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MATRIX VALUED ORTHOGONAL POLYNOMIALS OF JACOBI TYPE: THE ROLE OF GROUP REPRESENTATION THEORY

by F. Alberto GRÜNBAUM, Inés PACHARONI & Juan TIRAO (*)

1. Introduction.

Starting with the work of M. G. Krein, [K1] and [K2], as well as more recent contributions, including for instance [D1], [D2], [D3], [D4], [DP], [DvA], [Ge], and [SvA], there is a nice and general theory of matrix valued orthogonal polynomials. These are bound to play an important part in many areas of mathematics and its applications, just as their scalar counterparts. This should be particularly true for those matrix valued orthogonal polynomials that have some *extra property*, such as the one singled out for further study in [DG] and generally known under the label *the bispectral property*. The search for concrete instances enjoying these two properties has received a certain amount of recent attention, following earlier work started in [D1]. The collection of known examples enjoying this extra property is still very small. For a family of examples (in arbitrary dimension), not reducible to the scalar case, see [GPT1], [GPT2] and the closing paragraph in [GPT4]. For recent progress in this area, including a general method to attack this problem and a relevant hierarchy of examples,

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see [DG1] and [DG2]. For a different source of examples one can consult [G] and [GPT5].

In this paper we restrict our attention to examples that are of the *Jacobi type*, meaning that the differential operator is given by the Gauss hypergeometric one, see [T2].

Even a cursory look at the emerging family of examples reveals two distinct tools: either one poses and solves a certain set of matrix valued differential equations (along with certain boundary conditions) as in [DG1] and [GPT5], or one ignores these equations altogether, starts with a symmetric space G/K, as in [GPT1], [GPT2],[GPT3], [GPT4] and eventually arrives at explicit examples. No one familiar with the history of these developments in the scalar case should be surprised with the useful role played by these symmetric spaces. For a good general reference see [VK]. None of the two routes mentioned above gives an easy path to examples. For this reason alone we consider it very important, at this early stage of this search for examples, to exploit all possible avenues. For the examples obtained in [GPT1] one starts with the theory of matrix valued spherical functions, see [T1] and [GV]. Specifically one takes $G = \mathrm{SU}(3)$ and $K = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)) \simeq \mathrm{U}(2)$. The corresponding symmetric space is then the complex projective plane. As noticed in [GPT1], see page 355, when talking about *properly packaged* matrix valued spherical functions one is not quite dealing with matrix valued orthogonal polynomials. The step needed to make this connection is given in the last section of [GPT4]. To better understand the point of view of this paper it is important to review in more detail some of the developments discussed above. The results in [GPT1] give instances of matrix valued *classical pairs* $\{W, D\}$ of *arbitrary* size, the only restriction is that the value of the parameter β needs to be 1. The classical scalar valued case corresponds to the further specialization $\ell = 0$. Since the value of α in [GPT1] is only restricted to be an integer, it is not hard to see how to extend this beyond *group values*. The issue of extending beyond the case of $\beta = 1$ is a completely different game. This was achieved in [G] for square matrices of size 2 by postulating a certain structure for the weight matrix which was consistent with all the examples known up to that point. Later in [GPT5] three one parameter families of classical pairs of the same size were constructed, one of which extends the example in [G]. The extension of this search of classical pairs to larger size is not an easy matter. In this paper we use some preliminaries results from [PT] where the complex projective plane considered in [GPT1] is replaced by the n-dimensional complex projective space. This gives us a group theoretical framework which provides enough integers parameters m and n. Extending the resulting classical pairs to arbitrary values of the parameters α and β is then completely straightforward.

2. Matrix valued orthogonal polynomials and symmetric differential operators.

Here we recall some standard facts. Given a self adjoint positive definite matrix valued smooth weight function $W(t)$ with finite moments we can consider the skew symmetric bilinear form defined for any pair of matrix valued polynomial functions $P(t)$ and $Q(t)$ by the numerical matrix

$$(P, Q) = (P, Q)_W = \int_{\mathbb{R}} P(t)W(t)Q^*(t)dt,$$

where $Q^*(t)$ denotes the conjugate transpose of $Q(t)$. By the usual construction this leads to the existence of a sequence of matrix valued orthogonal polynomials with non singular leading coefficient. The skew symmetric bilinear form introduced above is not the only possible such choice, as noticed for instance in [SvA]. In this section we will also consider the form

$$\langle P, Q\rangle = (P^*, Q^*)^*.$$

The reason for considering this form can be traced back to [GPT1] as will be noticed below. Observe that a sequence $\{P_n\}_{n\geqslant 0}$ of matrix valued polynomials is orthogonal with respect to $(\cdot,\cdot)$ if and only if the sequence $\{P_n^*\}_{n\geqslant 0}$ is orthogonal with respect to $\langle\cdot,\cdot\rangle$. Given an orthonormal sequence $\{P_n(t)\}_{n\geqslant 0}$ one gets by the usual argument a three term recursion relation

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + C_{n-1}P_{n-1}(t), \tag{1}$$

where A_{n+1} is nonsingular, $B_n^* = B_n$ and $C_{n-1} = A_n^*$. We now turn our attention to an important class of orthogonal polynomials which we will call *classical matrix valued orthogonal polynomials*. As in [GPT5] we say that the weight function is *classical* if there exists a second order ordinary differential operator D with matrix valued polynomial coefficients $A_j(t)$ of degree less or equal to j of the form

$$D = A_2(t)\frac{d^2}{dt^2} + A_1(t)\frac{d}{dt} + A_0(t), \tag{2}$$

such that

$$\langle DP, Q\rangle = \langle P, DQ\rangle \tag{3}$$

for all matrix valued polynomial functions P and Q. We refer to such a pair $\{W, D\}$ as a *classical pair*. If $\{W, D\}$ is a classical pair then there exists an orthonormal sequence $\{P_n\}$, with respect to $(\cdot,\cdot)$, of matrix valued polynomials such that

$$DP_n^* = P_n^* \Lambda_n, \tag{4}$$

where Λ_n is a real valued diagonal matrix. This form of the eigenvalue equation (4) appears naturally in [GPT1] and corresponds to the fact that the *rows* of P_n are eigenfunctions of D. One could avoid the introduction of the inner product $\langle\ ,\rangle$ in (3) by introducing *right handed* differential operators as in [D1]. Either choice has its own drawbacks. This is a consequence of the fact that in the matrix valued case D and the difference operator in the right hand side of (1) do not commute. Assume that the weight function $W = W(t)$ is supported in the interval (a, b). We recall that in [GPT5] and in [DG1] (except for a change due to the fact that the differential operator there is acting on the right hand side) we have proved that the condition of symmetry (3) is equivalent to the following three differential equations

$$\begin{aligned} A_2^* W &= W A_2, \\ A_1^* W &= -W A_1 + 2\frac{d}{dt}(W A_2), \\ A_0^* W &= W A_0 - \frac{d}{dt}(W A_1) + \frac{d^2}{dt^2}(W A_2), \end{aligned} \tag{5}$$

with the boundary conditions

$$\lim_{t\to x} W(t) A_2(t) = 0 = \lim_{t\to x} \big(W(t) A_1(t) - A_1^*(t) W(t) \big),$$

for $x = a, b$.

The second condition can be replaced by

$$\lim_{t\to x} \Big(W(t) A_1(t) - \frac{d}{dt}\big(W(t) A_2(t)\big) \Big) = 0. \tag{6}$$

These equations are quite general and do not depend on the assumptions that the matrix valued coefficients of D are polynomials nor on the fact that the interval (a, b) is finite. Finding explicit solutions of these equations is a highly non trivial task. As noted before, in [GPT1] we computed explicitly the matrix valued spherical functions of any K-type associated to the complex projective plane. These results were connected in [GPT4] with the established theory of matrix valued orthogonal polynomials. We will see later that by using just the *first few steps* of the analysis in [GPT1] and [PT] one gets solutions of equations (5) and (6). More explicitly the Casimir of G is symmetric with respect to the L^2-inner product between matrix valued functions on G. The method of [GPT1] allows one to replace this differential operator by a second order differential operator in the variable $t \in (0,1)$. By an appropriate conjugation as in [GPT4] one converts this operator into a matrix valued differential operator D of the hypergeometric type. Similarly the L^2-inner product on G leads to a matrix valued weight function W on $(0,1)$. The symmetry of the Casimir mentioned above makes D symmetric with respect to W, hence we obtain a classical pair $\{W, D\}$. This program will be carried out in the next sections.

3. Some background material from representation theory.

In the forthcoming paper [PT] one tackles the problem of determining the matrix valued spherical functions associated to the n-dimensional complex projective space $P_n(\mathbb{C})$. This space can be realized as the homogeneous space G/K, where $G = \mathrm{SU}(n+1)$ and $K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1)) \simeq \mathrm{U}(n)$. By going from the complex projective plane as in [GPT1] to the n-dimensional complex projective space one gets a plethora of new phenomena. Here we recall a few facts from [PT] obtained in a similar way as those corresponding to the case $n = 2$ given in [GPT1]. Let (V_π, π) be any irreducible representation of K. An irreducible spherical function can be characterized as a function $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ such that

i) Φ is analytic.

ii) $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.

iii) $[\Delta\Phi](g) = \Phi(g)[\Delta\Phi](e)$, for all $g \in G$ and $\Delta \in D(G)^G$

where $D(G)^G$ denotes the algebra of all left and right invariant differential operators on G. We observe that the Casimir operator Δ_2 of G belongs to $D(G)^G$. For our purposes it is convenient to consider a larger class of functions, namely the vector space of all functions Φ that satisfy

i) Φ is analytic.

ii) $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$.

iii) $[\Delta_2\Phi](g) = \Phi(g)[\Delta_2\Phi](e)$, for all $g \in G$.

The study of these spaces of functions is carried out in [PT] by using the approach of [GPT1]. This naturally lead us to a very rich collection of classical pairs $\{W, D\}$.

For any $g \in \mathrm{SU}(n+1)$ we denote by $A(g)$ the left upper $n \times n$ block of g, and we consider the open dense subset $\mathcal{A} = \{g \in G : \det(A(g)) \neq 0\}$. Then $\mathcal{A}$ is left and right invariant under K. We introduce the following function defined on $\mathcal{A}$:

$$\Phi_\pi(g) = \pi(A(g)),$$

where π above denotes the unique holomorphic representation of $\mathrm{GL}(n, \mathbb{C})$ which extends the given representation of $\mathrm{U}(n)$. For any function Φ in our class we associate to it a function $H : \mathcal{A} \longrightarrow \mathrm{End}(V_\pi)$ defined by

$$H(g) = \Phi(g)\Phi_\pi(g)^{-1}.$$

Then H satisfies that $H(e) = I$ and

i) $H(gk) = H(g)$, for all $g \in \mathcal{A}, k \in K$.

ii) $H(kg) = \pi(k)H(g)\pi(k^{-1})$, for all $g \in \mathcal{A}, k \in K$.

The canonical projection $p : G \longrightarrow P_n(\mathbb{C})$ maps the open dense subset $\mathcal{A}$ onto the affine space $\mathbb{C}^n$ of those points in $P_n(\mathbb{C})$ whose last homogeneous coordinate is not zero. Thus, property i) says that H may be considered as a function on $\mathbb{C}^n$. The fact that Φ is an eigenfunction of Δ_2 makes H into an eigenfunction of a differential operator D on $\mathbb{C}^n$. The explicit computation of D is carried out in [PT]: For $H \in C^\infty(\mathbb{C}^n) \otimes \mathrm{End}(V_\pi)$ we have

$$\begin{aligned} D(H)(z_1, \dots, z_n) = &\Big(1 + \sum_{1 \leqslant j \leqslant n} |z_j|^2\Big)\Big(\sum_{1 \leqslant i \leqslant n} \Big(\tfrac{\partial^2 H}{\partial x_i^2} + \tfrac{\partial^2 H}{\partial y_i^2}\Big)(1 + |z_i|)^2 \\ &+ 2 \sum_{1 \leqslant i \leqslant n} \sum_{j > i} \Big(\tfrac{\partial^2 H}{\partial x_i \partial x_j} + \tfrac{\partial^2 H}{\partial y_i \partial y_j}\Big) \operatorname{Re}(z_i \overline{z}_j) \\ &- 2 \sum_{1 \leqslant i \leqslant n} \sum_{j > i} \Big(\tfrac{\partial^2 H}{\partial x_i \partial y_j} - \tfrac{\partial^2 H}{\partial x_j \partial y_i}\Big) \operatorname{Im}\,(z_i \overline{z}_j)\Big) \\ &- 2 \sum_{1 \leqslant i \leqslant n} \Big(\tfrac{\partial H}{\partial x_i} - \tfrac{\partial H}{\partial y_i}\Big) \dot{\pi}(P_i), \end{aligned}$$

where $z_j = x_j + iy_j$ and P_j is the $n \times n$ matrix whose (r, s) element is $(P_j)_{rs} = z_r(\delta_{js} + z_j \overline{z}_s)$. Here $\dot{\pi}$ denotes the representation of the Lie algebra of K obtained by taking the derivative at the identity of the representation π. By property ii), H is determined by its restriction $H = H(r)$ to the cross section $\{(r, 0, \dots, 0) : r \geqslant 0\}$ of the K-orbits in $\mathbb{C}^n$, which are the spheres of radius $r \geqslant 0$. Then $H = H(r)$ becomes an eigenfunction of the following differential operator

$$
\begin{aligned}
DH(r) &= (1+r^2)^2\frac{d^2H}{dr^2} + \frac{(1+r^2)}{r}\frac{dH}{dr}\left(2n{-}1 + r^2 - 2r^2\dot\pi(E_{11})\right)\\
&\quad + \frac{4}{r^2}\Bigg(\sum_{2\leqslant j\leqslant n}\dot\pi(E_{j1})H(r)\dot\pi(E_{1j}) - H(r)\sum_{2\leqslant j\leqslant n}\dot\pi(E_{j1})\dot\pi(E_{1j})\Bigg)\\
&\quad + \frac{4(1{+}r^2)}{r^2}\Bigg(\sum_{2\leqslant j\leqslant n}\dot\pi(E_{1j})H(r)\dot\pi(E_{j1}) - H(r)\sum_{2\leqslant j\leqslant n}\dot\pi(E_{1j})\dot\pi(E_{j1})\Bigg),
\end{aligned}
\tag{7}
$$

where E_{ij} denotes the $n\times n$ matrix with entry (i,j) equal to 1 and 0 elsewhere.

The irreducible representations of $\mathrm{U}(n)$ are restrictions of irreducible holomorphic representations of $\mathrm{GL}(n,\mathbb{C})$, which are parameterized, up to equivalence, by n-tuples of integers

$$\pi = (m_1, m_2, \dots, m_n) \quad \text{such that} \quad m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n.$$

As $\mathrm{GL}(n{-}1,\mathbb{C})$-module, V_π decomposes as a direct sum of irreducible representations, each one with multiplicity one, namely

$$V_\pi = \bigoplus_{\mu \text{ interlace } \pi} V_\mu,$$

where the sum is over all $(n{-}1)$-tuples that satisfy

$$\mu = (m_1^\mu, \dots, m_{n-1}^\mu) \in \mathbb{Z}^{n-1}, \ \text{ with } \ m_i \geqslant m_i^\mu \geqslant m_{i+1},\ i = 1, \dots, n{-}1.$$

The above facts are well known and can be found in [VK].

The subgroup M of all matrices in K of the form $\left(\begin{smallmatrix} a & 0\\ 0 & A\end{smallmatrix}\right)$, with $A \in \mathrm{U}(n{-}1)$ fixes all points $(r,0,\dots,0) \in \mathbb{C}^n$. Then since H satisfies property ii) above, it follows that the linear transformation $H(r)$ commutes with $\pi(M)$, for all $r \geqslant 0$. Thus $H(r)$ is a scalar $h_\mu(r)$ on each V_μ. Therefore, after a choice of an ordering of the interlacing $\mu's$, we can identify the function $H(r)$ with the vector valued function $(h_\mu(r))_\mu \in \mathbb{C}^L$, where L is the number of all $(n{-}1)$-tuples μ which interlace π.

We observe that the linear transformations

$$\sum_{2\leqslant j\leqslant n}\dot\pi(E_{j1})H(r)\dot\pi(E_{1j}), \quad \sum_{2\leqslant j\leqslant n}\dot\pi(E_{j1})\dot\pi(E_{1j}),$$

$$\sum_{2\leqslant j\leqslant n}\dot\pi(E_{1j})H(r)\dot\pi(E_{j1}) \ \text{ and } \ \sum_{2\leqslant j\leqslant n}\dot\pi(E_{1j})\dot\pi(E_{j1}),$$

which appear in (7) commute with $\pi(M)$. Therefore they are scalar multiples of the identity on each V_μ. These scalars are computed in [PT] by looking at the fine structure of the representation π going along a Gelfand-Cetlin basis of V_π. In the next section we will carry out this computation in the particular case $L=3$.

The fact that $H=H(r)$ is an eigenfunction of the differential operator (7), and after the change of variable $t=(1+r^2)^{-1}$, implies that the function $H(t)=(h_\mu(t))_\mu$ associated to the spherical function Φ satisfies the following system of differential equations

$$\begin{aligned} t(1-t)\,h''_\mu(t) &+ \big(s_\pi - s_\mu + 1 - t(s_\pi - s_\mu + n + 1)\big)\,h'_\mu(t)\\ &+\frac{1}{1-t}\left(\sum_{j=1}^{n-1} t_{j,\mu}\big(h_{\mu+e_j}(t)-h_\mu(t)\big)\right)\\ &+\frac{t}{1-t}\left(\sum_{j=1}^{n-1} s_{j,\mu}\big(h_{\mu-e_j}(t)-h_\mu(t)\big)\right)=\lambda\,h_\mu(t), \end{aligned}\tag{8}$$

where e_j denotes the j-th canonical basis vector in $\mathbb{R}^{n-1}$, $s_\pi=\sum_{i=1}^n m_i$, $s_\mu=\sum_{i=1}^{n-1} m_i^\mu$,

$$t_{j,\mu}=\frac{\prod_{i=1}^n |m_i-m_j^\mu-i+j|}{\prod_{\substack{1\leqslant i\leqslant n-1,\\ i\neq j}} |m_i^\mu-m_j^\mu-i+j|}$$

and

$$s_{j,\mu}=\frac{\prod_{i=1}^n |m_i-m_j^\mu-i+j+1|}{\prod_{\substack{1\leqslant i\leqslant n-1,\\ i\neq j}} |m_i^\mu-m_j^\mu-i+j|}.$$

As mentioned in Section 2 the group representation theory recalled above has given us in (8) a differential operator in the variable t which is symmetric with respect to the inner product given below.

The L^2-inner product for matrix valued functions Φ and Ψ in the class introduced above gives rise to the following inner product of the corresponding functions H and K on $(0,1)$,

$$\langle H,K\rangle=\sum_{\mu\ \text{interlace}\ \pi} 2n\dim(V_\mu)\int_0^1 (1-t)^{n-1}t^{s_\pi-s_\mu}\,h_\mu(t)\overline{k_\mu(t)}\,dt.\tag{9}$$

Explicitly the dimension of V_μ can be computed by using the Weyl's formula

$$\dim V_\mu=\prod_{1\leqslant i<k\leqslant n-1}\frac{m_i^\mu-m_k^\mu+k-i}{k-i}.$$

4. The new Jacobi type examples.

In this section we continue with the program advertised at the end of Section 2, i.e. we will give explicit classical pairs $\{\widetilde{W}, \widetilde{D}\}$ with

$$\widetilde{W}(t) = t^{\alpha}(1-t)^{\beta}F(t) \qquad \alpha, \beta > -1 \tag{10}$$

and

$$\widetilde{D} = t(1-t)\frac{d^2}{dt^2} + (X - tU)\frac{d}{dt} + V, \tag{11}$$

where $F(t)$ is a polynomial function and X, U, V are constant matrices.

The important step involved in going from (8) to (11) and from the weight implicit in (9) to the one in (10) will be carried out for two special kinds of representations π in Sections 4.1 and 4.2 below.

4.1. Examples of size 3×3.

We consider representations π of $\mathrm{GL}(n, \mathbb{C})$ that correspond to n-tuples of the form

$$\pi = (\underbrace{m+2, \dots, m+2}_{k},\ \underbrace{m, \dots, m}_{n-k}),$$

with $1 \leqslant k \leqslant n{-}1$. We have

$$\dim V_\pi = \prod_{j=0}^{k-1} \frac{(n{-}j)(n{-}j+1)}{(k-j)(k-j+1)}.$$

As $\mathrm{GL}(n{-}1, \mathbb{C})$-modules one has the decomposition

$$V_\pi = V_{\mu_1} \oplus V_{\mu_2} \oplus V_{\mu_3},$$

where

$$\mu_1 = (\underbrace{m+2, \dots, m+2}_{k-1},\ m,\ \underbrace{m, \dots, m}_{n-k-1}),$$

$$\mu_2 = (\underbrace{m+2, \dots, m+2}_{k-1},\ m+1,\ \underbrace{m, \dots, m}_{n-k-1}),$$

$$\mu_3 = (\underbrace{m+2, \dots, m+2}_{k-1},\ m+2,\ \underbrace{m, \dots, m}_{n-k-1}).$$

It is important to note that

$$\dim V_{\mu_1} = \prod_{j=0}^{k-2} \frac{(n-j-1)(n-j)}{(k-j-1)(k-j)},$$

$$\dim V_{\mu_2} = k(n-k) \prod_{j=0}^{k-2} \frac{(n-j-1)(n-j)}{(k-j)(k-j+1)},$$

$$\dim V_{\mu_3} = \prod_{j=0}^{k-1} \frac{(n-j-1)(n-j)}{(k-j)(k-j+1)}.$$

In this particular case the derivation of (8) from (7) is much simpler than in the general case. Next we give an idea of what is involved in this process. We recall that we need to compute in each V_{μ_i} $(i = 1, 2, 3)$ the linear transformations

$$\sum_{2\leqslant j\leqslant n} \dot\pi(E_{j1})H(r)\dot\pi(E_{1j}), \qquad \sum_{2\leqslant j\leqslant n} \dot\pi(E_{j1})\dot\pi(E_{1j}),$$

$$\sum_{2\leqslant j\leqslant n} \dot\pi(E_{1j})H(r)\dot\pi(E_{j1}) \quad \text{and} \quad \sum_{2\leqslant j\leqslant n} \dot\pi(E_{1j})\dot\pi(E_{j1}).$$

The highest weight of V_{μ_i} is

$$\mu_i = (m+3-i)x_1 + (m+2)(x_2+\cdots+x_k) + (m+i-1)x_{k+1} + m(x_{k+2}+\cdots+x_n).$$

Therefore all weights in V_{μ_i} are of the form $\mu = \mu_i - \sum_{r=2}^{n-1} n_r\alpha_r$ with $\alpha_r = x_r - x_{r+1}$, thus $\mu = (m+3-i)x_1 + \cdots$. Now we observe that for the representations π considered here, for all $j = 2, \dots, n$ we have

$$\dot\pi(E_{1j})(V_{\mu_i}) \subset V_{\mu_{i-1}} \quad \text{and} \quad \dot\pi(E_{j1})(V_{\mu_i}) \subset V_{\mu_{i+1}}.$$

In fact, if $v \in V_{\mu_i}$ is a vector of weight μ then $\dot\pi(E_{1j})v$ is a vector of weight $\mu + x_1 - x_j = (m+3-(i-1))x_1 + \cdots$. Similarly $\dot\pi(E_{j1})v$ is a vector of weight $\mu + x_j - x_1 = (m+3-(i+1))x_1 + \cdots$. Therefore

$$\sum_{2\leqslant j\leqslant n} \dot\pi(E_{j1})H(r)\dot\pi(E_{1j})v = h_{\mu_{i-1}}(r) \sum_{2\leqslant j\leqslant n} \dot\pi(E_{j1})\dot\pi(E_{1j})v$$

and

$$\sum_{2\leqslant j\leqslant n} \dot\pi(E_{1j})H(r)\dot\pi(E_{j1})v = h_{\mu_{i+1}}(r) \sum_{2\leqslant j\leqslant n} \dot\pi(E_{1j})\dot\pi(E_{j1})v.$$

The Casimir element of $\mathrm{GL}(n,\mathbb{C})$ is

$$\Delta_2^{(n)} = \sum_{1\leqslant i,j\leqslant n} E_{ij}E_{ji} = \sum_{1\leqslant i\leqslant n} E_{ii}^2 + \sum_{1\leqslant i<j\leqslant n} (E_{ii} - E_{jj}) + 2\sum_{1\leqslant i<j\leqslant n} E_{ji}E_{ij}.$$

Similarly the Casimir operator of $\mathrm{GL}(n-1,\mathbb{C}) \subset \mathrm{GL}(n,\mathbb{C})$ is

$$\Delta_2^{(n-1)} = \sum_{2\leqslant i\leqslant n} E_{ii}^2 + \sum_{2\leqslant i<j\leqslant n} (E_{ii}-E_{jj}) + 2\sum_{2\leqslant i<j\leqslant n} E_{ji}E_{ij}.$$

Hence

$$\sum_{2\leqslant j\leqslant n} E_{j1}E_{1j} = \frac{1}{2}\Big(\Delta_2^{(n)} - \Delta_2^{(n-1)} - E_{11}^2 - \sum_{2\leqslant j\leqslant n}(E_{11}-E_{jj})\Big). \tag{12}$$

To compute the scalar linear transformation $\sum_{2\leqslant j\leqslant n}\dot\pi(E_{j1})\dot\pi(E_{1j})$ on V_{μ_i} it is enough to apply it to a highest weight vector v_i of V_{μ_i}. The highest weight of V_π is $(m+2)(x_1+\cdots+x_k)+m(x_{k+1}+\cdots+x_n)$, and the weight of v_i is $(m+3-i)x_1+(m+2)(x_2+\cdots+x_k)+(m+i-1)x_{k+1}+m(x_{k+2}+\cdots+x_n)$. Then we have that, $\dot\pi(\Delta_2^{(n-1)})$ acting on V_{μ_i} is the scalar

$$\begin{aligned}
\dot\pi(\Delta_2^{(n)})v_i &= \big(k(m+2)^2+(n-k)m^2+2k(n-k)\big)v_i,\\
\dot\pi(\Delta_2^{(n-1)})v_i &= \big((k-1)(m+2)^2+(m+i-1)^2\\
&\quad + (n-k-1)m^2+(3-i)(k-1)\\
&\quad + (i-1)(n-k-1)+2(k-1)(n-k-1)\big)v_i,\\
\dot\pi(E_{11})v_i &= (m+3-i)v_i,\\
\dot\pi\Big(\sum_{2\leqslant j\leqslant n}(E_{11}-E_{jj})\Big)v_i &= \big(2(n-k)-n(i-1)\big)v_i.
\end{aligned}$$

Therefore, by using (12), we obtain for all $v\in V_{\mu_i}$

$$\sum_{2\leqslant j\leqslant n}\dot\pi(E_{j1})\dot\pi(E_{1j})v = \dot\pi\Big(\sum_{2\leqslant j\leqslant n}E_{j1}E_{1j}\Big)v = (i-1)(3+k-i)v.$$

Similarly for all $v\in V_{\mu_i}$ we have

$$\sum_{2\leqslant j\leqslant n}\dot\pi(E_{1j})\dot\pi(E_{j1})v = \dot\pi\Big(\sum_{2\leqslant j\leqslant n}E_{1j}E_{j1}\Big)v = (3-i)(n+i-1-k)v.$$

Hence, the differential operator D in (7) becomes, for $i=1,2,3$

$$\begin{aligned}
(DH)_i(r) =&(1+r^2)^2h_i''(r) + \frac{1+r^2}{r}\big(2n-1+r^2-2(m+3-i)r^2\big)h_i'(r)\\
&+\frac{4}{r^2}(i-1)(3+k-i)\big(h_{i-1}(r)-h_i(r)\big)\\
&+\frac{4(1+r^2)}{r^2}(3-i)(n+i-1-k)\big(h_{i+1}(r)-h_i(r)\big).
\end{aligned}$$

After the change of variable $t=(1+r^2)^{-1}$, one gets the matrix valued differential operator

$$D = t(1-t)\frac{d^2}{dt^2} + \left[\begin{pmatrix} 3+m & 0 & 0 \\ 0 & 2+m & 0 \\ 0 & 0 & 1+m \end{pmatrix} - t\begin{pmatrix} 3+m+n & 0 & 0 \\ 0 & 2+m+n & 0 \\ 0 & 0 & 1+m+n \end{pmatrix}\right]\frac{d}{dt}$$
$$+ \frac{1}{1-t}\begin{pmatrix} -2(n-k) & 2(n-k) & 0 \\ 0 & -n+k-1 & n-k+1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t}{1-t}\begin{pmatrix} 0 & 0 & 0 \\ k+1 & -k-1 & 0 \\ 0 & 2k & -2k \end{pmatrix}$$

and the corresponding weight function

$$W(t) = t^m(1-t)^{n-1}\begin{pmatrix} w_1t^2 & 0 & 0 \\ 0 & w_2t & 0 \\ 0 & 0 & w_3 \end{pmatrix},$$

with $w_1 = \dim V_{\mu_1}$, $w_2 = \dim V_{\mu_2}$ y $w_3 = \dim V_{\mu_3}$.

To put this in the framework of matrix valued orthogonal polynomials we proceed as in [GPT4] by finding an appropriate conjugation of the differential operator and of the corresponding weight function. We take

$$\Psi^*(t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & 0 \\ 0 & 0 & (1-t)^2 \end{pmatrix}.$$

Then we get

$$\widetilde{W}(t) = \Psi(t)W(t)\Psi^*(t)$$

for the new weight function, and the new differential operator $\widetilde{D}F = (\Psi^*)^{-1}D(\Psi^*F)$ is

$$\widetilde{D} = \widetilde{A}_2(t)\frac{d^2}{dt^2} + \widetilde{A}_1(t)\frac{d}{dt} + \widetilde{A}_0(t),$$

with $\widetilde{A}_2$, $\widetilde{A}_1$, $\widetilde{A}_0$ given by

$$\widetilde{A}_2(t) = t(1-t),$$

$$\widetilde{A}_1(t) = \begin{pmatrix} m+3 & 0 & 0 \\ -1 & m+2 & 0 \\ 0 & -2 & m+1 \end{pmatrix} - t\begin{pmatrix} (n+m+3) & 0 & 0 \\ 0 & (n+m+4) & 0 \\ 0 & 0 & (m+n+5) \end{pmatrix},$$

$$\widetilde{A}_0(t) = \begin{pmatrix} 0 & 2(n-k) & 0 \\ 0 & -(n+m+1-k) & n+1-k \\ 0 & 0 & -2(n+m+2-k) \end{pmatrix}.$$

From the program outlined at the end of Section 2 it follows that for any $m \in \mathbb{N}_0$ and $n = 2, 3, \ldots$ the following equations are satisfied

$$\begin{aligned} \widetilde{A}_0^*\widetilde{W} - \widetilde{W}\widetilde{A}_0 + (\widetilde{W}\widetilde{A}_1)' - (\widetilde{W}\widetilde{A}_2)'' &= 0, \\ \widetilde{A}_1^*\widetilde{W} + \widetilde{W}\widetilde{A}_1 - 2(\widetilde{W}\widetilde{A}_2)' &= 0, \\ \widetilde{W}\widetilde{A}_2\big|_{t=0} = \widetilde{W}\widetilde{A}_2\big|_{t=1} &= 0, \\ (\widetilde{W}\widetilde{A}_1 - \widetilde{A}_1^*\widetilde{W})\big|_{t=0} = (\widetilde{W}\widetilde{A}_1 - \widetilde{A}_1^*\widetilde{W})\big|_{t=1} &= 0. \end{aligned}$$

A look at the dependence of $\widetilde{A}_2$, $\widetilde{A}_1$, $\widetilde{A}_0$ and $\widetilde{W}$ on the parameters m and n makes it clear that these equations are satisfied if one replaces m by any $\alpha \in \mathbb{R}$ and $n-1$ by any $\beta \in \mathbb{R}$. In conclusion for any $\alpha, \beta > -1$ we have exhibited a classical pair $\{\widetilde{W}, \widetilde{D}\}$.

4.2. Examples of size 4×4.

We consider representations of $\mathrm{GL}(n, \mathbb{C})$ that correspond to n-tuples of the form

$$\pi = (\underbrace{m+2, \ldots, m+2}_{k_1}, \underbrace{m+1, \ldots, m+1}_{k_2-k_1}, \underbrace{m, \ldots, m}_{n-k_2})$$

with $1 \leqslant k_1 < k_2 \leqslant n-1$. We have

$$\dim V_\pi = \frac{k_2 - k_1 + 1}{k_2 + 1}\binom{n}{k_2}\binom{n+1}{k_1}.$$

As $\mathrm{GL}(n-1, \mathbb{C})$-modules one has the decomposition

$$V_\pi = V_{\mu_1} \oplus V_{\mu_2} \oplus V_{\mu_3} \oplus V_{\mu_4},$$

where

$$\mu_1 = (\underbrace{m+2,\dots,m+2}_{k_1-1},\ m+1,\ \underbrace{m+1,\dots,m+1}_{k_2-k_1-1},\ m,\ \underbrace{m,\dots,m}_{n-k_2-1}),$$
$$\mu_2 = (\underbrace{m+2,\dots,m+2}_{k_1-1},\ m+1,\ \underbrace{m+1,\dots,m+1}_{k_2-k_1-1},\ m+1,\ \underbrace{m,\dots,m}_{n-k_2-1}),$$
$$\mu_3 = (\underbrace{m+2,\dots,m+2}_{k_1-1},\ m+2,\ \underbrace{m+1,\dots,m+1}_{k_2-k_1-1},\ m,\ \underbrace{m,\dots,m}_{n-k_2-1}),$$
$$\mu_4 = (\underbrace{m+2,\dots,m+2}_{k_1-1},\ m+2,\ \underbrace{m+1,\dots,m+1}_{k_2-k_1-1},\ m+1,\ \underbrace{m,\dots,m}_{n-k_2-1}),$$

It is important to note that

$$\begin{aligned}
\dim V_{\mu_1} &= \frac{k_2-k_1+1}{k_2}\binom{n-1}{k_2-1}\binom{n}{k_1-1},\\
\dim V_{\mu_2} &= \frac{k_2-k_1+2}{k_2+1}\binom{n-1}{k_2}\binom{n}{k_1-1},\\
\dim V_{\mu_3} &= \frac{k_2-k_1}{k_2}\binom{n-1}{k_2-1}\binom{n}{k_1},\\
\dim V_{\mu_4} &= \frac{k_2-k_1+1}{k_2+1}\binom{n-1}{k_2}\binom{n}{k_1}.
\end{aligned}$$

By using (8) and choosing the order μ_1, μ_2, μ_3, μ_4 of the subrepresentations μ's one gets the matrix valued differential operator

$$D = t(1-t)\frac{d^2}{dt^2} + \left[\begin{pmatrix} 3+m & 0 & 0 & 0\\ 0 & 2+m & 0 & 0\\ 0 & 0 & 2+m & 0\\ 0 & 0 & 0 & 1+m\end{pmatrix}\right.$$
$$\left. -t\begin{pmatrix} 3+m+n & 0 & 0 & 0\\ 0 & 2+m+n & 0 & 0\\ 0 & 0 & 2+m+n & 0\\ 0 & 0 & 0 & 1+n+m\end{pmatrix}\right]\frac{d}{dt}$$
$$+\frac{1}{1-t}\begin{pmatrix} k_2+k_1-2n & \frac{(k_2-k_1+2)(n-k_2)}{k_2-k_1+1} & \frac{(k_2-k_1)(n-k_1+1)}{k_2-k_1+1} & 0\\ 0 & -n+k_1-1 & 0 & n-k_1+1\\ 0 & 0 & -n+k_2 & n-k_2\\ 0 & 0 & 0 & 0\end{pmatrix}$$
$$+\frac{t}{1-t}\begin{pmatrix} 0 & 0 & 0 & 0\\ k_2+1 & -k_2-1 & 0 & 0\\ k_1 & 0 & -k_1 & 0\\ 0 & \frac{k_1(k_2-k_1+2)}{k_2-k_1+1} & \frac{(k_2-k_1)(k_2+1)}{k_2-k_1+1} & -k_1-k_2\end{pmatrix}$$

and the corresponding weight

$$W(t) = t^m(1-t)^{n-1}\begin{pmatrix} w_1t^2 & 0 & 0 & 0 \\ 0 & w_2t & 0 & 0 \\ 0 & 0 & w_3t & 0 \\ 0 & 0 & 0 & w_4 \end{pmatrix},$$

with $w_1 = \dim V_{\mu_1}$, $w_2 = \dim V_{\mu_2}$, $w_3 = \dim V_{\mu_3}$ and $w_4 = \dim V_{\mu_4}$. To obtain out of this a classical pair $\{\widetilde{W}, \widetilde{D}\}$ we use the conjugating function

$$\Psi^*(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & \frac{k_2-k_1+2}{k_2-k_1+1} & \frac{k_2-k_1}{k_2-k_1+1} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & 0 & 0 \\ 0 & 0 & 1-t & 0 \\ 0 & 0 & 0 & (1-t)^2 \end{pmatrix}.$$

Then we get

$$\widetilde{W}(t) = \Psi(t)W(t)\Psi^*(t)$$

for the new weight function, and the new differential operator $\widetilde{D}F = (\Psi^*)^{-1}D(\Psi^*F)$ is

$$\widetilde{D} = \widetilde{A}_2(t)\frac{d^2}{dt^2} + \widetilde{A}_1(t)\frac{d}{dt} + \widetilde{A}_0(t),$$

with $\widetilde{A}_2$, $\widetilde{A}_1$, $\widetilde{A}_0$ given by

$$\widetilde{A}_2(t) = t(1-t)$$

$$\widetilde{A}_1(t) = \begin{pmatrix} m+3 & 0 & 0 & 0 \\ -1 & m+2 & 0 & 0 \\ -1 & 0 & m+2 & 0 \\ 0 & -\frac{k_2-k_1+2}{k_2-k_1+1} & -\frac{k_2-k_1}{k_2-k_1+1} & m+1 \end{pmatrix}$$

$$- t\begin{pmatrix} (n+m+3) & 0 & 0 & 0 \\ 0 & (n+m+4) & 0 & 0 \\ 0 & 0 & (n+m+4) & 0 \\ 0 & 0 & 0 & (n+m+5) \end{pmatrix}$$

$$\widetilde{A}_0(t) = \begin{pmatrix} 0 & \frac{(k_2-k_1+2)(n-k_2)}{k_2-k_1+1} & \frac{(k_2-k_1)(n-k_1+1)}{k_2-k_1+1} & 0 \\ 0 & -(n+m+1)+k_2 & 0 & n+1-k_1 \\ 0 & 0 & -(n+m+2)+k_1 & n-k_2 \\ 0 & 0 & 0 & -2(n+m+2)+k_1+k_2 \end{pmatrix}.$$

A look at the dependence of $\widetilde{A}_2$, $\widetilde{A}_1$, $\widetilde{A}_0$ and $\widetilde{W}$ on the parameters m and n makes it clear that $\{\widetilde{W}, \widetilde{D}\}$ remains a classical pair if one replaces m by any $\alpha \in \mathbb{R}$ and $n-1$ by any $\beta \in \mathbb{R}$. In conclusion for any $\alpha, \beta > -1$ $\{\widetilde{W}, \widetilde{D}\}$ is a classical pair.

4.3. Closing remarks.

We close with a few remarks pertaining to all examples discussed so far that arise from a group theoretical situation. For concreteness we limit ourselves to Example 4.2. First we note that $\widetilde{W}(t)$ has the factorization

$$\widetilde{W}(t) = \frac{\rho(t)}{\rho(1/2)} T(t)\widetilde{W}(1/2)T^*(t)$$

with $T(1/2) = I$ and $\rho(t) = t^\alpha(1-t)^\beta$. The matrix $T(t)$, introduced in [DG1], solves the equation

$$T'(t) = \left(\frac{A^*}{t} + \frac{B^*}{t-1}\right) T(t)$$

with

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{k_1-k_2-2}{2(k_2-k_1+1)} & \frac{k_1-k_2}{2(k_2-k_1+1)} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & \frac{k_2-k_1+2}{2(k_2-k_1+1)} & \frac{k_2-k-1}{2(k_2-k_1+1)} & 2 \end{pmatrix}.$$

The matrices A, B do not depend on α nor β and satisfy

$$[A, B] = I - A - B/2$$

independently of the others free parameters k_1, k_2. A deeper understanding of commutation relations of the type given above for all the examples in [GPT1], [GPT2], [GPT3], [GPT4], [GPT5] remains an interesting challenge. Finally we turn to some monodromy type considerations. The operator $T(t)$ in Example 4.2 has a nice expression

$$T(t) = I + \sum_{i=1}^{4} \left(\sqrt{t} - \frac{1}{\sqrt{2}}\right)^i T_i$$

with very simple upper triangular T_i's. The appropriate operator $\chi(t)$, also introduced in [DG1], has in this instance a nice expression

$$\chi(t) = P + \frac{Q}{t} + \frac{R}{\sqrt{t}-1} + \frac{S}{\sqrt{t}+1}$$

with simple constant and symmetric matrices P, Q, R, S. It would be of interest to find some group representation interpretation for the matrices T_i as well as the matrices P, Q, R, S.

BIBLIOGRAPHY

[DG] J. J. Duistermaat and F. A. Grünbaum, Differential equations in the spectral parameter, Comm. Math. Phys., 103 (1986), 177–240.

[D1] A. Duran, Matrix inner product having a matrix symmetric second order differential operators, Rocky Mountain Journal of Mathematics, 27-2 (Spring 1997), 585–600.

[D2] A. Duran, On orthogonal polynomials with respect to a positive definite matrix of measures, Canadian J. Math., 47 (1995), 88–112.

[D3] A. Duran, Markov Theorem for orthogonal matrix polynomials, Canadian J. Math. 4i, 48 (1996), 1180–1195.

[D4] A. Duran, Ratio Asymptotics for orthogonal matrix polynomials, J. Approx. Th., 100 (1999), 304–344.

[DP] A. Duran and B. Polo, Gaussian quadrature formulae for matrix weights, Linear Algebra and Appl., 355 (2002), 119–146.

[DG1] A. J. Duran and F. A. Grünbaum, Orthogonal matrix polynomials satisfying second order differential equations, International Math. Research Notices, 2004 : 10 (2004), 461–484.

[DG2] A. J. Duran and F. A. Grünbaum, A characterization for a class of weight matrices with orthogonal matrix polynomials satisfying second order differential equations. To appear in Math. Research Notices.

[DvA] A. Duran and van W. Assche, Orthogonal matrix polynomials and higher order recurrence relations, Linear Algebra and Its Applications, 219 (1995), 261–280.

[Ge] J. S. Geronimo, Scattering theory and matrix orthogonal polynomials on the real line, Circuits Systems Signal Process, 1 (1982), 471–495.

[G] F. A. Grünbaum, Matrix valued Jacobi polynomials, Bull. Sciences Math., 127-3 (May 2003), 207–214.

[GPT1] F. A. Grünbaum, I. Pacharoni and J. Tirao, Matrix valued spherical functions associated to the complex projective plane, J. Functional Analysis, 188 (2002), 350–441.

[GPT2] F. A. Grünbaum, I. Pacharoni and J. Tirao, A matrix valued solution to Bochner's problem, J. Physics A: Math. Gen., 34 (2001), 10647–10656.

[GPT3] F. A. Grünbaum, I. Pacharoni and J. Tirao, Matrix valued spherical functions associated to the three dimensional hyperbolic space, Internat. J. of Mathematics, 13 (2002), 727–784.

[GPT4] F. A. Grünbaum, I. Pacharoni and J. Tirao, An invitation to matrix valued spherical functions: Linearization of products in the case of the complex projective space $P_2(\mathbb{C})$, MSRI publication, Modern Signal Processing, D. Healy and D. Rockmore, editors, 46 (2003), 147–160.

[GPT5] F. A. Grünbaum, I. Pacharoni and J. Tirao, Matrix valued orthogonal polynomials of the Jacobi type, Indag. Mathem., 14-3,4 (2003), 353–366.

[GV] R. Gangolli and V. S. Varadarajan, Harmonic analysis of spherical functions on real reductive groups, Springer-Verlag, Berlin, New York, 1988. Series title: Ergebnisse der Mathematik und ihrer Grenzgebiete, 101.

[K1] M. G. Krein, Fundamental aspects of the representation theory of Hermitian operators with deficiency index (m, m), AMS Translations, Series 2, 97, Providence, Rhode Island (1971), 75–143.

[K2] M. G. Krein, Infinite J-matrices and a matrix moment problem, Dokl. Akad. Nauk SSSR, 69-2 (1949), 125–128.

[PT] I. Pacharoni and J. Tirao, Matrix valued spherical functions associated to the complex projective space, in preparation.

[SvA] A. Sinap and van W. Assche, Orthogonal matrix polynomials and applications, J. Comput. and Applied Math., 66 (1996), 27–52.

[T1] J. Tirao, Spherical functions, Rev. de la Unión Matem. Argentina, 28 (1977), 75–98.

[T2] J. Tirao, The matrix valued hypergeometric equation, Proc. Nat. Acad. Sci. U.S.A., 100-14 (2003), 8138–8141.

[VK] N. Vilenkin and A. Klimyk, Representation of Lie Groups and Special functions, vol. 3, Kluwer Academic, Dordrecht, MA, 1992.

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