(s)\;,$$

and

$$\begin{aligned}\mathcal{B}_{j,h} &:= \langle \chi_{j,h}u_h^{(1)}\,|\,\mathcal{H}\chi_{j,h}u_h^{(1)}\rangle \\ &= \int e[\chi_{j,h}u_h^{(1)}](s,t)\,ds\,dt\;,\end{aligned} \tag{4.14}$$

with

$$e[f] := a^{-1}|(hD_s-\tilde{A}_1)f|^2 + a|hD_tf|^2\;. \tag{4.15}$$

Similarly, we define

$$\kappa_{j,h}=\kappa(s_{j,h})\;,\quad \tilde{A}_{1,j,h}(s,t) = -t(1-t\kappa_{j,h}/2)\;,\quad a_{j,h} = 1-t\kappa_{j,h}\;,$$

and

$$\mathcal{A}_{j,h} := \int e_{j,h}[\chi_{j,h}u_h^{(1)}](s,t)\,ds\,dt\;, \tag{4.16}$$

with

$$e_{j,h}[f] := a_{j,h}^{-1}|(hD_s-\tilde{A}_{1,j,h})f|^2 + a_{j,h}|hD_tf|^2\;. \tag{4.17}$$

Then we will compare $\mathcal{B}_{j,h}$ and $\mathcal{A}_{j,h}$. We clearly have

$$e[\chi_{j,h}u_h^{(1)}](s,t) = e_{j,h}[\chi_{j,h}u_h^{(1)}](s,t) + f_1(s,t)+f_2(s,t)+f_3(s,t)\;, \tag{4.18}$$

and

$$\begin{aligned}f_1 &= (a^{-1}-a_{j,h}^{-1})|(hD_s-\tilde{A}_1)\chi_{j,h}u_h^{(1)}|^2 + (a-a_{j,h})|hD_t(\chi_{j,h}u_h^{(1)})|^2\;,\\ f_2 &= a_{j,h}^{-1}|(\tilde{A}_1-\tilde{A}_{1,j,h})\chi_{j,h}u_h^{(1)}|^2\;,\\ f_3 &= 2a_{j,h}^{-1}\operatorname{Re}\Big\{(\tilde{A}_1-\tilde{A}_{1,j,h})\chi_{j,h}\overline{u_h^{(1)}}(hD_s-\tilde{A}_{1,j,h})\chi_{j,h}u_h^{(1)}\Big\}\;.\end{aligned}$$

Notice that for $s\in s_{j,h}+[-h^\rho,h^\rho]$, we have, since $\kappa'(0)=0$,

$$\begin{aligned}|\kappa(s)-\kappa_{j,h}| &= |s-s_{j,h}|\cdot\Big|\int_0^1\kappa'((1-\ell)s_{j,h}+\ell s)\,d\ell\Big|\\ &\leqslant Ch^\rho\big(|s_{j,h}|+h^\rho\big)\;.\end{aligned} \tag{4.19}$$

Thus,

$$\begin{aligned}|a-a_{j,h}| &= t|\kappa(s)-\kappa_{j,h}| \leqslant Ch^\rho\big(|s_{j,h}|+h^\rho\big)t\;,\\ |a^{-1}-a_{j,h}^{-1}| &\leqslant Ch^\rho\big(|s_{j,h}|+h^\rho\big)t\;,\ \text{ for } t<2h^\rho\;.\end{aligned} \tag{4.20}$$

We estimate, using (4.20), for any $\varepsilon > 0$,

$$
\begin{aligned}
|f_1(s,t)| &\leqslant Ch^\rho\big(|s_{j,h}| + h^\rho\big)t\, e[\chi_{j,h}u_h^{(1)}](s,t)\\
&\leqslant C'\varepsilon h^{1/4+2\rho}\big(|s_{j,h}| + h^\rho\big)^2\, e[\chi_{j,h}u_h^{(1)}](s,t)\\
&\quad + C'\varepsilon^{-1}h^{-1/4}t^2\, e[\chi_{j,h}u_h^{(1)}](s,t)\\
&\leqslant C''\varepsilon h^{1/4+2\rho}\big(|s_{j,h}| + h^\rho\big)^2\, e_{j,h}[\chi_{j,h}u_h^{(1)}](s,t)\\
&\quad + C'\varepsilon^{-1}h^{-1/4}t^2\, e[\chi_{j,h}u_h^{(1)}](s,t)\,. \qquad (4.21)
\end{aligned}
$$

We also estimate f_2 and f_3 by

$$
f_2(s,t) \leqslant Ch^{2\rho}t^4|\chi_{j,h}u_h^{(1)}|^2\,, \qquad (4.22)
$$

and

$$
\begin{aligned}
|f_3(s,t)| &\leqslant 2Ct^2h^\rho\big(|s_{j,h}| + h^\rho\big)a_{j,h}^{-1}\Big|\chi_{j,h}\overline{u_h^{(1)}}(hD_s - \tilde{A}_{1,j,h})\chi_{j,h}u_h^{(1)}\Big|\\
&\leqslant C'\varepsilon^{-1}h^{-1/4}t^4|\chi_{j,h}u_h^{(1)}|^2\\
&\quad + C'\varepsilon h^{1/4+2\rho}\big(|s_{j,h}| + h^\rho\big)^2 e_{j,h}[\chi_{j,h}u_h^{(1)}](s,t)\,. \qquad (4.23)
\end{aligned}
$$

Thus, we get by combining (4.18) with (4.21), (4.22), and (4.23) and integrating,

$$
\begin{aligned}
\mathcal{B}_{j,h} &\geqslant \{1 - C\varepsilon h^{1/4+2\rho}\big(|s_{j,h}| + h^\rho\big)^2\}\mathcal{A}_{j,h}\\
&\quad - C\varepsilon^{-1}h^{-1/4}\int t^2 e[\chi_{j,h}u_h^{(1)}](s,t)\,ds\,dt\\
&\quad - C(h^{2\rho} + \varepsilon^{-1}h^{-1/4})\int t^4|\chi_{j,h}u_h^{(1)}|\,ds\,dt\,. \qquad (4.24)
\end{aligned}
$$

From Theorem 4.5 we get the estimate

$$
\begin{aligned}
\mathcal{A}_{j,h} &\geqslant \Big(\Theta_0 h - C_1\kappa_{j,h}h^{3/2} - Ch^2\Big)\|\chi_{j,h}u_h^{(1)}\|^2\\
&\geqslant \Big(\Theta_0 h - C_1\big(k_{\max} - c_0\min(|s_{j,h}|^2,1)\big)h^{3/2} - Ch^2\Big)\|\chi_{j,h}u_h^{(1)}\|^2\,, \qquad (4.25)
\end{aligned}
$$

for some $c_0 > 0$, using the non-degeneracy of the maximum. Therefore, using that $|s_{j,h}| \leqslant |\partial\Omega|/2$, we get that, for ε sufficiently small and $\rho = 1/8$,

$$
\begin{aligned}
&\big\{1 - C\varepsilon h^{1/4+2\rho}\big(|s_{j,h}| + h^\rho\big)^2\big\}\mathcal{A}_{j,h} \qquad (4.26)\\
&\qquad\qquad \geqslant \big(\Theta_0 h - C_1 k_{\max}h^{3/2} - Ch^{7/4}\big)\|\chi_{j,h}u_h^{(1)}\|^2.
\end{aligned}
$$

Therefore, we get from (4.24) and (4.13), for the given choice of ε and $\rho = 1/8$,

$$\begin{aligned}\sum_j \mathcal{B}_{j,h} \geqslant \Big(\Theta_0 h - C_1 k_{\max} h^{3/2} - C h^{7/4}\Big)\|u_h^{(1)}\|^2 \\ - C h^{-1/4}\int \sum_j t^2 e[\chi_{j,h} u_h^{(1)}](s,t)\,ds\,dt \\ - C h^{-1/4}\int t^4 |u_h^{(1)}|^2\,ds\,dt\,.\end{aligned} \tag{4.27}$$

The weak normal Agmon estimates and easy manipulations (as in [15, Section 10]) give that the last two terms in (4.27) are bounded by $Ch^{7/4}$. Therefore, the theorem follows from (4.27) and (4.12), remembering that $\rho = 1/8$. □

Remark 4.6. — The above proof actually extends to the case where $k_{\max}$ is a non-degenerate maximum of higher order, i.e.

$$\kappa(s) = k_{\max} + a s^{2N} + \mathcal{O}(|s|^{2N+1})\,,$$

with $N \geqslant 2$ and $a \neq 0$. In that case (4.19) becomes

$$|\kappa(s) - \kappa_{j,h}| \leqslant C h^{\rho}\big(|s_{j,h}|^{2N-1} + h^{\rho}\big)\,. \tag{4.28}$$

This implies that (4.24) becomes

$$\begin{aligned}\mathcal{B}_{j,h} \geqslant \big\{1 - C\varepsilon h^{1/4+2\rho}\big(|s_{j,h}|^{2N-1} + h^{\rho}\big)^2\big\}\mathcal{A}_{j,h} \\ - C\varepsilon^{-1} h^{-1/4}\int t^2 e[\chi_{j,h} u_h^{(1)}](s,t)\,ds\,dt \\ - C(h^{2\rho} + \varepsilon^{-1} h^{-1/4})\int t^4 |\chi_{j,h} u_h^{(1)}|^2\,ds\,dt\,.\end{aligned} \tag{4.29}$$

But instead of (4.25), we get

$$\begin{aligned}\mathcal{A}_{j,h} \geqslant \\ \big\{\Theta_0 h - C_1\big(k_{\max} - c_0 \max(|s_{j,h}|^{2N}, 1)\big) h^{3/2} - C h^2\big\}\|\chi_{j,h} u_h^{(1)}\|^2\,,\end{aligned} \tag{4.30}$$

Since $|s|^{2N} \geqslant |s|^{2(2N-1)}$ for small s, this implies the result. Actually, in this case one should be able to optimize the proof above (in particular choose $\rho < 1/8$) and get a better error bound than $\mathcal{O}(h^{7/4})$ in (4.4).

It is convenient to have a lower bound of the operator $\mathcal{H}$ in terms of a potential U_h. That is our next statement.

THEOREM 4.7. — *There exist* ε_0 , $C_0 > 0$ *such that, if*

$$\tilde{\kappa}(s) := k_{\max} - \varepsilon_0 s^2 \ , \tag{4.31}$$

and

$$U_h(x) = \begin{cases} h \ , & \text{if } t(x) > 2h^{1/8} \ , \\ \Theta_0 h - C_1\tilde{\kappa}(s)h^{3/2} - C_0 h^{7/4} \ , & \text{if } t(x) \leqslant 2h^{1/8} \ , \end{cases} \tag{4.32}$$

then

$$\langle u \,|\, \mathcal{H}u\rangle \geqslant \int_\Omega U_h(x)|u(x)|^2\,dx \ , \tag{4.33}$$

for all $u \in D(\mathcal{H})$ *and all* $h \in (0,1]$.

Remark 4.8. — It is very likely that one could replace $\tilde{\kappa}$ by κ in (4.32) (see also [14, Proposition 10.2]). However, we do not need this improvement.

Proof. — With $\theta_{1,h}$, $\theta_{2,h}$ as in (4.9) and $\rho = 1/8$, we have

$$\begin{aligned}\langle u \,|\, \mathcal{H}u\rangle &= \langle \theta_{1,h}u \,|\, \mathcal{H}\theta_{1,h}u\rangle + \langle \theta_{2,h}u \,|\, \mathcal{H}\theta_{2,h}u\rangle \\ &\quad - Ch^{7/4}\int_{\{h^{1/8}\leqslant t(x)\leqslant 2h^{1/8}\}} |u|^2\,dx \ .\end{aligned}$$

Since $\langle \theta_{2,h}u \,|\, \mathcal{H}\theta_{2,h}u\rangle \geqslant h\|\theta_{2,h}u\|^2$, it therefore suffices to prove that

$$\langle u \,|\, \mathcal{H}u\rangle \geqslant \int_\Omega \widetilde{U}_h(x)|u(x)|^2\,dx \ , \tag{4.34}$$

for all $u \in H^1(\Omega)$ and all $h \in (0,1]$, where

$$\widetilde{U}_h(x) = \begin{cases} \gamma h \ , & \text{if } t(x) > 2h^{1/8} \ , \\ \Theta_0 h - C_1\tilde{\kappa}(s)h^{3/2} - C_0' h^{7/4} \ , & \text{if } t(x) \leqslant 2h^{1/8} \ , \end{cases} \tag{4.35}$$

and $\gamma = (1+\Theta_0)/2$ and C_0' is some positive constant.

Let $\tilde{u}_h^{(1)}$ be a ground state for $\mathcal{H} - \widetilde{U}_h$ with ground state energy $\tilde{\mu}^{(1)}(h)$. We will prove that $\tilde{\mu}^{(1)}(h) \geqslant 0$.

Since $\Theta_0 < \gamma < 1$, the normal Agmon estimates, Theorem 4.1, are also valid for $\tilde{u}_h^{(1)}$.

Using the 'IMS'-formula, and notations as in the proof of Theorem 4.4, we get

$$\begin{aligned}\tilde{\mu}^{(1)}(h) \geqslant \sum_j \langle \chi_{j,h}\tilde{u}_h^{(1)} \,|\, \big(\mathcal{H} - \widetilde{U}_h\big)\chi_{j,h}\tilde{u}_h^{(1)}\rangle + \int (h - \widetilde{U}_h)|\theta_{2,h}\tilde{u}_h^{(1)}|^2\,dx \\ - C\,h^{7/4}\int_{\{h^{1/8}\leqslant t(x)\leqslant 2h^{1/8}\}} |\tilde{u}_h^{(1)}|^2\,dx \ .\end{aligned}$$

Modulo choosing C_0' sufficiently big, it therefore suffices to prove that

$$\sum_j \left\langle \chi_{j,h}\tilde{u}_h^{(1)} \,|\, \big(\mathcal{H} - \Theta_0 h + C_1\tilde{\kappa}(s)h^{3/2}\big)\chi_{j,h}\tilde{u}_h^{(1)}\right\rangle \geqslant -Ch^{7/4}\int \sum_j |\chi_{j,h}\tilde{u}_h^{(1)}|^2\,dx\,. \tag{4.36}$$

Since the normal Agmon estimates hold for $\tilde{u}_h^{(1)}$, we can now go through the proof of Theorem 4.4 with $u_h^{(1)}$ replaced everywhere by $\tilde{u}_h^{(1)}$. We replace $u_h^{(1)}$ everywhere by $\tilde{u}_h^{(1)}$, in particular in the definition of $\mathcal{A}_{j,h}$ and $\mathcal{B}_{j,h}$, which are then denoted by $\tilde{\mathcal{A}}_{j,h}$ and $\tilde{\mathcal{B}}_{j,h}$. In particular, we get as in (4.25)

$$\tilde{\mathcal{A}}_{j,h} \geqslant \big\{\Theta_0 h - C_1\big(k_{\max} - c_0\min(|s_{j,h}|^2,1)\big)h^{3/2} - Ch^2\big\}\|\chi_{j,h}\tilde{u}_h^{(1)}\|^2\,. \tag{4.37}$$

So

$$\big\{1 - C\varepsilon h^{1/4+2\rho}\big(|s_{j,h}| + h^{\rho}\big)^2\big\}\tilde{\mathcal{A}}_{j,h} - \big(\Theta_0 h - C_1\tilde{\kappa}(s)h^{3/2}\big)\|\chi_{j,h}\tilde{u}_h^{(1)}\|^2 \geqslant -Ch^{7/4}\|\chi_{j,h}\tilde{u}_h^{(1)}\|^2\,. \tag{4.38}$$

Therefore,

$$\begin{aligned}
\sum_j \left\langle \chi_{j,h}\tilde{u}_h^{(1)} \,|\, \big(\mathcal{H} - \Theta_0 h + C_1\tilde{\kappa}(s)h^{3/2}\big)\chi_{j,h}\tilde{u}_h^{(1)}\right\rangle
&= \sum_j \Big(\tilde{\mathcal{B}}_{j,h} - \big(\Theta_0 h - C_1\tilde{\kappa}(s)h^{3/2}\big)\|\chi_{j,h}\tilde{u}_h^{(1)}\|^2\Big)\\
&\geqslant -Ch^{7/4}\sum_j \|\chi_{j,h}\tilde{u}_h^{(1)}\|^2\\
&\quad - Ch^{-1/4}\int \sum_j t^2\, e[\chi_{j,h}\tilde{u}_h^{(1)}](s,t)\,ds\,dt\\
&\quad - Ch^{-1/4}\int t^4\,|\tilde{u}_h^{(1)}|^2\,ds\,dt\,.
\end{aligned}$$

Using the weak normal Agmon estimates to bound the last terms by $\mathcal{O}(h^{7/4})$, this implies (4.36) and therefore finishes the proof of Theorem 4.7. □

4.3. Agmon estimates in the tangential direction

Theorem 4.7 can be used to obtain exponential localization estimates in the tangential (s-)variable.

THEOREM 4.9 (Tangential Agmon estimates). — *Let $h_0 > 0$, $M > 0$. Then there exist C, $\alpha > 0$ and $h_1 \in (0, h_0]$, such that if $(u_h)_{h\in]0,h_0]}$ is a family of normalized eigenfunctions of $\mathcal{H}$ with corresponding eigenvalue $\mu(h)$ satisfying the bound*

$$\mu(h) \leqslant \Theta_0 h - C_1 k_{\max} h^{3/2} + M h^{7/4} \quad , \quad \forall h \in (0, h_0] \, ,$$

and if $\chi_1 \in C_0^\infty$ is the function from (4.7), *then, for all $h \in (0, h_1]$,*

$$\int_\Omega e^{2\alpha|s(x)|^2/h^{1/4}} \chi_1^2(t(x)/h^{1/8}) \times \Big\{ |u_h(x)|^2 + h^{-1} \big|(-ih\nabla - A(x))u_h(x)\big|^2 \Big\} \, dx \leqslant C \, . \tag{4.39}$$

Proof. — First we observe that there exists $\beta > 0$ such that, for all $S > 0$, we have (with χ_2 from (4.7))

$$\begin{aligned} \mu(h) \Big\| \chi_2(\tfrac{s}{Sh^{1/8}}) \chi_1(\tfrac{t}{h^{1/8}}) e^{\alpha|s|^2/h^{1/4}} u_h \Big\|^2 & \\ = \Big\langle \chi_2^2(\tfrac{s}{Sh^{1/8}}) \chi_1^2(\tfrac{t}{h^{1/8}}) e^{2\alpha|s|^2/h^{1/4}} u_h \,|\, \mathcal{H} u_h \Big\rangle & \\ \geqslant \Big\langle \chi_2(\tfrac{s}{Sh^{1/8}}) \chi_1(\tfrac{t}{h^{1/8}}) e^{\alpha|s|^2/h^{1/4}} u_h \,|\, \mathcal{H} \chi_2(\tfrac{s}{Sh^{1/8}}) \chi_1(\tfrac{t}{h^{1/8}}) e^{\alpha|s|^2/h^{1/4}} u_h \Big\rangle & \\ - Ch^2 \int \Big| \nabla (\chi_2(\tfrac{s}{Sh^{1/8}}) \chi_1(\tfrac{t}{h^{1/8}})) \Big|^2 e^{2\alpha|s|^2/h^{1/4}} |u_h|^2 \, dx & \\ - \alpha^2 \beta h^{3/2} \int \chi_2^2(\tfrac{s}{Sh^{1/8}}) \chi_1^2(\tfrac{t}{h^{1/8}}) s^2 e^{2\alpha|s|^2/h^{1/4}} |u_h|^2 \, dx \, . & \end{aligned} \tag{4.40}$$

But it follows from Theorem 4.7 and (4.9) that if α is chosen such that $\beta\alpha^2 \leqslant \varepsilon_0 C_1/2$, then

$$\begin{aligned} \Big\langle \chi_2(\tfrac{s}{Sh^{1/8}}) \chi_1(\tfrac{t}{h^{1/8}}) e^{\alpha|s|^2/h^{1/4}} u_h \,\Big|\, \big(\mathcal{H} - \mu(h) - \beta\alpha^2 s^2 h^{3/2}\big) & \\ \times \chi_2(\tfrac{s}{Sh^{1/8}}) \chi_1(\tfrac{t}{h^{1/8}}) e^{\alpha|s|^2/h^{1/4}} u_h \Big\rangle & \\ \geqslant (\frac{\varepsilon_0 C_1 S^2}{2} - C_0 - M) h^{7/4} \big\| \chi_2(\tfrac{s}{Sh^{1/8}}) \chi_1(\tfrac{t}{h^{1/8}}) e^{\alpha|s|^2/h^{1/4}} u_h \big\|^2 \, . & \end{aligned} \tag{4.41}$$

Therefore, it follows from (4.40) and (4.41) that, for α sufficiently small and S sufficiently big,

$$\begin{aligned} \big\| \chi_2(\tfrac{s}{Sh^{1/8}}) \chi_1(\tfrac{t}{h^{1/8}}) e^{\alpha|s|^2/h^{1/4}} u_h \big\|^2 & \\ \leqslant Ch^{1/4} \int \big| \nabla (\chi_2(\tfrac{s}{Sh^{1/8}}) \chi_1(\tfrac{t}{h^{1/8}})) \big|^2 e^{2\alpha|s|^2/h^{1/4}} |u_h|^2 \, dx \, . & \end{aligned} \tag{4.42}$$

Now,

$$h^{1/4}\int \big|\nabla(\chi_2(\tfrac{s}{Sh^{1/8}})\chi_1(\tfrac{t}{h^{1/8}}))\big|^2 e^{2\alpha|s|^2/h^{1/4}}|u_h|^2\,dx \leqslant I + II\ , \tag{4.43}$$

with

$$I := C\int |\chi_2'(\tfrac{s}{Sh^{1/8}})|^2\chi_1^2(\tfrac{t}{h^{1/8}})e^{2\alpha|s|^2/h^{1/4}}|u_h|^2\,dx\ ,$$

and

$$II := C\int_{\{h^{1/8}\leqslant t\leqslant 2h^{1/8}\}} e^{2\alpha|s|^2/h^{1/4}}|u_h|^2\,dx\ .$$

On $\{h^{1/8}\leqslant t\leqslant 2h^{1/8}\}$ we have $|s|\leqslant|\partial\Omega|t/h^{1/8}$, and clearly $|s|\leqslant|\partial\Omega|/2$, so we have

$$II \leqslant C\int_{\{h^{1/8}\leqslant t\leqslant 2h^{1/8}\}} e^{\alpha|\partial\Omega|^2 t/h^{3/8}}|u_h|^2\,dx\ .$$

By the normal Agmon estimates (Theorem 4.1), this implies that

$$II \leqslant C\ .$$

To estimate I, we use that $|\chi_2'(\frac{s}{Sh^{1/8}})|^2 e^{2\alpha|s|^2 h^{1/4}}$ is bounded uniformly in h and get

$$I \leqslant C\int |u_h|^2\,dx = C\ .$$

Since also $\chi_1^2(\frac{s}{Sh^{1/8}})e^{2\alpha|s|^2/h^{1/4}}$ is bounded uniformly in h, (4.42) implies that

$$\big\|\chi_1(\tfrac{t}{h^{1/8}})e^{\alpha|s|^2/h^{1/4}}u_h\big\|^2 \leqslant C\ . \tag{4.44}$$

The bound on

$$\int_\Omega e^{2\alpha|s|^2/h^{1/4}}\chi_1^2(t/h^{1/8})\big|(-ih\nabla - A(x))u_h(x)\big|^2\,dx$$

in (4.39) now follows in the same way by inserting (4.44) in (4.40). □

Corollary 4.10 (Weak tangential Agmon estimates). — *Let the assumptions be as in Theorem 4.9. Let $\chi\in C_0(\mathbb{R})$, $\operatorname{supp}\chi\subset(-t_0,t_0)$, with the constant t_0 from the definition of the boundary coordinates in Appendix B. Then, for all $k>0$, there exists $C>0$ such that*

$$\int_\Omega |s(x)|^k\chi(t(x))|u_h(x)|^2\,dx \leqslant Ch^{k/8}\ .$$

The proof of the corollary is immediate.

5. A phase space bound

For our analysis of the low lying eigenvalues of $\mathcal{H}$, we need, apart from the localizations in s and t, to have a precise localization in D_s. This is the goal of the present section. Remember that we work in the gauge chosen in Definition 1.4.

5.1. Main statement and main step of the proof

THEOREM 5.1 (Localization in D_s). — *Let $M > 0$, $h_0 > 0$ and $\chi_1, \chi_2 \in C^\infty(\mathbb{R})$ be a standard partition of unity as in* (4.7). *Let (s,t) be the boundary coordinates introduced in Appendix B chosen such that $\kappa(0) = k_{\max}$ and let ε in $(0, 3/8)$. Then for all $N > 0$ there exists $C_N > 0$ such that if $(u_h)_{h\in(0,h_0]}$ is a family of normalized eigenfunctions of $\mathcal{H} = \mathcal{H}(h)$ with eigenvalue $\mu(h)$ satisfying*

$$\mu(h) \leqslant \Theta_0 h - C_1 k_{\max} h^{3/2} + M h^{7/4}\ , \tag{5.1}$$

and the operator W_s acting on functions localized near the boundary is defined by

$$\begin{aligned} W_s\chi_1(4t/t_0) & \\ &= \chi_1(4s/|\partial\Omega|)\chi_2\left(\frac{|h^{1/2}D_s - \xi_0|}{h^\varepsilon}\right)\chi_1(4s/|\partial\Omega|)\chi_1(4t/t_0) \end{aligned} \tag{5.2}$$

with t_0 from (B.1), *then*

$$\left\|W_s\chi_1(4t/t_0)u_h\right\|_{L^2} \leqslant C_N h^N\ . \tag{5.3}$$

and

$$\left|\left\langle W_s\chi_1(4t/t_0)u_h \;\middle|\; \mathcal{H}(h)W_s\chi_1(4t/t_0)u_h\right\rangle\right| \leqslant C_N h^N\ . \tag{5.4}$$

Let us be more explicit about the meaning of the operator $W_s\chi_1(t/t_0)$. On the support of $t \mapsto \chi_1(4t/t_0)$, we can use boundary coordinates (s,t) (see Appendix B). Thus, for each $\phi \in L^2(\Omega)$, $f(s,t) := \chi_1(4t/t_0)\phi$ is a $|\partial\Omega|$-periodic function in s. After multiplication by $\chi_1(4s/|\partial\Omega|)$ we find a function with support in $(-|\partial\Omega|/2, |\partial\Omega|/2) \times \overline{\mathbb{R}_+}$ which we extend by zero to a function (with compact support) on $\mathbb{R} \times \overline{\mathbb{R}_+}$. This function we still denote by $\chi_1(4s/|\partial\Omega|)\chi_1(t/t_0)\phi$. On $\mathbb{R} \times \overline{\mathbb{R}_+}$ the meaning of the operator $\chi_2(|h^{1/2}D_s - \xi_0|/h^\varepsilon)$ is obvious (for example using the Fourier transformation). After multiplying a second time by $\chi_1(4s/|\partial\Omega|)$ we get a new function $\chi_1(4s/|\partial\Omega|)\chi_2(|h^{1/2}D_s - \xi_0|/h^\varepsilon)\chi_1(4s/|\partial\Omega|)\chi_1(t/t_0)\phi$ with support in $(-|\partial\Omega|/2, |\partial\Omega|/2) \times \overline{\mathbb{R}_+}$ which we may reinterpret as a function on a neighborhood of the boundary of Ω, expressed in boundary coordinates.

Thus, $W_s\chi_1(t/t_0)$ is an h-pseudodifferential operator (or rather $h^{1/2-\varepsilon}$-pseudodifferential operator). We will use elementary commutation properties of such operators. The relevant results (and much more) can be found in introductions to the subject, such as [10] and [28].

Remark 5.2. — As a shorter notation, instead of (5.3) and (5.4) we will write

$$\|W_s\chi_1(4t/t_0)u_h\|_{L^2} + \left|\langle W_s\chi_1(4t/t_0)u_h \,\big|\, \mathcal{H}(h)W_s\chi_1(4t/t_0)u_h\rangle\right| = \mathcal{O}_{\rm unif}(h^\infty)\,.$$

Here the subscript 'unif' is included to remind us that the constants (in (5.3) and (5.4)) are uniform for eigenfunctions corresponding to eigenvalues in a suitable energy interval (as given in (5.1)).

Proof of Theorem 5.1. — Let $0 < \delta < 1/2$ and define W, χ_s, $\chi_{s,0}$, χ_t, and χ_0 by

$$W := \chi_2\big(|h^{1/2}D_s - \xi_0|/h^\varepsilon\big)\,, \qquad \chi_t := \chi_1(t/h^{1/2-\delta})\,, \tag{5.5}$$

$$\chi_0 := \chi_1(4t/t_0)\,, \qquad \chi_{s,0} := \chi_1(4s/|\partial\Omega|)\,, \tag{5.6}$$

$$\chi_s := \chi_1(s/h^{1/8-\delta})\,.$$

We will choose δ small such that:

$$0 < \delta < (\tfrac{3}{8} - \varepsilon)/4\,. \tag{5.7}$$

By using the normal Agmon estimates (Theorem 4.1), it suffices to prove the following localized versions of (5.3) and (5.4):

$$\|W_s\chi_t u_h\|_{L^2} = \mathcal{O}_{\rm unif}(h^\infty)\,, \tag{5.8}$$

$$\langle W_s\chi_t u_h \,\big|\, \mathcal{H}(h)W_s\chi_t u_h\rangle = \mathcal{O}_{\rm unif}(h^\infty)\,. \tag{5.9}$$

We start the proof of (5.8) and (5.9) by the easy identities

$$\mu(h)\|W_s\chi_t u_h\|^2 = \mathrm{Re}\langle \chi_t W_s^* W_s\chi_t u_h \,|\, \mathcal{H}u_h\rangle = T_1(u_h) + T_2(u_h)\,, \tag{5.10}$$

with(2)

$$T_1(u_h) := \langle W_s\chi_t u_h \,|\, \mathcal{H} W_s\chi_t u_h\rangle \,, \tag{5.11}$$

$$\begin{aligned} T_2(u_h) := \frac{1}{2}\big\langle u_h \,\big|\, \big(&\chi_t W_s^* W_s \chi_t \mathcal{H} \\ &- 2\chi_t W_s^* \mathcal{H} W_s \chi_t + \mathcal{H}\chi_t W_s^* W_s \chi_t\big) u_h \big\rangle \,. \end{aligned} \tag{5.12}$$

We will also use the following estimates

$$\|\chi_t u_h\|^2 \leqslant 1 \,, \qquad \langle \chi_t u_h \mid \mathcal{H}\chi_t u_h\rangle \leqslant Ch \,. \tag{5.13}$$

Only the second estimate in (5.13) deserves comment. It is however an easy consequence of the standard identity

$$\chi_t \mathcal{H} \chi_t = \tfrac{1}{2}(\chi_t^2 \mathcal{H} + \mathcal{H}\chi_t^2) + h^2 |\nabla \chi_t|^2 \,,$$

and of the estimate:

$$\mu(h) \leqslant \tilde{C} h \,, \tag{5.14}$$

resulting from Assumption (5.1).

Induction argument. The proof of (5.8) and (5.9) will be obtained by proving by induction that $p(N)$ is satisfied for any $N \in \mathbb{N}$, where $p(N)$ is the following statement.

Statement $p(N)$. — For any χ_t and W as in (5.5), then

$$\|W_s \chi_t u_h\|_{L^2} = \mathcal{O}_{\text{unif}}(h^{3N(\frac{3}{8}-\varepsilon-\delta)}) \,, \tag{5.15}$$

$$\langle W_s\chi_t u_h \,\big|\, \mathcal{H}(h) W_s \chi_t u_h\rangle = \mathcal{O}_{\text{unif}}(h^{3N(\frac{3}{8}-\varepsilon-\delta)+1}) \,. \tag{5.16}$$

Initialization $N = 0$. — The estimate (5.15) is trivially satisfied for $N = 0$ and (5.16) is a consequence of (5.11), (5.10), (5.14), (5.7) and Proposition 5.7 (Proposition 5.7 is somewhat stronger than needed at this step).

From N *to* $N+1$. — Suppose now that we have proved $p(N)$ for some $N \geqslant 0$. Given χ_t and W, choose $\tilde{\chi}_t$ and $\widetilde{W}$ satisfying the same assumptions, but being slightly 'larger', i.e.

$$\tilde{\chi}_t \chi_t = \chi_t \,, \qquad \widetilde{W} W = W \,.$$

We introduce $\widetilde{W}_s := \chi_{s,0} \widetilde{W} \chi_{s,0}$. Then we consider $\phi_h := \tilde{\chi}_t \widetilde{W}_s u_h$ instead of u_h and assume $p(N)$, with the pair $(\widetilde{W}_s, \tilde{\chi}_t)$. We come back to (5.10) and observe, using the rough h-pseudodifferential calculus, that

(2) Notice that T_j depends also on a choice of a pair $(\chi_t\,,\, W_s)$ and that we will have to consider different pairs in the induction argument.

$T_j(u_h) = T_j(\phi_h) + \mathcal{O}_{\rm unif}(h^\infty)$ $(j = 1,2)$. We notice also that Proposition 5.6 implies that one can take $\varphi_h = \phi_h$ in Propositions 5.3 and 5.7. Therefore, (5.10) and Proposition 5.7 applied with $\varphi_h = \phi_h$ together with $p(N)$ leads to $(5.16)_{N+1}$. Finally, Proposition 5.3 and $(5.16)_{N+1}$ give (using (5.1)) $(5.15)_{N+1}$ and this finishes the induction.

Thus, (5.3) and (5.4) are proved and we have reduced the proof of Theorem 5.1 to the proof of the three Propositions 5.3, 5.6, and 5.7, which will be given in the next subsections. □

5.2. Step 2: Lower bound for the local energy $T_1(\varphi_h)$

PROPOSITION 5.3. — *Let $\Xi \in C_0^\infty(\mathbb{R})$, $\Xi \equiv 1$ on $[-1/2,1/2]$, $\Xi \equiv 0$ on $\mathbb{R}\setminus[-1,1]$. Suppose that $\varepsilon \in (0,3/8)$, that δ satisfies (5.7) and let $C>0$. Then there exists $c_0>0$ (depending also on the constants implicit in $\mathcal{O}(h^\infty)$ in (5.19) and (5.20) below) and for all $N\in\mathbb{N}$ there exists $C_N>0$ such that if $\varphi_h \in D(\mathcal{H})$ satisfies*

$$\operatorname{supp}\varphi_h \subset \{t(x) \leqslant 2h^{1/2-\delta}\}\ , \tag{5.17}$$

$$\int_\Omega \big(|\varphi_h|^2 + h^{-1}|(-ih\nabla - A(x))\varphi_h|^2\big)\,dx \leqslant C\ , \tag{5.18}$$

$$\int_\Omega \chi_2(s/h^{1/8-\delta})\big\{|\varphi_h|^2 + h^{-1}|(-ih\nabla - A(x))\varphi_h|^2\big\}\,dx = \mathcal{O}(h^\infty)\ , \tag{5.19}$$

and

$$\Big\|\Xi(\frac{4s}{|\partial\Omega|})\Xi(\frac{h^{1/2}D_s-\xi_0}{h^\varepsilon})\Xi(\frac{4s}{|\partial\Omega|})\varphi_h\Big\|_{L^2(\Omega)} = \mathcal{O}(h^\infty)\ . \tag{5.20}$$

Then,

$$\begin{aligned} T_1(\varphi_h) &:= \langle \varphi_h \,|\, \mathcal{H}\varphi_h\rangle \\ &\geqslant \Big(\Theta_0 h - C_1 k_{\max} h^{3/2} + c_0 h^{1+2\varepsilon}\Big)\|\varphi_h\|^2 - C_N h^N\ . \end{aligned} \tag{5.21}$$

Proof of Proposition 5.3. — The term $T_1(\varphi_h)$ is an integral

$$T_1(\varphi_h) = \int \Big(a^{-1}|(hD_s - \tilde{A}_1)\varphi_h|^2 + a|(hD_t)\varphi_h|^2\Big)\,ds\,dt\ . \tag{5.22}$$

We introduce a localized version of $T_1(\varphi_h)$,

$$\tilde{T}_1(\varphi_h) = \int \chi_s^2\Big(a^{-1}|(hD_s - \tilde{A}_1)\varphi_h|^2 + a|(hD_t)\varphi_h|^2\Big)\,ds\,dt\ . \tag{5.23}$$

Using the localization estimates in s (see (5.19)), we obtain:

$$T_1(\varphi_h) = \tilde{T}_1(\varphi_h) + \mathcal{O}(h^\infty)\ . \tag{5.24}$$

On the set

$$\{|s| \leqslant h^{1/8-\delta}\} \cap \{t \leqslant 2h^{1/2-\delta}\}\ ,$$

we have

$$\begin{aligned} a(s,t) &= 1 - tk_{\max} + \mathcal{O}(h^{3/4-3\delta})\ ,\\ a^{-1}(s,t) &= 1 + tk_{\max} + \mathcal{O}(h^{3/4-3\delta})\ ,\\ \tilde{A}_1(s,t) &= -t(1-tk_{\max}) + \mathcal{O}(h^{5/4-4\delta})\ . \end{aligned} \tag{5.25}$$

Therefore, with

$$\begin{aligned} a_1(t) &:= 1 - tk_{\max}\ ,\\ a_2(t) &:= 1 + 2tk_{\max}\ ,\\ A(t) &:= -t(1-tk_{\max})\ , \end{aligned} \tag{5.26}$$

we get, using (5.24),

$$T_1(\varphi_h) \geqslant (1-h^{3/4-3\delta})\tilde{Q}[\varphi_h] + \mathcal{O}(h^{7/4-5\delta})\|\varphi_h\|^2 + \mathcal{O}(h^\infty)\ , \tag{5.27}$$

where

$$\tilde{Q}[f] := \int \chi_s^2\Big(a_2(t)\big|(hD_s - A(t))f\big|^2 + \big|(hD_t)f\big|^2\Big)a_1(t)\,ds\,dt\ . \tag{5.28}$$

It is clear, using again (5.19), that we can remove the localization χ_s and get

$$\begin{aligned} T_1(\varphi_h) \geqslant (1-h^{3/4-3\delta})&Q[\Xi(\frac{4s}{|\partial\Omega|})\varphi_h]\\ &+ \mathcal{O}(h^{7/4-5\delta})\|\Xi(\frac{4s}{|\partial\Omega|})\varphi_h\|^2 + \mathcal{O}(h^\infty)\ , \end{aligned} \tag{5.29}$$

where

$$Q[f] := \int_{\mathbb{R}^2_+} \Big(a_2(t)\big|(hD_s - A(t))f\big|^2 + \big|(hD_t)f\big|^2\Big)a_1(t)\,ds\,dt\ . \tag{5.30}$$

Now the coefficients in Q do not depend on s, so we can make a Fourier decomposition of the quadratic form. Let us define

$$\begin{aligned} \tilde{f}(s,\tau) &:= h^{1/4}\chi_1(h^\delta\tau)f(s,h^{1/2}\tau)\ ,\\ \ell &:= k_{\max}h^\delta\tau\chi_1(h^\delta\tau/2)\ ,\\ g_\zeta(\tau) &:= (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\zeta s}\tilde{f}(s,\tau)\,ds\ . \end{aligned} \tag{5.31}$$

Notice that the function ℓ is uniformly bounded on $\mathbb{R}^+$. Then

$$Q[f] = h \int_{\mathbb{R}} q_\zeta[g_\zeta]\, d\zeta \,,$$

$$q_\zeta[g] := \int_0^\infty \Big\{ (1 + 2h^{1/2-\delta}\ell(\tau))\big[h^{1/2}\zeta/2)\big]^2 |g(\tau)|^2$$

$$+ \tau(1 - h^{1/2-\delta}\ell(\tau) + |g'(\tau)|^2 \Big\}(1 - h^{1/2-\delta}\ell(\tau))\, d\tau \,. \tag{5.32}$$

The form q_ζ is the quadratic form on

$$H^1\big(\mathbb{R}_+, (1 - h^{1/2-\delta}\ell)\, d\tau\big) \cap L^2\big(\mathbb{R}_+, \tau^2(1 - h^{1/2-\delta}\ell)\, d\tau\big)$$

defining a selfadjoint unbounded operator $\mathfrak{h}(\zeta)$ on the space

$$L^2\big(\mathbb{R}_+, (1 - h^{1/2-\delta}\ell)\, d\tau\big) :$$

$$\mathfrak{h}(\zeta) = -\frac{1}{(1 - h^{1/2-\delta}\ell)} \frac{d}{d\tau}\{(1 - h^{1/2-\delta}\ell)\}\frac{d}{d\tau}$$
$$+ (1 + 2h^{1/2-\delta}\ell(\tau)) \times \big[h^{1/2}\zeta + \tau(1 - h^{1/2-\delta}\ell(\tau)/2)\big]^2.$$

Similarly, we can introduce the quadratic form on $H^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+, \tau^2 d\tau)$:

$$q_\zeta^0[g] := \int_0^\infty \Big\{ (h^{1/2}\zeta + \tau)^2 |g(\tau)|^2 + |g'(\tau)|^2 \Big\}\, d\tau \,.$$

with associated operator $\mathfrak{h}_0(\zeta)$ on $L^2(\mathbb{R}_+, d\tau)$ which is the Neumann selfadjoint realization of:

$$\mathfrak{h}_0(\zeta) := -\frac{d^2}{d\tau^2} + (h^{1/2}\zeta + \tau)^2 \,.$$

In the two cases, the form domain is the same space and the operator domain involves the Neumann condition at $\tau = 0$.

Lemma 5.4. — *There exists c_0 , C , $M > 0$, such that if $|h^{1/2}\zeta - \xi_0| \geqslant Mh^{1/4-3\delta/2}$, then*

$$\inf \operatorname{Spec} \mathfrak{h}(\zeta) \geqslant \Theta_0 + c_0 \min\big(1, |h^{1/2}\zeta - \xi_0|^2\big), \tag{5.33}$$

and if $|h^{1/2}\zeta - \xi_0| \leqslant Mh^{1/4-3\delta/2}$, then

$$\inf \operatorname{Spec} \mathfrak{h}(\zeta) \geqslant \Big\{ \Theta_0 + 3C_1|\xi_0|(h^{1/2}\zeta - \xi_0)^2 - C_1 k_{\max} h^{1/2} \Big\}$$
$$- C\big(|h^{1/2}\zeta - \xi_0|^3 + h^{1/2}|h^{1/2}\zeta - \xi_0|\big) \,. \tag{5.34}$$

Proof of Lemma 5.4. — The proof is similar to a calculation given in [15, Section 11], so we will be rather brief. Since $0 \leqslant \ell \leqslant C$, $0 \leqslant \ell\tau \leqslant h^{-\delta}$ and $\delta \in (0, 1/2)$, we get for all f in the form domain of $\mathfrak{h}(\zeta)$,

$$q_\zeta[f] \leqslant (1 + Ch^{1/2-\delta}) q_\zeta^0[f] + h^{1/2-3\delta}\|f\|^2 \,, \tag{5.35}$$

and the same inequality is true by exchanging q_ζ and q_ζ^0. Thus, by the variational characterization of the eigenvalues,

$$\big|\mu_j(\mathfrak{h}(\zeta)) - \mu_j(\mathfrak{h}_0(\zeta))\big| \leqslant Ch^{1/2-3\delta}\big\{1 + \mu_j(\mathfrak{h}_0(\zeta))\big\}\,. \tag{5.36}$$

Here $\mu_j(\mathfrak{h})$ denotes the jth eigenvalue of the self-adjoint operator $\mathfrak{h}$ (with $\mathfrak{h} = \mathfrak{h}(\zeta)$ or $\mathfrak{h}_0(\zeta)$) . Now it follows from (A.4) that, for some $c_0 > 0$,

$$\mu_1(\mathfrak{h}_0(\zeta)) \geqslant \Theta_0 + c_0 \min\big(1, |h^{1/2}\zeta - \xi_0|^2\big)\,,$$

and therefore we get from (5.36), if M is chosen sufficiently big, that

$$\mu_1(\mathfrak{h}(\zeta)) \geqslant \Theta_0 + \frac{c_0}{2} \min\big(1, |h^{1/2}\zeta - \xi_0|^2\big) \tag{5.37}$$

for $|h^{1/2}\zeta - \xi_0| \geqslant Mh^{1/4-3\delta/2}$. Note that from (5.7), $1/4 - 3\delta/2 \geqslant 0$.

For $|h^{1/2}\zeta - \xi_0| < Mh^{1/4-3\delta/2}$, we will construct an explicit trial function for $\mathfrak{h}(\zeta)$. With P_0^{-1} being the regularized resolvent from (A.14) we write

$$\begin{aligned} f_\zeta(\tau) &= u_0(\tau) - 2(h^{1/2}\zeta - \xi_0)P_0^{-1}[(t+\xi_0)u_0(t)](\tau) \\ &\quad + 4(h^{1/2}\zeta - \xi_0)^2 P_0^{-1}\Big\{(t+\xi_0)P_0^{-1}\big[(t'+\xi_0)u_0(t')\big](t) - I_2 u_0(t)\Big\}(\tau)\,. \end{aligned} \tag{5.38}$$

We note that $f_\zeta(\tau)$ belongs to the domain of $\mathfrak{h}(\zeta)$ and a straightforward calculation gives that

$$\Big\|\{\mathfrak{h}_0(\zeta) - [\Theta_0 + (h^{1/2}\zeta - \xi_0)^2(1 - 4I_2)]\}f_\zeta\Big\| \leqslant C|h^{1/2}\zeta - \xi_0|^3\,, \tag{5.39}$$

where I_2 is the constant from (3.37), i.e.

$$I_2 := \int_0^\infty (\tau + \xi_0)u_0(\tau)P_0^{-1}[(t+\xi_0)u_0(t)](\tau)\,d\tau\,. \tag{5.40}$$

Define now

$$\tilde{f}_\zeta := f_\zeta - h^{1/2}k_{\max}P_0^{-1}\Big[\big\{\frac{d}{d\tau} + 2\tau(\xi_0+\tau)^2 - \tau^2(\xi_0+\tau)\big\}u_0\Big]. \tag{5.41}$$

Again a straightforward calculation, using the decomposition

$$\begin{aligned} \mathfrak{h}(\zeta) - \mathfrak{h}_0(\zeta) := h^{1/2-\delta}\Bigg(&\frac{\ell'}{1 - h^{1/2-\delta}\ell}\frac{d}{d\tau} \\ &+ 2\ell(h^{1/2}\zeta + \tau - h^{1/2-\delta}\ell\tau/2)^2 \\ &-(1 - h^{1/2-\delta}\ell) \times \Big\{(h^{1/2}\zeta + \tau)\ell\tau + \tau^2 h^{1/2-\delta}\frac{\ell^2}{4}\Big\}\Bigg)\,, \end{aligned}$$

gives that

$$\begin{aligned} &\Big\|\{\mathfrak{h}(\zeta) - [\Theta_0 + (h^{1/2}\zeta - \xi_0)^2(1 - 4I_2) + h^{1/2}k_{\max}N]\}\tilde{f}_\zeta\Big\| \\ &\qquad\qquad \leqslant C\big(|h^{1/2}\zeta - \xi_0|^3 + h^{1/2}|h^{1/2}\zeta - \xi_0|\big)\,, \end{aligned} \tag{5.42}$$

with

$$N := \int_0^{\infty} u_0(\tau)\Big\{\frac{d}{d\tau} + 2\tau(\xi_0+\tau)^2 - \tau^2(\xi_0+\tau)\Big\}u_0(\tau)\,d\tau\ . \tag{5.43}$$

Using Propositions A.2 and A.3, we get

$$N = -C_1\ , \qquad\qquad 1 - 4I_2 = -3C_1\xi_0 = 3C_1|\xi_0|\ . \tag{5.44}$$

This, together with (5.36) which permits to have a lower bound of $\mu_2(h(\zeta))$, finishes the proof of Lemma 5.4 (see [15, Section 11] for a similar argument). We have actually obtained the better

$$\begin{aligned}\mu_1(\mathfrak{h}(\zeta)) \sim \Big\{\Theta_0 + 3C_1|\xi_0|(h^{1/2}\zeta - \xi_0)^2 - C_1 k_{\max} h^{1/2}\Big\} \\ - C\big(|h^{1/2}\zeta - \xi_0|^3 + h^{1/2}|h^{1/2}\zeta - \xi_0|\big)\ .\end{aligned} \tag{5.45}$$

□

Lemma 5.4 has the following consequence

LEMMA 5.5. — *Let φ_h, ε and δ satisfying the assumptions of Proposition 5.3. Then there exists $c_0 > 0$ such that*

$$\begin{aligned}Q[\Xi(\frac{4s}{|\partial\Omega|})\varphi_h] \geqslant \big(\Theta_0 h - C_1 k_{\max} h^{3/2} + c_0 h^{1+2\varepsilon}\big) \\ \times \int |\Xi(\frac{4s}{|\partial\Omega|})\varphi_h|^2\,(1 - tk_{\max})\,ds\,dt + \mathcal{O}_{\text{unif}}(h^{\infty})\ .\end{aligned}$$

The proof of Lemma 5.5 is immediate.

End of the proof of Proposition 5.3. — Using (5.17) and (5.19), we get that

$$\int_{\mathbb{R}^2_+} |\Xi(\frac{4s}{|\partial\Omega|})\varphi_h|^2(1 - tk_{\max})\,ds\,dt = (1 + \mathcal{O}(h^{3/4-3\delta}))\|\varphi_h\|^2_{L^2(\Omega)}\ .$$

Therefore, Lemma 5.5 implies that

$$\begin{aligned}Q[\Xi(\frac{4s}{|\partial\Omega|})\varphi_h] \\ \geqslant \big(\Theta_0 h - C_1 k_{\max} h^{3/2} + c_0 h^{1+2\varepsilon} - C h^{7/4-3\delta}\big)\|\varphi_h\|^2_{L^2(\Omega)}\ .\end{aligned} \tag{5.46}$$

Combining (5.46) with (5.29), and using the choice of δ from (5.7), yields (5.21). □

5.3. Step 3: Preservation of localization

PROPOSITION 5.6. — *Let $\varepsilon \in (0,3/8)$ and let $\delta \in (0,\frac{1}{2})$. Then there exist $\alpha, C > 0$ such that if $(u_h)_{h\in(0,h_0)}$ is the family of functions from Theorem 5.1 and χ_t, W are as in (5.5), then $\phi_h := \chi_t W_s u_h$ satisfies*

$$\int e^{\alpha t(x)/h^{1/2}}\{|\phi_h|^2 + h^{-1}|(-ih\nabla - A(x)\phi_h|^2\}\,dx \leqslant C\ ,$$

$$\int \chi_2^2(s/h^{1/8-\delta})\big\{|\phi_h|^2 + h^{-1}|(-ih\nabla - A(x))\phi_h|^2\big\}\,dx = \mathcal{O}_{\rm unif}(h^\infty)\ .$$

Proof of Proposition 5.6. — We only consider the localization in s, since the localization in t is much simpler. Let us define

$$T := \int \chi_2^2(s/h^{1/8-\delta})|(-ih\nabla - A(x))\phi_h|^2\,dx\ .$$

We will only prove that $T = \mathcal{O}_{\rm unif}(h^\infty)$, the remaining estimate in Proposition 5.6 being easier. We write, with $\overline{\chi}_s := \chi_2(s/h^{1/8-\delta})$,

$$\begin{aligned} T\ &= R_1 + R_2\ , \\ R_1 &:= \int \overline{\chi}_s^2 a^{-1}|(hD_s - \tilde{A}_1)\phi_h|^2\,ds\,dt\ , \\ R_2 &:= \int \overline{\chi}_s^2 a|(hD_t)\phi_h|^2\,ds\,dt\ . \end{aligned} \tag{5.47}$$

Since a is a bounded function on $\operatorname{supp}\chi_t$ and W_s commutes with D_t, we have, with $\langle\cdot\mid\cdot\rangle$ being the inner product on $L^2([-|\partial\Omega|/2,|\partial\Omega|/2]\times\mathbb{R};ds\,dt)$,

$$|R_2| \leqslant C\langle(hD_t)(\chi_t u_h)\,|\,W_s\overline{\chi}_s^2 W_s(hD_t)(\chi_t u_h)\rangle\ . \tag{5.48}$$

Now, with $\tilde{\chi}_j$ defined by

$$\tilde{\chi}_j(s) = \chi_j(2s/h^{1/8-\delta})\ ,$$

$$W_s\overline{\chi}_s^2 W_s = (\tilde{\chi}_1^2 + \tilde{\chi}_2^2)W_s\overline{\chi}_s^2 W_s(\tilde{\chi}_1^2 + \tilde{\chi}_2^2)\ . \tag{5.49}$$

Since $\varepsilon < 3/8$, we see by repeated commutations that

$$\tilde{\chi}_1^2 W_s \overline{\chi}_s^2 = \mathcal{O}_{\rm unif}(h^\infty),$$

so

$$\begin{aligned} |R_2| \leqslant C\big\langle\tilde{\chi}_2^2(hD_t)(\chi_t u_h)\,|\,W_s\overline{\chi}_s^2 W_s\,\tilde{\chi}_2^2(hD_t)(\chi_t u_h)\big\rangle \\ + \mathcal{O}_{\rm unif}(h^\infty)\|(hD_t)(\chi_t u_h)\|^2\ . \end{aligned} \tag{5.50}$$

Now the Agmon estimates in s and t easily imply that $R_2 = \mathcal{O}_{\rm unif}(h^\infty)$.

The estimate of R_1 is similar but slightly more complicated, since W_s does not commute with $\tilde{A}_1$. We estimate

$$
\begin{aligned}
(5.51)\qquad & |R_1| \leqslant R_1^{(1)} + R_1^{(2)} , \\
& R_1^{(1)} := C \int \overline{\chi}_s^2 |(hD_s + t)(\chi_t W_s u_h)|^2 \, ds \, dt , \\
& R_1^{(2)} := C \int \overline{\chi}_s^2 |\tilde{A}_1 - t|^2 |\chi_t W_s u_h|^2 \, ds \, dt ,
\end{aligned}
$$

On $\operatorname{supp} \chi_t$, $|\tilde{A}_1 - t| \leqslant C$, so $R_1^{(2)}$ can be controlled like R_2 by $\mathcal{O}_{\mathrm{unif}}(h^\infty)$.

For $R_1^{(1)}$ we use that W commutes with $(hD_s + t)$ and the Agmon estimates in s, to write

$$
(5.52)\qquad R_1^{(1)} = C \langle (hD_s + t)\chi_t u_h \,|\, W_s \overline{\chi}_s^2 W_s (hD_s + t)\chi_t u_h \rangle + \mathcal{O}_{\mathrm{unif}}(h^\infty) .
$$

Notice that

$$
(5.53)\qquad \|(hD_s + t)\chi_t u_h\|^2 \leqslant C \|(hD_s + \tilde{A}_1)\chi_t u_h\|^2 + C \leqslant Const .
$$

So we can, like for the control of R_2, localize modulo $\mathcal{O}_{\mathrm{unif}}(h^\infty)$ on the support of $\tilde{\chi}_2$ and get

$$
\begin{aligned}
|R_1^{(1)}| &\leqslant C \langle \tilde{\chi}_2^2 (hD_s + t)\chi_t u_h \,|\, W_s \overline{\chi}_s^2 W_s \tilde{\chi}_2^2 (hD_s + t)\chi_t u_h \rangle + \mathcal{O}_{\mathrm{unif}}(h^\infty) \\
(5.54)\qquad &\leqslant C' \|\tilde{\chi}_2^2 (hD_s + t)\chi_t u_h\|^2 + \mathcal{O}_{\mathrm{unif}}(h^\infty) \\
&\leqslant C'' \Big(\|\tilde{\chi}_2^2 (hD_s - \tilde{A}_1)\chi_t u_h\|^2 + \|\tilde{\chi}_2^2 \chi_t u_h\|^2 \Big) + \mathcal{O}_{\mathrm{unif}}(h^\infty) \\
&= \mathcal{O}_{\mathrm{unif}}(h^\infty) .
\end{aligned}
$$

Here we used the tangential Agmon estimates to get the last inequality. This finishes the proof of Proposition 5.6. □

5.4. Step 4: Control of the commutator $T_2(\varphi_h)$

5.4.1. Main statement

Proposition 5.7. — *Suppose that* $\varepsilon \in (0, 3/8)$, *that* δ *satisfies* (5.7) *and let* α, $C > 0$. *Then there exists* $c_0 > 0$ *(depending also on the constants implicit in* (5.56) *below) and for all* $N \in \mathbb{N}$ *there exists* $C_N > 0$ *such that if* $\varphi_h \in D(\mathcal{H})$ *is such that*

$$
(5.55)\qquad \int e^{\alpha t(x)/h^{1/2}} \{ |\varphi_h|^2 + h^{-1} |(-ih\nabla - A(x))\varphi_h|^2 \} \, dx \leqslant C ,
$$

$$
(5.56)\qquad \int \chi_2(s/h^{1/8-\delta}) \{ |\varphi_h|^2 + h^{-1} |(-ih\nabla - A(x))\varphi_h|^2 \} \, dx = \mathcal{O}(h^\infty) .
$$

Then, with W_s *and* χ_t *from* (5.5) *and*

$$T_2(\varphi_h) := \frac{1}{2}\langle \varphi_h \,|\, \big(\chi_t W_s^* W_s \chi_t \mathcal{H} - 2\chi_t W_s^* \mathcal{H} W_s \chi_t + \mathcal{H}\chi_t W_s^* W_s \chi_t\big)\varphi_h\rangle\ , \tag{5.57}$$

we have

$$|T_2(\varphi_h)| \leqslant c_0 h^{9/8-\varepsilon-\delta}\Big(\langle \varphi_h \,|\, \mathcal{H}\varphi_h\rangle + h\|\varphi_h\|^2\Big) + C_N h^N\ . \tag{5.58}$$

The proof of Proposition 5.7 is based on successive decompositions of the 'commutator'.

5.4.2. First decomposition for $T_2(\varphi_h)$

Since χ_t localizes near the boundary, we can use boundary coordinates (s,t). Thus, we get, with $a = 1 - t\kappa(s)$,

$$\begin{aligned} W_s^* &= a^{-1} W_s a = W_s + \widehat{W}_s\ ; \qquad \widehat{W}_s := \chi_{s,0}\widehat{W}\chi_{s,0}\ ,\\ \widehat{W} &:= \frac{-t}{a}[W, \kappa(s)]\ . \end{aligned} \tag{5.59}$$

Remember that $W_s = \chi_{s,0} W \chi_{s,0}$. Let $\Xi \in C_0^\infty(\mathbb{R})$ satisfy

$$\Xi(s)\chi_{s,0}(s) = \chi_{s,0}(s), \qquad \operatorname{supp}\Xi \subset \Big(-\frac{|\partial\Omega|}{2}, \frac{|\partial\Omega|}{2}\Big).$$

Clearly, $T_2(\varphi_h) = T_2(\Xi\varphi_h)$. Now we can calculate the 'commutator' in $T_2(\Xi\varphi_h)$:

$$\chi_t W_s^* W_s \chi_t \mathcal{H} - 2\chi_t W_s^* \mathcal{H} W_s \chi_t + \mathcal{H}\chi_t W_s^* W_s \chi_t$$

as an (pseudodifferential) operator on $L^2(\mathbb{R}^2_+)$, where we extend the curvature function $\kappa(s)$ (appearing, for instance, in the expression for $\mathcal{H}$ in boundary coordinates) as a periodic function of $s \in \mathbb{R}$.

Using the localization estimates in s from (5.56) combined with the fact that χ_t commutes with W and $\widehat{W}$, we therefore get

$$T_2(\varphi_h) = \frac{1}{2}\langle \chi_{s,0}^2\varphi_h \,|\, (\mathcal{C}_1 + \mathcal{C}_2)\chi_{s,0}^2\varphi_h\rangle + \mathcal{O}_{\rm unif}(h^\infty)\ , \tag{5.60}$$

$$\mathcal{C}_1 := [\chi_t W, [\chi_t W, \mathcal{H}]]\ , \qquad \mathcal{C}_2 := \mathcal{C}_{2,1} + \mathcal{C}_{2,2} + \mathcal{C}_{2,3}\ , \tag{5.61}$$

$$\mathcal{C}_{2,1} := \widehat{W}[\chi_t, [\chi_t, \mathcal{H}]]W\ ,\ \mathcal{C}_{2,2} := \widehat{W}\chi_t^2[W, \mathcal{H}]\ ,\ \mathcal{C}_{2,3} := [\mathcal{H}, \widehat{W}]W\chi_t^2. \tag{5.62}$$

We will calculate and estimate these commutators. We write

$$\begin{aligned} \mathcal{H}(h) &= a^{-1}(hD_s - \tilde{A}_1)a^{-1}(hD_s - \tilde{A}_1) + a^{-1}(hD_t)a(hD_t)\\ &= \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4\ , \end{aligned} \tag{5.63}$$

with

$$\begin{aligned}\mathcal{H}_1 &:= (hD_s - \tilde{A}_1)a^{-2}(hD_s - \tilde{A}_1)\,, & \mathcal{H}_2 &:= -ih\frac{\partial_s a}{a^3}(hD_s - \tilde{A}_1)\,,\\ \mathcal{H}_3 &:= (hD_t)^2\,, & \mathcal{H}_4 &:= -ih\frac{\partial_t a}{a}(hD_t)\,.\end{aligned} \tag{5.64}$$

Remark 5.8. — The commutator $\mathcal{C}_{2,3}$ gives the leading order term. This can be understood by a 'back-of-the-envelope' calculation replacing $\mathcal{H}$ by the leading terms from (3.14), i.e. $\mathcal{H} \approx P_0 + h^{3/8}P_1 + h^{1/2}P_2$. We do not give this formal calculation here, since we do not justify this approximation.

5.4.3. Control of $\frac{1}{2}\langle \mathcal{C}_1 \chi_{s,0}^2 \varphi_h \mid \chi_{s,0}^2 \varphi_h \rangle$

The terms with derivatives in D_s are the most involved.

Commutation with $\mathcal{H}_1$. — Since χ_t commutes with W and $\mathcal{H}_1$, we find

$$[\chi_t W, [\chi_t W, \mathcal{H}_1]] = \chi_t^2 [W, [W, \mathcal{H}_1]]\,. \tag{5.65}$$

The inner commutator becomes

$$[W, \mathcal{H}_1] =: Q_1 + Q_2\,, \tag{5.66}$$

with

$$Q_1 := (hD_s - \tilde{A}_1)[W, a^{-2}](hD_s - \tilde{A}_1)\,, \tag{5.67}$$

$$Q_2 := -\Big\{[W, \tilde{A}_1]a^{-2}(hD_s - \tilde{A}_1) + (hD_s - \tilde{A}_1)a^{-2}[W, \tilde{A}_1]\Big\}\,. \tag{5.68}$$

We calculate the double commutators separately

$$\begin{aligned}[W, Q_1] &= (hD_s - \tilde{A}_1)[W, [W, a^{-2}]](hD_s - \tilde{A}_1)\\ &\quad - \Big\{[W, \tilde{A}_1][W, a^{-2}](hD_s - \tilde{A}_1) + (hD_s - \tilde{A}_1)[W, a^{-2}][W, \tilde{A}_1]\Big\}\,,\\ [W, Q_2] &= -[W, [W, \tilde{A}_1]]a^{-2}(hD_s - \tilde{A}_1) - (hD_s - \tilde{A}_1)a^{-2}[W, [W, \tilde{A}_1]]\\ &\quad - [W, \tilde{A}_1][W, a^{-2}](hD_s - \tilde{A}_1) - (hD_s - \tilde{A}_1)[W, a^{-2}][W, \tilde{A}_1]\\ &\quad + 2[W, \tilde{A}_1]a^{-2}[W, \tilde{A}_1]\,.\end{aligned} \tag{5.69}$$

Remember that $\tilde{A}_1(s,t) = -t(1 - t\kappa(s)/2)$, $a(s,t) = 1 - t\kappa(s)$. Therefore,

$$[W, \tilde{A}_1] = t^2[W, \kappa(s)]/2 = t^2 h^{1/2-\varepsilon}\mathcal{O}_1\,,\quad [W, [W, \tilde{A}_1]] = t^2 h^{1-2\varepsilon}\mathcal{O}_2\,, \tag{5.70}$$

$$[W, a^{-2}] = th^{1/2-\varepsilon}\mathcal{O}_3\,, \qquad [W, [W, a^{-2}]] = th^{1-2\varepsilon}\mathcal{O}_4\,, \tag{5.71}$$

where, after a right multiplication by a cutoff function localizing in $[0, t_0)$, the $\mathcal{O}_j$'s are bounded (pseudodifferential) operators commuting with the multiplication by functions $\psi(t)$.

Using (5.70), (5.71), the Cauchy-Schwarz inequality and that $|t| \leqslant 2h^{1/2-\delta}$ on $\operatorname{supp}\chi_t$, we find from (5.69),

$$\begin{aligned}
&|\langle \chi_{s,0}^2\varphi_h \mid [\chi_t W, [\chi_t W, \mathcal{H}_1]]\chi_{s,0}^2\varphi_h\rangle| \\
&\qquad \leqslant C\Big(h^{\frac{3}{2}-2\varepsilon-\delta}\|(hD_s - \tilde{A}_1)\chi_t\chi_{s,0}^2\varphi_h\|^2 \\
&\qquad\quad + h^{2-2\varepsilon-2\delta}\|(hD_s - \tilde{A}_1)\chi_t\chi_{s,0}^2\varphi_h\| \, \|\chi_t\chi_{s,0}^2\varphi_h\| \\
&\qquad\quad + h^{3-2\varepsilon-4\delta}\|\chi_t\chi_{s,0}^2\varphi_h\|^2\Big) \\
&\qquad \leqslant \tilde{C}h^{3/2-2\varepsilon-2\delta}\big(\langle \chi_t\varphi_h \mid \mathcal{H}(h)\chi_t\varphi_h\rangle + (h + h^{3/2-2\delta})\|\chi_t\varphi_h\|^2\big) \, . \qquad (5.72)
\end{aligned}$$

Using the condition satisfied by δ from (5.7), and the localization estimate in s (5.56), this implies (5.58) for the expectation value

$$\langle \chi_{s,0}^2\varphi_h \mid [\chi_t W, [\chi_t W, \mathcal{H}_1]]\chi_{s,0}^2\varphi_h\rangle.$$

Commutation with $\mathcal{H}_2$. — The commutation with $\mathcal{H}_2$ is similar, but easier.

$$\begin{aligned}
\big[\chi_t W, [\chi_t W, \mathcal{H}_2]\big] &= -ih\chi_t^2\big[W, [W, \frac{\partial_s a}{a^3}](hD_s - \tilde{A}_1) - \frac{\partial_s a}{a^3}[W, \tilde{A}_1)]\big] \\
&= -ih\chi_t^2[W, [W, \frac{\partial_s a}{a^3}]](hD_s - \tilde{A}_1) \\
&\quad + 2ih\chi_t^2[W, \frac{\partial_s a}{a^3}][W, \tilde{A}_1)] + ih\chi_t^2\frac{\partial_s a}{a^3}[W, [W, \tilde{A}_1)]] \, . \qquad (5.73)
\end{aligned}$$

Now, $\partial_s a = -t\kappa'(s)$, so the new terms to control are

$$[W, \frac{\partial_s a}{a^3}] = h^{1/2-\varepsilon}t\,\mathcal{O}_5, \qquad [W, [W, \frac{\partial_s a}{a^3}]] = h^{1-2\varepsilon}t\,\mathcal{O}_6 \, , \qquad (5.74)$$

with bounded[(3)] $\mathcal{O}_j$'s.

Thus, for $[\chi_t W, [\chi_t W, \mathcal{H}_2]]$, we get

$$\begin{aligned}
&|\langle \chi_{s,0}^2\varphi_h \mid [\chi_t W, [\chi_t W, \mathcal{H}_2]]\chi_{s,0}^2\varphi_h\rangle| \\
&\qquad \leqslant Ch^{2-2\varepsilon}(\|t\chi_t\chi_{s,0}^2\varphi_h\| \, \|(hD_s - \tilde{A}_1)\chi_t\chi_{s,0}^2\varphi_h\| \\
&\qquad\quad + \|t^2\chi_t\chi_{s,0}^2\varphi_h\| \, \|t\chi_t\chi_{s,0}^2\varphi_h\|) \\
&\qquad \leqslant \tilde{C}\left(h^{2-2\varepsilon-\delta} + h^{\frac{5}{2}-2\varepsilon-3\delta}\right)\big(\langle \chi_t\varphi_h \mid \mathcal{H}(h)\chi_t\varphi_h\rangle + h\|\chi_t\varphi_h\|^2\big) \, . \qquad (5.75)
\end{aligned}$$

This implies (5.58) for $\langle \chi_{s,0}^2\varphi_h \mid [\chi_t W, [\chi_t W, \mathcal{H}_2]]\chi_{s,0}^2\varphi_h\rangle$.

---

(3) after multiplication by a cutoff function localizing in $[0, t_0)$,

Commutation with $\mathcal{H}_3$. — Since W commutes with χ_t and D_t, we can calculate

$$
\begin{aligned}
[\chi_t W,[\chi_t W,\mathcal{H}_3]] &= h^2 W^2 [\chi_t,[\chi_t,D_t^2]] = -2h^2 W^2 |\partial_t \chi_t|^2 \\
&= -2h^{1+2\delta} W^2 |\chi_1'(\frac{t}{h^{1/2-\delta}})|^2 \,. \qquad (5.76)
\end{aligned}
$$

The expectation value of this term in the state $\chi_{s,0}^2\varphi_h$ will be exponentially small, due to the normal Agmon estimates, (5.55). Explicitly,

$$
\begin{aligned}
&|\langle \chi_{s,0}^2\varphi_h \mid [\chi_t W,[\chi_t W,\mathcal{H}_3]]\chi_{s,0}^2\varphi_h\rangle| \\
&\quad = 2h^{1+2\delta}\Big|\langle e^{\alpha t/h^{1/2}}\chi_{s,0}^2\varphi_h \mid W^2 e^{-2\alpha t/h^{1/2}} \\
&\qquad\quad \times |\chi_1'(\frac{t}{h^{1/2}-\delta})|^2 e^{\alpha t/h^{1/2}}\chi_{s,0}^2\varphi_h\rangle\Big| \\
&\quad \leqslant 2h^{1+2\delta}\|W\|^2 \|e^{-2\alpha t/h^{1/2}}|\chi_1'(\frac{t}{h^{1/2-\delta}})|^2\|_\infty \|e^{\alpha t/h^{1/2}}\varphi_h\|^2 \\
&\quad \leqslant Ch^{1+2\delta}e^{-2\alpha h^{-\delta}} = \mathcal{O}_{\mathrm{unif}}(h^\infty)\,. \qquad (5.77)
\end{aligned}
$$

In particular, (5.58) is satisfied for the term

$$
\langle \chi_{s,0}^2\varphi_h \mid [\chi_t W,[\chi_t W,\mathcal{H}_3]]\chi_{s,0}^2\varphi_h\rangle.
$$

Commutation with $\mathcal{H}_4$. — When we calculate $[\chi_t W,[\chi_t W,\mathcal{H}_4]]$, we will use the discussion of the previous paragraph to conclude that if a derivative falls on χ_t, then the resulting expectation value becomes exponentially small. Thus,

$$
\begin{aligned}
&\langle \chi_{s,0}^2\varphi_h \mid [\chi_t W,[\chi_t W,\mathcal{H}_4]]\chi_{s,0}^2\varphi_h\rangle \\
&\quad = \langle \chi_{s,0}^2\varphi_h \mid -ih[W,[W,\frac{\partial_t a}{a}]]\chi_t^2(hD_t)\chi_{s,0}^2\varphi_h\rangle + \mathcal{O}_{\mathrm{unif}}(h^\infty) \\
&\quad = -ih\langle \chi_t\chi_{s,0}^2\varphi_h \mid -ih[W,[W,\frac{\partial_t a}{a}]](hD_t)\chi_t\chi_{s,0}^2\varphi_h\rangle + \mathcal{O}_{\mathrm{unif}}(h^\infty)\,. \qquad (5.78)
\end{aligned}
$$

From the formula

$$
\frac{\partial_t a}{a} = \frac{-\kappa(s)}{1-t\kappa(s)}\,,
$$

we see that all derivatives (in s) of $\partial_t a/a$ are uniformly bounded on the support of χ_t. We therefore find

$$
[W,[W,\frac{\partial_t a}{a}]] = h^{1-2\varepsilon}\mathcal{O}_7\,, \qquad (5.79)
$$

where $\mathcal{O}_7$ is a bounded (pseudodifferential) operator. Thus

(5.80)
$$\begin{aligned}|\langle\chi_{s,0}^2\varphi_h \mid [\chi_t W,[\chi_t W,\mathcal{H}_4]]\chi_{s,0}^2\varphi_h\rangle| \\ \leqslant Ch^{2-2\varepsilon}\big(\|(hD_t)\chi_t\chi_{s,0}^2\varphi_h\|\|\chi_t\chi_{s,0}^2\varphi_h\|\big)+\mathcal{O}_{\mathrm{unif}}(h^\infty) \\ \leqslant \tilde{C}h^{\frac{3}{2}-2\varepsilon}\big(\langle\chi_t\varphi_h \mid \mathcal{H}(h)\chi_t\varphi_h\rangle+h\|\chi_t\varphi_h\|^2\big)+\mathcal{O}_{\mathrm{unif}}(h^\infty)\,.\end{aligned}$$

This implies (5.58) for $\langle\chi_{s,0}^2\varphi_h \mid [\chi_t W,[\chi_t W,\mathcal{H}_4]]\chi_{s,0}^2\varphi_h\rangle$, and therefore (5.58) is established for the expectation of $\mathcal{C}_1$.

5.4.4. Control of $\frac{1}{2}\langle\mathcal{C}_2\chi_{s,0}^2\varphi_h \mid \chi_{s,0}^2\varphi_h\rangle$

To estimate this term, we use similar calculations and the decomposition given in (5.61) and (5.62).

Estimate of $\langle\mathcal{C}_{2,1}\chi_{s,0}^2\varphi_h\,,\,\chi_{s,0}^2\varphi_h\rangle$. — For the first term

$$\widehat{W}[\chi_t,[\chi_t,\mathcal{H}]]W,$$

we clearly get, using (5.55) and as for (5.77),

$$\begin{aligned}&\langle\chi_{s,0}^2\varphi_h \mid \widehat{W}[\chi_t,[\chi_t,\mathcal{H}]]W\chi_{s,0}^2\varphi_h\rangle\\ &\qquad= -h^2\langle\chi_{s,0}^2\varphi_h \mid \widehat{W}|\partial_t\chi_t|^2W\chi_{s,0}^2\varphi_h\rangle\\ &\qquad= \mathcal{O}_{\mathrm{unif}}(h^\infty)\,.\end{aligned}\tag{5.81}$$

Thus, (5.58) holds for the expectation of $\mathcal{C}_{2,1}$.

Estimate of $\langle\mathcal{C}_{2,2}\chi_{s,0}^2\varphi_h \mid \chi_{s,0}^2\varphi_h\rangle$. — For this term, we notice that $[W,\mathcal{H}_3]=0$, and calculate, using the $\mathcal{O}_j$'s introduced previously.

$$\begin{aligned}\mathcal{C}_{2,2} &= \widehat{W}\chi_t^2[W,\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_4]\\ &= \widehat{W}\chi_t^2\Big((hD_s-\tilde{A}_1)[W,a^{-2}](hD_s-\tilde{A}_1)-[W,\tilde{A}_1]a^{-2}(hD_s-\tilde{A}_1)\\ &\quad -(hD_s-\tilde{A}_1)a^{-2}[W,\tilde{A}_1]-ih[W,\frac{\partial_s a}{a^3}](hD_s-\tilde{A}_1)\\ &\quad +ih\frac{\partial_s a}{a^3}[W,\tilde{A}_1]-ih[W,\frac{\partial_t a}{a}](hD_t)\Big)\end{aligned}$$

$$
\begin{aligned}
&= \widehat{W}\chi_t^2\Big((hD_s-\tilde{A}_1)th^{1/2-\varepsilon}\mathcal{O}_3(hD_s-\tilde{A}_1)\\
&\qquad - t^2h^{1/2-\varepsilon}\mathcal{O}_1a^{-2}(hD_s-\tilde{A}_1)\\
&\qquad - (hD_s-\tilde{A}_1)a^{-2}t^2h^{1/2-\varepsilon}\mathcal{O}_1 - ith^{3/2-\varepsilon}\mathcal{O}_5(hD_s-\tilde{A}_1)\\
&\qquad - ih^{3/2-\varepsilon}t\frac{\kappa'(s)}{a^3}t^2\mathcal{O}_1 - ih^{3/2-\varepsilon}\mathcal{O}_8(hD_t)\Big)\,. \qquad (5.82)
\end{aligned}
$$

Here $\mathcal{O}_8$ (which we used in the last line) and $\mathcal{O}_9$ (that will be used below), are

$$
[W, \frac{-\kappa(s)}{a}] = h^{1/2-\varepsilon}\mathcal{O}_8\,, \qquad \widehat{W} = th^{1/2-\varepsilon}\mathcal{O}_9\,. \qquad (5.83)
$$

Therefore, we get (using the pseudo-differential calculus for showing that $[D_s,\mathcal{O}_9]=\mathcal{O}_{10}$)

$$
\begin{aligned}
&|\langle \chi_{s,0}^2\varphi_h \,|\, \mathcal{C}_{2,2}\chi_{s,0}^2\varphi_h\rangle|\\
&\quad\leqslant C\Big(h^{1-2\varepsilon}h^{1-2\delta}\|(hD_s-\tilde{A}_1)\mathcal{O}_9\chi_{s,0}^2\varphi_h\|\,\|(hD_s-\tilde{A}_1)\chi_{s,0}^2\varphi_h\|\\
&\qquad + h^{1-2\varepsilon}h^{3/2-3\delta}\|\chi_{s,0}^2\varphi_h\|\,\|(hD_s-\tilde{A}_1)\chi_{s,0}^2\varphi_h\|\\
&\qquad + h^{1-2\varepsilon}h^{3/2-3\delta}\|(hD_s-\tilde{A}_1)\mathcal{O}_9\chi_{s,0}^2\varphi_h\|\,\|\chi_{s,0}^2\varphi_h\|\\
&\qquad + h^{2-2\varepsilon}h^{1-2\delta}\|\chi_{s,0}^2\varphi_h\|\,\|(hD_s-\tilde{A}_1)\chi_{s,0}^2\varphi_h\|\\
&\qquad + h^{2-2\varepsilon}h^{2-4\delta}\|\chi_{s,0}^2\varphi_h\|^2\\
&\qquad + h^{2-2\varepsilon}h^{1/2-\delta}\|\chi_{s,0}^2\varphi_h\|\,\|hD_t\chi_{s,0}^2\varphi_h\|\Big)\\
&\quad\leqslant \tilde{C}h^{3/2-2\varepsilon}\big(\langle\varphi_h\,|\,\mathcal{H}\varphi_h\rangle + h\|\varphi_h\|^2\big)\,. \qquad (5.84)
\end{aligned}
$$

So, by (5.7), (5.58) also holds for the expectation of $\mathcal{C}_{2,2}$.

Estimate of $\langle \mathcal{C}_{2,3}\chi_{s,0}^2\varphi_h \,|\, \chi_{s,0}^2\varphi_h\rangle$. — For this last term the approach is equally direct. We calculate:

$$
\begin{aligned}
[\widehat{W},\mathcal{H}] &= [\widehat{W},(hD_s-\tilde{A}_1)]a^{-2}(hD_s-\tilde{A}_1)\\
&\quad + (hD_s-\tilde{A}_1)[\widehat{W},a^{-2}](hD_s-\tilde{A}_1)\\
&\quad + (hD_s-\tilde{A}_1)a^{-2}[\widehat{W},(hD_s-\tilde{A}_1)]\\
&\quad - ih[\widehat{W},\frac{\partial_s a}{a^3}](hD_s-\tilde{A}_1) - ih\frac{\partial_s a}{a^3}[\widehat{W},(hD_s-\tilde{A}_1)]\\
&\quad - [\frac{t}{a},(hD_t)^2][W,\kappa(s)]\\
&\quad + ih\frac{t}{a}\big[[W,\kappa(s)],\frac{\partial_t a}{a}\big](hD_t) + ih\frac{\partial_t a}{a}\big[\frac{t}{a},(hD_t)\big][W,\kappa(s)]\,. \qquad (5.85)
\end{aligned}
$$

The commutators in the above expression can be estimated:

$$\begin{aligned}[\widehat{W},(hD_s-\tilde{A}_1)] &= -t\big[a^{-1}[W,\kappa(s)],hD_s\big]+t\big[a^{-1}[W,\kappa(s)],\tilde{A}_1\big]\\ &= -iht\frac{\partial_s a}{a^2}[W,\kappa(s)]+hta^{-1}[[W,\kappa(s)],D_s]\\ &\quad -t^3a^{-1}[[W,\kappa(s)],\kappa(s)]/2\\ &= -iht^2\frac{\kappa'(s)}{a^2}[W,\kappa(s)]+hta^{-1}[W,[\kappa(s),D_s]]\\ &\quad -t^3a^{-1}[[W,\kappa(s)],\kappa(s)]/2\\ &= t^2h^{3/2-\varepsilon}\mathcal{U}_{(1)}+th^{3/2-\varepsilon}\mathcal{U}_{(2)}+t^3h^{1-2\varepsilon}\mathcal{U}_{(3)}\ ,\end{aligned}\tag{5.86}$$

with bounded $\mathcal{U}_{(j)}$'s when composed with t-cut-off functions. Thus, when we localize in $\{|t|\leqslant Ch^{1/2-\delta}\}$, we get

$$[\widehat{W},(hD_s-\tilde{A}_1)]=h^{2-\delta-\varepsilon}\mathcal{U}_1\ ,\tag{5.87}$$

where $\mathcal{U}_1$ is uniformly bounded. Similarly,

$$\begin{aligned}[\widehat{W},a^{-2}] &= [\widehat{W},\frac{1-a^2}{a^2}]\\ &= t^2a^{-1}\big[[W,\kappa(s)],\frac{2\kappa(s)-t\kappa(s)}{a^2}\big]\\ &= h^{1-2\delta}h^{1-2\varepsilon}\mathcal{U}_2\ ,\end{aligned}\tag{5.88}$$

$$\big[\widehat{W},\frac{\partial_s a}{a^3}\big]=t^2a^{-1}\big[[W,\kappa(s)],\frac{\kappa'(s)}{a^3}\big]=h^{1-2\delta}h^{1-2\varepsilon}\mathcal{U}_3\ ,\tag{5.89}$$

$$\big[\frac{t}{a},(hD_t)\big]=ih\partial_t(\frac{t}{a})\ .\tag{5.90}$$

$$\begin{aligned}\big[\frac{t}{a},(hD_t)^2\big] &= ih(hD_t)\partial_t(\frac{t}{a})+ih\partial_t(\frac{t}{a})(hD_t)\\ &= 2ih(hD_t)\partial_t(\frac{t}{a})-h^2\partial_t^2(\frac{t}{a})\\ &= (hD_t)h\mathcal{U}_4+h^2\mathcal{U}_5\ ,\end{aligned}\tag{5.91}$$

$$\big[[W,\kappa(s)],\frac{\partial_t a}{a}\big]=h^{1-2\varepsilon}\mathcal{U}_6\ ,\tag{5.92}$$

Thus, we can estimate, after additional controls of commutators,

$$
\begin{aligned}
&|\langle \chi_{s,0}^2 \varphi_h \,|\, \mathcal{C}_{2,3} \chi_{s,0}^2 \varphi_h \rangle| \\
&\quad \leqslant C\Big(h^{2-\delta-\varepsilon} \|\chi_{s,0}^2 \varphi_h\| \, \|(hD_s - \tilde{A}_1) W \chi_t^2 \chi_{s,0}^2 \varphi_h\| \\
&\qquad + h^{1-2\delta} h^{1-2\varepsilon} \|(hD_s - \tilde{A}_1) \chi_{s,0}^2 \varphi_h\| \, \|(hD_s - \tilde{A}_1) W \chi_t^2 \chi_{s,0}^2 \varphi_h\| \\
&\qquad + h^{2-\delta-\varepsilon} \|(hD_s - \tilde{A}_1) \chi_{s,0}^2 \varphi_h\| \, \|\chi_{s,0}^2 \varphi_h\| \\
&\qquad + h^{2-2\delta} h^{1-2\varepsilon} \|\chi_{s,0}^2 \varphi_h\| \, \|(hD_s - \tilde{A}_1) W \chi_t^2 \chi_{s,0}^2 \varphi_h\| \\
&\qquad + h^{3-\delta-\varepsilon} \|\chi_{s,0}^2 \varphi_h\|^2 + h |\langle \chi_{s,0}^2 \varphi_h \,|\, (hD_t) \mathcal{U}_4 [W, \kappa(s)] \chi_{s,0}^2 \varphi_h \rangle| \\
&\qquad + h^{3/2-\delta} h^{1-2\varepsilon} \|\chi_{s,0}^2 \varphi_h\| \, \|(hD_t) W \chi_t^2 \chi_{s,0}^2 \varphi_h\| \\
&\qquad + h^{5/2-\varepsilon} \|\chi_{s,0}^2 \varphi_h\|^2 \Big) \\
&\quad \leqslant \tilde{C} \Big(h^{3/2-\delta-\varepsilon} \langle \varphi_h \,|\, \mathcal{H} \varphi_h \rangle + h^{5/2-\delta-\varepsilon} \|\varphi_h\|^2 \\
&\qquad + h |\langle \chi_{s,0}^2 \varphi_h \,|\, (hD_t) \mathcal{U}_4 [W, \kappa(s)] \chi_{s,0}^2 \varphi_h \rangle| \Big) . \qquad (5.93)
\end{aligned}
$$

Remark 5.9. — If, in (5.93), we estimate $[W, \kappa(s)]$ by the easy pseudo-differential result, i.e. use the bound

$$\big\|[W, \kappa(s)]\big\| = \mathcal{O}(h^{1/2-\varepsilon}) ,$$

we can continue (5.93) to get

$$|\langle \chi_{s,0}^2 \varphi_h \,|\, \mathcal{C}_{2,3} \chi_{s,0}^2 \varphi_h \rangle| \leqslant C\big(h^{1-\varepsilon} \langle \varphi_h \,|\, \mathcal{H} \varphi_h \rangle + h^{2-\varepsilon} \|\varphi_h\|^2 \big) .$$

This would only allow us to take $\varepsilon \in (0, 1/3)$ instead of $\varepsilon \in (0, 3/8)$ as claimed in Theorem 5.1. In order to get the optimal range for ε, we estimate the commutator $[W, \kappa(s)]$ slightly better using the 'Agmon estimates' in s, i.e. (5.56).

We write $\kappa(s) = k_{\max} - s^2 \hat{\kappa}(s)$, with $\hat{\kappa}$ a smooth function with bounded derivatives of all orders. Now,

$$
\begin{aligned}
[W, \kappa(s)] &= [\hat{\kappa}(s) s^2, W] = \hat{\kappa}(s)[s, [s, W]] + 2\hat{\kappa}(s)[s, W] s + [\hat{\kappa}(s), W] s^2 \\
&= \widetilde{\mathcal{U}}_1 h^{1-2\varepsilon} + \widetilde{\mathcal{U}}_2 h^{1/2-\varepsilon} s + \widetilde{\mathcal{U}}_3 h^{1/2-\varepsilon} s^2 . \qquad (5.94)
\end{aligned}
$$

Therefore, using the estimate (5.56), we get the inequality,

$$\big\|[W, \kappa(s)] \chi_{s,0}^2 \varphi_h\big\| \leqslant C h^{5/8-\varepsilon-\delta} \|\varphi_h\| + \mathcal{O}_{\mathrm{unif}}(h^\infty) , \qquad (5.95)$$

and finally, (5.58) follows (for the final term $\mathcal{C}_{2,3}$) by inserting (5.95) in (5.93).

This ends the proof of Proposition 5.7.

6. Lower bounds in Grushin's approach

In this section we finish the proof of Theorem 1.1.

THEOREM 6.1. — *Let $\mu^{(n)}(h)$ be the n-th eigenvalue of $\mathcal{H}(h)$ and let $z_\infty^{(n)}(h)$ be the asymptotic sum given in* (3.4)*. Then $\mu^{(n)}(h)$ has $z_\infty^{(n)}(h)$ as asymptotic expansion.*

It is clear that Theorem 6.1 implies Theorem 1.1. From the estimates in the previous sections 4 and 5, we know the low-lying eigenfunctions to be localized in phase space, around the following set (where we denote by ξ the variable in phase space dual to s):

- $s \in (-2h^{1/8-\varepsilon_1}\ ,\ 2h^{1/8-\varepsilon_1})$;
- $t \in (-2h^{1/2-\varepsilon_2}\ ,\ 2h^{1/2-\varepsilon_2})$;
- $\xi \in (-2h^{\varepsilon_3-1/2} + \xi_0 h^{-1/2}\ ,\ 2h^{\varepsilon_3-1/2} + \xi_0 h^{-1/2})$.

We will choose $\varepsilon_1 = \varepsilon_2 =: \eta$, $\varepsilon_3 = 3/8 - \eta$, where $\eta < 1/8$. Thus, in the remainder of the section, η will be a fixed (small) constant satisfying,

$$\eta \in (0, 1/8)\ . \tag{6.1}$$

Remark 6.2. — We only need a space localization in t. There is no localization in the corresponding frequency. This is pleasant, since it avoids possible complications due to the boundary condition.

More precisely, the localization of $u_h^{(n)}$ is analyzed in the following lemma.

LEMMA 6.3. — *Let $M > 0$, $h_0 > 0$ and let χ be a standard cut-off function:*

$$\begin{aligned} &\chi \in C_0^\infty(\mathbb{R})\ , \quad \operatorname{supp}\chi \subset (-2,2)\ , \\ &\chi(x) = 1 \text{ on a nbd. of } [-\tfrac{3}{2}, +\tfrac{3}{2}]\ . \end{aligned} \tag{6.2}$$

Then for all $K > 0$ there exists $b_K > 0$ such that if $(u_h)_{h\in(0,h_0)}$ is a family of normalized eigenfunctions of $\mathcal{H}$ with eigenvalue $\mu(h)$ satisfying

$$\mu(h) \leqslant \Theta_0 h - C_1 k_{\max} h^{3/2} + M h^{7/4}\ , \tag{6.3}$$

then, with η from (6.1),

$$\Big\| u_h - \chi\big(\frac{t}{h^{1/2-\eta}}\big)\chi\big(\frac{s}{h^{1/8-\eta}}\big)\chi\big(\frac{|h^{1/2}D_s - \xi_0|}{h^{3/8-\eta}}\big)\chi\big(\frac{4s}{|\partial\Omega|}\big)u_h\Big\|_2 \leqslant b_K h^K\ . \tag{6.4}$$

Proof of Lemma 6.3. — Define $\chi_2 = 1-\chi$. From Theorems 4.1 and 4.9 we know that

$$(1-\chi(\frac{t}{h^{1/2-\eta}})\chi(\frac{s}{h^{1/8-\eta}}))u_h = \mathcal{O}_{\rm unif}(h^\infty)\,.$$

So it suffices to prove that

$$\Big\|\chi(\frac{t}{h^{1/2-\eta}})\chi(\frac{s}{h^{1/8-\eta}})u_h - \chi(\frac{t}{h^{1/2-\eta}})\chi(\frac{s}{h^{1/8-\eta}}) \times \chi(\frac{|h^{1/2}D_s-\xi_0|}{h^{3/8-\eta}})\chi(\frac{4s}{|\partial\Omega|})u_h\Big\|_2 = \mathcal{O}_{\rm unif}(h^\infty)\,. \tag{6.5}$$

By writing

$$\begin{aligned}\chi(\frac{t}{h^{1/2-\eta}})\chi(\frac{s}{h^{1/8-\eta}})u_h &= \chi(\frac{t}{h^{1/2-\eta}})\chi(\frac{s}{h^{1/8-\eta}})\chi(\frac{4s}{|\partial\Omega|})u_h\\ &= \chi(\frac{t}{h^{1/2-\eta}})\chi(\frac{s}{h^{1/8-\eta}})\{\chi(\frac{|h^{1/2}D_s-\xi_0|}{h^{3/8-\eta}})\\ &\qquad + \chi_2(\frac{|h^{1/2}D_s-\xi_0|}{h^{3/8-\eta}})\}\chi(\frac{4s}{|\partial\Omega|})u_h\,,\end{aligned}$$

and appealing to Theorem 5.1 we get (6.5) and thereby Lemma 6.3. □

Proof of Theorem 6.1. — For a definite choice of χ (fixed once and for all) as in Lemma 6.3, and u_h an eigenfunction of $\mathcal{H}$ satisfying (6.3), define

$$\tilde\psi_h = e^{-is\xi_0/h^{1/2}}\chi(\frac{t}{h^{1/2-\eta}})\chi(\frac{s}{h^{1/8-\eta}}) \times \chi(\frac{|h^{1/2}D_s-\xi_0|}{h^{3/8-\eta}})\chi(\frac{4s}{|\partial\Omega|})u_h\,, \tag{6.6}$$

and

$$\psi_h(\sigma,\tau) = h^{5/16}\tilde\psi_h(h^{1/8}\sigma, h^{1/2}\tau)\,.$$

Calculations will from now on be carried out in the variables (σ,τ). All functions considered will be localized on a scale of order $h^{-\eta}$ in the (σ,τ)-variables. This implies (in particular) that they are localized to a tubular neighborhood of size $h^{1/2-\eta}$ near the boundary in the original coordinates $x\in\Omega$. The natural measure in (σ,τ)-variables, inherited from $L^2(\Omega,dx)$ by implementing unitarily the change of coordinates, is $(1-h^{1/2}\tau\kappa(h^{-1/8}\sigma))d\sigma d\tau$. However, due to the localization of our functions (and the boundedness of κ) we can replace this measure by $d\sigma d\tau$ (since $h^{1/2}\tau\kappa(h^{-1/8}\sigma) = \mathcal{O}(h^{1/2-\eta})$ on $|\tau|\leqslant Ch^{-\eta}$), without changing our estimates up to multiplicative h-independent constants. Therefore we can (and will !) do all our estimates by choosing the norms in $L^2(\mathbb{R}\times\mathbb{R}_+, d\sigma d\tau)$ or in

$\mathcal{L}(L^2(\mathbb{R}\times\mathbb{R}_+, d\sigma d\tau))$. Thus all L^2-norms below refer to $L^2(\mathbb{R}\times\mathbb{R}_+, d\sigma d\tau)$ and similarly for operator norms.

With these conventions we have, using Theorems 4.1, 4.9 and 5.1, for all eigenfunctions u_h corresponding to eigenvalues $\mu(h)$ satisfying (6.3), that the corresponding ψ_h (given in (6.6)) satisfies (with error terms $\mathcal{O}_{\rm unif}(h^\infty)$ uniform for eigenfunctions u_h as long as (6.3) is satisfied),

$$\|\psi_h\|_{L^2} = 1 + \mathcal{O}_{\rm unif}(h^\infty)\ , \qquad \text{and } L\psi_h = \psi_h + \mathcal{O}_{\rm unif}(h^\infty)\ , \tag{6.7}$$

for all

$$L = \tilde\chi(\frac{\tau}{h^{-\eta}})\tilde\chi(\frac{\sigma}{h^{-\eta}})\tilde\chi(h^\eta D_\sigma)\ , \tag{6.8}$$

with $\tilde\chi$ satisfying (6.2) and with η from (6.1). Let us fix an L_0 as in (6.8) in the rest of this Section.

Let $H_{\rm harm}$ be the harmonic oscillator on $L^2(\mathbb{R})$ defined by

$$H_{\rm harm} := 3C_1\sqrt{\Theta_0}D_\sigma^2 + C_1\tfrac{k_2\sigma^2}{2}\ , \tag{6.9}$$

(compare with Section 3). Clearly, $H_{\rm harm}$ has eigenvalues

$$e_\ell := C_1\Theta_0^{1/4}\sqrt{\tfrac{3k_2}{2}}(2\ell-1)\ , \tag{6.10}$$

with $\ell\in\mathbb{N}\setminus\{0\}$. Let v_ℓ be the corresponding (unique up to scalar multiple) normalized eigenfunction. For $N\in\mathbb{N}\setminus\{0\}$, the value $C_1\Theta_0^{1/4}\sqrt{6k_2}N$ is right in the middle between two eigenvalues (e_N and e_{N+1}). We define the vector space $V_N\subset L^2(\mathbb{R})$ as the space spanned by eigenfunctions of $H_{\rm harm}$ corresponding to eigenvalues below $C_1\Theta_0^{1/4}\sqrt{6k_2}N$, i.e.

$$V_N := \operatorname{Ran} 1_{[0, C_1\Theta_0^{1/4}\sqrt{6k_2}N]}(H_{\rm harm}) = \operatorname{Span}\{v_1,\dots,v_N\}\ . \tag{6.11}$$

Clearly, $\dim V_N = N$.

Similarly, we define $U_N(h)\subset L^2(\Omega)$ as the spectral subspace attached to the interval $I_N(h)$, with

$$I_N(h) = \big(-\infty, \Theta_0 h - k_{\max}C_1h^{3/2} + C_1\Theta_0^{1/4}\sqrt{6k_2}Nh^{7/4}\big]\ . \tag{6.12}$$

Let $\Pi_{V_N}: L^2(\mathbb{R})\to V_N$ and $\Pi_{U_N}: L^2(\Omega)\to U_N(h)$ be the orthogonal projections. We define a linear map $\mathcal{M}_1^{(N)}(h)$ from V_N to $U_N(h)$ by

$$\mathcal{M}_1^{(N)}(h)v_\ell = \Pi_{U_N}\phi_{M_0}^{(\ell)}\ ,$$

(where $\phi_M^{(n)}$ was defined in (3.2)) and extended by linearity. The number M_0 is chosen fixed, but suffficiently large—the choice $M_0 = 10$ would suffice.

Furthermore, we define a linear map $\mathcal{M}_2^{(N)}(h)$ from $U_N(h)$ to V_N by

$$\mathcal{M}_2^{(N)}(h)u_h = \Pi_{V_N}R_0^- L_0\psi_h\ ,$$

where ψ_h is defined from u_h by (6.6). We will prove the following lemma.

LEMMA 6.4. — *Let $N \in \mathbb{N}\setminus\{0\}$. Then there exists $h_0 > 0$, and $\eta > 0$ (as in* (6.1) *and used in* (6.6)*) such that for all $h < h_0$, $\mathcal{M}_1^{(N)}(h)$ and $\mathcal{M}_2^{(N)}(h)$ are bijective.*

We will prove Lemma 6.4 below. First we apply it to finish the proof of Theorem 6.1.

Lemma 6.4 implies that, for sufficiently small h,

$$\dim U_N(h) = \dim V_N = N\ .$$

But Corollary 3.3 describes N distinct points of $\mathrm{Spec}(\mathcal{H})$ below

$$\Theta_0 h - k_{\max} C_1 h^{3/2} + C_1 \Theta_0^{1/4} \sqrt{\tfrac{3k_2}{2}}\, 2N h^{7/4}\ .$$

This finishes the proof of Theorem 6.1. □

Proof of Lemma 6.4. — We only need to prove that $\mathcal{M}_1^{(N)}(h)$ and $\mathcal{M}_2^{(N)}(h)$ are both injective. Injectivity of $\mathcal{M}_1^{(N)}(h)$ is clear from Section 3, so we only consider injectivity of $\mathcal{M}_2^{(N)}(h)$.

The key to the proof of injectivity of $\mathcal{M}_2^{(N)}(h)$ is the following lemma.

LEMMA 6.5. — *There exists $\eta_0 > 0$ such that if $\eta < \eta_0$ in* (6.1)*, then there exists $C > 0$ such that for all normalized eigenfunctions $u_h \in U_N(h)$ with corresponding eigenvalue $\mu(h)$, we have*

$$\big\|\big(\nu(h) - H_{\mathrm{harm}}\big) R_0^- L_0 \psi_h\big\| \leqslant C h^{1/16}\ , \tag{6.13}$$

where ψ_h is related to u_h by (6.6) *and $\nu(h)$ is defined by*

$$\nu(h) := h^{-7/4}\big\{\mu(h) - (\Theta_0 h - k_{\max} C_1 h^{3/2})\big\}\ . \tag{6.14}$$

Proof of Lemma 6.5. — With P from (3.10) and

$$\lambda(h) = h^{-1}(\mu(h) - \Theta_0 h)\ , \tag{6.15}$$

we have (using Theorems 4.1, 4.9, and 5.1), uniformly for normalized eigenfunctions $u_h \in U_N(h)$,

$$(P - \lambda(h))\psi_h = \mathcal{O}_{\mathrm{unif}}(h^\infty)\ , \text{ and } (P - \lambda(h)) L\psi_h = \mathcal{O}_{\mathrm{unif}}(h^\infty)\ . \tag{6.16}$$

In the rest of the proof of Lemma 6.5 we will often have estimates like (6.16). We will generally not repeat the phrase 'uniformly for normalized eigenfunctions $u_h \in U_N(h)$', but the estimates are meant to have such uniformity.

Using (6.16) and the notation from (3.24), we get:

$$\begin{pmatrix} P - \lambda(h) & R_0^+ \\ R_0^- & 0 \end{pmatrix} \begin{pmatrix} L_0\psi_h \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ R_0^- L_0 \psi_h \end{pmatrix} + \mathcal{O}_{\mathrm{unif}}(h^\infty)\ . \tag{6.17}$$

Furthermore, with $\mathcal{E}_0$ from (3.24),

$$\mathcal{E}_0 \begin{pmatrix} P-\lambda(h) & R_0^+ \\ R_0^- & 0 \end{pmatrix} \begin{pmatrix} L_0\psi_h \\ 0 \end{pmatrix} = \left\{ L_0 + \begin{pmatrix} E_0(\partial P)L_0 & 0 \\ R_0^-(\partial P)L_0 & 0 \end{pmatrix} \right\} \begin{pmatrix} \psi_h \\ 0 \end{pmatrix} . \tag{6.18}$$

Here we have introduced

$$\partial P = (P-P_0) - \lambda(h) . \tag{6.19}$$

In order to proceed we need a bound on the matrix in $\{\cdot\}$ in (6.18).

LEMMA 6.6. — *There exists a constant $C>0$ such that (with η from (6.1))*

$$\|E_0(P-P_0)L_0\| + \|R_0^-(P-P_0)L_0\| \leqslant Ch^{3/8-\eta} . \tag{6.20}$$

More precisely, with P_1, P_2 and P_3 from (3.16)-(3.18), *there exists $N_0 \in \mathbb{N}$ such that, for η satisfying* (6.1),

$$\begin{aligned} \|E_0(P-P_0)L_0 - E_0\big(h^{3/8}P_1 + h^{1/2}P_2 + h^{3/4}P_3\big)L_0\| &\leqslant Ch^{7/8-N_0\eta} , \\ \|R_0^-(P-P_0)L_0 - R_0^-\big(h^{3/8}P_1 + h^{1/2}P_2 + h^{3/4}P_3\big)L_0\| &\leqslant Ch^{7/8-N_0\eta} . \end{aligned} \tag{6.21}$$

Furthermore,

$$\begin{aligned} h^{3/8}\|E_0P_1L_0\| + h^{3/8}\|R_0^-P_1L_0\| &\leqslant Ch^{\frac{3}{8}-2\eta} , \\ h^{1/2}\|E_0P_2L_0\| + h^{1/2}\|R_0^-P_2L_0\| &\leqslant Ch^{1/2-3\eta} , \\ h^{3/4}\|E_0P_3L_0\| + h^{3/4}\|R_0^-P_3L_0\| &\leqslant Ch^{3/4-5\eta} . \end{aligned} \tag{6.22}$$

Proof of Lemma 6.6. — With $\tilde a$ and $\tilde a_2$ from (3.9) and omitting the tilda's on the a's, we have by definition

$$\begin{aligned} P = {} & a^{-1}\big[(\tau+\xi_0) + h^{3/8}D_\sigma - \tau(1-a_2)\big] \\ & \times a^{-1}\big[(\tau+\xi_0) + h^{3/8}D_\sigma - \tau(1-a_2)\big] \\ & + a^{-1}D_\tau a D_\tau - \Theta_0 , \end{aligned} \tag{6.23}$$

and therefore

$$\begin{aligned} P - P_0 = {} & \Big\{ a^{-1}\big[(\tau+\xi_0) + h^{3/8}D_\sigma - \tau(1-a_2)\big] \\ & \times a^{-1}\big[(\tau+\xi_0) + h^{3/8}D_\sigma - \tau(1-a_2)\big] - (\tau+\xi_0)^2 \Big\} \\ & - iD_\tau \frac{\partial_\tau a}{a} + \partial_\tau\big(\frac{\partial_\tau a}{a}\big) . \end{aligned} \tag{6.24}$$

We will use the property that L localizes to $\{\tau < 2h^{-\eta}, |\sigma| < 2h^{-\eta}\}$ and that E_0 , $E_0 D_\tau$, R_0^-, and $R_0^- D_\tau$ are bounded. We introduce

$$f := (\tau + \xi_0) - \tau(1 - a_2)$$

and calculate

$$\begin{aligned} a^{-1}\big(f + h^{3/8}D_\sigma\big)a^{-1}\big(f + h^{3/8}D_\sigma\big) - (\tau+\xi_0)^2 \\ = \big(f^2/a^2 - (\tau+\xi_0)^2\big) + ih^{3/8}\frac{\partial_\sigma a}{a^3} f + 2a^{-2} f h^{3/8} D_\sigma \\ + i\frac{\partial_\sigma a}{a^3} h^{3/4} D_\sigma + a^{-2}h^{3/4}D_\sigma^2 \;. \end{aligned}$$

Thus,

$$P - P_0 = \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 \;,$$

where

$$\begin{aligned} \mathcal{Q}_1 &= f^2/a^2 - (\tau+\xi_0)^2 + ih^{3/8}\frac{\partial_\sigma a}{a^3} f + \partial_\tau\big(\frac{\partial_\tau a}{a}\big) \;, \\ \mathcal{Q}_2 &= -iD_\tau \frac{\partial_\tau a}{a} \;, \\ \mathcal{Q}_3 &= 2a^{-2} f (h^{3/8} D_\sigma) + i\frac{\partial_\sigma a}{a^3}(h^{3/4}D_\sigma) + a^{-2}(h^{3/4}D_\sigma^2) \;. \end{aligned}$$

Now, on $\{\tau < 2h^{-\eta}, |\sigma| < 2h^{-\eta}\}$, we have

$$a = 1 + \mathcal{O}(h^{1/2-\eta}) \;, \qquad 1 - a_2 = \mathcal{O}(h^{1/2-\eta}) \;. \tag{6.25}$$

Therefore,

$$\begin{aligned} f^2/a^2 &= (\tau+\xi_0)^2 + \mathcal{O}(h^{1/2-2\eta}) \;, & \partial_\sigma a &= \mathcal{O}(h^{1/2+1/8-\eta}) \;, \\ f &= \mathcal{O}(h^{-\eta}) \;, & \partial_\tau a &= \mathcal{O}(h^{1/2}) \;, \\ \partial_\tau\big(\frac{\partial_\tau a}{a}\big) &= \mathcal{O}(h) \;. \end{aligned}$$

Thus, $\mathcal{Q}_1 = \mathcal{O}(h^{1/2-2\eta})$, so

$$\|E_0 \mathcal{Q}_1 L_0\| + \|R_0^- \mathcal{Q}_1 L_0\| = \mathcal{O}(h^{\frac{1}{2}-2\eta}) \;.$$

Furthermore, for all $j \geqslant 0$,

$$D_\sigma^j \tilde\chi(h^\eta\sigma)\tilde\chi(h^\eta D_\sigma) = \tilde\chi(2h^\eta\sigma) D_\sigma^j \tilde\chi(h^\eta\sigma)\tilde\chi(h^\eta D_\sigma) \;, \tag{6.26}$$

and

$$\|D_\sigma^j \tilde\chi(h^\eta\sigma)\tilde\chi(h^\eta D_\sigma)\| \leqslant c_j (h^{-\eta})^j \;. \tag{6.27}$$

Using (6.26) and (6.27), we get

$$\|E_0\mathcal{Q}_2L_0\| + \|R_0^-\mathcal{Q}_2L_0\| = \mathcal{O}(h^{1/2}) ,$$
$$\|E_0\mathcal{Q}_3L_0\| + \|R_0^-\mathcal{Q}_3L_0\| = \mathcal{O}(h^{-\eta}h^{3/8}) + \mathcal{O}(h^{1/2+1/8-\eta}h^{-\eta}) + \mathcal{O}(h^{3/4-2\eta}) .$$

This finishes the proof of (6.20).

The more precise estimates, (6.21) and (6.22), follow in the same manner, using that on $\{\tau < 2h^{-\eta}, |\sigma| < 2h^{-\eta}\}$,

$$a = 1 - h^{1/2}\tau\big(\kappa(0) - \tfrac{1}{2}h^{1/4}\sigma^2\kappa''(0) + \mathcal{O}(h^{3(1/8-\eta)})\big) ,$$
$$a_2 = 1 - h^{1/2}\tau\tfrac{1}{2}\big(\kappa(0) - \tfrac{1}{2}h^{1/4}\sigma^2\kappa''(0) + \mathcal{O}(h^{3(1/8-\eta)})\big) ,$$

instead of (6.25). We omit the details.

This finishes the proof of Lemma 6.6. □

Combining (6.17) and (6.18), we get:

$$L_0\psi_h = R_0^+R_0^-L_0\psi_h - E_0(\partial P)L_0\psi_h + \mathcal{O}_{\mathrm{unif}}(h^\infty) , \tag{6.28}$$

and

$$R_0^-(\partial P)L_0\psi_h = \mathcal{O}_{\mathrm{unif}}(h^\infty) . \tag{6.29}$$

We now introduce an additional localization through an operator L as in (6.8), which is chosen (slightly 'bigger' than L_0, i.e.) such that

$$LL_0 = L_0 + \mathcal{O}(h^\infty) . \tag{6.30}$$

We observe that (6.29) is also valid with L_0 replaced by LL_0 and, applying the (uniformly) bounded operator $R_0^-(\partial P)L$ to (6.28), we obtain

$$R_0^-(\partial P)LR_0^+R_0^-L_0\psi_h - R_0^-(\partial P)LE_0(\partial P)L_0\psi_h = \mathcal{O}_{\mathrm{unif}}(h^\infty) . \tag{6.31}$$

We again apply Lemma 6.6, (6.31) and the comparison estimates (6.34) and (6.33) (to be proved below) and obtain that, for all $\delta > 0$, there exists $\eta_0 \in (0, \frac{1}{8})$, such that if $\eta < \eta_0$, then

$$- R_0^-\big(h^{3/8}P_1 + h^{1/2}P_2 + h^{3/4}P_3 - \lambda(h)\big)LR_0^+R_0^-L_0\psi_h + R_0^-\big(h^{3/8}P_1\big)LE_0\big(h^{3/8}P_1\big)R_0^+R_0^-L_0\psi_h = \mathcal{O}_{\mathrm{unif}}(h^\infty) + \mathcal{O}_{\mathrm{unif}}(h^{7/8-\delta})\|R_0^+R_0^-L_0\psi_h\| .$$

Using (6.29), the rapid decay of the function u_0, the support properties (in τ) of L and the pseudo-differential calculus in the σ variable for controlling commutators (in order to push L to the right), we finally get:

$$\begin{aligned}(6.32)\quad & - R_0^-\big(h^{3/8}P_1 + h^{1/2}P_2 + h^{3/4}P_3 - \lambda(h)\big)R_0^+R_0^-L_0\psi_h \\ & \qquad + R_0^-\big(h^{3/8}P_1\big)E_0\big(h^{3/8}P_1\big)R_0^+R_0^-L_0\psi_h \\ & \qquad\qquad = \mathcal{O}_{\rm unif}(h^\infty) + \mathcal{O}_{\rm unif}(h^{7/8-\delta})\|R_0^+R_0^-L_0\psi_h\| \,.\end{aligned}$$

We get (6.13) from (6.32) by calculations similar to those leading to the expressions for E_1, E_2, E_3 in Subsection 3.4. We just recall that

$$R_0^-P_1R_0^+ = 0\,, \qquad\qquad -R_0^-P_2R_0^+ = \kappa(0)C_1\,,$$

and

$$-R_0^-P_3R_0^+ + R_0^-P_1E_0P_1R_0^+ = -H_{\rm harm}\,.$$

This finishes the proof of Lemma 6.5 □

We now compare (as already used above) various norms and observe:

Lemma 6.7. — *Let $N \in \mathbb{N}\setminus\{0\}$. There exists $c > 0$ and $h_0 > 0$ such that if ψ_h is associated (as in (6.6)) to a normalized eigenfunction u_h of $\mathcal{H}$ with $u_h \in U_N(h)$, then for all $h \in (0, h_0]$,*

$$(6.33)\qquad \|R_0^-L_0\psi_h\| - ch^{1/4} \leqslant \|\psi_h\| \leqslant \|R_0^-L_0\psi_h\| + ch^{1/4}\,.$$

Proof. — Since clearly $\|R_0^-\| = 1$, we get from (6.7),

$$\|R_0^-L_0\psi_h\| \leqslant \|L_0\psi_h\| = \|\psi_h\| + \mathcal{O}_{\rm unif}(h^\infty)\,.$$

This implies the first inequality in Lemma 6.7. To get the second inequality, we apply (6.28), Lemma 6.6 and (6.7), and get

$$(6.34)\qquad \psi_h = R_0^+R_0^-L_0\psi_h + \mathcal{O}_{\rm unif}(h^{3/8-\eta})\,.$$

Since $\|R_0^+\| = 1$ and ψ_h satisfies (6.7), this implies the lemma. □

Using the self-adjointness of the harmonic oscillators, we get the following proposition.

Proposition 6.8. — *Let $N \in \mathbb{N}\setminus\{0\}$. There exist $h_0 > 0$ and $C > 0$ such that if $(\mu(h))_{h\in(0,h_0]}$ is an eigenvalue of $\mathcal{H}$ satisfying with $\mu(h) \in I_N(h)$ (see (6.12)), then $\nu(h)$ (defined by (6.14)) satisfies*

$$(6.35)\qquad \nu(h) \in \cup_{\ell=1}^N\{e_\ell\} + [-Ch^{1/16}, +Ch^{1/16}]\,.$$

Proof. — Using Lemma 6.7 above, Lemma 6.5 implies that

$$\operatorname{dist}\big(\nu(h), \operatorname{Spec}\{H_{\rm harm}\}\big) = \mathcal{O}_{\rm unif}(h^{1/16})\,.$$

□

LEMMA 6.9. — *Let $N \in \mathbb{N} \setminus \{0\}$. There exists $h_0 > 0$ such that if $h \in (0, h_0]$, then $\dim U_N(h) = N$.*

Proof. — We know from Section 3 that $\dim U_N(h) \geqslant N$. In order to prove Lemma 6.9 we only have to prove that the eigenspace attached to some interval $\nu(h) \in [e_\ell - Ch^{1/16}, e_\ell + Ch^{1/16}]$, with e_ℓ from (6.10), and $\ell \leqslant N$, is necessarily of dimension $\leqslant 1$. If it was not the case, let $u_{1,h}$, $u_{2,h}$ be normalized orthogonal eigenfunctions corresponding to eigenvalues $\mu_1(h)$ and $\mu_2(h)$ in the interval

$$\Theta_0 h - C_1 k_{\max} h^{3/2} + h^{7/4}[e_\ell - Ch^{1/16}, e_\ell + Ch^{1/16}] \,,$$

for some $\ell = \ell(h)$. Let $\psi_{1,h}$, $\psi_{2,h}$ be defined as in (6.6) and let ν_1, ν_2 be as in (6.14). Let e_ℓ and v_ℓ be as in (6.10) and below.

For $a, b \in \mathbb{C}$ (a, b will depend on h) with $|a|^2 + |b|^2 = 1$. We have, using the almost orthonormality of $\psi_{1,h}$, $\psi_{2,h}$ and (6.7),

$$\begin{aligned}(6.36)\quad 1 + \mathcal{O}_{\text{unif}}(h^\infty) &= \|a\psi_{1,h} + b\psi_{2,h}\|^2 \\ &\leqslant \|R_0^- L_0(a\psi_{1,h} + b\psi_{2,h})\|^2 + \mathcal{O}_{\text{unif}}(h^{1/8-\eta}) \,.\end{aligned}$$

With $\ell = \ell(h)$ as above, we may choose a, b such that

$$\int_{-\infty}^{\infty} \overline{v_\ell(\sigma)} R_0^- L_0(a\psi_{1,h} + b\psi_{2,h})\, d\sigma = 0 \,. \tag{6.37}$$

Lemma 6.5 implies that

$$\begin{aligned}(e_\ell - H_{\text{harm}}) R_0^- L_0(a\psi_{1,h} + b\psi_{2,h}) &= a(\nu_1(h) - H_{\text{harm}}) R_0^- L_0 \psi_{1,h} \\ &\quad + b(\nu_2(h) - H_{\text{harm}}) R_0^- L_0 \psi_{2,h} \\ &\quad + \mathcal{O}_{\text{unif}}(h^{1/16}) \\ &= \mathcal{O}_{\text{unif}}(h^{1/16}) \,. \qquad (6.38)\end{aligned}$$

Using (6.37), (6.38) implies that

$$\|R_0^- L_0(a\psi_{1,h} + b\psi_{2,h})\| = \mathcal{O}_{\text{unif}}(h^{1/16}) \,, \tag{6.39}$$

which is in contradiction to (6.36). This finishes the proof of Lemma 6.9. □

Thus, for sufficiently small h, $U_N(h) = \operatorname{Span}\{u_h^{(j)}\}_{j=1}^N$.

LEMMA 6.10. — *Let $N \in \mathbb{N} \setminus \{0\}$. There exists $h_0 > 0$ such that*

$$\mathcal{M}_2^{(N)} u_h^{(j)} = v_j + \mathcal{O}_{\text{unif}}(h^{1/16}) \,,$$

for all $h < h_0$ and all $j \in \{1, \dots, N\}$.

Proof. — By induction it suffices to prove the lemma for $j = N$. By Lemma 6.5 and the spectral theorem there exists $\ell(h) \in \{1, \dots, N\}$ such that (with ψ_N being associated to $u_h^{(N)}$ as in (6.6))

$$R_0^- L_0 \psi_N - \langle v_{\ell(h)}, R_0^- L_0 \psi_N \rangle v_{\ell(h)} = \mathcal{O}_{\text{unif}}(h^{1/16}) \,.$$

Suppose $\ell(h_n) < N$ for a sequence $\{h_n\}$ with $h_n \searrow 0$.

Then $u_{h_n}^{(N)} \in U_{N-1}(h_n)$ and therefore $\dim U_{N-1}(h_n) \geqslant N$, in contradiction to Lemma 6.9. Thus,

$$R_0^- L_0 \psi_N - \langle v_N, R_0^- L_0 \psi_N \rangle v_N = \mathcal{O}_{\text{unif}}(h^{1/16}) \,,$$

and therefore

$$\mathcal{M}_2^{(N)} u_h^{(N)} = \langle v_N, R_0^- L_0 \psi_N \rangle v_N + \mathcal{O}_{\text{unif}}(h^{1/16}) \,.$$

Lemma 6.10 now follows from Lemma 6.7. □

The injectivity of $\mathcal{M}_2^{(N)}$ clearly follows from Lemma 6.10. This finishes the proof of Lemma 6.4. □

Appendix A. On an important family of ordinary differential equations

Let us recall for the comfort of the reader the main properties (mainly due to [9] and [3]) concerning the Neumann realization of $H^{N,\xi}$ in $L^2(\mathbb{R}^+)$ associated to $D_x^2 + (x+\xi)^2$. We denote by $\hat{\mu}^{(1)}(\xi)$ the lowest eigenvalue of $H^{N,\xi}$ and by φ_ξ the corresponding strictly positive normalized eigenfunction. More simply we will write $\mu(\xi)$ instead of $\hat{\mu}^{(1)}(\xi)$ in this appendix. It has been proved that the infimum $\inf_{\xi\in\mathbb{R}} \inf \operatorname{Spec}(H^{N,\xi})$ is actually a minimum. Then one can show that there exists $\xi_0 < 0$ such that $\mu(\xi)$ decays monotonically to a minimum value $\Theta_0 < 1$ and then increases monotonically again. So it can be proved that:

$$\Theta_0 = \inf_\xi \big(\inf \ \operatorname{Spec}(H^{N,\xi})\big) = \inf \ \operatorname{Spec}(H^{N,\xi_0}) \,, \tag{A.1}$$

and moreover that:

$$\Theta_0 = \xi_0^2 \,. \tag{A.2}$$

It is indeed proved in [9] that

$$\mu'(\xi) = [\mu(\xi) - \xi^2]\varphi_\xi(0)^2 \,. \tag{A.3}$$

From (A.3), we get that

$$\mu''(\xi_0) = -2\xi_0 \varphi_{\xi_0}^2(0) > 0 \,. \tag{A.4}$$

We will write u_0 instead of ϕ_{ξ_0}, and define the constant C_1 by (1.3).

Let us now recall some formulas appearing in [3]. Define M_k to be the k'th moment, centered at $-\xi_0$, of the measure $u_0^2(x)\,dx$:

$$M_k = \int_{\mathbb{R}_+} (x+\xi_0)^k u_0^2(x)\,dx\;. \tag{A.5}$$

These moments were calculated in [3].

Lemma A.1. — *The first moments can be expressed by the following formulas:*

$$M_0 = 1\;, \qquad M_1 = 0\;, \qquad M_2 = \frac{\Theta_0}{2}\;, \qquad M_3 = \frac{u_0^2(0)}{6} > 0\;. \tag{A.6}$$

We will also need a few other results on the model operator.

Proposition A.2. — *We have the following identities*

$$\int_0^\infty [2\tau(\tau+\xi_0)^2 - \tau^2(\tau+\xi_0)]u_0^2(\tau)\,d\tau = \frac{u_0^2(0)}{6} = \frac{C_1}{2} > 0\;,$$

$$i\int_0^\infty u_0(\tau)D_\tau u_0(\tau)\,d\tau = -\frac{u_0^2(0)}{2} = -\frac{3}{2}C_1\;.$$

Proof. — The first identity clearly follows from the known moments of u_0^2 and (1.3). The second identity follows from partial integration. □

Proposition A.3. — *For $z \in \mathbb{R}$, let $E(z)$ be defined as the ground state energy of the Neumann realization of*

$$H(z) = -\frac{d^2}{d\tau^2} + (\tau+\xi_0+z)^2\;,$$

on $L^2(\mathbb{R}_+)$. Then $E(z)$ is a smooth function and satisfies

$$E''(0) = 2(1-4I_2)\;, \tag{A.7}$$

with I_2 from (3.37).

Furthermore,

$$E''(0) = 6C_1\sqrt{\Theta_0}\;. \tag{A.8}$$

Proof. — By analytic perturbation theory, $E(z)$ is analytic and there exists an analytic function $\mathbb{R} \ni z \mapsto \phi(z) \in L^2(\mathbb{R}_+)$ such that

$$\|\phi(z)\| = 1\;, \qquad H(z)\phi(z) = E(z)\phi(z)\;, \qquad \phi(0) = u_0\;. \tag{A.9}$$

By differentiating the identity $\|\phi(z)\|^2 = 1$ twice with respect to z, we find

$$2\,\mathrm{Re}\langle \phi'(0) \mid u_0\rangle = 0\;, \qquad -\|\phi'(0)\|^2 = \mathrm{Re}\langle \phi''(0) \mid u_0\rangle\;. \tag{A.10}$$

From the equation $H(z)\phi(z) = E(z)\phi(z)$, and the fact that $E(z)$ is minimal at $z = 0$, we get, with $P_0 = H(0) - \Theta_0$,

$$P_0\phi'(0) = -2(\tau + \xi_0)u_0 \ ,$$

which implies with P_0^{-1} from (A.13) and (A.14), since $u_0 \perp (\tau + \xi_0)u_0$ (by (A.6)),

$$\phi'(0) = -2P_0^{-1}\Big((\tau + \xi_0)u_0\Big) + cu_0 \ , \tag{A.11}$$

for some $c \in i\mathbb{R}$. Finally, differentiating the relation

$$E(z) = \langle \phi(z) \mid H(z)\phi(z)\rangle$$

twice gives us the formula:

$$\begin{aligned}E''(0) = 2\Theta_0 \operatorname{Re}\langle \phi''(0) \mid u_0\rangle + 8 \operatorname{Re}\langle \phi'(0) \mid (\tau + \xi_0)u_0\rangle \\ + 2\langle \phi'(0) \mid H(0)\phi'(0)\rangle + 2 \ .\end{aligned} \tag{A.12}$$

Upon inserting (A.10) and (A.11) in (A.12), we get (A.7). The final identity, (A.8) is a rephrasing of (A.4) . □

We also have the following easy observation:

Lemma A.4. — *Let R_0^+ be the operator from* (3.21). *Suppose $\phi \in \mathcal{S}(\mathbb{R})$, then $R_0^+\phi \in \mathcal{S}(\mathbb{R} \times \overline{\mathbb{R}_+})$.*

Proof. — This is an easy consequence of the regularity and decay of u_0 . □

Finally, we will need the following mapping properties of the regularized resolvent.

Lemma A.5. — *Let P_0 be the Neumann realization of*

$$-\frac{d^2}{d\tau^2} + (\tau + \xi_0)^2 - \Theta_0 \ ,$$

on $L^2(\mathbb{R}_+)$. For $\phi \perp u_0$ we can define $P_0^{-1}\phi$ as the unique solution f to

$$P_0 f = \phi \ , \qquad\qquad f \perp u_0 \ . \tag{A.13}$$

Let $P_0^{-1} \in \mathcal{L}(L^2(\mathbb{R}_+))$ be the regularized resolvent:

$$P_0^{-1}\phi = \begin{cases} 0 \ , & \phi \parallel u_0 \\ P_0^{-1}\phi, & \phi \perp u_0 \ , \end{cases} \tag{A.14}$$

(and extended by linearity). Then P_0^{-1} is continuous from $\mathcal{S}(\overline{\mathbb{R}_+})$ into $\mathcal{S}(\overline{\mathbb{R}_+})$. Moreover, for any $\alpha \geqslant 0$, P_0^{-1} is continuous in $L^2(\mathbb{R}^+ \ ; \ \exp -\alpha\tau)$.

Proof. — Using the local regularity up to the boundary of P_0, one first gets that P_0^{-1} sends $\mathcal{S}(\overline{\mathbb{R}_+})$ into $C^\infty(\overline{\mathbb{R}_+})$. For the control at ∞, one then observes, after cutting away from 0, that the problem is reduced to the analysis of inverting the harmonic oscillator $-\frac{d^2}{d\tau^2} + (\tau)^2 - \Theta_0$ on $\mathcal{S}(\mathbb{R})$, which is a standard result.

For the last statement, we can also observe that, for any real α, the operator

$$\exp -\alpha\sqrt{1+\tau^2} \cdot (-\frac{d^2}{d\tau^2} + (\tau+\xi_0)^2 - \Theta_0)^{-1} \cdot \exp \alpha\sqrt{1+\tau^2}$$

extends continuously on $L^2(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$.

One can also show by the same technique that

$$\tau^j u_0^{(k)}(\tau) \in L^2(\mathbb{R}^+ \,;\, \exp -\alpha\tau)\ , \text{ for all } \alpha \geqslant 0\ , \quad \text{and for all } j, k\ . \tag{A.15}$$

With additional work, one could actually get a better decay. □

Appendix B. Coordinates near the boundary

It is convenient in most calculations to straighten out the boundary by a coordinate transformation. This, quite standard, procedure, will be defined below. Let $z_0 \in \partial\Omega$ and let ℓ be the length of the boundary $\partial\Omega$ and $I =]-\ell/2, \ell/2]$. Let $M \in C^\infty(I; \partial\Omega)$ be a parametrization of $\partial\Omega$ such that $M(0) = z_0$ and s is the distance inside $\partial\Omega$ between $M(s)$ and z_0. We denote by

$$T(s) := M'(s)\ ,$$

the unit tangent vector of $\partial\Omega$ at $M(s)$ and the scalar curvature by $\kappa(s)$, which can be defined by

$$T'(s) = \kappa(s)\,\nu(s)\ ,$$

where $\nu(s)$ is the interior normal unit vector of $\partial\Omega$ at $M(s)$.

Moreover the parametrization is chosen positive:

$$\det\,(T(s), \nu(s)) = 1,\ \forall\, s\, \in\, I\ .$$

For any $z \in \overline{\Omega}$, we denote by $t(z)$ the standard distance of z to $\partial\Omega$:

$$t(z) = \inf_{\omega\in\partial\Omega} |z - \omega|\ .$$

So, there exists $t_0 > 0$ and a diffeomorphism of class C^∞:

$$\psi : \Omega_{t_0} \ \rightarrow\ S^1_{\ell/(2\pi)} \times (0, t_0)\ , \tag{B.1}$$

such that $\psi(z) = w = (s(z), t(z))$ and $|z - M(s(z))| = t(z)$.

We have denoted, for small enough ε, by Ω_ε the tubular neighborhood of $\partial\Omega$:

$$\Omega_\varepsilon := \{z \in \Omega;\ \mathrm{dist}(z, \partial\Omega) < \varepsilon\}$$

and S^1_r is the circle of radius r is identified with $[-\pi r, \pi r[$. So we have the identity

$$z = M(s(z)) + t(z)\nu(s(z)), \quad \forall\, z \,\in\, \Omega_{\varepsilon_0}\ . \tag{B.2}$$

From this equality, it is easy to check that

$$T(s(z)) = [1 - t(z)\kappa(s(z))]\nabla s(z) \text{ and } \nu(s(z)) = \nabla t(z)\ . \tag{B.3}$$

So for all $u \,\in\, H^1(\Omega)$ such that $\mathrm{supp}(u) \subset \Omega_{\varepsilon_0}$,

$$\begin{aligned}\int_\omega |(hD_z - A)u|^2\, dz = \int_K \Big[&|(hD_t - \widetilde{A}_2)v|^2 + (1 - t\kappa(s))^{-2} \\ &\times |(hD_s - \widetilde{A}_1)v|^2 \Big] (1 - t\kappa(s))\, dw\end{aligned} \tag{B.4}$$

and

$$\int_\omega |u|^2\, dz = \int_K |v|^2 (1 - t\kappa(s))\, dw\ , \tag{B.5}$$

with $v(w) = u(\psi^{-1}(w))$, $K = I \times (0, t_0)$, $w = (s,t)$ and $dw = ds\, dt$.

The magnetic potential $\widetilde{A}$ satisfies

$$\widetilde{A}_1\, ds + \widetilde{A}_2\, dt = A_1\, dx + A_2\, dy\ .$$

So

$$\begin{aligned}[\frac{\partial \widetilde{A}_2}{\partial s}(w) - \frac{\partial \widetilde{A}_1}{\partial t}(w)]\, ds \wedge dt &= B(z)\, dx \wedge dy \\ &= \hat{B}(w)[1 - t\kappa(s)]\, ds \wedge dt\ ,\end{aligned} \tag{B.6}$$

with $\psi(z) = w$ and $\hat{B}$ defined as:

$$\hat{B}(w) = B(z)\ . \tag{B.7}$$

This gives:

$$\begin{aligned}\frac{\partial \widetilde{A}_2}{\partial s}(w) - \frac{\partial \widetilde{A}_1}{\partial t}(w) &= B(\psi^{-1}(w))[1 - t\kappa(s)] \\ &= \hat{B}(t,s)(1 - t\kappa(s))\ .\end{aligned} \tag{B.8}$$

Then we get the identity between differential operators

$$\begin{aligned}(hD_z - A)^2 = a^{-1}[&(hD_s - \widetilde{A}_1)a^{-1}(hD_s - \widetilde{A}_1) \\ &+ (hD_t - \widetilde{A}_2)a(hD_t - \widetilde{A}_2)]\ ,\end{aligned} \tag{B.9}$$

where $a(w) = 1 - t\kappa(s)$.

The usual Hilbert space $L^2(\Omega_{t_0})$ is transformed to $L^2(K; a\,dw)$.

In the new coordinates and using a gauge transform, we can always assume that the magnetic potential has no normal component in a neighborhood of $\partial\Omega$:

$$\widetilde{A}_2 = 0\,. \tag{B.10}$$

In this case, we have:

$$\partial_t \widetilde{A}_1 = -\hat{B}(t,s)(1 - t\kappa(s))\,, \tag{B.11}$$

where $\hat{B}$ was introduced in (B.7). So we can choose a convenient gauge.

Lemma B.1. — *Suppose Ω is a bounded, simply connected domain with smooth boundary, let t_0 be the constant from* (B.1) *and let $[s_0, s_1]$ be a subset of $\mathbb{R}/|\partial\Omega|$ with $s_1 - s_0 < |\partial\Omega|$. Then there exists a constant $C > 0$ such that, if $\vec{A}$ is a vector potential in Ω with*

$$\operatorname{curl} \vec{A} = 1 \text{ in } \Omega\,, \tag{B.12}$$

and with $\tilde{A}$ defined above, then there exists a gauge function $\varphi(s,t)$ on $(s_0, s_1) \times (0, t_0)$ such that

$$\bar{A}(s,t) = \begin{pmatrix} \bar{A}_1(s,t) \\ \bar{A}_2(s,t) \end{pmatrix} := \tilde{A} - \nabla_{(s,t)}\varphi = \begin{pmatrix} -t + \frac{t^2 k(s)}{2} \\ 0 \end{pmatrix}\,. \tag{B.13}$$

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BIBLIOGRAPHY

[1] S. Agmon, *Lectures on exponential decay of solutions of second order elliptic equations*, Math. Notes, vol. 29, Princeton University Press, 1982.

[2] P. Bauman, D. Phillips & Q. Tang, "Stable nucleation for the Ginzburg-Landau system with an applied magnetic field", *Arch. Rational Mech. Anal.* **142** (1998), p. 1-43.

[3] A. Bernoff & P. Sternberg, "Onset of superconductivity in decreasing fields for general domains", *J. Math. Phys.* **39** (1998), p. 1272-1284.

[4] C. Bolley & B. Helffer, "An application of semi-classical analysis to the asymptotic study of the supercooling field of a superconducting material", *Ann. Inst. H. Poincaré (Section Physique Théorique)* **58** (1993), no. 2, p. 169-233.

[5] V. Bonnaillie, "Analyse mathématique de la supraconductivité dans un domaine à coins : méthodes semi-classiques et numériques", PhD Thesis, Université Paris 11, 2003.

[6] ———, "On the fundamental state for a Schrödinger operator with magnetic fields in domains with corners", *Asymptotic Anal.* **41** (2005), no. 3-4, p. 215-258.

[7] V. Bonnaillie-Noël & M. Dauge, "Asymptotics for the low-lying eigenstates of the Schrödinger operator with magnetic field near corners", preprint, 2005.

[8] H. Cycon, R. Froese, W. Kirsch & B. Simon, *Schrödinger operators*, Springer Verlag, Berlin, 1987.

[9] M. Dauge & B. Helffer, "Eigenvalues variation I, Neumann problem for Sturm-Liouville operators", *J. Differential Equations* **104** (1993), no. 2, p. 243-262.

[10] M. Dimassi & J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, 1999.

[11] S. Fournais & B. Helffer, "Energy asymptotics for type II superconductors", preprint, 2004.

[12] V. V. Grušhin, "Hypoelliptic differential equations and pseudodifferential operators with operator-valued symbols", *Mat. Sb. (N.S.)* **88** (1972), no. 130, p. 504-521 (russian).

[13] B. Helffer, *Introduction to the semiclassical analysis for the Schrödinger operator and applications*, Lecture Notes in Math., vol. 1336, Springer Verlag, 1988.

[14] B. Helffer & A. Mohamed, "Semiclassical analysis for the ground state energy of a schrödinger operator with magnetic wells", *J. Funct. Anal.* **138** (1996), no. 1, p. 40-81.

[15] B. Helffer & A. Morame, "Magnetic bottles in connection with superconductivity", *J. Funct. Anal.* **185** (2001), no. 2, p. 604-680.

[16] ———, "Magnetic bottles for the Neumann problem : curvature effect in the case of dimension 3 (general case)", *Ann. Sci. École Norm. Sup.* **37** (2004), p. 105-170.

[17] B. Helffer & X. Pan, "Upper critical field and location of surface nucleation of superconductivity", *Ann. Inst. H. Poincaré (Section Analyse non linéaire)* **20** (2003), no. 1, p. 145-181.

[18] B. Helffer & J. Sjöstrand, "Multiple wells in the semiclassical limit I", *Comm. Partial Differential Equations* **9** (1984), no. 4, p. 337-408.

[19] ———, "Puits multiples en limite semi-classique II – Interaction moléculaire – Symétries – Perturbations", *Ann. Inst. H. Poincaré (Section Physique théorique)* **42** (1985), no. 2, p. 127-212.

[20] ———, "Puits multiples en limite semiclassique V – le cas des minipuits", in *Current topics in partial differential equations*, Kinokuniya, Tokyo, 1986, p. 133-186.

[21] ———, "Effet tunnel pour l'équation de Schrödinger avec champ magnétique", *Ann. Scuola Norm. Sup. Pisa* **14** (1987), no. 4, p. 625-657.

[22] K. Lu & X.-B. Pan, "Eigenvalue problems of Ginzburg-Landau operator in bounded domains", *J. Math. Phys.* **40** (1999), no. 6, p. 2647-2670.

[23] ———, "Estimates of the upper critical field for the equations of superconductivity", *Physica D* **127** (1999), p. 73-104.

[24] ———, "Gauge invariant eigenvalue problems on $\mathbb{R}^2$ and $\mathbb{R}^2_+$", *Trans. Amer. Math. Soc.* **352** (2000), no. 3, p. 1247-1276.

[25] ———, "Surface nucleation of superconductivity in 3-dimension", *J. of Differential Equations* **168** (2000), no. 2, p. 386-452.

[26] M. d. Pino, P. Felmer & P. Sternberg, "Boundary concentration for eigenvalue problems related to the onset of superconductivity", *Comm. Math. Phys.* **210** (2000), p. 413-446.

[27] M. Reed & B. Simon, *Methods of modern mathematical physics, IV: Analysis of operators*, Academic Press, New York, 1978.

[28] D. Robert, *Autour de l'approximation semi-classique*, Birkhäuser, Boston, 1987.

[29] D. Saint-James, G. Sarma & E. Thomas, *Type II superconductivity*, Pergamon, Oxford, 1969.

[30] B. Simon, "Semi-classical analysis of low lying eigenvalues I", *Ann. Inst. H. Poincaré (Section Physique Théorique)* **38** (1983), no. 4, p. 295-307.

[31] J. Sjöstrand, "Operators of principal type with interior boundary conditions", *Acta Math.* **130** (1973), p. 1-51.

[32] P. Sternberg, "On the normal/superconducting phase transition in the presence of large magnetic fields", in *Connectivity and Superconductivity* (J. Berger & J. Rubinstein, eds.), Lect. Notes in Physics, vol. M 62, Springer Verlag, 2000, p. 188-199.

[33] D. Tilley & J. Tilley, *Superfluidity and superconductivity*, 3rd ed., Institute of Physics Publishing, Bristol and Philadelphia, 1990.

[34] M. Tinkham, *Introduction to superconductivity*, McGraw-Hill Inc, New York, 1975.

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Soeren FOURNAIS & Bernard HELFFER
Université Paris-Sud
CNRS & Laboratoire de Mathématiques
UMR 8628 — Bât 425
91405 Orsay Cedex (France)
soeren.fournais@math.u-psud.fr
bernard.helffer@math.u-psud.fr